Point Preimages under Ball Non-Collapsing Mappings*

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Summary. We study three classes of Lipschitz mappings of the plane: Lipschitz quotient mappings, ball non-collapsing mappings and locally ball non-collapsing mappings. For each class, we estimate the maximum cardinality of point preimage in terms of the ratio of two characteristic constants of the mapping. For Lipschitz quotients and for Lipschitz locally BNC mappings, we provide a complete scale of such estimates, while for the intermediate class of BNC mappings the answer is not complete yet.

1. Let X and Y be metric spaces. The class of Lipschitz mappings $f\colon X \to Y$ is defined by the condition: $f(B_r(x)) \subset B_{Lr}(f(x))$ for all points x of X and all positive r (by $B_r(x)$ we denote an open ball of radius r, centered at x). Here L is a constant depending on the mapping f but not on the point x; the infimum of all possible such L is called the Lipschitz constant of f.

In a similar way, co-Lipschitz mappings $f\colon X \to Y$ are defined by the condition $f(B_r(x)) \supset B_{cr}(f(x))$, where the positive constant c is independent of x and r; the supremum of all such c is called the co-Lipschitz constant of the mapping f. (In some fundamental papers, e.g. [JLPS], the co-Lipschitz constant of the mapping is defined as infimum over all c', such that $f(B_r(x)) \supset B_{r/c'}(f(x))$.)

By definition, a Lipschitz quotient mapping is a mapping that satisfies both of the above conditions, i.e. is L-Lipschitz and c-co-Lipschitz for some constants $0 < c \le L < \infty$.

The recently developed theory of Lipschitz quotient mappings between Banach spaces raised many interesting questions about the properties of these mappings. Here we are interested in the case when X and Y are finite dimensional Banach spaces.

The paper [JLPS] contains far-reaching results for Lipschitz quotient mappings $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. In particular, it is proved there that the preimage of each point under such an f is finite. The question whether the same is true for Lipschitz quotients $f\colon \mathbb{R}^n \to \mathbb{R}^n$ for $n \ge 3$ is still open, although the following result concerning this was obtained in [M]: There is a $\rho_n < 1$ such that if the ratio of co-Lipschitz and Lipschitz constants of such a mapping is greater than ρ_n, then the mapping is one-to-one. It was also proved in [M] that the cardinality of the preimage of a point under a Lipschitz quotient mapping

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of the plane does not exceed the ratio between its Lipschitz constant L and co-Lipschitz constant c with respect to the Euclidean norm.

In section 2 of the present paper, we generalize this result to the case of arbitrary norm. One important situation is when the ratio c/L is greater than $1/2$, then the mapping is a homeomorphism. In section 3, we discuss the question whether the bound $c/L \le 1/\max_x \#f^{-1}(x)$ is tight.

In section 4, we study so-called ball non-collapsing (BNC) mappings. We say that a mapping $f\colon X \to Y$ is C-ball non-collapsing, if for any $x \in X$ and $r > 0$ one has

$$f\big(B_r(x)\big) \supset B_{Cr}(y) \qquad (*)$$

for some $y \in Y$. This property generalizes co-Lipschitzness. We will say that a mapping is C locally BNC, if for any $x \in X$ there exists $\varepsilon = \varepsilon(x) > 0$ such that $(*)$ holds for all $r \le \varepsilon$.

Note that ball non-collapsing mappings can be very far from being co-Lipschitz: e.g., the mapping $F(x,y) = (x, |y|)$ from $\mathbb{R}^2$ to itself is $1/2$ BNC, but is not co-Lipschitz (its image is not the whole plane).

The local ball non-collapsing property does not imply in general the global property, as demonstrated by another plane-folding example: $F_1(x,y) = (x, |y - [y + \frac{1}{2}]|)$, where $[t]$ stands for the integer part of t. This mapping is locally $1/2$ ball non-collapsing, but is not globally ball non-collapsing for any constant.

However, it turns out that in particular cases, the local BNC property may even imply co-Lipschitzness, though with smaller constant: it is easy to show (see Lemma 4, section 4 that if the Lipschitz constant of a Lipschitz, locally BNC mapping f is less than twice the BNC constant, then f is a Lipschitz quotient mapping. For the mappings of the plane this immediately yields finiteness of point preimages. But we obtain a stronger result. In Theorem 2 we show that such a mapping f is a bi-Lipschitz homeomorphism, that is, the preimage of each point consists of one point. On the other hand, the above example of locally BNC mapping $F_1(x,y)$ shows that as soon as the ratio of constants is less than or equal to one half, the locally BNC mapping may have infinite point preimages.

The idea of folding the plane infinitely many times has to be modified in order to construct an example of a Lipschitz globally BNC mapping of the plane with infinite point preimage. In section 5 we discuss the modified construction, but it yields the BNC constant less than (and arbitrarily close to) one third of the Lipschitz constant. Thus, we do not know exactly how large the point preimages in the global BNC case can be, when the ratio of constants is in the interval $[1/3, 1/2]$.

2. This section is devoted to Lipschitz quotient mappings. We would like to prove the following theorem, which is a generalization of a similar result in [M] to the case of arbitrary norm.

Theorem 1. *If* $f\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ *is an* L*-Lipschitz and* c*-co-Lipschitz mapping with respect to any norm* $\|\cdot\|$ *and*

$$\max_{x\in\mathbb{R}^2} \# f^{-1}(x) = n,$$

then $c/L \le 1/n$.

Proof. The proof will follow the same scheme as the proof of [M, Theorem 2]. We will only explain the details needed for the argument to work in case of arbitrary norm. We consider the decomposition $f = P \circ h$, where $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism and $P(z)$ is a polynomial of one complex variable (see [JLPS]). Clearly, $\deg P = \max_{x\in\mathbb{R}^2} \# f^{-1}(x) = n$. We may also assume that $f(0) = 0$ and $L = \mathrm{Lip}(f) = 1$.

Assume $c > 1/n$, then there exists $\varepsilon > 0$ such that $c_1 = c(1-\varepsilon) > 1/n$.

We omit the proof of the following lemma, since it would in fact repeat the proof of [M, Lemma 1]:

Lemma 1. *There exists an* R *such that for any* x *with* $\|x\| \ge R$ *one has* $\|f(x)\| \ge c_1\|x\|$. □

Let us show that for large enough r the index of the image $f(\partial B_r^{\|\cdot\|}(0))$ around zero is equal to n.

Lemma 2. *There exists* $d > 1$ *such that for any* $\rho > d$

$$\mathrm{Ind}_0\, f\big(\partial B_\rho^{\|\cdot\|}(0)\big) = \mathrm{Ind}_0\, P\Big(h\big(\partial B_\rho^{\|\cdot\|}(0)\big)\Big) = n.$$

Proof. Denote the Euclidean norm of $x \in \mathbb{R}^2$ by $|x|$. By [M, Lemma 3] there exists such σ that $\mathrm{Ind}_0\, f(\partial B_\sigma^{|\cdot|}(0)) = n$, and all preimages of zero under f lie in $B_\sigma^{|\cdot|}(0)$. Take d such that $\|x\| \ge d$ implies $|x| \ge \sigma$, and let $\rho \ge d$. Since the set $B_\rho^{\|\cdot\|}(0) \setminus B_\sigma^{|\cdot|}(0)$ does not contain preimages of zero, one has

$$\mathrm{Ind}_0\, f\big(\partial B_\rho^{\|\cdot\|}(0)\big) = \mathrm{Ind}_0\, f\big(\partial B_\sigma^{|\cdot|}(0)\big) = n.$$

□

The last lemma in the proof of Theorem 1 is rather obvious in the Euclidean case, but needs some technical work in the case of arbitrary norm and the corresponding Hausdorff measure. By the k-dimensional Hausdorff measure of a Borel set A we mean

$$\mathcal{H}_k(A) = \sup_{\delta>0} \inf \left\{ \sum_{j=1}^{\infty} (\mathrm{diam}\ C_j)^k \;\Big|\; A \subset \bigcup_{j=1}^{\infty} C_j,\ \mathrm{diam}\ C_j \le \delta \right\}$$

(cf. [F, 2.8.15]). The diameter in this definition is with respect to the metric given by the norm $\|\cdot\|$. Note that $\mathcal{H}_k$ is so normalized that the 1-Hausdorff measure of a segment $[x, y]$ is equal to $\|x - y\|$.

Lemma 3. *If $\Gamma\colon [0,1] \to \mathbb{R}^2$ is a closed curve with $\|\Gamma(t)\| \geq r$ for all $t \in [0,1]$ and $\operatorname{Ind}_0 \Gamma = n$, then the length of Γ in the sense of the 1-dimensional Hausdorff measure $\mathcal{H}_1$ is at least $n\mathcal{H}_1(\partial B_r(0))$.*

Proof. In order to prove Lemma 3, it suffices to prove it in the case $n = 1$, since a closed curve of index n can be split into n closed curves of index 1.

Note first that there exist convex polygons inscribed in the sphere $\partial B_r(0)$ with perimeter arbitrarily close to $\mathcal{H}_1(\partial B_r(0))$.

Indeed, fix positive ε and take $\delta > 0$ such that for any covering of $\partial B_r(0)$ by balls of diameters less than δ, the sum of the diameters is at least $\mathcal{H}_1(\partial B_r(0)) - \varepsilon$. Consider the family of all balls with centers on $\partial B_r(0)$ and diameters less than δ. By the Besicovitch Covering Theorem (see [F, 2.8.15]) there exists a countable subfamily of disjoint balls $\{B_i\}$, which covers almost all of $\partial B_r(0)$. Since the remaining part of $\partial B_r(0)$ is of $\mathcal{H}_1$ measure zero, it can be covered by a collection of balls with diameters less than δ and sum of diameters less than ε. Therefore, $\sum_i \operatorname{diam}(B_i) \geq \mathcal{H}_1(\partial B_r(0)) - 2\varepsilon$.

Choose m such that $\sum_{i\leq m} \operatorname{diam}(B_i) \geq \mathcal{H}_1(\partial B_r(0)) - 3\varepsilon$. The perimeter of the convex polygon whose vertices are the centers of $B_1, \ldots, B_m$ is then at least $\mathcal{H}_1(\partial B_r(0)) - 3\varepsilon$, since the balls are disjoint.

Thus it is enough to consider a convex polygon γ inside the ball $B_r(0)$, and to prove that $\mathcal{H}_1(\Gamma) \geq \mathcal{H}_1(\gamma)$.

Let us note that the $\mathcal{H}_1$-length of a planar curve is at least the $\|\cdot\|$-distance between its endpoints. This can be shown by replacing the curve by a broken line of nearly the same $\mathcal{H}_1$-length (which may be achieved by a procedure similar to inscribing a polygon in a sphere as above) and using the triangle inequality. Therefore, if we replace an arc of a curve by a straight line segment, we do not make the curve longer (this is similar to the case of Euclidean norm, except that in some norms a curve may have length equal to the distance between its endpoints even if it is not a straight line).

Successively replacing arcs of the curve Γ by straight line segments containing sides of the polygon γ, we do not increase the $\mathcal{H}_1$-length, and in a finite number of steps will replace Γ by γ. □

To conclude the proof of Theorem 1, note that 1-Lipschitz mappings do not increase the Hausdorff measure. Therefore the $\mathcal{H}_1$-length of $\Gamma = f(\partial B_\rho(0))$ cannot exceed $\mathcal{H}_1(\partial B_\rho(0))$. On the other hand, if ρ is sufficiently large, then by Lemma 2, $\operatorname{Ind}_0 \Gamma = n$, and by Lemma 1, $\|y\| \geq c_1\rho$ for any $y \in \Gamma$. So by Lemma 3 the $\mathcal{H}_1$-length of Γ is at least $nc_1\mathcal{H}_1(\partial B_\rho(0))$. Since $nc_1 > 1$, this is a contradiction which finishes the proof of the theorem. □

3. Having proved such a theorem, one would like to know if the $1/n$ bounds are precise. In the case of Euclidean norm the mappings $\phi_n(re^{i\theta}) = re^{ni\theta}$ have the ratio of constants equal to $1/n$ and maximum cardinality of a point preimage equal to n. Unfortunately, this does not immediately generalize to the case of arbitrary norm.

We are able to construct examples of such mappings in the situation when the unit ball is a regular polygon (or, of course, its affine equivalent). The ℓ_∞ norm is then a particular case of this. The idea of construction is as follows. Let V_0 be a vertex of the unit sphere $S = \{x\colon \|x\| = 1\}$. If x is a point on S, let $\arg_{\|\cdot\|}(x)$ be the length of the arc of S between V_0 and x in the counter-clockwise direction, measured by the Hausdorff measure $\mathcal{H}_1$ corresponding to the metric defined by the norm $\|\cdot\|$. We define $\psi_n(rx) = ry$, where $r \geq 0$, and y is such a point on S that $\arg_{\|\cdot\|}(y) = n \arg_{\|\cdot\|}(x)$. One easily checks that the Lipschitz constant of ψ_n is equal to n. To check that the co-Lipschitz constant is equal to 1, one may consider a local inverse of ψ_n (see Lemma 5 below) and satisfy oneself that this inverse does not increase the $\|\cdot\|$-distance.

We do not know of such examples for other norms, so despite the feeling that the converse of the theorem holds for any norm (that is, there exist mappings with maximum of n point preimages and the ratio of constants equal to $1/n$), this question remains open.

4. Now we would like to switch from Lipschitz quotient mappings to more general locally BNC mappings of $\mathbb{R}^2$ with the distance defined by an arbitrary norm $\|\cdot\|$. Our next goal will be to obtain a result which links the maximum cardinality of a point preimage to the ratio of the BNC constant C and the Lipschitz constant L of the mapping. This result, which is Theorem 2 below, deals only with the case $C/L > 1/2$. Recall that if $C/L \leq 1/2$, point preimages can be infinite (an example is given in Section 1). However, we know this only for Lipschitz, locally BNC mappings of the plane. See the next section for a discussion of the case $C/L \leq 1/2$ for Lipschitz, globally BNC mappings of $\mathbb{R}^2$.

We start with a simple lemma for BNC mappings between metric spaces.

Lemma 4. *If a mapping f between two normed spaces X and Y is L-Lipschitz and is locally C-BNC with $C/L > 1/2$ then f is $c = (2C - L)$ co-Lipschitz.*

Proof. Consider any point x and radius $R \leq \varepsilon(x)$, where $\varepsilon(x)$ is from the definition $(*)$ of local BNC property of the mapping f. There exists a point y such that $B_{CR}(y) \subset fB_R(x) \subset B_{LR}(f(x))$. Then the distance $\operatorname{dist}(y, f(x))$ does not exceed $(L-C)R < CR$. Now since $B_{CR-\operatorname{dist}(y,f(x))}(f(x))$ is contained in $B_{CR}(y)$, we conclude that the mapping f is locally $C - (L - C) = (2C - L)$ co-Lipschitz. This implies that f is globally $(2C-L)$ co-Lipschitz. For a proof that local co-Lipschitzness at every point implies global co-Lipschitzness see, for example, [C, Section 4]. □

We proved in Theorem 1 that for an L-Lipschitz and c-co-Lipschitz mapping from the plane to itself, the cardinality of a point preimage is not greater than L/c. We thus have a

Corollary. *If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is L-Lipschitz and C locally BNC with $C/L > 1/2$ then*

$$\max_{x\in\mathbb{R}^2} \# f^{-1}(x) \le \tfrac{L}{2C-L}.$$

The bound on the right blows up when C/L is larger than but close to $1/2$. Our aim now is to improve the bound to the best possible one, that is, to prove that a C locally BNC and L-Lipschitz mapping with $C/L > 1/2$ is in fact a homeomorphism, i.e. the preimage of each point is a single point.

We will need several lemmas.

Lemma 5 (Local invertibility of a Lipschitz quotient mapping). *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz quotient mapping. There exists a finite subset $\mathcal{A}$ of $\mathbb{R}^2$ such that if Ω is a connected simply connected open domain which does not intersect with $\mathcal{A}$, then for any point x such that $y = f(x) \in \Omega$ there exists a mapping $\phi = \phi_{x,y}\colon \Omega \to \mathbb{R}^2$ which satisfies $\phi(y) = x$ and $f \circ \phi = Id_\Omega$. This mapping ϕ is open and is locally $1/c$-Lipschitz, where c is the co-Lipschitz constant of f.*

Proof. By [JLPS] any such f is a composition $P \circ h$ of a polynomial P with a homeomorphism h. Let $\mathcal{A}$ be the finite set $\{P(z) \mid P'(z) = 0\}$. If Ω is a connected simply connected open domain which does not intersect with $\mathcal{A}$, then the polynomial P has a unique inverse, which is an analytic function p defined on Ω such that $p(y) = h(x)$. Define $\phi = h^{-1} \circ p$. It is clear that $\phi(y) = x$ and $f \circ \phi = \mathrm{Id}_\Omega$.

Since ϕ is a composition of a homeomorphism h^{-1} and an analytic function p, whose derivative $p'(\omega) = \frac{1}{P'(p(\omega))}$ is nonzero, we conclude that ϕ is open.

Suppose $\omega \in \Omega$ and $r > 0$ is so small that $B_{cr}(\omega) \subset \Omega$ and $B_r(\phi(\omega)) \subset \phi(\Omega)$. Then co-Lipschitzness of f implies that $\phi B_{cr}(\omega) \subset B_r(\phi(\omega))$, so ϕ is locally c^{-1}-Lipschitz, where c is the co-Lipschitz constant of f. $\square$

Lemma 6. *Assume that a mapping f between two finite dimensional normed spaces X and Y is C locally BNC and is differentiable at a point a. Then for any $\epsilon > 0$ there exists $r = r(\epsilon, a)$ such that $fB_\rho(a) \supset B_{(C-\epsilon)\rho}(f(a))$ for $\rho \le r$.*

Proof. Let $d_a f$ be the differential of f at a, so that $f(a+h) = f(a) + (d_a f)h + o(h)$. We will show now that $(d_a f)B_1(0) \supset B_C(0)$. Then for every $\epsilon > 0$ one can find r such that $\|o(h)\| < \epsilon\|h\|$ for $\|h\| \le r$. It follows that for $\rho \le r$ the image $fB_\rho(a)$ contains the ball centered at $f(a)$ of radius $C\rho - \epsilon\rho = \rho(C-\epsilon)$.

Assume $C_1 = \min_{\|x\|=1} \|d_a f(x)\| < C$. Then $(d_a f)B_1(0) \not\supset B_{C_1(1+\epsilon)}(0)$ for every $\epsilon > 0$ (thus, in particular, $(d_a f)B_1(0) \not\supset B_C(0)$).

It follows that $(d_a f)B_1(0) \not\supset B_{C_1(1+\epsilon)}(x)$ for any $x \in (d_a f)B_1(0)$ and $\epsilon > 0$. Indeed, assuming $(d_a f)B_1(0) \supset B_R(x)$ one gets $(d_a f)B_1(0) \supset -B_R(x) = B_R(-x)$ and thus

$$(d_a f)B_1(0) \supset \mathrm{conv}\big(B_R(x), B_R(-x)\big) \supset B_R(0).$$

Take r such that $\|o(h)\| < \frac{C-C_1}{2}\|h\|$ for $\|h\| \le r$. Then for any $\rho \le r$ one has

$$fB_\rho(a) \subset \Sigma = f(a) + \rho(d_a f)B_1(0) + B_{\rho\frac{C-C_1}{2}}(0).$$

The latter does not contain a ball of radius greater than $\frac{C+C_1}{2}\rho$ (the proof of this uses that $(d_a f)B_1(0)$ is convex), and in particular we conclude that Σ (and therefore $fB_\rho(a)$) does not contain a ball of radius $C\rho$, in contradiction to the local C-BNC property of f. □

In what follows we will assume that $f(0) = 0$.

The next key lemma is an analogue of Lemma 1 for Lipschitz quotient mappings, but in the case of BNC mappings the proof becomes technically more complicated.

Lemma 7. *If a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is L-Lipschitz and is locally C-BNC with $C/L > 1/2$ and $f(0) = 0$, then for any $C' < C$ there exists $R > 0$ such that $\|f(x)\| \ge C'\|x\|$ for any $\|x\| \ge R$. Consequently, $fB_r(0) \supset B_{C'r}(0)$ for all $r \ge R$.*

Proof. Assume $L = 1$, set $M = 1 + \max_{f(z)=0} \|z\|$ and consider $R = 4M/(C - C')$. Assume that there exists a point x_0 such that $\|x_0\| = r \ge R$ and $\|f(x_0)\| < C'r$. There exists $\varepsilon > 0$ such that for all $y \in U(x_0, \varepsilon) = \{y \colon \|y\| = \|x_0\| \text{ and } \|y - x_0\| < \varepsilon\}$ one has $\|f(y)\| < C'r$.

Note that there exists $x_1 \in U(x_0, \varepsilon)$ and $\varepsilon' > 0$ such that $U(x_1, \varepsilon') \subset U(x_0, \varepsilon)$ and

$$\Omega = \cup_{y \in U(x_1, \varepsilon')} \big(0, 2f(y)\big)$$

is such a domain as was described in Lemma 5 (i.e., Ω does not contain $P(z)$ such that $P'(z) = 0$). Here $(0, a)$ is the straight line interval between 0 and a in $\mathbb{R}^2$. Let $\phi = \phi_{x_1, f(x_1)} : \Omega \to \mathbb{R}^2$ be the mapping from Lemma 5. Note that $\phi(\Omega)$, being open, contains an open neighbourhood of x_1, so there exists $\varepsilon_1 : 0 < \varepsilon_1 < \varepsilon'$, such that $U(x_1, \varepsilon_1) \subset \phi(\Omega)$. Then $\phi f(y) = y$ for any $y \in U(x_1, \varepsilon_1)$, since $\phi|_\Omega$ is a 1-1 mapping.

Since ϕ is locally Lipschitz, and is defined in an open cone, $\phi(0)$ is also well-defined.

In what follows, we are going to use both the Lebesgue measure $\mathcal{L}_k$ and the Hausdorff measure $\mathcal{H}_k$ for $k = 1, 2$. Recall that in $\mathbb{R}^k$ the measure $\mathcal{L}_k$ coincides with $\mathcal{H}_k$ on Borel sets. But the measure $\mathcal{H}_k$ is defined also in spaces of dimension different from k; if ψ is a Lipschitz mapping and A is such a set that $\mathcal{H}_k(A) = 0$, then $\mathcal{H}_k(\psi(A)) = 0$. In particular, if A is a Borel set in $\mathbb{R}^k$ such that $\mathcal{L}_k(A) = 0$, and $\psi \colon \mathbb{R}^k \to \mathbb{R}^k$ is Lipschitz, then $\mathcal{L}_k(\psi(A)) = 0$.

We know that f is $\mathcal{L}_2$-almost everywhere differentiable on $\phi(\Omega)$. Let $\mathcal{D} = \{t \in \phi(\Omega) \mid f \text{ is differentiable at } t\}$. Since $\mathcal{H}_2(\phi(\Omega) \setminus \mathcal{D}) = 0$ and f is Lipschitz, we conclude that the set $\Omega \setminus f(\mathcal{D})$ is also of $\mathcal{L}_2$ measure zero. Then by Fubini's theorem there exists a point y in $U(x_1, \varepsilon_1)$, such that almost every point of the interval $(0, 2f(y))$ with respect to $\mathcal{L}_1$ measure is in $f(\mathcal{D})$. Now consider the restriction of ϕ onto the segment $[0, f(y)]$. This restriction is a Lipschitz mapping from $[0, f(y)]$ to $\mathbb{R}^2$; therefore $\mathcal{H}_1$-almost every point of the curve $\gamma = \phi([0, f(y)])$ is in $\mathcal{D}$, that is f is $\mathcal{H}_1$-almost everywhere differentiable on γ. Let $\mathcal{B} = \mathcal{D} \cap \gamma$ be the set of points on γ where f is differentiable.

Since $\frac{C+C'}{2} < C$, by Lemma 6 for each differentiability point $z \in \mathcal{B}$ there exists $r_z > 0$ such that $fB_\rho(z) \supset B_{\rho(C+C')/2}(f(z))$ for any $\rho \le r_z$.

Let $\mathcal{H}_1(\gamma)$ be the 1-Hausdorff measure of γ. There exists $\tau > 0$ such that if almost all of γ is covered by balls of diameter at most τ, then the sum of diameters of the balls is at least $\mathcal{H}_1(\gamma) - \frac{M}{2}$ (we defined M in the beginning of the proof). Without loss of generality we may assume that $\tau < M/2$.

Consider $\mathcal{F} = \{B_\rho(z) \mid z \in \mathcal{B},\ \rho \le \min\{r_z, \tau/2\}\}$. By the Besicovitch Covering Theorem (see [F, 2.8.15]) there exists a countable disjoint subcollection $\mathcal{F}_0$ of $\mathcal{F}$, which covers almost all of $\mathcal{B}$, therefore almost all of γ, with respect to the measure $\mathcal{H}_1$. Then

$$\sum_{B \in \mathcal{F}_0} \text{diam } B \ge \mathcal{H}_1(\gamma) - \frac{M}{2}.$$

On the other hand the f-image of each ball $B \in \mathcal{F}_0$ contains a ball with center on $[0, f(y)]$ and of radius $r(B)\frac{C+C'}{2}$. Note that $\mathcal{F}_1 = \{B_{\rho(C+C')/2}(f(z)) \mid B_\rho(z) \in \mathcal{F}_0\}$ is a family of nonintersecting balls with centers on the interval $[0, f(y)]$, therefore

$$\frac{C+C'}{2} \sum_{B \in \mathcal{F}_0} \text{diam } B = \sum_{B \in \mathcal{F}_1} \text{diam } B \le \|f(y)\| + \tau \frac{C+C'}{2}.$$

Thus

$$\|f(y)\| \ge \left(\mathcal{H}_1(\gamma) - \frac{M}{2}\right)\frac{C+C'}{2} - \tau\frac{C+C'}{2} \ge \big(\mathcal{H}_1(\gamma) - M\big)\frac{C+C'}{2}.$$

Note also that $\mathcal{H}_1(\gamma) \ge \|y\| - \|\phi(0)\| \ge r - M$ (see the explanation in the proof of Lemma 3), so $\|f(y)\| \ge (r - 2M)\frac{C+C'}{2}$. But we assumed that $\|f(y)\| < C'r$, so one gets

$$C'r > \frac{C+C'}{2}r - 2M,$$

or, equivalently, $2M > \frac{C-C'}{2}r$, which contradicts $r \ge R = \frac{4M}{C-C'}$. $\square$

Theorem 2. *Let $\mathbb{R}^2$ be equipped with an arbitrary norm $\|\cdot\|$. If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an L-Lipschitz and C locally ball non-collapsing mapping with $C/L > 1/2$, then*

$$\# f^{-1}(x) = 1$$

for any point $x \in \mathbb{R}^2$.

Proof. By Lemma 4, such a mapping f is a Lipschitz quotient mapping. Let $n = \max_{x\in\mathbb{R}^2} \# f^{-1}(x)$. We may assume $f(0) = 0$.

Fix any C', such that $L/2 < C' < C$. Then by Lemma 7 there exists R such that $\|f(x)\| \geq C'\|x\|$ for all $\|x\| \geq R$. By Lemma 2, there exists $r > R$ such that $|\operatorname{Ind}_0 f(\partial B_r(0))| = n$.

Then by Lemma 4 the $\mathcal{H}_1$-length of $f(\partial B_r(0))$ is at least $nC'\mathcal{H}_1(\partial B_r(0))$, which is strictly greater than $\frac{nL}{2}\mathcal{H}_1(\partial B_r(0))$. But since f is L-Lipschitz, the length of $f(\partial B_r(0))$ is at most $L\mathcal{H}_1(\partial B_r(0))$. Hence $\frac{nL}{2} < L$, therefore $n = 1$. This finishes the proof of the theorem. □

5. The last question we would like to discuss here is what happens when a globally BNC mapping has a ratio of constants less than or equal to $1/2$. The plane folding example, $F(x, y) = (x, |y|)$, where $C/L = 1/2$, shows that such a mapping neither has to be co-Lipschitz, nor is necessarily 1-1. However, the mapping in this example has point preimages of finite maximum cardinality 2.

On the other hand a mapping with ratio C/L less than $1/3$ may have infinite point preimages. An example to this end is the following. For an interval $I = [a, b]$ in $\mathbb{R}^1$ define the "hat function" $h_I(x)$ by $\frac{b-a}{2} - |x - \frac{a+b}{2}|$. Now let the mapping $\zeta_A : \mathbb{R}^1 \to \mathbb{R}^1$, where $A > 1$, be defined by

$$\zeta_A(x) = \begin{cases} x, & \text{if } x \leq 0, \\ (-1)^k h_{[A^{-k}, A^{-k+1}]}(x), & \text{if } A^{-k} \leq x \leq A^{-k+1},\ k \text{ a positive integer}, \\ x - 1, & \text{if } x > 1. \end{cases}$$

Obviously, ζ_A is a 1-Lipschitz function. One can check that ζ_A is BNC with constant $C = \frac{1-A^{-2}}{3-A^{-2}}$. Then the function $f(x, y) = (x, \zeta_A(y))$ is a Lipschitz and BNC mapping of the plane, with infinite point preimages, and the ratio of constants less than but arbitrarily close to $1/3$ (at least with respect to a norm $\|\cdot\|$ for which $\|(x, y)\| = \|(x, -y)\|$).

Note that a point preimage under a Lipschitz BNC mapping may even be uncountable. For example, if

$$E = [0, 1] \setminus \bigcup_{k,n\geq 0} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

is a Cantor set on $[0, 1]$, the mapping $g(x) = \operatorname{dist}(x, E)$ is 1-Lipschitz and is globally BNC, whose zeros set is E.

We also have a proof that in 1-dimensional space the bound of $1/3$ cannot be improved (that is, if a Lipschitz and BNC mapping has infinite point preimages, then the ratio of constants C/L is strictly less than $1/3$). Thus, we have no definite results concerning point preimages under Lipschitz globally BNC mappings of the plane whose ratio of constants is between $1/3$ and $1/2$.

Let us summarize the results concerning the estimates of the maximum cardinality of the preimage of a point under the three classes of Lipschitz mappings of the plane. Let L be the Lipschitz constant of a mapping. If a mapping is Lipschitz quotient with co-Lipschitz constant c, the preimage of a point consists of at most L/c points. If a mapping is (globally) BNC with BNC constant C, then in the case $C/L > 1/2$ a point preimage is a single point, in the case $C/L < 1/3$ it can be infinite, and in the case $1/3 \leq C/L \leq 1/2$ we have no definite answer. And if a mapping is locally BNC with BNC constant C, the complete answer is as follows. If $C/L > 1/2$, a point preimage is a single point, and in the case $C/L \leq 1/2$ a point preimage can be infinite.

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