LIPSCHITZ QUOTIENT MAPPINGS WITH GOOD RATIO OF CONSTANTS

OLGA MALEVA
DEPARTMENT OF MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE

ABSTRACT. We show that Lipschitz quotient mappings between finite dimensional spaces behave nicely (e.g. are bijective in the case of equal dimensions) if the Lipschitz and co-Lipschitz constants are close to each other. For Lipschitz quotient mappings of the plane, a bound for the cardinality of the preimage of a point in terms of the ratio of the constants is obtained.

Let X and Y be metric spaces. The class of Lipschitz mappings $f: X \to Y$ is defined by the condition: $f(B_r(x)) \subset B_{Lr}(f(x))$ for all points x of X and all positive r (by $B_r(x)$ we denote an open ball of radius r, centered at x). Here L is a constant depending on the mapping f but not on the point x; the infimum of all possible such L is called the Lipschitz constant of f.

In a similar way, co-Lipschitz mappings $f: X \to Y$ are defined by the condition $f(B_r(x)) \supset B_{cr}(f(x))$, where the positive constant c is independent of x and r; the supremum of all such c is called the co-Lipschitz constant of the mapping f.

By definition, Lipschitz quotient mapping is a mapping that satisfies both of the above conditions, i.e. is L-Lipschitz and c-co-Lipschitz for some constants $0 < c \leq L < \infty$.

The recently developed theory of Lipschitz quotient mappings between Banach spaces raised many interesting questions about properties of these mapping. Here we are interested in the case when X and Y are finite dimensional Banach spaces.

Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz quotient mapping. It is immediate that the existence of such mapping implies that $m \geq n$, and f is necessarily surjective. What else can be said of f? Are the properties of f similar to those of linear quotient mappings?

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The paper [JLPS] contains far-reaching results for Lipschitz quotient mappings $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. In particular, it is proved there that the preimage of each point under such an f is finite. The question whether the same is true for Lipschitz quotients $f\colon \mathbb{R}^n \to \mathbb{R}^n$ for $n \geq 3$ is still open. On the other hand, there are irregular examples of Lipschitz quotients between different dimensions: as was shown in [C], there exists a Lipschitz quotient mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that the inverse image of zero contains a 2-dimensional plane.

In this paper we try to approach regularity properties of Lipschitz quotient mappings which depend on the ratio c/L of the co-Lipschitz and Lipschitz constants of the mapping, where the domain and range $\mathbb{R}^n$ are equipped with two general norms. This ratio is obviously not greater than 1, and it is not hard to prove that if $c/L = 1$ then $f\colon (\mathbb{R}^n, \|\cdot\|) \to (\mathbb{R}^n, |\!|\!|\cdot|\!|\!|)$ is affine (see Corollary below).

The idea is to look what happens if the two constants are close to each other. It turns out that in this case Lipschitz quotient mapping behaves in a regular way.

In the proof of the first theorem, we use the notion of n-dimensional Hausdorff measure:

$$\mathcal{H}_n(A) = \sup_{\delta>0} \inf \Big\{ \sum_{j=1}^{\infty} \big(\tfrac{\operatorname{diam} C_j}{2}\big)^n \mid A \subset \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \leq \delta \Big\}.$$

Of course, the diameter in this definition is with respect to the metric given by the norm. Note that $\mathcal{H}_n$ is so normalized that the measure of a unit ball is equal to 1.

Theorem 1. For each n there exists a universal constant $0 < \rho_n < 1$ such that if $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are two norms on $\mathbb{R}^n$ and $f\colon (\mathbb{R}^n, \|\cdot\|) \to (\mathbb{R}^n, |\!|\!|\cdot|\!|\!|)$ is an L-Lipschitz and c-co-Lipschitz mapping and $\rho_n < c/L \leq 1$, then the preimage of a point under f is a single point (so that f is a bi-Lipschitz homeomorphism). The constant ρ_n does not depend on the norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$.

Proof. Since we may rescale the mapping, multiplying it by a constant, we may assume that the Lipschitz constant L is equal to 1.

Assume that $f(x) = f(y)$ with $\|x - y\| = R > 0$. Let $r = R/(1 + \frac{1}{c})$ and $z = \frac{c}{c+1}x + \frac{1}{c+1}y$. Consider the open ball $B_R(z)$. We claim that the image of

$B_R(z) \setminus B_r(x)$ coincides with the image of the whole $B_R(z)$, and therefore contains the ball $B_{cR}(f(z))$.

Indeed, $B_R(z) \setminus B_r(x)$ contains $B_{r/c}(y)$, and $f(B_{r/c}(y))$ contains $B_r(f(y))$ which is the same as $B_r(f(x))$, and contains $f(B_r(x))$; thus $f(B_r(x))$ is already contained in $f(B_R(z) \setminus B_r(x))$. This implies that $f(B_R(z) \setminus B_r(x))$ is equal to $f(B_R(z))$.

Recall that a 1-Lipschitz function does not increase the n-dimensional Hausdorff measure of a set (which can be easily seen from the definition of $\mathcal{H}_n$). Applying this to the set $B_R(z) \setminus B_r(x)$, whose image contains a ball of radius cR, we get that $R^n - r^n \geq (cR)^n$. This is equivalent to $\psi_n(c) = (c+1)^n - c^n - c^n(c+1)^n \geq 0$.

Thus, if f is non-injective then $\psi_n(c) \geq 0$. But $\psi_n(1) = -1$, so there exists $\rho_n < 1$ such that $\psi_n(c) < 0$ for $\rho_n < c \leq 1$. □

Corollary. *Let* $\|\cdot\|$ *and* $|\!|\!|\cdot|\!|\!|$ *be two norms on* $\mathbb{R}^n$ *and let* $f\colon (\mathbb{R}^n, \|\cdot\|) \to (\mathbb{R}^n, |\!|\!|\cdot|\!|\!|)$ *be a Lipschitz quotient mapping whose Lipschitz and co-Lipschitz constants are equal. Then* f *is an affine isometry up to a constant factor; in particular, the norms* $\|\cdot\|$ *and* $|\!|\!|\cdot|\!|\!|$ *are essentially the same.*

Proof. Consider a ball $B_r(x)$ and its image $B_{Lr}(f(x))$. The image of each interior point of $B_r(x)$ is an interior point of $B_{Lr}(f(x))$, so $f(\partial B_r(x)) \supset \partial B_{Lr}(f(x))$. Since the ratio c/L is equal to 1, the mapping f, by the theorem above, is one-to-one. The inverse mapping f^{-1} is also a Lipschitz quotient whose Lipschitz and co-Lipschitz constants are equal, so $f^{-1}(\partial B_{Lr}(f(x))) \supset \partial B_r(x)$. Therefore, $f(\partial B_r(x)) = \partial B_{Lr}(f(x))$.

This means that $|\!|\!|f(x) - f(y)|\!|\!| = L\|x - y\|$ for any x and y, so f/L is an isometry, and by a classical theorem of Mazur and Ulam [MU], f is affine. □

The proof of Theorem 1 raises several questions already in the case when $n = 2$ and $\|\cdot\| = |\!|\!|\cdot|\!|\!|$ is the Euclidean norm. As we mentioned earlier, it was proved in this case that for any ratio c/L of the co-Lipschitz and Lipschitz constants of the mapping, the preimage of a point is finite. Now the proof of the theorem above yields some constant $\rho_2 \approx 0.839287$ such that for $c/L > \rho_2$ the mapping is a homeomorphism. On the other hand, the basic examples of non-bijective Lipschitz quotients of the plane are $f_n(re^{i\theta}) = re^{ni\theta}$, $n \geq 2$ with ratios of constants equal to $1/n$, so the maximal value is one half.

Question 1. Is it true that if the ratio of the co-Lipschitz constant and the Lipschitz constant of a Lipschitz quotient mapping from the plane to itself is greater than 0.5, then the mapping is a homeomorphism, i.e. the preimage of each point is a single point?

Question 2. Is there a scale $0 < \cdots < \rho_2^{(n)} < \cdots < \rho_2^{(1)} = \rho_2 < 1$ such that for any Lipschitz quotient mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ the condition $c/L > \rho_2^{(n)}$ implies $\#f^{-1}(x) \leq n$ for any $x \in \mathbb{R}^2$?

It turns out that the answers to both these questions are positive. The next theorem asserts that the appropriate values of the scale are $\rho_2^{(n)} = 1/(n+1)$, so in particular $\rho_2 = \rho_2^{(1)} = 1/2$.

Theorem 2. If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an L-Lipschitz and c-co-Lipschitz mapping with respect to the Euclidean norm and

$$\max_{x \in \mathbb{R}^2} \#f^{-1}(x) = n,$$

then $c/L \leq 1/n$.

Proof. Without loss of generality we may assume that $f(0) = 0$ and $L = \mathrm{Lip}(f) = 1$. By [JLPS] there exist a homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ and a polynomial $P(z)$ of one complex variable, such that $f = P \circ h$. Clearly, $\deg P = \max_{x \in \mathbb{R}^2} \#f^{-1}(x) = n$. If $n = 1$ then the statement is obvious. Assume $n \geq 2$.

Changing h by a transformation of the form $h \to ah + b$, we may assume that $h(0) = 0$ and the leading coefficient a_n of $P(z)$ is 1. Then $P(0) = f(0) = 0$ and $P(z)$ has the form $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z$.

We consider $\mathbb{R}^2$ as the complex plane, and use the notation $|x|$ for the norm of $x \in \mathbb{R}^2$.

Let $\{z_1 = 0, z_2 \ldots, z_k\}$ be the set of preimages of zero under f, denote $M = \max_{1 \leq i \leq k} |z_i|$. Assume $c > 1/n$, then there exists $\varepsilon > 0$ such that $c_1 = c(1-\varepsilon) > 1/n$.

Lemma 1. *There exists an R such that for any x with $|x| \geq R$ one has $|f(x)| \geq c_1|x|$.*

Proof. Take $R > M/\varepsilon$, then for any $r \geq R$ one has $r - M > r(1-\varepsilon)$. Fix any point x with $|x| \geq R$. Then 0 belongs to $\overline{B}_{|f(x)|}(f(x))$, which, by the co-Lipschitz property, is a subset of $f(\overline{B}_{|f(x)|/c}(x))$. This implies that there exists a preimage z_i of zero, such that $z_i \in \overline{B}_{|f(x)|/c}(x)$. Then $|z_i - x| \leq |f(x)|/c$, so $|f(x)| \geq c(|x| - |z_i|) \geq c(|x| - M) > c|x|(1-\varepsilon) = c_1|x|$. □

Let us show that for large enough r the index of the image $f(\partial B_r(0))$ around zero is equal to n.

Lemma 2. *For any $r > 1$ there exists $r' > r$ such that $|h(x)| > r$ for all $|x| = r'$.*

Proof. Take R from Lemma 1, let $r' = \max\{R, r, \frac{1}{c_1}\sum_{k=1}^{n}|a_k|r^k\} + 1$ and suppose that $|h(x)| \leq r$ for some $|x| = r'$. Then $|f(x)| = |P(h(x))| \leq \sum_{k=1}^{n}|a_k||h(x)|^k \leq \sum_{k=1}^{n}|a_k|r^k < c_1 r'$ – this is a contradiction with Lemma 1. By definition, $r' > r$. □

Lemma 3. *For any $d > 1$ there exists $\rho > d$ such that*

$$\mathrm{Ind}_0 f(\partial B_\rho(0)) = \mathrm{Ind}_0 P(h(\partial B_\rho(0))) = n.$$

Proof. Take R_1 such that $\sum_{k=1}^{n-1}\frac{|a_k|}{r^{n-k}} < 0.5$ for any $r \geq R_1$. Let $r = \max(d, R_1)$ and $\rho = r' > r$ from Lemma 2. Then $|h(x)| > r$ for $|x| = \rho$. Therefore, if the curve γ is $h(\partial B_\rho(0))$, then for any $z \in \gamma$ one has $P(z) = (z^n + |z|^n(\sum_{k=1}^{n-1}\frac{a_k z^k}{|z|^n})) = z^n + |z|^n\phi(z)$ with $|\phi(z)| \leq \sum_{k=1}^{n-1}\frac{|a_k|}{|z|^{n-k}} < 0.5$, since $|z| > r \geq R_1$. This implies $\mathrm{Ind}_0 P(\gamma) = \mathrm{Ind}_0\limits_{z\in\gamma} z^n = n$, since $\mathrm{Ind}_0\gamma = 1$. □

It is enough to note, as we shall in Lemma 4, that the length of $f(\partial B_\rho(0))$ for such $\rho > R$, where R is from Lemma 1, is at least $n \cdot 2\pi c_1 \rho > 2\pi\rho$. This is a contradiction, since 1-Lipschitz mappings do not increase the length. This finishes the proof of the theorem. □

Lemma 4. *If $\gamma\colon [0,1] \to \mathbb{R}^2$ is a closed curve with $|\gamma(t)| \geq r$ for all $t \in [0,1]$ and $\mathrm{Ind}_0\gamma = n$, then the length of γ is at least $2\pi rn$.*

Proof. Fix any $\varepsilon > 0$ and consider a regular polygon $\mathbb{A} = A_0A_1\ldots A_{m-1}$, centered at zero, of perimeter at least $2\pi r - \varepsilon$, inscribed in the circle of radius r.

We consider $\mathbb{R}^2$ as a complex plane. For simplicity assume that $\gamma(0)$ and A_0 lie on the positive real semiaxis, so that $A_0 = r$ and $\mathrm{Ind}_0(A_0A_1\ldots A_{m-1}A_0) = 1$. Let

$0 = t_0 < t_1 < \cdots < t_{mn} = 1$ be a set of points $t \in [0, 1]$ such that $\gamma(t_j)$ lies on the ray $\mathbb{R}^+ A_{j \mod m}$ (in other words, $\arg \gamma(t_j) = 2\pi j/m$). Consider a closed broken line $\gamma' = \gamma(t_0)\gamma(t_1)\ldots\gamma(t_{mn})$. Note that $\mathrm{length}(\gamma') \leq \mathrm{length}(\gamma)$ and $\mathrm{Ind}_0 \gamma' = n$. Also $|\gamma(t_j) - \gamma(t_{j+1})| \geq |A_0 - A_1|$ for all j. This means that $\mathrm{length}(\gamma') \geq nm|A_0 - A_1| \geq n(2\pi r - \varepsilon)$. Since this holds for arbitrary ε, we get $\mathrm{length}(\gamma) \geq 2\pi rn$. $\square$

Recently, M. Csörnyei [C] constructed an example of a Lipschitz quotient mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that the preimage of zero contains a 2-dimensional plane. It is natural to conjecture that such phenomena cannot occur if the Lipschitz and co-Lipschitz constants are close to each other.

The rest of this paper will be devoted to the case of different dimensions. We will consider only the case of the standard Euclidean norm, using the notion of orthogonal projection.

Lemma 5. *For a finite set $\mathcal{A}$, let $\rho(\mathcal{A})$ be the minimal Euclidean distance between two different points of $\mathcal{A}$. Let $d_{N,M}$ be the maximum of $\rho(\mathcal{A})$ over all N-point subsets $\mathcal{A}$ of $\overline{B}_1^M(0)$, the Euclidean unit ball of $\mathbb{R}^M$. Then*

1. *if $\mathcal{A} \subset \overline{B}_1^n(0)$, $|\mathcal{A}| = n+1$, then $\rho(\mathcal{A}) = d_{n+1,n}$ if and only if $\mathcal{A}$ is the set of vertices of an equilateral n-dimensional simplex Δ_{n+1} inscribed in the $(n-1)$-dimensional sphere $S_1^{n-1}(0) = \partial B_1^n(0)$;*
2. *$d_{n+1,n} > d_{n+1,n-1}$.*

Proof. 1. Let us show that $\rho(\Delta_{n+1}) = d_{n+1,n}$. Take any set $\mathcal{A} = \{a_1, \ldots, a_{n+1}\} \subset \overline{B}_1^n(0)$ such that $\rho(\mathcal{A}) = d_{n+1,n}$. Then by definition $d_{n+1,n}^2 = \min_{i \neq j} \|a_i - a_j\|^2 \leq 2 - 2\max_{i \neq j}\langle a_i, a_j \rangle$, since $\|a_i\| \leq 1$ for all i. But

$$\begin{aligned} \max_{i \neq j}\langle a_i, a_j \rangle \geq \operatorname*{Ave}_{i \neq j}\langle a_i, a_j \rangle &= \tfrac{1}{n(n+1)}(\|a_1 + \cdots + a_{n+1}\|^2 - \sum_{i=1}^{n+1}\|a_i\|^2) \\ &\geq -\tfrac{\sum_{i=1}^{n+1}\|a_i\|^2}{n(n+1)} \geq -1/n. \end{aligned}$$

It follows that $d_{n+1,n}^2 \leq 2 + 2/n = \rho(\Delta_{n+1})^2 \leq d_{n+1,n}^2$, so $\rho(\Delta_{n+1}) = d_{n+1,n}$. Moreover, if $\rho(\mathcal{A}) = \rho(\Delta_{n+1})$, then $\langle a_i, a_j \rangle$ must equal $-1/n$ for all $i \neq j$, and thus $\|a_i\| = 1$ for all i (otherwise $\min_{i \neq j}\|a_i - a_j\|^2 < 2 + 2/n$). This means that $\mathcal{A}$ is an equilateral n-dimensional simplex whose vertices lie on the unit sphere.

2. It is enough to note that a subset of $\overline{B}_1^{n-1}(0)$ can be regarded as a subset of $\overline{B}_1^n(0)$, and as such, it is not an equilateral n-simplex. $\square$

Theorem 3. There exists a universal constant $0 < \sigma_n < 1$ such that if $f\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ is L-Lipschitz and c-co-Lipschitz mapping with $\sigma_n < c/L \le 1$, then the preimage of a point under f cannot contain an $(m+1)$-dimensional surface having at least one point with tangent $(m+1)$-plane.

Proof. We again assume that $L = 1$. Let us prove first that the preimage of a point under f cannot contain an $(m+1)$-dimensional ball. Assume the opposite is true: $f(\overline{B}_R^{m+1}(0)) = 0$. Let $B = \overline{B}_R^{n+m}(0)$ and $? = \partial f(B)$. Denote by π the projection of $\mathbb{R}^{n+m}$ on the first $m+1$ coordinates (so that $\pi B = \overline{B}_R^{m+1}(0)$), let $\widetilde{\pi} = I - \pi$.

Since for any interior point x of B its image is an interior point of $f(B)$, the preimage of a point of $?$ lies on the sphere $S_R^{n+m-1}(0)$. Note that $\|y\| \ge cR$ for any point $y \in ? \subset \mathbb{R}^n$, since $f(B) \supset \overline{B}_{cR}^n(0)$. But $\|f(x)\| < cR$ if $\|\widetilde{\pi}x\| < cR$ for $\|x\| \le R$, so we get that $?$ is contained in $f(\widetilde{S})$, where

$$\begin{aligned}\widetilde{S} &= \{x \in \mathbb{R}^{n+m} \colon \|x\| = R \text{ and } \|\widetilde{\pi}x\| \ge cR\}\\ &= \{x \in \mathbb{R}^{n+m} \colon \|x\| = R \text{ and } \|\pi x\|^2 \le (1-c^2)R^2\}.\end{aligned}$$

Let us use the notation $d_{N,M}$ from Lemma 5. Note that $?$ contains $n+1$ points with pairwise distances at least $cRd_{n+1,n}$ ($?$ is the boundary of a set which contains $\overline{B}_{cR}^n(0)$; consider an equilateral n-simplex inscribed in $S_{cR}^{n-1}(0)$ and take the intersections of the rays from zero through the vertices of the simplex with $?$). Let $A_1, \ldots, A_{n+1}$ be their preimages in $\widetilde{S}$. Since f is 1-Lipschitz, we conclude that $\|A_i - A_j\| \ge cd_{n+1,n}R$ for $i \ne j$.

Furthermore, $\|A_i - A_j\|^2 = \|\pi A_i - \pi A_j\|^2 + \|\widetilde{\pi}A_i - \widetilde{\pi}A_j\|^2$. Since $A_k \in \widetilde{S}$, the first summand is not greater than $4(1-c^2)R^2$. Consider $\{\widetilde{\pi}A_k\}$ as $n+1$ points in the $(n-1)$-dimensional ball of radius R. Then $\min_{i\ne j} \|\widetilde{\pi}A_i - \widetilde{\pi}A_j\| \le d_{n+1,n-1}R$.

Thus

$$c^2 d_{n+1,n}^2 R^2 \le \min_{i\ne j} \|A_i - A_j\|^2 \le 4(1-c^2)R^2 + d_{n+1,n-1}^2 R^2,$$

which implies

$$c^2 \le \tfrac{4+d_{n+1,n-1}^2}{4+d_{n+1,n}^2}.$$

If we put $\sigma_n = \sqrt{\frac{4+d^2_{n+1,n-1}}{4+d^2_{n+1,n}}}$, then $\sigma_n < 1$ by Lemma 5 and for $c > \sigma_n$ we get a contradiction.

Now assume that $f(T) = 0$ where T is an $(m+1)$-dimensional surface with tangent $(m+1)$-plane L at the point $u \in T$. We may assume that L is a plane spanned by the first $(m+1)$ basis vectors, $L = Ox_1 \dots x_{m+1}$. If $c > \sigma_n$ there exists $\varepsilon > 0$ such that $c - \varepsilon > \sigma_n$. For this fixed ε there exists $R > 0$ such that for each point v in $\overline{B}^{n+m}_R(u) \cap L$ there is a point t on T with $\|v - t\| \le \varepsilon R$. Consider a ball $B = \overline{B}^{n+m}_R(u)$ and as before denote by π the projection of $\mathbb{R}^{n+m}$ on L (so that $\pi B = \overline{B}^{n+m}_R(u) \cap L = \overline{B}^{m+1}_R(u)$), $\widetilde{\pi} = I - \pi$. If $\|\widetilde{\pi} x\| < (c - \varepsilon)R$, then $\|f(x)\| < cR$, so $\partial f(B)$ is contained in $f(\widetilde{S})$, where $\widetilde{S} = \{x \colon \|x\| = R \text{ and } \|\pi x - u\|^2 \le (1 - (c-\varepsilon)^2)R^2\}$.

Let the points A_i be as before, then

$$c^2 d^2_{n+1,n} R^2 \le \min_{i \neq j} \|A_i - A_j\|^2 \le 4(1 - (c-\varepsilon)^2)R^2 + d^2_{n+1,n-1} R^2.$$

This implies $4 + d^2_{n+1,n-1} \ge c^2 d^2_{n+1,n} + 4(c-\varepsilon)^2 > \sigma_n^2 d^2_{n+1,n} + 4\sigma_n^2 = 4 + d^2_{n+1,n-1}$ by the definition of the constant σ_n. This contradiction finishes the proof. □

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Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel,
email:maleva@wisdom.weizmann.ac.il