The probability that the hyperbolic random graph is connected

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March 6, 2014

Abstract

This work is a study of a family of random geometric graphs on the hyperbolic plane. In this setting, N points are chosen randomly on the hyperbolic plane and any two of them are joined by an edge if they are within hyperbolic distance R. The value of R is suitably chosen so that the disc in which the N points are chosen has volume that is proportional to N^2 and is further controlled by a parameter ν that scales the average degree. The N points are distributed according to a *quasi-uniform* distribution, which is a distorted version of the uniform distribution and is controlled by a parameter α – when $\alpha = 1$ this coincides with the uniform distribution. This model was introduced recently by Krioukov et al. [13] and for certain values of α it exhibits basic properties of complex networks such as power law degree distribution as well as clustering exist.

The present paper focuses on the probability of connectivity of the graph, which is determined by the parameters α and ν. We show that for $\alpha > \frac{1}{2}$, the graph is not connected with high probability. When α crosses $\frac{1}{2}$ the graph exhibits connectivity with high probability. The value $1/2$ is the connectivity threshold. When $\alpha = \frac{1}{2}$ we give the critical value $\nu_0 = \pi$ for the parameter ν such that when $\nu \geq \nu_0$ then the random graph is connected with high probability. For $\nu < \nu_0$ we show that the probability for connectivity is bounded away from zero and one and we determine it precisely.

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1 Introduction

The theory of geometric random graphs was initiated by Gilbert [9] already in 1961 in the context of what is called *continuum percolation*. In 1972, Hafner [11] focused on the typical properties of large but finite random geometric graphs. Here N points are sampled within a certain region of $\mathbb{R}^d$ following a certain distribution and any two of them are joined when their Euclidean distance is smaller than some threshold which, in general, is a function of N. In the last two decades, this kind of random graphs was studied extensively – see the monograph of Penrose [15] and the references therein. Numerous typical properties of such random graphs have been investigated, such as the chromatic number [14] or Hamiltonicity [2].

However, what structural characteristics emerge when one considers these points distributed on a curved space where distances are measured through some (non-Euclidean) metric? This setting has been considered in the context of percolation theory by Benjamini and Schramm [3]. Already there, the change of the underlying geometry gives rise to a radically different behaviour compared to that of similar models which are set on the Euclidean plane.

A finite-scale model on the hyperbolic plane was introduced only recently by Krioukov et al. in [13]. The aim of that work was the development of a geometric framework for the analysis of properties of the so-called complex networks. This term summarizes a large class of networks that emerge in a range of human activities which includes social networks, scientific collaborator networks as well as computer networks, such as the Internet, and the power grid – see for example [1]. These are networks that mainly consist of a very large number of heterogeneous nodes (nowadays social networks such as the Facebook or the Twitter have billions of users), and they are locally very sparse. This means that the number of neighbours of a typical node (also called the degree of the node) is much smaller than the total number of nodes in the network. However, this is not the case for all nodes. In fact, there are nodes that have a number of neighbours that is much larger than that of a typical node. These are the *hubs* of the network and in fact most of the typical nodes are within a small distance from them, keeping the distance between most pairs of nodes small. This phenomenon has been known as the *small world effect*. The existence of hubs is made possible by the distribution of the degrees. Measurements on several examples of networks suggest that this follows a *power law* – see [1] and the references therein. That is, the fraction of nodes of degree k scales like $k^{-\gamma}$, where γ is the exponent of the power law and in most cases this has been measured to be less than 3.

The basic hypothesis of Krioukov et al. [13] is that the hyperbolic geometry underlies these networks. In particular, the heterogeneity of the nodes which is expressed through the power law degree distribution is in fact the expression of an underlying hyperbolic geometry. They defined the associated random graph model, which we will describe in detail shortly, and analysed some of their typical properties through methods which are largely heuristic. More specifically, they considered their degree distribution suggesting a power law as well as clustering properties. Clustering is also a typical feature of complex networks, which is related to the density of the neighbourhoods of the nodes. These characteristics have been verified rigorously by Gugelmann et al. [10] as well as by the second author [8] and Candellero and the second author [5]. (for a "smooth" version of the model that we will describe).

In this work, we want to focus on the connectivity of such a graph. In particular, we give ranges of the parameters that ensure connectivity. The connectivity depends primarily on the distribution of the vertices, and the parameter controlling the average degree is crucial only in the critical case. Similar to the classical Erdős-Rényi model (see [4]) for random graphs and the standard model for random graphs on Euclidean spaces (see [15]), we are able to prove that connectivity is given whenever there are no isolated vertices in the graph. In the critical case for the distribution, we give a range for the parameter controlling the average degree where the probability of connectivity is bounded away from one and zero, a behaviour uncommon to the other random graphs.

The original model of Krioukov et al. [13] involves a parameter ζ that controls the curvature of the space. We prove that this parameter is not necessary, the parameters ν and α suffice to yield the same degrees of freedom for the model.

1.1 Random geometric graphs on the hyperbolic plane

The most common representations of the hyperbolic plane are the upper-half plane model $\{z \,:\, \Im z > 0\}$ and the Poincaré disc model which is simply the open disc of radius one, that is, $\{(u,v) \in \mathbb{R}^2 \,:\, 1-u^2-v^2>0\}$. Both spaces are equipped with the hyperbolic metric; in the former case this is determined by the differential form $\frac{1}{y^2}dy^2$ whereas in the latter case by the differential form $4\,\frac{du^2+dv^2}{(1-u^2-v^2)^2}$.

It can be shown that the (Gaussian) curvature in both cases is equal to -1 and the two spaces are isometric, that is, there exists a bijection between the two spaces which preserves (hyperbolic) distances. In fact, there are more representations of the 2-dimensional hyperbolic space of curvature -1 which are isometrically equivalent to the above two. We will denote by $\mathbb{H}$ the class of these spaces.

In this paper, following the definitions in [13], we shall be using the native representation of $\mathbb{H}$. Under this representation, the ground space of $\mathbb{H}$ is $\mathbb{R}^2$ and every point $x \in \mathbb{R}^2$ whose polar coordinates are (r,θ) has hyperbolic distance from the origin O equal to r. Also, a circle of radius r around the origin has length equal to $2\pi\sinh(r)$ and area equal to $2\pi(\cosh(r)-1)$.

Let $N = \nu e^{R/2}$, where ν is a positive real number, which controls the average degree of the random graph. We create a random graph by selecting randomly N points from the disc of radius R centred at the origin O, which we denote by $\mathcal{D}_R$, according to the following probability distribution. If the random point u has polar coordinates (r,θ), then θ, r are independent, θ is uniformly distributed in $(0,2\pi]$ and the probability distribution of r has density function given by:

$$\rho(r) = \begin{cases} \alpha\frac{\sinh \alpha r}{\cosh \alpha R - 1} & \text{if } 0 \le r \le R, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Here $\alpha > 0$ is a parameter. When $\alpha = 1$, then we simply obtain the uniform distribution on $\mathcal{D}_R$. This set of points will be the vertex set of the random graph and we denote it by V_N. The random graph $\mathcal{G}(N;\alpha,\nu)$ is formed when we join two vertices, if they are within (hyperbolic) distance R. Another way to derive this distribution is to consider a disc of radius R on a hyperbolic plane of curvature $-\alpha^2$, generate N vertices uniformly at random and then project them to $\mathcal{D}_R$, preserving the polar coordinates.

The distribution of angles and radii is exactly the uniform distribution on a disc of radius R in a space with curvature $-\alpha^2$. The parameter α thus plays the role of the "growth factor" of the distribution. Since this is an exponential growth away from the centre, vertices are always much more likely to lie close to the periphery than close to the centre of the disc. The difference in distribution is that, with a smaller α, the ratio of the central density to the peripheral density gets bigger. Thus we can expect vertices to generally be closer to the centre the smaller α is.

Krioukov et al. define this model on the hyperbolic plane of curvature $-\zeta^2$ defining the radius of $\mathcal{D}_R$ through the relation $N = \nu e^{\zeta R/2}$ – in this case the random graphs is denoted by $\mathcal{G}(N;\zeta,\alpha,\nu)$. All their results are stated in terms of the ratio ζ/α. However, the following lemma, which we prove in the appendix, shows that this is not necessary.

Lemma 1.1 *Suppose that $\zeta/\alpha = \zeta'/\alpha'$. For every ν, there exists a coupling between $\mathcal{G}(N;\zeta,\alpha,\nu)$ and $\mathcal{G}(N;\zeta',\alpha',\nu)$ such that*

$$G(N;\zeta,\alpha,\nu) = G(N;\zeta',\alpha',\nu).$$

Krioukov et al. [13] focus on the degree distribution of $\mathcal{G}(N;\alpha,\nu)$, showing that when $\alpha > \frac{1}{2}$ this follows a power law with exponent $2\alpha+1$. They also discuss clustering on a smooth version of the above model. Their results have been verified rigorously by Gugelmann et al. [10]. Note that when $\alpha = 1$, that is, when the N vertices are uniformly distributed in $\mathcal{D}_R$, the exponent of the power law is equal to 3. When $\frac{1}{2} < \alpha < 1$, the exponent is between 2 and 3, as is the case in a number of networks that emerge in applications such as several computer networks, social networks and biological networks (see for example [1]). They have also shown that the average degree of the random graph can be "tuned" through the parameter ν.

1.2 Connectivity of $\mathcal{G}(N;\alpha,\nu)$

This paper focuses on the connectivity of $\mathcal{G}(N;\alpha,\nu)$. In their seminal paper on random graphs, Erdős and Rényi [6] find the threshold for connectivity in a uniformly chosen graph on N vertices and M edges, usually denoted by $G(N,M)$. It turns out that the critical number of edges for the connectivity in an N vertex graph is $(N/2)\ln(N)$. More precisely, if for some $\varepsilon > 0$ we have $\frac{M}{N} < \frac{1-\varepsilon}{2}\ln(N)$, then $G(N,M)$ is not connected with probability $1-o(1)$, while for $\frac{M}{N} > \frac{1+\varepsilon}{2}\ln(N)$, we have that $G(N,M)$ is connected with probability $1-o(1)$. In fact, much more is true. If we add edges into an N vertex graph one by one, with probability $1-o(1)$ the edge that removes the last isolated vertex is precisely the one that makes the graph connected. See [4] for a details on the connectivity in random graphs.

For random geometric graphs on the Euclidean plane there are similar results (see [15]). Consider a random graph that is formed by placing N vertices in the 2-dimensional unit cube uniformly at random. Then two vertices are connected, if their Euclidean distance is at most r_N, a parameter chosen for the model. It turns out that the threshold function for connectivity is $r_c(N) = (\ln(N)/N)^{1/2}$. If r_N is asymptotically less than $r_c(N)$ then with probability $1-o(1)$ the graph is disconnected. On the other hand, if it is asymptotically greater than $r_c(N)$ then with probability $1-o(1)$ the graph is connected. Moreover, as in the Erdős-Rényi model, the threshold for connectivity coincides with the threshold for having isolated vertices. See the monograph of [15] for more detailed results on the connectivity in random geometric graphs.

In this paper we deal with the probability that the random graph $\mathcal{G}(N;\alpha,\nu)$ is connected. A straightforward consequence of the results on the degree sequence [10], is that the graph is a.a.s. disconnected whenever $\alpha > \frac{1}{2}$ (since there will then be vertices of degree zero). As it happens, the probability that the graph is connected becomes bounded away from zero exactly when α crosses $\frac{1}{2}$. Let us also remark that the case $\alpha \leq \frac{1}{2}$ is a "dense" regime in the sense that the average degree of $\mathcal{G}(N;\alpha,\nu)$ is no longer constant, but grows with N.

Theorem 1.2 *Let $\alpha, \nu > 0$ be arbitrary. Then the following hold*

(i) *If $\alpha > \frac{1}{2}$ then $G(N; \alpha, \nu)$ is a.a.s. disconnected;*

(ii) *If $\alpha < \frac{1}{2}$ then $G(N; \alpha, \nu)$ is a.a.s. connected.*

(iii) *If $\alpha = \frac{1}{2}$ then*

$$\lim_{N\to\infty} \mathbb{P}(G(N; \alpha, \nu) \text{ is connected }) = f(\nu),$$

where $f : (0, \infty) \to (0, 1]$ is a continuous function satisfying **(a)** *$f(\nu) = 1$ for all $\nu \geq \pi$;* **(b)** *$f(\nu)$ is strictly increasing for $0 < \nu < \pi$; and* **(c)** *$\lim_{\nu\downarrow 0} f(\nu) = 0$.*

The connectivity of $G(N; \alpha, \nu)$ is determined by the configuration nearby the centre of $\mathcal{D}_R$, to be precise only vertices of constant radius are relevant. A very small central configuration of vertices (indeed, a single vertex in the very centre of the disc will suffice) can already cover the whole disc, thus yielding a diameter of at most three. For $\alpha < \frac{1}{2}$, such a configuration exists with high probability. For $\alpha = \frac{1}{2}$, the probability of such a configuration depends on ν. If ν is large enough the probability becomes $1 - o(1)$, while otherwise it is some constant between 0 and 1 that becomes smaller the smaller ν is. If vertices within the disc of radius $\frac{R}{2}$ do not cover the whole disc, then with probability $1 - o(1)$ there will be an isolated vertex close to the periphery of the disc. Thus for $\alpha > \frac{1}{2}$, where with high probability no covering configuration exists, the probability of being connected is $o(1)$.

Note the curious behaviour in part **(iii)**. In particular the limiting probability of connectedness is bounded away from zero and one for all $0 < \nu < \pi$, while it equals one for $\nu \geq \pi$. We are not aware of any similar results in the literature.

2 Proof of Theorem 1.2: parts (i) and (ii)

Part **(i)** is a direct corollary of Theorem 2.2 in [10], since this theorem implies that when $\alpha > \frac{1}{2}$ there are isolated vertices a.a.s.

For Part **(ii)** of Theorem 1.2 we argue as follows. We would like to show that if $0 < \alpha < 1/2$, then for any $\nu > 0$ the random graph $G(N; \alpha, \nu)$ is connected with probability $1 - o(1)$ as $N \to \infty$. Let $|x|_{2\pi} = \min\{|x|, 2\pi - |x|\}$ for $-2\pi \leq x \leq 2\pi$ and consider a point $v_1 = (r_1, \vartheta_1)$ with $r_1 \leq \varepsilon$ and a point $v_2 = (R, \vartheta_2)$. The hyperbolic cosine rule implies that if

$$|\vartheta_1 - \vartheta_2|_{2\pi} < \arccos\left(\frac{\cosh(r_1)\cosh(R) - \cosh(R)}{\sinh(r_1)\sinh(R)}\right),$$

then the hyperbolic distance between v_1 and v_2 is at most R. The function on the right-hand side of the above is decreasing in r_1, whereby it follows that if

$$|\vartheta_1 - \vartheta_2|_{2\pi} < \arccos\left(\frac{\cosh(\varepsilon)\cosh(R) - \cosh(R)}{\sinh(\varepsilon)\sinh(R)}\right),$$

then the hyperbolic distance between v_1 and v_2 is at most R. In turn, if ε is small we have

$$\arccos\left(\frac{\cosh(\varepsilon)\cosh(R) - \cosh(R)}{\sinh(\varepsilon)\sinh(R)}\right) = \arccos\left((1 - o(1)\varepsilon)\right) > \pi/2 - \delta,$$

for some $\delta \in (0, \pi/6)$ and N large enough. Thus, for any such N, if

$$|\vartheta_1 - \vartheta_2|_{2\pi} < \pi/2 - \delta,$$

then the hyperbolic distance between v_1 and v_2 is at most R. In turn, this implies that if the relative angle between points v_1 and v_2 is at most $\pi/2 - \delta$, then there are within hyperbolic distance R. In other words, the set $D_{v_1} = \{v_2 \ : \ v_2 = (R, \vartheta_2),\ |\vartheta_1 - \vartheta_2|_{2\pi} < \pi/2 - \delta\}$ is contained in the disc of radius R around v_1.

Consider now three domains in the ball of radius ε around O (which we denote by $B_{\mathbb{H}}(O, \varepsilon)$), namely $D_i = \{v \ : \ v = (r, \vartheta), r \le \varepsilon, \vartheta \in (2\pi i/3 - \delta, 2\pi i/3 + \delta)\}$, for $i = 1, 2, 3$. Note that if each one of them contains at least one vertex, v_1, v_2, v_3, respectively, then $\mathcal{D}_R \subseteq D_{v_1} \cup D_{v_2} \cup D_{v_3}$.

For any $\varepsilon > 0$, the probability that a given vertex belongs to $B_{\mathbb{H}}(O, \varepsilon)$ is

$$\alpha \int_0^{\varepsilon} \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1} dr = \Theta\left(N^{-2\alpha}\right).$$

Hence, using a Chernoff-type bound it follows that there exists a $c = c(\varepsilon) > 0$ such that with probability $1 - o(1)$, as $N \to \infty$, the number of vertices in $B_{\mathbb{H}}(O, \varepsilon)$ is at least $cN^{1-2\alpha}$. As the angles of these points are uniformly distributed over $(0, 2\pi]$, this implies that with probability $1 - o(1)$ D_i in fact contains at least $c'N^{1-2\alpha}$ vertices, for some c'. Thereby, any point in $\mathcal{D}_R$ is within distance R from a point in D_i, for $i = 1, 2, 3$. Hence, $G(N; \alpha, \nu)$ is connected and, in fact, its diameter is at most 3.

3 The critical case

In this section, we consider an auxiliary random process that is closely related to the hyperbolic random geometric graph with $\alpha = 1/2$. In the rest of the paper, $\mathcal{P} = \mathcal{P}_\nu$ will be a Poisson process on the *entire* hyperbolic plane with intensity function:

$$g(r, \theta) = g_\nu(r, \theta) := (\nu/4\pi) \cdot \sinh(r/2), \tag{2}$$

where (r, θ) represents a point of $\mathbb{H}$ under the native model. We let $\mathbb{P}_\nu(.)$ denote the associated probability measure. $\mathbb{E}_\nu(.)$ denotes the expected values of random variables over the probability space. We say that an event $E(N)$ is realized with high probability (w.h.p.), if $\mathbb{P}_\nu(E(N)) \to 1$ as $n \to \infty$.

We set

$$\gamma(r) = \gamma_\lambda(r) := \lambda \cdot \arccos\left(\frac{\cosh(r) - 1}{\sinh(r)}\right), \tag{3}$$

where $\lambda > 0$ is a parameter. We will see in the proof of Lemma 3.3 that if two points $x_1 = (r_1, \theta_1), x_2 = (r_2, \theta_2) \in \mathcal{D}_R$ have $|\theta_1 - \theta_2|_{2\pi} \le \gamma(r_1)$ with $\lambda = 1 - \delta$, then x_1 and x_2 are within distance R (provided N is large). Let us remark that $\gamma(r)$ is strictly decreasing in r. (This can be easily seen from the facts that arccos(.) is strictly decreasing and that $(\cosh(r) - 1)/\sinh(r) = 1 - \frac{2}{e^r + 1}$ is strictly increasing.) Let us say that an angle $\vartheta \in [0, 2\pi)$ is *covered* by a point $(r, \theta) \in \mathbb{H}$ if

$$|\vartheta - \theta|_{2\pi} \le \gamma_\lambda(r).$$

We say that a set $A \subseteq \mathbb{H}$ is a *cover* if every angle is covered by some point of A. For $s > 0$, we denote by $\mathcal{C}_s(\lambda)$ the event that $\mathcal{P} \cap B_{\mathbb{H}}(O, s)$ is a cover. The event $\mathcal{C}(\lambda)$ will denote that $\mathcal{C}_s(\lambda)$ is realized for some (finite) $s < \infty$. Note that $\mathcal{C}(\lambda) = \bigcup_{s>0} \mathcal{C}_s(\lambda)$. We now define:

$$\Psi(\nu, \lambda) := \mathbb{P}_\nu(\mathcal{C}(\lambda)). \tag{4}$$

As we will see, $f(\nu) := \Psi(\nu, 1)$ has the properties claimed in Theorem 1.2 and the probability that $G(N; 1/2, \nu)$ is connected tends to $f(\nu)$ as $N \to \infty$. The following theorem, which is proved later, is crucial for the proof of the main theorem.

Theorem 3.1 *The function Ψ defined in* (4) *has the following properties:*

(i) $\Psi(\nu, \lambda)$ *is continuous in both parameters;*

(ii) $\Psi(\nu, \lambda) = 1$ *if* $\nu \cdot \lambda \geq \pi$*;*

(iii) $\Psi(\nu, \lambda)$ *is strictly increasing in* ν *for* $0 < \nu < \pi/\lambda$*;*

(iv) *For every fixed* $\lambda > 0$ *we have* $\lim_{\nu \downarrow 0} \Psi(\nu, \lambda) = 0$.

We will prove Theorem 3.1 in section 5. The following lemmas use this theorem to prove the main result.

Lemma 3.2 *Let $\mathcal{P} = \mathcal{P}_\nu$ be as defined earlier. For every $\varepsilon > 0$ there is a coupling such that $\mathcal{P}_{\nu-\varepsilon} \cap B_{\mathbb{H}}(O; R) \subseteq V_N \subseteq \mathcal{P}_{\nu+\varepsilon} \cap B_{\mathbb{H}}(O; R)$ w.h.p. as $N \to \infty$.*

Proof: Let $X_1, X_2, \ldots$ be an infinite supply of i.i.d. points distributed as in (1). Then we can set $V = \{X_1, \ldots, X_N\}$. Now let $Z_1 \stackrel{d}{=} \mathrm{Po}((1-\delta)N), Z_2 \stackrel{d}{=} \mathrm{Po}((1+\delta)N)$ and set $V_i := \{X_1, \ldots, X_{Z_i}\}$ for $i = 1, 2$. It follows from the Chernoff bound that

$$\mathbb{P}_\nu(Z_1 \leq N \leq Z_2) = 1 - o(1).$$

Put differently, this proves that $V_1 \subseteq V_N \subseteq V_2$ w.h.p.

Now observe that V_1 is a Poisson process with intensity function:

$$\begin{aligned} h_1(r,\theta) &= (1-\delta)N \cdot (1/2\pi) \cdot \tfrac{(1/2)\cdot\sinh(r/2)}{\cosh(R/2)-1} \cdot 1_{\{r \leq R\}} \\ &= (1-\delta)\nu e^{R/2} \cdot (1/2\pi) \cdot \tfrac{(1/2)\cdot\sinh(r/2)}{\cosh(R/2)-1} \cdot 1_{\{r \leq R\}} \\ &= (1-\delta+o(1)) \cdot (\nu/4\pi) \cdot \sinh(r/2) \cdot 1_{\{r \leq R\}}. \end{aligned}$$

So, provided we chose $\delta = \delta(\varepsilon)$ sufficiently small, we have $h_1(r,\theta) \geq g_{\nu-\varepsilon}(r,\theta) 1_{\{r \leq R\}}$ for all r, θ if N is sufficiently large (where g is the density of $\mathcal{P}$ defined in (2)). Similarly the density h_2 of V_2 satisfies $h_2 \leq g_{\nu+\varepsilon} 1_{\{r \leq R\}}$ for N sufficiently large. The statement follows. ∎

Lemma 3.3 *For every $\nu > 0$ we have* $\liminf_{N\to\infty} \mathbb{P}(G(N; 1/2, \nu)$ *is connected* $) \geq \Psi(\nu, 1)$.

Proof: Let us pick a $\delta > 0$ such that $\Psi(\nu - \delta, 1 - \delta) > \Psi(\nu, 1) - \varepsilon/3$. For convenience we write $\mu := \nu - \delta, \lambda := 1 - \delta$. Next, let us pick $s > 0$ such that $\mathbb{P}_\mu(\mathcal{C}_s(\lambda)) \geq \Psi(\mu, \lambda) - \varepsilon/3$. This is possible as $\mathcal{C}_s \subseteq \mathcal{C}_{s'}$ for $s < s'$, so $\mathbb{P}_\mu(\mathcal{C}_s(\lambda))$ is a monotone increasing function with limit $\mathbb{P}_\mu(\mathcal{C}(\lambda))\Psi(\mu, \lambda)$. Let us consider the coupling from the previous lemma. Taking N suffiently large, we can assume that the probability that it fails is at most $\varepsilon/3$ and that $s < R/2$. (Recall that $R = R(N)$ depends on and is growing with N.)

We claim that, if $\mathcal{C}_s(\lambda)$ occurs with respect to μ, and the coupling succeeds (i.e. $\mathcal{P}_\mu \cap B_{\mathbb{H}}(O,R) \subseteq V_N$), then the graph $G(N;1/2,\nu)$ will be connected. To see this suppose that $\mathcal{C}_s(\lambda)$ occurs with respect to μ, and pick an arbitrary point $X_i = (\rho_i,\theta_i) \in V_N$. There is some point $X_j = (\rho_j,\theta_j) \in V_N$ with $\rho_j \le s$ such that $|\rho_i - \rho_j|_{2\pi} < \gamma(\rho_i) = \lambda \cdot \arccos\left(\frac{\cosh(\rho_i)-1}{\sinh(\rho_i)}\right)$. We claim that X_i and X_j must have distance less than R. To see this, note first that we are done when $\rho_i \le R/2$ (using as $\rho_j \le s < R/2$ and the triangle inequality). By the hyperbolic cosine rule we have that the distance between X_i and X_j is less than R if and only if

$$|\theta_i - \theta_j|_{2\pi} < \arccos\left(\frac{\cosh(\rho_i)\cosh(\rho_j) - \cosh(R)}{\sinh(\rho_i)\sinh(\rho_j)}\right).$$

Now notice that

$$\begin{aligned}\arccos\left(\tfrac{\cosh(\rho_i)\cosh(\rho_j)-\cosh(R)}{\sinh(\rho_i)\sinh(\rho_j)}\right) &\le \arccos\left(\tfrac{\cosh(\rho_i)\cosh(\rho_j)-\cosh(\rho_i)}{\sinh(\rho_i)\sinh(\rho_j)}\right)\\ &= \arccos\left(\tfrac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right).\end{aligned}$$

Recall that $(\cosh(r)-1)/\sinh(r) = 1 - 2e^{-r} + o(e^{-r})$ and note that $\cosh(\rho_i)/\sinh(\rho_i) = 1 + O(e^{-2\rho_i}) = 1 + O(e^{-R})$. Using Taylor's expansion $\arccos(x+y) = \arccos(x) - y/\sqrt{1-x^2} + O(xy^2/(1-x^2)^{3/2})$, we see that

$$\begin{aligned}\arccos\left(\tfrac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) &= \arccos\left(\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)} + O(e^{-R})\right)\\ &= \arccos\left(\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) + O(e^{\rho_j - R}).\end{aligned}$$

Before we continue, it will be helpful to derive some asymptotics. Observe that

$$\frac{\cosh(r)-1}{\sinh(r)} = 1 - 2e^{-r}\left(\frac{1-e^{-r}}{1-e^{-2r}}\right). \tag{5}$$

Recall that $\cos(y) = 1 - y^2/2 + O(y^4)$. This implies that if $y = \arccos(1-x)$ then $y = \sqrt{2x}\cdot\big(1+O(x^2)\big)$. Combining this with (5) gives:

$$\gamma(r) = \lambda\cdot\arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right) = 2\lambda e^{-r/2}(1+O(e^{-r})) \quad \text{as } r\to\infty. \tag{6}$$

Using these equations, we find that

$$\arccos\left(\frac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) = (1+o(1))\cdot\arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right).$$

Since $|\theta_i - \theta_j|_{2\pi} \le \gamma(\rho_j) = (1-\delta)\cdot\arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right)$, we do find that X_i, X_j have distance at most R (for N sufficiently large).

This shows that, provided $\mathcal{C}_s(\lambda)$ occurs with respect to μ and the coupling succeeds (i.e. $\mathcal{P}_\mu \cap B_{\mathbb{H}}(O,R) \subseteq V_N$), then every vertex of $G(N;1/2,\nu)$ will be at distance less that R from some vertex of radius $< R/2$. So the graph will have diameter at most three, and in particular it will be connected. That is, we have shown

$$\liminf_{N\to\infty} \mathbb{P}(G(N;1/2,\nu) \text{ is connected }) \ge \mathbb{P}_\mu(\mathcal{C}_s(\lambda)) - \mathbb{P}(\text{the coupling fails}) \ge \Psi(\nu,1) - \varepsilon.$$

Sending ε to zero proves the lemma. ■

The following lemma completes the proof of the limiting value of the probability of connectivity.

Lemma 3.4 *For every $\nu > 0$ we have $\limsup_{N\to\infty} \mathbb{P}(G(N; 1/2, \nu)$ is connected $) \leq \Psi(\nu, 1)$.*

The proof of this lemma needs a few technical results which will be developed in the following section and we postpone it until these have been established.

Also Theorem 3.1 implies the properties of $f(\nu)$ as these are described in Theorem 1.2**(iii)**.

4 Preliminary results: multitype Galton-Watson processes

We shall make use of a classical result on Galton-Watson branching processes with finitely many types. If there are $t < \infty$ types, then such a process is described by a sequence $Z_0, Z_1, \ldots$ of random vectors, where $Z_n := (Z_n^1, \ldots, Z_n^t)$ denotes the vector of the number of particles (individuals) of each type in the n-th generation. In each generation, each of the particles replaces itself with a random set of "children", independently of all other particles and the previous history of the process and always according to the same probability distribution (which typically depends on the type of the particle). We denote

$$p(i; z_1, \ldots, z_t) := \mathbb{P}_\nu(Z_1 = (z_1, \ldots, z_t)|Z_0 = e_i).$$

That is, $p(i; z_1, \ldots, z_t)$ is the probability that a particle of type i fathers z_1 children of type 1, z_2 children of type of type 2, and so on until type t. We will say that "extinction" occurs if $Z_n = (0, \ldots, 0)$ for some n. Otherwise we say "survival" occurs.

We also set

$$m_{ij} := \mathbb{E}(Z_1^j|Z_0 = e_i).$$

That is, m_{ij} is equal to the expected number of children of type j of a particle of type i; and we write $M := (m_{ij})_{1\leq i,j\leq t}$ for the "matrix of first moments". Let us also remark that, for every $k \in \mathbb{N}$ and $1 \leq i, j \leq t$ we have $(M^k)_{ij} = \mathbb{E}(Z_k^j|Z_0 = e_i)$ (the expected number of type-j particles in the k-th generation if we start with a single particle of type i). We say that the process is *positive regular* if there exists a $k \in \mathbb{N}$ such that every entry of M^k is positive. By the Perron-Frobenius theorem a positive regular matrix has a real, positive eigenvalue ρ that is larger in absolute value than all other eigenvalues (see for instance [12], Chapter II, Section 5, page 37). A multitype Galton-Watson process is called *singular* if each particle has exactly one child (with probability one). Otherwise it is *nonsingular*.

A proof of the following standard result can for instance be found in the book by Harris [12] (Theorem 7.1, Chapter II, page 41), who attributes it to Sevast'yanov [16] and independently Everett and Ulam [7].

Theorem 4.1 *Consider a positive regular, non-singular multitype Galton-Watson process with finitely many types, and let ρ denote the largest eigenvalue of its first moment matrix M. Then the following hold:*

(i) *If $\rho \leq 1$ then $\mathbb{P}_\nu(extinction|Z_0 = e_i) = 1$ for all types $1 \leq i \leq t$;*

(ii) *If $\rho > 1$ then $\mathbb{P}_\nu(extinction|Z_0 = e_i) < 1$ for all types $1 \le i \le t$.*

If $Z_0, Z_1, \ldots$ is as in Theorem 4.1 and ρ is the largest eigenvalue of M then we say the process is *subcritical* if $\rho < 1$, we say it is *critical* if $\rho = 1$ and we say it is *supercritical* if $\rho > 1$.

The following straightforward observation will be used in the sequel. For completeness we spell out a short proof.

Lemma 4.2 *Suppose that $Z_0, Z_1, \ldots$ is a positive regular, nonsingular, supercritical Galton Watson process with $t < \infty$ types. Then there exists another t-type Galton-Watson process $Y_0, Y_1, \ldots$ such that*

- **(i)** *$p_Y(i; z_1, \ldots, z_t) = 0$ if $p_Z(i; z_1, \ldots, z_t) = 0$;*
- **(ii)** *$p_Y(i; z_1, \ldots, z_t) < p_Z(i; z_1, \ldots, z_t)$ if $p_Z(i; z_1, \ldots, z_t) > 0$ and $(z_1, \ldots, z_t) \ne (0, \ldots, 0)$;*

and Y is positive regular, nonsingular and supercritical.

Proof: Let us fix a $0 < \delta < 1$, to be made specific later, and let us define the offspring distributions of Y by:

$$p_Y(i; z_1, \ldots, z_t) = \begin{cases} (1-\delta) \cdot p_Z(i; z_1, \ldots, z_t) & \text{if } (z_1, \ldots, z_t) \ne (0, \ldots, 0), \\ p_Z(i; 0, \ldots, 0) + \delta \cdot (1 - p_Z(i; 0, \ldots, 0)) & \text{if } (z_1, \ldots, z_t) = (0, \ldots, 0). \end{cases} .$$

It is easy to see that this way Y is nonsingular and that $m_{ij}^Y = (1-\delta) m_{ij}^Z$. So in particular Y is also positive regular, and the largest eigenvalue of its first moment matrix satisfies $\rho_Y = (1-\delta)\rho_Z$. Hence we can choose δ so that $\rho_Y > 1$, in which case Y is as required. ■

Let us say that a Galton-Watson process $Z_0, Z_1, \ldots$ *stochastically dominates* a process $Y_0, Y_1, \ldots$ if there is a coupling such that $Z_n^i \ge Y_n^i$ for all $n \in \mathbb{N}$ and all types i. (Note that if the two processes do not have the same number of types then we can formally add types to the one with fewer types and redefine the offspring distributions in such a way that no particle ever gives birth to a child of the new types.) It is for instance easily seen that the process Y from the previous lemma is stochastically dominated by the original process Z.

We say that *explosion* occurs, if the total number of particles grows without bounds. In other words,

$$\{\text{explosion}\} = \left\{ \lim_{n\to\infty} \left(Z_n^1 + \cdots + Z_n^t \right) = \infty \right\}.$$

If $Z_0, Z_1, \ldots$ is as in Theorem 4.1 above, then Theorem 6.1 on page 39 of [12] states that for every vector $z = (z_1, \ldots, z_t)$ other than the all-zero vector there are only finitely many generations n for which $Z_n = z$ (with probability one). This has the following immediate corollary.

Theorem 4.3 *If $Z_0, Z_1, \ldots$ is a positve regular, nonsingular multitype Galton-Watson process with finitely many types, then*

$$\mathbb{P}_\nu(extinction|Z_0 = z) + \mathbb{P}_\nu(explosion|Z_0 = z) = 1,$$

for every initial state z.

It is natural to also consider multitype Galton-Watson processes with countably many types. In this case the state of the i-th generation is of course a random vector $Z_i = (Z_i^1, Z_i^2, \dots)$ of countably many nonnegative numbers. We define $p(i; z_1, z_2, \dots)$ and m_{ij} analogously to the case of finitely many types. For $t \in \mathbb{N}$, the t-restriction of a Galton-Watson process $Z_0, Z_1, \dots$ with countably many types is the t-type Galton-Watson process $Y_0, Y_1, \dots$ with offspring distributions given by:

$$p_Y(i; z_1, \dots, z_t) := p_Z(i; z_1, \dots, z_t, 0, 0, \dots).$$

That is, the probability that a particle of type i in the Y process of a z_1 children of type 1, z_2 children of type 2 and so on up to type t, is the probability the a particle of type i under the Z process has exactly these children and none of type bigger than t. We can think of the t-restricted process as a version of the old process where a particle and its potential children die during labour if at least one of the potential children has a type $> t$.

Observe that the original process stochastically dominates the t-restricted process.

Lemma 4.4 *Suppose $Z_0, Z_1, \dots$ is a multitype Galton-Watson process with countably many types, that satisfies the following conditions:*

- **(i)** *There exists a $c > 1$ such that, for every $i \in \mathbb{N}$, we have $\sum_{j=1}^{\infty} j \cdot m_{ij} \geq c \cdot i$;*
- **(ii)** *For every $i \in \mathbb{N}$ and $j \leq 2i$ we have $m_{ij} > 0$;*
- **(iii)** *Whenever $p(i; z_1, z_2, \dots) > 0$ we have $\sum_{j=1}^{\infty} j \cdot z_j \leq 2i$. (for every $i \in \mathbb{N}, z_1, z_2, \dots \geq 0$);*
- **(iv)** *We have*

$$\lim_{i \to \infty} \sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1} + z_{i+2} + \dots > 0}} p(i; z_1, z_2, \dots) = 0.$$

(That is, the probability that a particle of type i has at least one child of a strictly larger type is small for large i.)

Then there exists a $t \in \mathbb{N}$ such that the t-restricted process is positive regular, nonsingular and supercritical.

Proof: Observe that, by part **(ii)**, the t-restricted process is positive regular and nonsingular for every $t \geq 1$. Let $\varepsilon > 0$ be arbitrary, to be determined later. By part **(iv)**, there exists a t_0 such that the probability that a particle of type i has a child of type greater than i amongst its children is at most ε. That is:

$$\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1} + z_{i+2} + \dots > 0}} p(i; z_1, z_2, \dots) < \varepsilon \quad \text{(for all } i \geq t_0).$$

We now set $t := 2t_0$. Then we have that

$$\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{t+1} + z_{t+2} + \dots > 0}} p(i; z_1, z_2, \dots) = 0 \quad \text{if } i < t_0,$$

by condition **(iii)** of the lemma. And, if $t_0 \leq i \leq t$ then we have:

$$\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{t+1}+z_{t+2}+\dots > 0}} p(i; z_1, z_2, \dots) \leq \sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1}+z_{i+2}+\dots > 0}} p(i; z_1, z_2, \dots) < \varepsilon. \tag{7}$$

Let $M = (m_{ij})_{i,j \geq 1}$ denote the matrix of first moments of the original process, and let $M' = (m'_{ij})_{1 \leq i,j \leq t}$ denote that of the t-restricted process. We have that, for every $1 \leq i \leq t$:

$$\sum_{j=1}^{t} j \cdot m'_{ij} \geq \sum_{j=1}^{\infty} j \cdot m_{ij} - \varepsilon \cdot 2i \geq (c - 2\varepsilon) i,$$

using conditions **(i)**, **(iii)** of the lemma and (7). Thus, if we chose ε small enough so that $c' := c - 2\varepsilon > 1$, then we see that if $v := (1, 2, \dots, t)$ then $(M')^k v \geq (c')^k v$ coordinatewise. Since $(c')^k$ grows without bounds, it follows that M' must have an eigenvalue that is strictly larger than one in absolute value. So in particular (invoking Perron-Frobenius) the eigenvalue of largest absolute value is a real number strictly larger than one. This concludes the proof of the lemma. ■

In a time-inhomogeneous multitype Galton-Watson process, the offspring distibutions depend on n, the generation. We now denote by $p_n(i; z_1, z_2, \dots) := \mathbb{P}_\nu(Z_{n+1} = (z_1, z_2, \dots) | Z_n = e_i)$ the probability that a particle of type i, in generation n, fathers exactly z_j children of type j (for $j = 1, 2, \dots$).

Lemma 4.5 *Suppose that $Z_0, Z_1, \dots$ is a time-inhomogeneous multi-type Galton-Watson process with countably many types such that the limits*

$$\lim_{n \to \infty} p_n(i; z_1, z_2, \dots) =: p(i; z_1, z_2, \dots),$$

exist for all $i \in \mathbb{N}$ and $z_1, z_2, \dots \geq 0$. Suppose further that the limits p belong to a (time-homogeneous) multitype Galton-Watson process satisfying the conditions of Lemma 4.4. Then

$$\liminf_{n \to \infty} \mathbb{P}_\nu(\mathit{explosion} | Z_n = e_1) > 0.$$

Proof: Let $Z'_0, Z'_1, \dots$ denote the Galton-Watson process belonging to the limiting probabilities $p(i; \underline{z})$ and let us pick t according to Lemma 4.4 with respect to Z'. Let $Y_0, Y_1, \dots$ denote the t-restricted process.

Let $X_0, X_1, \dots$ denote a process that Lemma 4.2 provides if we apply it to $Y_0, Y_1, \dots$. Let $\mathcal{I} := \{(z_1, \dots, z_t) \neq (0, \dots, 0) : p_X(i; z_1, \dots, z_t) > 0\}$. Observe that $\mathcal{I}$ is finite, so that there is an n such that $p_{n+m}(i; z_1, \dots, z_t, 0, 0, \dots) \geq p_X(i; z_1, \dots, z_t)$, for all $m \geq 0$ and all $(z_1, \dots, z_t) \in \mathcal{I}$. This means that $Z_n, Z_{n+1}, \dots$ stochastically dominates $X_0, X_1, \dots$, if we condition on $Z_n = X_0 = e_1$. So in particular:

$$\liminf_{n \to \infty} \mathbb{P}_\nu(Z \text{ explodes} | Z_n = e_1) \geq \mathbb{P}_\nu(X \text{ explodes} | X_0 = e_1) > 0.$$

This concludes the proof of the lemma. ■

5 Proof of Theorem 3.1

We will split the proof of this theorem up into a sequence of lemmas.

Lemma 5.1 $\Psi(\nu, \lambda) > 0$ *for all* $\nu, \lambda > 0$.

Proof: Let us set $m := \min\{4, \lceil 4/\lambda \rceil\}$, and let E be the event that each of the $2m$ sets $[0, \pi/m) \times [0, 1], \dots, [(2m-1)\pi/m, 2\pi) \times [0, 1]$ contains at least one point of $\mathcal{P}$. The expected number of points of $\mathcal{P}$ in each of these sets is $\frac{1}{2m} \cdot \int_0^{2\pi} \int_0^1 g(r, \theta) \mathrm{d}r\mathrm{d}\theta = (\nu/2m) \cdot (\cosh(1/2) - 1)$.

It is easily checked that $\arccos((\cosh(1) - 1)/\sinh(1)) > \pi/4$, so that $\gamma(r) > \lambda\pi/4$ for all $r \leq 1$. We claim the event E implies $\mathcal{C}(\lambda)$. To see this, suppose E is realized and pick an arbitrary angle $\theta \in [0, 2\pi)$. By symmetry, we can assume without loss of generality $\theta \in [0, \pi/m)$. Since E holds, there is a point $(r, \vartheta) \in \mathcal{P} \cap [0, \pi/m) \times [0, 1]$. We find that $|\theta - \vartheta|_{2\pi} < \pi/m \leq \lambda\pi/4 < \gamma(r)$. Thus, the event E indeed inplies $\mathcal{C}(\lambda)$.

We therefore have

$$\Psi(\nu, \lambda) \geq \mathbb{P}_\nu(E) = \left(1 - e^{-(\nu/m)\cdot(\cosh(1/2)-1)}\right)^m > 0,$$

as required. ■

Lemma 5.2 *For all* $a, b, \lambda > 0$ *we have* $\Psi(a + b, \lambda) \geq \Psi(a, \lambda) + (1 - \Psi(a, \lambda)) \cdot \Psi(b, \lambda)$.

Proof: Since $\mathcal{P}_{a+b}$ can be seen as a *superposition* of $\mathcal{P}_a$ and $\mathcal{P}_b$ for every $a, b > 0$, the probability that $\mathcal{C}(\lambda)$ occurs in $\mathcal{P}_{a+b}$ is at least the probabilty that it occurs in $\mathcal{P}_a$ plus the probability that is does not occur in $\mathcal{P}_a$ and it occurs in $\mathcal{P}_b$. ■

Note that the previous two lemmas show that $\Psi(\nu, \lambda)$ is strictly increasing in ν whenever $\Psi(\nu, \lambda) < 1$.

Corollary 5.3 *If* $\nu, \lambda > 0$ *are such that* $\Psi(\nu, \lambda) < 1$ *then* Ψ *is strictly increasing in* ν *at* (ν, λ).

It will be helpful to consider a process where we reveal $\mathcal{P}$ in "discrete steps". For $n \in \mathbb{N}$ let us denote

$$r_n := n \cdot 2 \ln 2. \tag{8}$$

Let us denote $\mathcal{B}_n := \mathcal{P} \cap B_{\mathbb{H}}(0, r_n)$ and $\mathcal{A}_n := \mathcal{B}_n \setminus \mathcal{B}_{n-1}$. ($\mathcal{B}_n$ is the set of points of $\mathcal{P}$ with radii at most r_n and $\mathcal{A}$ is the set of points with radii between r_{n-1} and r_n.)
Let us also recall that $\gamma(r)$ is strictly decreasing in r. (As $(\cosh(r)-1)/\sinh(r) = 1-2/(e^r+1)$ is strictly increasing and $\arccos(.)$ is strictly decreasing.) Using equation 6 we can now derive the following.

Lemma 5.4 *For every fixed* $\nu, \lambda > 0$ *we have that*

$$\mathbb{E}_\nu|\{p \in \mathcal{A}_n : p \textit{ covers the angle } 0\}| = (1 + O((1/4)^n)) \cdot (\nu\lambda/\pi) \cdot \ln 2,$$

and

$$\mathbb{P}_\nu(\mathcal{A}_n \textit{ does not cover } 0) = (1 + O((1/4)^n)) \cdot (1/2)^{\nu\lambda/\pi},$$

where the $O(.)$*-notation refers to* $n \to \infty$.

Proof: If μ_n denotes the expected number of points in $\mathcal{A}_n$ that cover the angle 0, then

$$\begin{aligned}
\mu_n &= \int_0^{2\pi}\int_{r_{n-1}}^{r_n} 1_{\{|\theta|_{2\pi}<\gamma(r)\}}\cdot g(r,\theta)\mathrm{d}r\mathrm{d}\theta \\
&= \int_{r_{n-1}}^{r_n} 2\gamma(r)\cdot g(r,\theta)\mathrm{d}r \\
&= \int_{r_{n-1}}^{r_n} 4\lambda(1+O(e^{-r}))e^{-r/2}\cdot(\nu/4\pi)\cdot\sinh(r/2)\mathrm{d}r \\
&= \int_{r_{n-1}}^{r_n} 4\lambda(1+O(e^{-r}))e^{-r/2}\cdot(\nu/4\pi)\cdot(1+O(e^{-r}))\frac{1}{2}e^{r/2}\mathrm{d}r \\
&= (1+O(e^{-r_n}))\cdot(\nu\lambda/2\pi)\int_{r_{n-1}}^{r_n} 1\ \mathrm{d}r \\
&= (1+O(4^{-n}))\cdot(\nu\lambda/\pi)\cdot\ln 2.
\end{aligned} \tag{9}$$

Here we used that $\sinh(x) = (1+O(e^{-x}))\cdot\frac{1}{2}e^x$ for large x.This proves the first statement of the lemma. The second statement follows immediately from the fact that $\mathbb{P}_\nu(\mathcal{A}_n \text{ covers } 0) = e^{-\mu_n}$. ■

Lemma 5.5 *We have $\gamma(r_n) > \lambda\cdot 2^{-n}$, for all $n\in\mathbb{N}$.*

Proof: It suffices to prove that

$$\varphi(r) := e^{r/2}\cdot\gamma(r)/\lambda = e^{r/2}\cdot\arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right),$$

is strictly larger than one for all $r\geq r_1 = 2\ln 2$. Observe that $\cos(y)\geq 1-y^2/2$ for all $y\in\mathbb{R}$. This implies that if $y=\arccos(1-x)$ then $y\geq\sqrt{2x}$. Combining this with (5) shows that

$$\varphi(r)\geq e^{r/2}\cdot 2e^{-r/2}\left(\frac{1-e^{-r}}{1-e^{-2r}}\right)^{1/2} = 2\left(\frac{1-e^{-r}}{1-e^{-2r}}\right)^{1/2}\geq 2\sqrt{1-e^{-r}}\geq\sqrt{3}>1,$$

using that $r\geq 2\ln 2$ for the penultimate inequality. ■

Lemma 5.6 *For every $\nu,\lambda>0$ there exists a $c=c(\nu,\lambda)>0$ such that*

$$\mathbb{P}_\nu\left[\mathcal{A}_n \textit{ covers } [0,\lambda 2^{-n})\right]\geq c,$$

(i.e., the probability that $[0,\lambda 2^{-n})$ is covered in its entirety by the points of $\mathcal{P}$ with radii between r_{n-1} and r_n is at least c) for all $n\in\mathbb{N}$.

Proof: It follows from Lemma 5.5 and the monotonicity of $\gamma(r)$ that if (r,θ) covers 0 and furthermore $\theta\in[0,\pi)$ and $r\leq r_n$ then (r,θ) in fact covers all of $[0,\lambda 2^{-n})$. It follows that

$$\mathbb{P}_\nu\left[\mathcal{A}_n \text{ covers } [0,\lambda 2^{-n})\right]\geq\frac{1}{2}\cdot\mathbb{P}_\nu\left[\mathcal{A}_n \text{ covers } 0\right] = \frac{1}{2}\cdot\left(1-(1+O((1/4)^n))\cdot(1/2)^{\nu\lambda/\pi}\right) = \Omega(1),$$

using Lemma 5.4. ■

Let us write $\mathcal{U}_n \subseteq [0, 2\pi)$ for the union of intervals of angles *not* covered by the points of $\mathcal{B}_n$. Then $\mathcal{U}_n$ clearly consists of a finite number of intervals. Let $\mathcal{U}_n^{\text{long}} \subseteq \mathcal{U}_n$ denote the union of all intervals of length at least $\lambda 2^{-n}$, and let $\mathcal{U}_n^{\text{short}} := \mathcal{U}_n \setminus \mathcal{U}_n^{\text{long}}$ denote the union of all intervals strictly shorter than $\lambda 2^{-n}$.

We now also define

$$L_n = L_n(\lambda) := \text{length}(\mathcal{U}_n) \cdot \lambda^{-1} \cdot 2^n, \quad L_n^{\text{long}} = L_n^{\text{long}}(\lambda) := \text{length}(\mathcal{U}_n^{\text{long}}) \cdot \lambda^{-1} \cdot 2^n,$$
$$L_n^{\text{short}} = L_n^{\text{short}}(\lambda) := \text{length}(\mathcal{U}_n^{\text{short}}) \cdot \lambda^{-1} \cdot 2^n. \tag{10}$$

The λ is omitted when it is clear from the context.
That is, L_n denotes total length of $\mathcal{U}_n$, multiplied by $\lambda^{-1} 2^n$ and $L^{\text{long}}, L^{\text{short}}$ are defined analogously. We let $\mathcal{N}_n^{\text{short}}$ denote the number of components of $\mathcal{U}_n^{\text{short}}$ (i.e. the number of intervals of length strictly less than $\lambda 2^{-n}$), and we set

$$Y_n := \mathcal{N}_n^{\text{short}} + L_n^{\text{long}}. \tag{11}$$

Recall that if $(E_n)_n$ is a sequence of events then we say the event "E_n *almost always*" holds if E_n holds for all but finitely many n. In other words $\{E_n \text{ almost always}\} = \liminf E_m = \bigcup_n \bigcap_{m>n} E_m$. We can for instance write

$$\{\mathcal{C}(\lambda)\} = \{L_n = 0 \text{ almost always}\} = \{Y_n = 0 \text{ almost always}\}.$$

Also recall that we say that the event "E_n *infinitely often*" holds if E_n holds for infinitely many n. In other words $\{E_n \text{ infinitely often}\} = \bigcap_n \bigcup_{m>n} E_m$.

Lemma 5.7 *For every $\nu, \lambda, K > 0$ we have $\mathbb{P}_\nu(Y_n > K$ almost always$) = 1 - \Psi(\nu, \lambda)$.*

Proof: Observe that $\mathbb{P}_\nu(Y_n = 0 \text{ almost always}) = \Psi(\nu, \lambda)$. Let us also observe that, for every $K > 0$:

$$\mathbb{P}_\nu(Y_n = 0 \text{ almost always}) + \mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often}) + \mathbb{P}_\nu(Y_n > K \text{ almost always}) = 1.$$

Hence, it suffices to show that $\mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often }) = 0$ for every $K > 0$. Observe that if $Y_n = y$, then $\mathcal{U}_n$ can be covered by at most $2\lceil y/\lambda \rceil$ intervals of length $\lambda 2^{-n}$. By Lemma 5.6, and positive correlation, there exists a $c > 0$ such that for all $y > 0$:

$$\mathbb{P}_\nu(Y_{n+1} = 0 | Y_n = y, Y_{n-1} = y_{n-1}, \ldots, Y_1 = y_1) \geq c^{2\lceil y/\lambda \rceil}, \tag{12}$$

for all $n \in \mathbb{N}$ and all $y, y_1, \ldots, y_{n-1} > 0$. Now let N_1 be the (random) $n \in \mathbb{N}$ for which $Y_n \in (0, K]$ for the first time. Similarly, let N_i be the i-th index n for which $Y_n \in (0, K]$. (Here we set $N_i = \infty$ if $Y_n \in (0, K]$ for less than i indices n.) It follows from (12) that $\mathbb{P}_\nu(N_{i+1} < \infty | N_i < \infty) \leq 1 - c^{2\lceil K/\lambda \rceil} =: x$. But then we also have that, for every $M \in \mathbb{N}$:

$$\begin{array}{rcl} \mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often }) & \leq & \mathbb{P}_\nu(N_i < \infty \text{ for all } 1 \leq i \leq M) \\ & = & \mathbb{P}_\nu(N_1 < \infty) \cdot \prod_{i=1}^{M-1} \mathbb{P}_\nu(N_{i+1} < \infty | N_i < \infty) \\ & \leq & 1 \cdot x^{M-1}. \end{array}$$

Sending $M \to \infty$ shows that $\mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often}) = 0$, as required. ∎

Lemma 5.8 *If $I \subseteq \mathcal{U}_n$ is an interval then $I \cap \mathcal{U}_{n+1}$ consists of at most $\left\lfloor \frac{\text{length}(I)}{\lambda 2^{-n}} \right\rfloor + 1$ intervals.*

Proof: Notice that, if the interval I is cut into $k+1$ disjoint, non-empty intervals by $\mathcal{A}_{n+1}$ then there must be k points $(\rho_1, \theta_1), \ldots, (\rho_k, \theta_k) \in A_{n+1}$ such that the intervals $(\theta_i - \gamma(\rho_i), \theta_i + \gamma(\rho_i))$ are disjoint and completely contained in I. Hence we must have that

$$\text{length}(I) > \sum_{i=1}^{k} 2\gamma(r_i) \geq 2k\gamma(r_n) > k\lambda 2^{-n},$$

using Lemma 5.5. The lemma follows. ■

Corollary 5.9 *If $I \subseteq \mathcal{U}_n$ is an interval of length at most $\lambda 2^{-n}$ then $I \cap \mathcal{U}_{n+1}$ is either empty or a single interval.*

Another relatively obvious, but key, observation is the following.

Lemma 5.10 *If $I, J \subseteq [0, 2\pi)$ are two sets such that $|x - y|_{2\pi} \geq 2\gamma(r_n)$ for all $x \in I, y \in J$, then $I \cap \mathcal{A}_m$ and $J \cap \mathcal{A}_m$ are independent for all $m > n$.*

Proof: This follows immediately from the fact that a point of radius bigger than r_n cannot simultaneously cover two angles that are more than $2\gamma(r_n)$ apart, and the fact that $\mathcal{P}_\nu \cap A$ and $\mathcal{P}_\nu \cap B$ are independent if $A, B \subseteq \mathbb{H}$ are disjoint. ■

Lemma 5.11 *For every $\nu, \lambda, K > 0$ we have $\mathbb{P}_\nu(L_n^{long} > K$ infinitely often$) = 1 - \Psi(\nu, \lambda)$.*

Proof: Recall that $\Psi(\nu, \lambda) = \mathbb{P}_\nu(L_n = 0 \text{ almost always})$, so that $\mathbb{P}_\nu(L_n > 0 \text{ almost always}) = 1 - \Psi(\nu, \lambda)$. It thus suffices to show that $\mathbb{P}_\nu(L_n > 0 \text{ and } L_n^{\text{long}} < K \text{ almost always}) = 0$, for every $K > 0$. Suppose that, on the contrary, for some $K > 0$ it holds that

$$\mathbb{P}_\nu(L_n > 0 \text{ and } L_n^{\text{long}} < K \text{ almost always}) > 0.$$

By Lemma 5.7 it must then also be the case that $\mathbb{P}_\nu(Y_n > K' \text{ and } L_n^{\text{long}} < K \text{ almost always}) > 0$, for every constant K'. And, since $Y_n = \mathcal{N}_n^{\text{short}} + L_n^{\text{long}}$, we must then also have that $\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K' \text{ and } L_n^{\text{long}} < K \text{ almost always}) > 0$, for every constant K'. Let us remark that, if E_n almost always holds, then there is a (random) N such that E_n holds for all $n \geq N$. Hence, to prove the lemma it suffices to show that for every $K > 0$ there exists a $K' = K'(K) > 0$ such that $\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K' \text{ and } L_n^{\text{long}} < K \text{ for all } n \geq n_0) = 0$, for all $n_0 \in \mathbb{N}$.

Let $K > 0$ thus be arbitrary. Let $c = c(\nu, \lambda)$ be as provided by Lemma 5.6, and let us choose K' such that $K' > 8K/c$ and

$$\mathbb{P}(\text{Bi}(a, c) > ac/2) \geq 2/3,$$

for all $a \geq K'$. (The existence of such a K' follows for instance from the Chebyschev bound.)

Observe that, by Lemma 5.8, if $L_n^{\text{long}} \leq K$ then the long components (intervals) of generation n will split into no more than $2K$ components in generation $n + 1$. On the other hand, the short intervals of generation n each disappear with probability $\geq c$ and if they don't disappear then they cannot split into two or more intervals by Lemma 5.8.

This shows that for all $a \geq K', b \leq K$ we have

$$\mathbb{P}_\nu(\mathcal{N}_{n+1}^{\text{short}} < (1-c/4)\mathcal{N}_n^{\text{short}} | \mathcal{N}_n^{\text{short}} = a, L_n^{\text{long}} = b) \geq 2/3.$$

(To see this note that, with probability 2/3, no more than $(1-c/2)a$ short intervals survive to the next generation, while the long intervals generate at most $2b \leq 2K < K' \cdot c/4 \leq ac/4$ short ones.)

On the other hand, if $\mathcal{N}_n^{\text{short}} = a$ and $L_n^{\text{long}} \leq K$ then a (deteministic) upper bound is $\mathcal{N}_{n+1}^{\text{short}} \leq a + 2K/\eta_0 \leq (1+c/4)\mathcal{N}_n^{\text{short}}$.

Let us now fix arbitrary $n_0 \in \mathbb{N}, a_0 > K', b_0 \leq K$. If $\mathcal{N}_{n_0}^{\text{short}} = a_0, L_{n_0}^{\text{long}} = b_0$ and $\mathcal{N}_n^{\text{short}} > K', L_n^{\text{long}} \leq K$ for all $n \geq n_0$ then, for every $m \geq 2\log(K'/a_0)/\log(1-c^2/16)$, there are more than $m/2$ indices $n \leq i \leq n+m-1$ such that $\mathcal{N}_{i+1}^{\text{short}} > (1-c/4)\mathcal{N}_i^{\text{short}}$. (Otherwise we would have that $\mathcal{N}_m^{\text{short}} < ((1-c/4)(1+c/4))^{m/2} \cdot a_0 = (1-c^2/16)^{m/2} \cdot a_0 < K'$.) Thus, we have

$$\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K', L_n^{\text{long}} \leq K \text{ for all } n \geq n_0 | \mathcal{N}_{n_0}^{\text{short}} = a_0, L_{n_0}^{\text{long}} = b_0) \leq \lim_{m\to\infty} \mathbb{P}(\text{Bi}(m, 1/3) \geq m/2) = 0.$$

(The last inequality follows for instance from the weak law of large numbers.) Since n_0, a_0, b_0 were arbitrary, it follows that

$$\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K' \text{ and } L_n^{\text{long}} < K \text{ for all } n \geq n_0) = 0 \quad \text{ for all } n_0 \in \mathbb{N},$$

as required. ■

Lemma 5.12 *If $\nu \cdot \lambda = \pi$ then there exists a constant $C = C(\nu, \lambda)$ such that $\mathbb{E}_\nu L_n \leq C$ for all n.*

Proof: For every $\nu, \lambda > 0$, we have that

$$\begin{aligned}
\mathbb{E}_\nu L_n &= 2^n \cdot \int_0^{2\pi} \mathbb{P}_\nu(\text{the angle } \theta \text{ is covered by } \mathcal{B}_n)\mathrm{d}\theta \\
&= 2^n \cdot 2\pi \cdot \mathbb{P}_\nu(\text{ the angle 0 is covered by } \mathcal{B}_n).
\end{aligned}$$

Hence, when $\nu \cdot \lambda = \pi$, we have

$$\mathbb{E}_\nu L_n = 2^n \cdot 2\pi \cdot \exp\left[-\sum_{i=1}^n (1 + O((1/4)^i)) \cdot \ln 2\right] = 2^n \cdot 2\pi \cdot \exp[-n\ln 2 + O(1)] = O(1),$$

using Lemma 5.4. ■

Lemma 5.13 *Let $\nu \cdot \lambda \leq \pi$ and suppose that $\Psi(\nu, \lambda) < 1$ then $\mathbb{E}_\nu L_n \to \infty$ as $n \to \infty$.*

Proof: It follows from Lemma 5.11 that $\mathbb{P}_\nu(L_n > K \text{ infinitely often }) = 1 - \Psi(\nu, \lambda)$, for every constant $K > 0$. Let us thus pick a K (to be made explicit later), and let N be the (random) first index n such that $L_n > K$. (Here $N = \infty$ if no such n exists. Note $N < \infty$

with probability $1 - \Psi(\nu, \lambda) > 0$.) Let n_0 be such that $\mathbb{P}_\nu(N < n_0) > (1 - \Psi(\nu, \lambda))/2$. By conditioning on the value of N, we find that for $n \geq n_0$:

$$\begin{array}{rcl}
\mathbb{E}L_n & \geq & \sum_{m=0}^{n_0} \mathbb{E}(L_n | N = m)\mathbb{P}_\nu(N = m) \\
& = & \sum_{m=0}^{n_0} K \cdot 2^{n-m} \cdot \exp[-\sum_{i=m+1}^{n}(1 + O((1/4)^i)) \cdot (\nu\lambda/\pi) \cdot \ln 2] \cdot \mathbb{P}_\nu(N = m) \\
& = & \sum_{m=0}^{n_0} K \cdot 2^{n-m} \cdot \exp[-(n-m) \cdot (\nu\lambda/\pi) \cdot \ln 2 + O(1)] \cdot \mathbb{P}_\nu(N = m) \\
& = & \Omega\left(K \cdot \sum_{m=0}^{n_0} 2^{(n-m)(1-\nu\lambda/\pi)}\mathbb{P}_\nu(N = m)\right) \\
& = & \Omega\left(K \cdot \sum_{m=0}^{n_0} \mathbb{P}_\nu(N = m)\right) \\
& = & \Omega(K \cdot (1 - \Psi(\nu, \lambda))/2).
\end{array}$$

Sending $K \to \infty$ proves the lemma. ■

It follows immediately from Lemmas 5.12 and 5.13 that:

Corollary 5.14 *If $\nu\lambda = \pi$ then $\Psi(\nu, \lambda) = 1$.*

This last corollary of course also implies that $\Psi(\nu, \lambda) = 1$ for all $\nu \cdot \lambda \geq \pi$.

Lemma 5.15 *For every $\nu, \lambda > 0$ with $\nu \cdot \lambda < \pi$ there exists an $\eta_0 = \eta(\nu, \lambda)$ such that for every $0 < \eta < \eta_0$ we have*

$$\liminf_{n\to\infty} \mathbb{P}_\nu\big([0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_n\big) > 1/2.$$

Proof: Let μ_n denote the expected number of points $(r, \theta) \in \mathcal{A}_n$ that cover 0, and let $\tilde{\mu}_n$ denote expected number of points $(r, \theta) \in \mathcal{A}_n$ that cover *some* point of $[0, \eta \cdot 2^{-n})$. Then we have, similar to the proof of Lemma 5.4:

$$\begin{array}{rcl}
\tilde{\mu}_n & = & \int_{r_{n-1}}^{r_n} \left(\eta \cdot 2^{-n} + 2\arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right)\right) \cdot g(r, \theta)\mathrm{d}r \\
& = & \eta \int_{r_{n-1}}^{r_n} 2^{-n} \cdot (1 + O(e^{-r})) \cdot e^{r/2}\mathrm{d}r + \mu_n \\
& = & (1 + o(1)) \cdot (\eta/2 + (\nu\lambda/\pi) \cdot \ln 2),
\end{array} \tag{13}$$

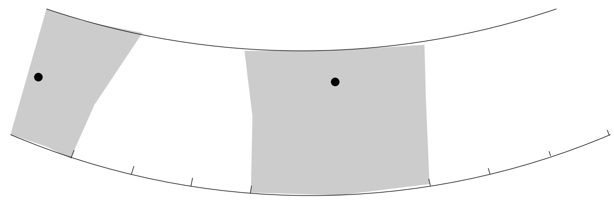
reusing the computations (9) in the second line. Since $\nu\lambda < \pi$ we can choose $\eta > 0$ such that $\eta/2 + (\nu\lambda/\pi) \cdot \ln 2 < \ln 2$. In that case we have

$$\liminf_{n\to\infty} \mathbb{P}_\nu\big([0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_n\big) = \liminf_{n\to\infty} e^{-\tilde{\mu}_n} > 1/2,$$

as required. ■

For the remainder of the section, we fix $\eta > 0$ such that the conclusion of the last lemma holds. Let us now consider the following random process. We start by dissecting $[0, 2\pi)$ into intevals $[0, \eta), [\eta, 2\eta), \ldots, [2\pi - \eta, 2\pi)$ of length η. (We assume without loss of generality that $\eta = \frac{2\pi}{k}$, for some k.) Each of these intervals "survives" if none of its points is covered by points of $\mathcal{P}$ of radius at most r_1. In each subsequent "generation", we split the surviving intervals in two, and these survive if none of their points are covered by a point of $\mathcal{P}$ of radius between r_{n-1} and r_n. This does produce a kind of branching process, but with the unfortunate property that the offspring of different intervals in generation n are not always independent (e.g., if two intervals share an endpoint then their offspring are dependent, or more generally if they are close enough for a point of radius bigger than r_n to cover a point in each of the two intervals.)

To deal with this problem, we group the surviving intervals into "particles" consisting of (maximal) sequences of intervals each sharing an endpoint with the next. The type of a particle will be the number of intervals it consists of. See Figure 5 for a depiction.



Note that, in generation n, the gap between different particles is at least $2 \cdot \gamma(r_n)$. So no point of radius $> r_n$ can cover points in two different particles of generation n. This implies that the offspring distributions are independent.

Thus, we have defined a time-inhomogeneous multitype Galton-Watson process $Z_0^\lambda, Z_1^\lambda, \dots$ with countably many types. Again, we drop the superscript if it is clear from the context. Let $p_n(i; z_1, z_2, \dots)$ denote the probability that a particle of type i in generation n produces z_1 children of type 1, z_2 children of type 2 and so on. (Note that strictly speaking we would also need to introduce types for the case when $\mathcal{U}_n = [0, 2\pi)$ in which case there is one particle that "wraps around". This situation however does not occur as soon as there is at least one point with radius $\leq r_n$. So this is not a real issue. We leave it to the reader to check that the proofs below can be adapted to work also with this more proper but also more cumbersome definition of the process.)

Lemma 5.16 *For every $i, z_1, z_2, \dots$ the limits*

$$p(i; z_1, z_2, \dots) := \lim_{n\to\infty} p_n(i; z_1, z_2, \dots),$$

exist.

Proof: Let us fix $i, z_1, z_2, \dots,$ and let E_n denote the event that $[0, i \cdot \eta \cdot 2^{-n})$ is split into a groups of intervals of length $2^{-(n+1)}$ in the required way by $\mathcal{A}_n$, i.e. among $[0, \eta \cdot 2^{-(n+1)}), \dots, [(2i - 1) \cdot \eta \cdot 2^{-(n+1)})$ there are z_1 intervals such that none of their points are covered by $\mathcal{A}_n$ but some point in each of the neighbouring intervals were covered, and so on.

Let $A_n \subseteq \mathbb{H}$ denote the set of all points (r, θ) with $r_{n-1} < r \leq r_n$ and $\theta \in (-10 \cdot 2^{-n}, (i \cdot \eta + 10) \cdot 2^{-n})$; and let $W_n := |\mathcal{P} \cap A_n|$ denote the number of points of $\mathcal{P}$ that fall inside A_n. By (6), for large enough n, whether or not E_n holds will only depend on the points of $\mathcal{P}$ that fall inside A_n. We have

$$p_n(i; z_1, z_2, \dots) = \mathbb{P}_\nu(E_n) = \sum_{t=0}^{\infty} \mathbb{P}_\nu(E_n | W_n = t) \mathbb{P}_\nu(W_n = t). \tag{14}$$

Let us observe that

$$\begin{aligned}
\mathbb{E}W_n &= \int_{A_n} g(r,\theta) \mathrm{d}r \mathrm{d}\theta = 2^{-n} \cdot (i \cdot \eta + 20) \cdot (\nu\lambda/2\pi) \cdot 2(\cosh(r_n/2) - \cosh(r_{n-1}/2)) \\
&= 2^{-n} \cdot (i \cdot \eta + 20) \cdot (\nu\lambda/2\pi) \cdot (e^{r_n/2} + e^{-r_n/2} - e^{r_{n-1}/2} + e^{-r_{n-1}/2})) \\
&= (1 + o(1)) \cdot (i \cdot \eta + 20) \cdot (\nu\lambda/\pi).
\end{aligned}$$

It follows also that W_n converges in distribution to a Po $\big((i\cdot\eta+20)^{-1}\cdot(\nu\lambda/\pi)\big)$-distribution random variable. Therefore, in the light of (14), in order to prove that $p_n(i;z_1,z_2,\dots)$ converges, it suffices to prove that the conditional probability $\mathbb{P}_\nu(E_n|W_n=t)$ converges for every fixed $t\in\mathbb{N}$. Let us thus fix a $t\in\mathbb{N}$.

Observe that if we condition on $W=t$ then $\mathcal{P}\cap A$ behaves like t i.i.d. random vectors $X_1=(\rho_1,\theta_1),\dots,X_t=(\rho_t,\theta_t)$ with common probability density:

$$\tilde g(\rho,\theta)=\frac{g(\rho,\theta)}{\int_A g(r',\theta')\mathrm{d}r'\mathrm{d}\theta'}=(1+o_n(1))\cdot(i\cdot\eta+20)^{-1}\cdot e^{\rho/2},$$

where we used that $g(\rho,\theta)=(\nu/4\pi)\sinh(\rho/2)=(1+O(e^{-\rho}))\cdot(\nu/4)\cdot e^{\rho/2}$.

For notational convenience we write $I_j:=[j\cdot\eta\cdot 2^{-(n+1)},(j+1)\cdot\eta\cdot 2^{-(n+1)}$. For $0\le j<2i$ and $1\le s\le t$ we set $F_n^{j,s}:=\{(\rho_s,\theta_s)$ covers a point of $I_j\}$ and for $J\subseteq\{0,\dots,2i-1\}\times\{1,\dots,t\}$ we define

$$F_n^J:=\left(\bigcap_{(j,s)\in J}F_n^{j,s}\right)\cap\left(\bigcap_{(j,s)\notin J}(F_n^{j,s})^c\right).$$

I.e., the event F_n^J prescribes precisely which of the t points covers which of the $2i$ intervals. Clearly there is some family of sets $\mathcal{J}\subseteq 2^{\{0,\dots,2i-1\}\times\{1,\dots,t\}}$ such that

$$\mathbb{P}_\nu(E_n|W_n=t)=\mathbb{P}_\nu\left(\bigcup_{J\in\mathcal{J}}F_n^J\right)=\sum_{J\in\mathcal{J}}\mathbb{P}_\nu(F_n^J).$$

It thus suffices to prove that the probabilities $\mathbb{P}_\nu(F_n^J)$ converge. Let us thus fix some $J\subseteq\{0,\dots,2i-1\}\times\{1,\dots,t\}$. Setting

$$\varphi_n^j(\rho,\theta):=\begin{cases}1 & \text{if } \theta\in\big(j\cdot\eta\cdot 2^{-(n+1)}-\gamma(\rho),(j+1)\cdot\eta\cdot 2^{-(n+1)}+\gamma(\rho)\big);\\ 0 & \text{otherwise.}\end{cases},$$

and $\ell:=-10\cdot 2^{-n}, u:=(i\cdot\eta+10)\cdot 2^{-n}$, we can write

$$\begin{aligned}
&\mathbb{P}_\nu(F_n^J)\\
=\;&\int_\ell^u\int_{r_{n-1}}^{r_n}\dots\int_\ell^u\int_{r_{n-1}}^{r_n}\prod_{(j,s)\in J}\varphi_n^j(\rho_s,\theta_s)\cdot\prod_{(j,s)\notin J}(1-\varphi_n^j(\rho_s,\theta_s))\cdot\prod_{s=1}^t\tilde g(\rho_s,\theta_s)\;\mathrm{d}\rho_1\mathrm{d}\theta_1\dots\mathrm{d}\rho_t\mathrm{d}\theta_t\\
=\;&\int_{-10}^{i\eta+10}\int_0^{2\ln 2}\dots\int_{-10}^{i\eta+10}\int_0^{2\ln 2}\prod_{(j,s)\in J}\varphi_n^j(r_{n-1}+x_s,2^{-n}\cdot\vartheta_s)\cdot\\
&\qquad\prod_{(j,s)\notin J}(1-\varphi_n^j(r_{n-1}+x_s,2^{-n}\vartheta_s))\cdot\prod_{s=1}^t\tilde g(r_{n-1}+x_s,2^{-n}\vartheta_s)\cdot 2^{-t\cdot n}\;\mathrm{d}x_1\mathrm{d}\vartheta_1\dots\mathrm{d}x_t\mathrm{d}\vartheta_t\\
=\;&\int_{-10}^{i\eta+10}\int_0^{2\ln 2}\dots\int_{-10}^{i\eta+10}\int_0^{2\ln 2}\prod_{(j,s)\in J}\varphi_n^j(r_{n-1}+x_s,2^{-n}\cdot\vartheta_s)\cdot\prod_{(j,s)\notin J}(1-\varphi_n^j(r_{n-1}+x_s,2^{-n}\vartheta_s))\cdot\\
&\qquad(1+o_n(1))\cdot(i\cdot\eta+20)^{-t}\cdot e^{\sum_{s=1}^t(r_{n-1}+x_i)/2}\cdot 2^{-t\cdot n}\;\mathrm{d}x_1\mathrm{d}\vartheta_1\dots\mathrm{d}x_t\mathrm{d}\vartheta_t\\
=\;&\int_{-10}^{i\eta+10}\int_0^{2\ln 2}\dots\int_{-10}^{i\eta+10}\int_0^{2\ln 2}\prod_{(j,s)\in J}\varphi_n^j(r_{n-1}+x_s,2^{-n}\cdot\vartheta_s)\cdot\prod_{(j,s)\notin J}(1-\varphi_n^j(r_{n-1}+x_s,2^{-n}\vartheta_s))\cdot\\
&\qquad(1+o_n(1))\cdot(i\cdot\eta+20)^{-t}\cdot 2^{-t}\cdot e^{(x_1+\dots+x_t)/2}\;\mathrm{d}x_1\mathrm{d}\vartheta_1\dots\mathrm{d}x_t\mathrm{d}\vartheta_t,
\end{aligned}$$

applying the substitutions $r_s = r_{n-1} + x_s, \theta_s = 2^{-n}\vartheta_s$ in the second line. Let us now define, for $0 \leq x \leq 2\ln 2$ and $-10 \leq \vartheta \leq i \cdot \eta + 10$:

$$\psi^j(x,\vartheta) := \begin{cases} 1 & \text{if } \vartheta \in (j \cdot \eta/2 - e^{-x/2}, (j+1) \cdot \eta/2 + e^{-x/2}), \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (6) that

$$\lim_{n\to\infty} \varphi_n^j(r_{n-1} + x, 2^{-n}\vartheta) = \psi^j(x,\vartheta) \quad \text{almost everywhere.}$$

(Recall that *almost everywhere* means "for all (x,ϑ) except for a set of Lebesgue measure zero".) Using the dominated convergence theorem we can now conclude that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}_\nu(F_n^J) \;=\; & (2i \cdot \eta + 40)^{-t} \int_{-10}^{i\eta+10} \int_0^{2\ln 2} \dots \int_{-10}^{i\eta+10} \int_0^{2\ln 2} \prod_{(j,s)\in J} \psi^j(x_s,\vartheta_s) \cdot \\ & \prod_{(j,s)\notin J} (1 - \psi^j(x_s,\vartheta_s)) \cdot e^{(x_1+\dots+x_t)/2}\, \mathrm{d}x_1 \mathrm{d}\vartheta_1 \dots \mathrm{d}x_t \mathrm{d}\vartheta_t. \end{aligned}$$

The lemma follows. ■

Lemma 5.17 *The limits $p(i; z_1, z_2, \dots)$ from Lemma 5.16 satisfy the conditions of Lemma 4.4.*

Proof: Let us first note that the expression $\sum_j m_{ij}$ simply counts the expected (total) number of intervals of length $\eta \cdot 2^{-(n+1)}$ in the offspring of a type i particle. An uncovered interval I of length $\eta \cdot 2^{-n}$ in generation n will get split into two uncovered intervals of length $\eta \cdot 2^{-(n+1)}$ in generation $n+1$ if no point of $\mathcal{A}_n$ covers a point of I. It thus follows immediately from the choice of η (cf. Lemma 5.15) that

$$\sum_j m_{ij} \geq \liminf_{n\to\infty} 2i \cdot \mathbb{P}_\nu([0, \eta \cdot 2^{-n}) \in \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \in \mathcal{U}_n) = c \cdot i,$$

where $c := 2 \cdot \liminf_{n\to\infty} \mathbb{P}_\nu([0, \eta \cdot 2^{-n}) \in \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \in \mathcal{U}_n) > 1$. This verifies the first condition of Lemma 4.4.

The third condition follows immediately from the fact that the total length of the offspring of a particle is never more than the length of the particle.

To see that the second condition holds, it suffices to show that the probability that a particle of type i gives birth to at least one particle of type j is bounded away from zero whenever $j \leq i$. To this end, let $\mu_n^{(i)}$ denote the expected number of points $(r,\theta) \in \mathcal{A}_n$ that cover some angle of $[0, j \cdot \eta \cdot 2^{-(n+1)})$. By an almost verbatim repeat of the computations (13) we have

$$\begin{aligned} \mu_n^{(i)} &= \int_{r_{n-1}}^{r_n} \left(j \cdot \eta \cdot 2^{-(n+1)} + 2\gamma(r)\right) \cdot g(r,\theta) \mathrm{d}r \\ &= (1 + o(1)) \cdot (j \cdot \eta/4 + (\nu\lambda/\pi) \cdot \ln 2), \end{aligned}$$

Let E denote the event that that $\mathcal{A}_n$ covers no angle of $[0, j \cdot \eta \cdot 2^{-(n+1)})$ but some angle of $[0, (j+1) \cdot \eta \cdot 2^{-(n+1)})$. Since the probability that a particle of type j is born among the offspring of a type i particle is at least the probability that E holds, we have that

$$
\begin{aligned}
m_{ij} &\geq \mathbb{P}_\nu(E) \\
&= \lim_{n\to\infty} \mathbb{P}(\mathrm{Po}(\mu_n^{(j)}) = 0) \cdot \mathbb{P}(\mathrm{Po}(\mu_n^{(j+1)} - \mu_n^{(j)}) > 0) \\
&= \lim_{n\to\infty} (\mu_n^{(j+1)} - \mu_n^{(j)}) \cdot e^{-\mu_n^{(j+1)}} \\
&= (\eta/4) \cdot e^{-(j+1)\cdot\eta/4 + (\nu\lambda/\pi)\cdot\ln 2} \\
&> 0.
\end{aligned}
$$

It remains to check that the fourth condition holds. To this end, observe that if we cut an interval of length $i \cdot \eta \cdot 2^{-n}$ into four equal parts, then if $\mathcal{A}_n$ covers at least one point in each part, then the offspring of the original type-I particle will consist of particles of types $\leq i$. Hence, we have:

$$
\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1}+z_{i+2}+\dots > 0}} p(i; z_1, z_2, \dots) \leq 1 - \liminf_{n\to\infty} (1 - e^{-\mu_n^{(\lfloor i/4 \rfloor)}})^4 = 1 - \left(1 - e^{-(\lfloor i/4 \rfloor \cdot \eta/4 + (\nu\lambda/\pi)\cdot \ln 2)}\right)^4.
$$

It is clear that if we send $i \to \infty$ then this last expression approaches zero. This proves that the fourth condition holds, and finishes the proof of the lemma. ■

Invoking Lemma 4.5, we have the following immediate corollary.

Corollary 5.18 *If $\nu \cdot \lambda < \pi$ then $\liminf_{n\to\infty} \mathbb{P}_\nu(Z$ explodes $|Z_n = e_1) > 0$.*

We are now also able to deduce:

Lemma 5.19 *If $\nu\lambda < \pi$ then $\Psi(\nu, \lambda) < 1$.*

Proof: Observe that the event that Z explodes is contained in the event that $\mathcal{C}(\lambda)$ does not occur. By Corollary 5.18 we can pick $n \in \mathbb{N}$ such that $\mathbb{P}_\nu(Z$ explodes $|Z_n = e_1) > 0$. Let E denote the event that $\mathcal{B}_n = \emptyset$, i.e. no point of $\mathcal{P}_\nu$ has radius $\leq r_n$. Then we have that

$$
\mathbb{P}_\nu(E) = \exp[-(\nu/2) \cdot (\cosh(r_n/2) - 1)] > 0.
$$

We have

$$
\begin{aligned}
1 - \Psi(\nu, \lambda) &= \mathbb{P}_\nu(\text{not } \mathcal{C}(\lambda)) \\
&\geq \mathbb{P}_\nu(E) \cdot \mathbb{P}_\nu(Z \text{ explodes} | E) \\
&\geq \mathbb{P}_\nu(E) \cdot \mathbb{P}_\nu(Z \text{ explodes} | Z_n = e_1) \\
&> 0,
\end{aligned}
$$

where the penultimate inequality holds by obvious monotonicity. ■

Lemma 5.20 *For every $\lambda > 0$ it holds that $\lim_{\nu \downarrow 0} \Psi(\nu, \lambda) = 0$*

Proof: The proof is very similar to the previous lemma. Let us first observe that for every fixed n the conditional probability $\mathbb{P}_\nu(Z$ explodes $|Z_n = e_1)$ is nonincreasing in ν. (This can for instance be seen by noting that a Poisson process with intensity function $g_{\nu+\delta}(r, \theta)$ is the superposition of one with density function g_ν and one with density function g_δ.) Hence we can find an $n_0 \in \mathbb{N}$ and $c > 0$ such that $\mathbb{P}_\nu(Z \text{ explodes}|Z_n = e_1) \geq c$ for all $n \geq n_0$ and all $0 < \nu < 1$. Now note that for every $K > 0$, there exists an n such that among $[0, \eta \cdot 2^{-n}), \ldots, [2\pi - \eta \cdot 2^{-n}, 2\pi)$ there are at least K intervals that are separated by pairwise distance of at least $2\gamma(r_n)$. Fix such an n, and let E denote the event that no point fell inside $\mathcal{B}_n$.

Then we have

$$1 - \Psi(\nu,\lambda) \geq \mathbb{P}_\nu(E) \cdot \big(1 - \mathbb{P}_\nu(Z \text{ dies out } |Z_n = e_1)^K\big) \geq e^{-(\nu/2)\cdot(\cosh(r_n/2)-1)} \cdot (1-(1-c)^K).$$

Let $\varepsilon > 0$ be arbitrary. By choosing K sufficiently large, we can ensure that $(1-c)^K < \varepsilon$. It follows that

$$\lim_{\nu \downarrow 0} \Psi(\nu,\lambda) \leq 1 - \lim_{\nu\downarrow 0} e^{-(\nu/2)\cdot(\cosh(r_n/2)-1)} \cdot (1-\varepsilon) = \varepsilon.$$

Sending ε to zero finishes the proof. ■

Let $L_n^\eta = L_n^\eta(\lambda)$ denote the total length of all components of $L_n(\lambda)$ that have length at least $\eta \cdot 2^{-n}$. As usual, when λ is clear from the context we omit it. A similar proof to that of the previous lemma also gives the following.

Lemma 5.21 *If $\nu\lambda < \pi$ and $K > 0$ arbitrary then $\mathbb{P}_\nu(L_n^\eta > K$ almost always $) = 1 - \Psi(\nu,\lambda)$.*

Proof: Observe that if $L_n^\eta > K$ almost always, then $\mathcal{C}(\lambda)$ certainly does not occur. This shows that

$$\mathbb{P}_\nu(L_n^\eta > K \text{ almost always }) \leq 1 - \Psi(\nu,\lambda).$$

Also observe that if Z explodes, then we also have that $L_n^\eta > K$ almost always.

Now let $\varepsilon > 0$ be arbitrary and let us fix a $K' = K'(\varepsilon, K)$, to be made precise later. By Lemma 5.11, we have that $\mathbb{P}_\nu(L_n^{\text{long}} > K' \text{ infinitely often }) = 1 - \Psi(\nu,\lambda)$. As in the proof of the previous lemma, we can pick $n_0, c > 0$ such that $\mathbb{P}_\nu(Z \text{ explodes } |Z_n = e_1) \geq c$ for all $n \geq n_0$.

Observe that if $L^{\text{long}} > K'$ then we can find a family of at least

$$M := \left\lceil \frac{K' \cdot 2^{-n}}{\eta \cdot 2^{-n} + 2\gamma(r_n)} \right\rceil,$$

intervals of length $\eta \cdot 2^{-n}$ in $\mathcal{U}_n$ that are separated by pairwise distance $2\gamma(r_n)$. By (6), we have that $M > K'/10$ for sufficiently large n.

Now consider the following setup. We let N denote the (random) first integer after n_0 for which $L_n^{\text{long}} > K'$, where $N = \infty$ if there is no such N. Note that the event $N = n$ is independent of $\mathcal{P} \setminus B_{\mathbb{H}}(O; r_n)$. This shows that

$$\begin{array}{rcl}
\mathbb{P}_\nu(Z \text{ explodes}) & \geq & \sum_{n=n_0}^\infty \mathbb{P}_\nu(N=n) \cdot \big(1 - \mathbb{P}_\nu(Z \text{ dies out}|Z_n = e_1)^M\big) \\
& \geq & \sum_{n=n_0}^\infty \mathbb{P}_\nu(N=n) \cdot \big(1 - (1-c)^M\big) \\
& \geq & \sum_{n=n_0}^\infty \mathbb{P}_\nu(N=n) \cdot (1-\varepsilon) \\
& = & \mathbb{P}_\nu(N < \infty) \cdot (1-\varepsilon) \\
& \geq & \mathbb{P}_\nu(L_n^{\text{long}} > K' \text{ infinitely often }) \cdot (1-\varepsilon) \\
& = & (1 - f(\nu)) \cdot (1-\varepsilon).
\end{array}$$

Sending ε to zero gives the lemma. ■

Let us define

$$\Psi_n(\nu, \lambda) := \mathbb{P}_\nu(\mathcal{C}_{r_n}(\lambda)).$$

In other words, Ψ_n is the probability that $\mathcal{B}_n$ is a cover.

Lemma 5.22 *Let $s > 0$ be fixed, but arbitrary. Let F be any event that depends only on $\mathcal{P}_\nu \cap B_{\mathbb{H}}(0, s)$ (i.e. F depends only on the points of radius less than s), and set $\varphi(\nu) := \mathbb{P}_\nu(F)$. Then φ is a continuous function of ν.*

Proof: Let Y denote the number of points of $\mathcal{P}$ with radius at most s. Then Y is Poisson-distributed with mean $\mathbb{E}Y = \nu \cdot (\cosh(s/2) - 1)$. Let us remark that

$$a_t := \mathbb{P}_\nu(F|Y = t),$$

is independent of ν. (To see this, note that if we condition on $Y = t$ then the points of $\mathcal{P}$ with radius $\leq s$ behave like an i.i.d. sample $X_1, \ldots, X_t$ with common density function

$$h(r, \theta) = \frac{g(r, \theta)}{\int_0^{2\pi} \int_0^s g(t, \beta) \mathrm{d}t\mathrm{d}\beta} = \frac{\sinh(r/2)}{2\pi \cdot (\cosh(s/2) - 1)}.$$

The function h is clearly independent of ν.) We clearly have

$$\varphi(\nu) = \sum_{t=0}^{\infty} a_t \cdot \mathbb{P}_\nu(Y = t).$$

Let us now fix an arbitrary $\varepsilon > 0$. Set $K := 1000 \cdot \mathbb{E}_\nu Y/\varepsilon$. By Markov's inequality we have $\mathbb{P}_\mu(Y \geq K) \leq \mathbb{E}_\mu Y/K \leq \varepsilon/2$, for all $\mu < 500\nu$. Hence, for all $\mu < 500\nu$ we have

$$\left| \varphi(\mu) - \sum_{t=0}^{K} a_t \cdot p_t(\mu) \right| < \varepsilon/2,$$

where $p_t(\mu) := \mathbb{P}_\mu(Y = t) = (\mu \cdot (\cosh(s/2) - 1))^t \cdot e^{-\mu \cdot (\cosh(s/2)-1)}/t!$. Now observe that p_t is a continuous function of μ for every (fixed) t. It follows that there is a $\delta > 0$ such that if $|\mu - \nu| < \delta$ then $|p_t(\mu) - p_t(\nu)| < \varepsilon/2(K + 1)$ for all $0 \leq t < K$. Hence we also have that $|\varphi(\mu) - \varphi(\nu)| < \varepsilon$ whenever $|\mu - \nu| < \min(\delta, 499\nu)$. This proves that φ is continuous as claimed. ■

Corollary 5.23 *For every $n \in \mathbb{N}$, the function Ψ_n is continuous in its first parameter, ν.*

Lemma 5.24 *For every $n \in \mathbb{N}$, the function Ψ_n is continuous in its second parameter, λ.*

Proof: Let us fix ν. Let us take $\lambda_1 < \lambda_2$ and let us write $\gamma_i(r) = \lambda_i \arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right)$ for $i = 1, 2$. Note that $\Psi_n(\nu, \lambda_2) - \Psi_n(\nu, \lambda_1)$ is precisely the probability of the event E that $\bigcup_{(r,\theta)\in\mathcal{B}_n}(\theta - \gamma_2(r), \theta + \gamma_2(r))$ covers all angles, but some angle is not covered by $\bigcup_{(r,\theta)\in\mathcal{B}_n}(\theta - \gamma_1(r), \theta + \gamma_1(r))$.

Next, let us observe that if E holds then there must exist two points $(r, \theta), (s, \vartheta) \in \mathcal{B}_n$ such that

$$\gamma_1(r) + \gamma_1(s) < |\theta - \vartheta|_{2\pi} < \gamma_2(r) + \gamma_2(s). \tag{15}$$

(Consider some component I of $\mathcal{U}_n$ under λ_1. The leftmost endpoint of this interval is the rightmost endpoint of $(\theta - \gamma_1(r), \theta + \gamma_2(r))$ for some $(r, \theta) \in \mathcal{B}_n$. Since $\mathcal{C}(\lambda)$ occurs at λ_2, it must be the case that $\theta + \gamma_2(r)$ is inside some interval $(\vartheta - \gamma_2(s), \vartheta + \gamma_2(s))$.) From this it follows that

$$\mathbb{P}_\nu(E) \leq (\mathbb{E}_\nu|\mathcal{B}_n|)^2 \cdot \mathbb{P}_\nu\left(|\theta - \vartheta|_{2\pi} \in (\gamma_1(r) + \gamma_1(s), \gamma_2(r) + \gamma_2(s)\right),$$

where $(r, \theta), (s, \vartheta)$ are chosen i.i.d. according to the distribution with density $g / \int_{B_{\mathbb{H}}(O,R)} \int_0^{2\pi} g$. (We used Palm Theory for counting the number of pairs with this property.)

Now note that the length of the interval $(\lambda_1(r) + \lambda_1(s), \lambda_2(r) + \lambda_2(s))$ is at most $2(\lambda_2 - \lambda_1) \lim_{x \downarrow 0} \arccos\left(\frac{\cosh(x)-1}{\sinh(x)}\right) = (\lambda_2 - \lambda_1) \cdot \pi$. It follows that

$$\mathbb{P}_\nu(E) \leq (\mathbb{E}_\nu|\mathcal{B}_n|)^2 \cdot \frac{\lambda_2 - \lambda_1}{2}.$$

Thus, by choosing λ_1, λ_2 such that $\lambda_2 - \lambda_1 < 2\varepsilon / (\mathbb{E}_\nu|\mathcal{B}_n|)^2$, we can ensure that $|\Psi_n(\nu, \lambda_2) - \Psi_n(\nu, \lambda_1)| \leq \mathbb{P}_\nu(E) < \varepsilon$. This proves that Ψ_n is indeed continuous in λ. ∎

Next, we define, for every $\eta, K > 0$ and $n \in \mathbb{N}$:

$$\Phi_{n,\eta,K}(\nu, \lambda) := \mathbb{P}_\nu(L_n^\eta > 0).$$

By an application of Lemma 5.22, we find that:

Corollary 5.25 $\Phi_{n,\eta,K}$ *is continuous in its first parameter, ν. (For every $\eta, K > 0$ and $n \in \mathbb{N}$.)*

Lemma 5.26 $\Phi_{n,\eta,K}$ *is continuous in its second parameter, λ. (For every $\eta, K > 0$ and $n \in \mathbb{N}$.)*

Proof: To begin, we fix $\nu, \lambda, \eta, K > 0$ and $n \in \mathbb{N}$. Observe that there exists some $\delta > 0$ such that

$$\mathbb{P}_\nu(L_n^\eta \geq K + \delta) \geq \Phi_{n,\eta,K}(\nu, \lambda_1) - \varepsilon/3. \tag{16}$$

Similarly, we may assume that δ is small enough so that

$$\mathbb{P}_\nu(\mathcal{U}_n \text{ has a component of length} \in [\eta 2^{-n} - \delta, \eta 2^{-n} + \delta]) < \varepsilon/3. \tag{17}$$

(Arguing as in the proof of Lemma 5.24, but now considering pair of points whose distance is close to $\gamma(r) + \gamma(s) + \eta 2^{-n}$.)

Finally let us pick some $\lambda' \neq \lambda$, and let X denote the sum $\sum_{(r,\theta) \in \mathcal{B}_n} 2|\lambda' - \lambda| \arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right)$. (I.e., X is the sum over all points in $\mathcal{B}_n$ of the difference in the covered length under the two choices of the parameter λ.) Using Markov's inequality, we have that

$$\mathbb{P}_\nu(X > \delta) \leq \frac{\mathbb{E}_\nu X}{\delta} \leq \mathbb{E}_\nu|\mathcal{B}_n| \cdot \pi \cdot |\lambda' - \lambda| < \varepsilon/3, \tag{18}$$

we the last inequality holds for $|\lambda' - \lambda|$ sufficiently small.

Observe that if $L_n^\eta \geq K + \delta$ with respect to λ, there are no components in $\mathcal{U}_n$ of length $\in [\eta 2^{-n} - \delta, \eta 2^{-n} + \delta]$, and $X \leq \delta$, then $L_n^\eta > K$ with respect to λ. Thus, combining (16), (17) and (18), we have proved the lemma. ■

Lemma 5.27 Ψ *is continuous.*

Proof: Let $\nu, \lambda > 0$ be abtritrary. We first assume that $\nu\lambda \geq \pi$. In this case $\Psi(\nu, \lambda) = 1$ by Corollary 5.14. Note that, since $\mathcal{C}(\lambda) = \bigcup_n \mathcal{C}_{r_n}(\lambda)$, there exists an n such that $\Psi_n(\nu, \lambda) \geq 1 - \varepsilon/2$. Since Ψ_n is continuous, there is a $\delta > 0$ such that

$$\Psi(\nu', \lambda') \geq \Psi_n(\nu', \lambda') \geq \Psi_n(\nu, \lambda) - \varepsilon/2 \geq 1 - \varepsilon,$$

for all $\nu' \in (\nu - \delta, \nu + \delta)$ and $\lambda' \in (\lambda - \delta, \lambda + \delta)$. This shows Ψ is continuous at ν, λ.

Let us then assume that $\nu\lambda < \pi$. Let us pick $\nu' > \nu, \lambda' > \lambda$ such that still $\nu'\lambda' < \pi$; and let $n_0 \in \mathbb{N}, c > 0$ be such that

$$\mathbb{P}_{\nu'}(Z^{\lambda'} \text{ explodes } | Z_n^{\lambda'} = e_1) \geq c,$$

for all $n \geq n_0$. Note that, by obvious monotonicity, this inequality also holds for all $\nu'' < \nu', \lambda'' < \lambda'$ (here we keep η, used in the definition of the process Z, fixed).

Let $\varepsilon > 0$ be arbitrary and let $K = K(\varepsilon)$ be fixed to be made precise later. Since $\Psi(\nu, \lambda) = \lim_{n\to\infty} \Psi_n(\nu, \lambda)$, we can find an n_1 such that $|\Psi_n(\nu, \lambda) - \Psi(\nu, \lambda)| < \varepsilon/2$ for all $n \geq n_1$. Similarly, since

$$1 - \Psi(\nu, \lambda) = \mathbb{P}_\nu(L_n^\eta(\lambda) > K \text{ almost always }) = \lim_{n\to\infty} \mathbb{P}_\nu(L_m^\eta(\lambda) > K \text{ for all } m \geq n \text{ }),$$

we can fix an n_2 such that $\Phi_{n,\eta,K}(\nu, \lambda) = \mathbb{P}_\nu(L_n^\eta > K) \geq 1 - \Psi(\nu, \lambda) - \varepsilon/2$ for all $n \geq n_2$.

Let us now fix $n := \max\{n_0, n_1, n_2\}$ and put $\varphi(\nu) := \mathbb{P}_\nu(L_n = 0), \psi(\nu) = \mathbb{P}_\nu(Z_n > K)$.

Since both Ψ_n and $\Phi_{n,\eta,K}$ are continuous, we can pick a $\delta > 0$ such that $|\Psi_n(\nu'', \lambda'') - \Psi_n(\nu, \lambda)| < \varepsilon/2$ and $|\Phi_{n,\eta,K}(\nu'', \lambda'') - \Phi_{n,\eta,K}(\nu, \lambda)| < \varepsilon/2$ for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$. We assume without loss of generality that $\delta < \min(\lambda' - \lambda, \nu' - \nu)$.

Now note that if $L_n^\eta(\lambda) > K$ then there are at least

$$M := \left\lceil \frac{K \cdot \eta \cdot 2^{-n}}{\eta \cdot 2^{-n} + 2\gamma(r_n)} \right\rceil = \Omega(K),$$

intervals of length at least $\eta \cdot 2^{-n}$ that are contained in $\mathcal{U}_n$ and that are separated by pairwise distance $2\gamma(r_n)$. It follows that, for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$, we have

$$\begin{array}{rcl}
\mathbb{P}_{\nu''}(L_m^\eta(\lambda'') > K \text{ almost always} | L_n^\eta(\lambda'') > K) & \geq & 1 - \mathbb{P}_{\nu''}(Z(\lambda'') \text{ dies out } | Z_n(\lambda'') = e_1)^M \\
& \geq & 1 - (1 - c)^M \\
& \geq & 1 - \varepsilon/2,
\end{array}$$

where the last inequality holds provided we chose K sufficiently large (which we can assume without loss of generality). We thus get that

$$\begin{array}{rcl} 1-\Psi(\nu'',\lambda'') & = & \mathbb{P}_{\nu'',\lambda''}(\text{not } \mathcal{C}(\lambda)) \\ & \geq & \mathbb{P}_{\nu'',\lambda''}(Z \text{ explodes } | L_n^\eta > K)\Phi_{n,\eta,K}(\nu'',\lambda'') \\ & \geq & (1-\varepsilon/2)\cdot(1-\Psi(\nu,\lambda)-\varepsilon/2) \\ & \geq & 1-\Psi(\nu,\lambda)-\varepsilon, \end{array}$$

for all $\nu'' \in (\nu-\delta,\nu+\delta)$ and $\lambda'' \in (\lambda-\delta,\lambda+\delta)$. In other words, $\Psi(\nu'',\lambda'') \leq \Psi(\nu,\lambda)+\varepsilon$ for all $\nu'' \in (\nu-\delta,\nu+\delta)$ and $\lambda'' \in (\lambda-\delta,\lambda+\delta)$. On the other hand we have

$$\Psi(\nu'',\lambda'') \geq \Psi_n(\nu'',\lambda'') \geq \Psi(\nu,\lambda)-\varepsilon,$$

for all $\nu'' \in (\nu-\delta,\nu+\delta)$ and $\lambda'' \in (\lambda-\delta,\lambda+\delta)$, by choice of n and δ. We have seen that Ψ is continuous at (ν,λ) as required. ■

We have already proved Theorem 3.1, but for completeness we collect our findings from this Section in an explicit proof.

Proof of Theorem 3.1: That Ψ is continuous was just established in the previous lemma. That $\Psi(\nu,\lambda)=1$ when $\nu\lambda \geq \pi$ was established in Corollary 5.14. That Ψ is strictly increasing at every point $(\nu,\lambda$ with $\nu\lambda < \pi$ follows from Corollary 5.3 together with Lemma 5.19. That $\lim_{\nu\downarrow 0}\Psi(\nu,\lambda)=0$ was established in Lemma 5.20. ■

We are now ready to prove Lemma 3.4. We need the following geometric result, which we prove in the appendix. We need the following geometric fact.

Lemma 5.28 *Suppose that $p=(r,\theta), q=(s,\vartheta)$ are two points in the hyperbolic plane satisfying* $\mathrm{dist}_{\mathbb{H}}(p,O), \mathrm{dist}_{\mathbb{H}}(q,O), \mathrm{dist}_{\mathbb{H}}(p,q) \leq R$ *and let $p'=(r',\theta), q'=(s',\vartheta)$ with $r' \leq r, s' \leq s$. Then* $\mathrm{dist}_{\mathbb{H}}(p',q') \leq R$.

Proof of Lemma 3.4: If $\nu > \pi$ then there is nothing to prove as $\Psi(\nu,1)=1$. Let us thus suppose that $\nu < \pi$ so that $\Psi(\nu,1)<1$. Reformulating, it suffices to show that

$$\liminf_{N\to\infty} \mathbb{P}(G(N;1/2,\nu) \text{ is NOT connected }) \geq 1-\Psi(\nu,1).$$

Pick a $\delta>0$ such that $\Psi(\nu+\delta,1+\delta) \leq \Psi(\nu,1)+\varepsilon/2$ and write $\mu := \nu+\delta, \lambda := 1+\delta$. Let K be large but fixed, to be made more precise later; and let $\eta=\eta(\mu,\lambda)$ be as in Lemma 5.15. By Lemma 5.21, there exist an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$:

$$\Phi_{n,\eta,K}(\mu,\lambda) = \mathbb{P}_\nu(L_n^\eta > K) \geq 1-\Psi(\mu,\lambda)-\varepsilon/2.$$

Now let $n := \lfloor R/2\ln 2 \rfloor - 1$, and let F denote the event that $L_n^\eta > K$ (with respect to μ,λ). Given that F holds, we can pick $M=\Omega(K)$ intervals $I_1,\dots,I_M \subseteq \mathcal{U}_n$ of length $\eta 2^{-n}$ such that the angle between a point in I_i and a point in I_j is at least $1000\cdot 2^{-n}$ (for all $1 \leq i \neq j \leq M$). Now let F_i denote the event that there is *exacltly one* point $X_\ell=(\rho_\ell,\theta_\ell) \in V_{\mathrm{Poi}}$ such that **1)** $R-\varepsilon < \rho_\ell \leq R$ and $\theta_\ell \in I_i$ and **2)** there is no point of $X_m=(\rho_m,\theta_m) \in \mathcal{P}_\mu$ with $\rho_m > r_n$ and θ_m within angle $10\cdot 2^{-n}$ of one of the endpoints of I_i. Observe that

$$\mathbb{P}_\nu(F_i|F) = \mathbb{P}(\mathrm{Po}(\mu_1)=1)\mathbb{P}(\mathrm{Po}(\mu_2)=0) = \Theta(1),$$

where $\mu_1 := \eta\cdot 2^{-n}\cdot(\nu/4\pi)\cdot(\cosh(R/2)-\cosh((R-\varepsilon)/2))$ and $\mu_2 := 20\cdot 2^{-n}\cdot(\nu/4\pi)\cdot(\cosh(R/2)-\cosh(r_n/2))-\mu_1$. (That both μ_1,μ_2 are $\Theta(1)$ follows from the fact that

$\cosh(R/2), \cosh((R-\varepsilon)/2), \cosh(r_n/2) = \Theta(2^n)$.) Note also that the event F_i-s are independent (given F). Hence we have

$$\mathbb{P}\left(\bigcup F_i | F\right) \geq 1 - (1-\Theta(1))^M > 1 - \varepsilon/2,$$

provided we chose K sufficiently large.

We now claim that, if F and some F_i hold, then there is a point $X_j \in W := \mathcal{P}_\mu \cap B_{\mathbb{H}}(O;R)$ that is at distance $> R$ from all other points in W (namely the sole vertex $X_j = (\rho_j, \theta_j)$ with angle in $\theta_j \in I_i$ and radius $\rho_j > R - \delta$). To see this, let $X_k = (\rho_k, \theta_k) \in W$ be an arbitrary other point. If $\rho_k > r_n$ we have $|\theta_j - \theta_k|_{2\pi} > 10 \cdot 2^{-n}$. On the other hand, we have $\operatorname{dist}_{\mathbb{H}}(X_j, X_k) \leq \operatorname{dist}_{\mathbb{H}}(X_j', X_k')$ where $X_j' = (r_n, \theta_j), X_k' = (r_n, \theta_k)$ by Lemma 5.28. Hence, by the hyperbolic cosine rule $\operatorname{dist}_{\mathbb{H}}(X_j, X_k) \leq R$ only if the difference in angle $|\theta_j - \theta_k|_{2\pi}$ is at most

$$\arccos\left(\frac{\cosh^2(r_n) - \cosh(R)}{\sinh^2(r_n)}\right) = \arccos\left(1 - O(e^{-r_n})\right) = (1+o(1))2e^{-r_n/2} = (1+o(1))\cdot 2^{-(n-1)}.$$

It follows $\operatorname{dist}_{\mathbb{H}}(X_j, X_k) > R$.

Now suppose that $\rho_k < r_n$. Since $\theta_j \in \mathcal{U}_n$ it follows that

$$|\theta_j - \theta_k|_{2\pi} > (1+\delta)\arccos\left(\frac{\cosh(r_k) - 1}{\sinh(r_k)}\right).$$

Now observe that, for $\operatorname{dist}_{\mathbb{H}}(X_j, X_k) < R$ to hold, the angle between them can be at most $\arccos\left(\frac{\cosh(r_j)\cosh(r_k) - \cosh(R)}{\sinh(r_j)\sinh(r_k)}\right)$, by the hyperbolic cosine rule. Since $r_j \in (R-\varepsilon, R)$ we have that $\cosh(r_j) = (1+O(\varepsilon))\cosh(R)$ and $\sinh(r_j) = (1+O(\varepsilon))\cosh(R)$. This also gives that

$$\frac{\cosh(r_j)\cosh(r_k) - \cosh(R)}{\sinh(r_j)\sinh(r_k)} = (1+O(\varepsilon) \cdot \frac{\cosh(r_k) - 1}{\sinh(r_k)}.$$

Using Taylor's expansion $\arccos(x+y) = arccos(x) + O(y/(1-x^2)^{1/2})$, we now find

$$\begin{aligned}\arccos\left(\tfrac{\cosh(r_j)\cosh(r_k) - \cosh(R)}{\sinh(r_j)\sinh(r_k)}\right) &= \arccos\left(\tfrac{\cosh(r_k)-1}{\sinh(r_k)}\right) + O(\varepsilon e^{-r_k/2}) \\ &= (1+O(\varepsilon)) \cdot \arccos\left(\tfrac{\cosh(r_k)-1}{\sinh(r_k)}\right).\end{aligned}$$

(Using that $(\cosh(r_k) - 1)/\sinh(r_k) = 1 - O(e^{-r})$. It follows that $\operatorname{dist}_{\mathbb{H}}(X_j, X_k) > R$, as claimed. Hence if $(\bigcup F_j) \cap F$ has been realized, then at least one point of W will have distance larger than R to all other points of W.

We wish now to deduce that in such a case, $G(N; 1/2, \nu)$ will have an isolated vertex, but as it happens V_N is a strict subset of W. To get around this problem, we use the coupling from Lemma 3.2, and symmetry. Suppose that $(\bigcup F_j) \cap F$ holds, and choose a point X_j of distance $> R$ to all other points (uniformly at random from all such points, say). By symmetry considerations, under the coupling from Lemma 3.2 the probability that X_j is also a point of $\mathcal{P}_{\nu-\delta}$ is $\frac{\nu-\delta}{\nu+\delta} = 1 - O(\delta)$. Putting everything together, we find that

$$\begin{aligned}\mathbb{P}(G(N; 1/2, \nu) \text{ has an isolated vertex}) &\geq \mathbb{P}\left(\bigcup F_i | F\right)\mathbb{P}_\nu(F) - O(\delta) - \mathbb{P}(\text{coupling fails}) \\ &\geq (1-\varepsilon/2)\cdot(1-\Psi(\mu,\lambda) - \varepsilon/2) - O(\delta) - o(1) \\ &\geq (1-\varepsilon/2)\cdot(1-\Psi(\nu,1) - \varepsilon) - O(\delta) - o(1).\end{aligned}$$

Sending ε, δ to zero gives the lemma. ■

A Appendix

Before giving the proof of Lemma 5.28, let us remind the reader that disks are convex, also in the hyperbolic plane. This means that if D is a disk in the hyperbolic plane and $x, y \in D$ then the geodesic between x, y is contained in D. One way to see this is by noting that every disk can be isometrically mapped to a disk with origin O, and that in the projective disk model of the hyperbolic plane (a.k.a. the Beltrami-Klein model) a hyperbolic disk with origin O looks like a Euclidean disk, while geodesics are just line segments in the projective disk model. (See for instance Section 4.8 of [17] for a description of the projective disk model.)

Proof of Lemma 5.28: It is enough to consider the case when $r' < r$ and $s' = s$. (Another application of this case will then give the full result.) Observe that the geodesic between O and p is just the line segment between them. So in particular, p' lies on the geodesic between O and p. Since $O, p \in B_{\mathbb{H}}(q; R)$ it follows that also $p' \in B_{\mathbb{H}}(q; R)$, as required. ■

Proof of Lemma 1.1: Note that in the above theorem R' is chosen such that $N = \nu e^{\zeta R/2} = \nu e^{\zeta' R'/2}$.

The desired coupling is constructed as follows. We pick $\theta_1, \dots, \theta_N$ i.i.d. uniform on $[0, 2\pi)$ and we pick $U_1, \dots, U_N$ i.i.d. uniform on $[0, 1]$.

We now let $\rho_1, \dots, \rho_N$ and $\rho_1', \dots, \rho_N'$ be defined by the equations:

$$F_{\alpha,R}(\rho_i) = F_{\alpha',R'}(\rho_i') = U_i \qquad \text{(for } i = 1, \dots, N.\text{)} \tag{19}$$

(Note that in this way the ρ_is have exactly the distribution with cdf $F_{\alpha,R}$ and the ρ_i's have cdf $F_{\alpha',R'}$.) The points used in the construction of $G(R, \zeta, \alpha, \nu)$ will be $(\theta_1, \rho_1), \dots, (\theta_N, \rho_N)$ while the points used in the construction of $G(R', \zeta', \alpha', \nu)$ will be $(\theta_1, \rho_1'), \dots, (\theta_N, \rho_N')$.

It remains to be seen that this way we get two isomorphic graphs.

Claim A.1 *We have $\rho_i' = (\alpha/\alpha')\rho_i$ for all i.*

Proof: Observe that

$$\alpha' R' = \alpha' \cdot (\zeta/\zeta')R = \alpha' \cdot (\alpha/\alpha')R = \alpha R.$$

Thus, the equation (19) defining ρ_i and ρ_i' yields:

$$\cosh(\alpha\rho_i) = \cosh(\alpha'\rho_i').$$

Since $\cosh(x)$ is stricly increasing for $x \geq 0$, it follows that we must have $\alpha\rho_i = \alpha'\rho_i'$. ■

Let us write d_{ij} for the distance between (θ_i, ρ_i) and (θ_j, ρ_j) in the curvature-ζ-surface, and let d_{ij}' be defined analogously.

Claim A.2 *For all i, j we have $d_{ij}' = (\alpha/\alpha')d_{ij}$.*

Proof: By the *hyperbolic cosine rule* we have that

$$\cosh(\zeta d_{ij}) = \cosh(\zeta\rho_i)\cosh(\zeta\rho_j) - \sinh(\zeta\rho_i)\sinh(\zeta\rho_j)\cos(|\theta_i - \theta_j|),$$

and

$$\cosh(\zeta' d'_{ij}) = \cosh(\zeta' \rho'_i)\cosh(\zeta' \rho'_j) - \sinh(\zeta' \rho'_i)\sinh(\zeta' \rho'_j)\cos(|\theta_i - \theta_j|).$$

Now observe that

$$\zeta \rho_i = \alpha \cdot (\zeta/\alpha) \cdot \rho_i = \alpha \cdot (\zeta'/\alpha')\rho_i = \zeta' \rho'_i,$$

using Claim A.1, and similarly $\zeta \rho_j = \zeta' \rho'_j$. It follows that

$$\cosh(\zeta' d'_{ij}) = \cosh(\zeta d_{ij}).$$

Again using that $\cosh(x)$ is strictly increasing for $x \geq 0$ (and the distances d_{ij}, d'_{ij} are non-negative), we see that $d'_{ij} = (\zeta/\zeta')d_{ij} = (\alpha/\alpha')d_{ij}$. ■

Since $R' = (\zeta/\zeta')R = (\alpha/\alpha')R$, we see that

$$d_{ij} \leq R \text{ iff. } d'_{ij} \leq R',$$

which proves the lemma. ■

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