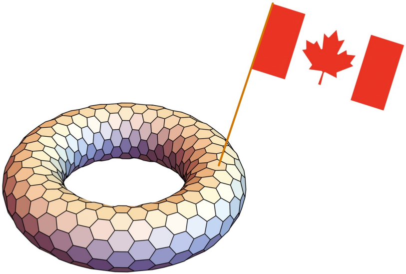
Canonical decompositions of 3-connected graphs

Jan Kurkofka

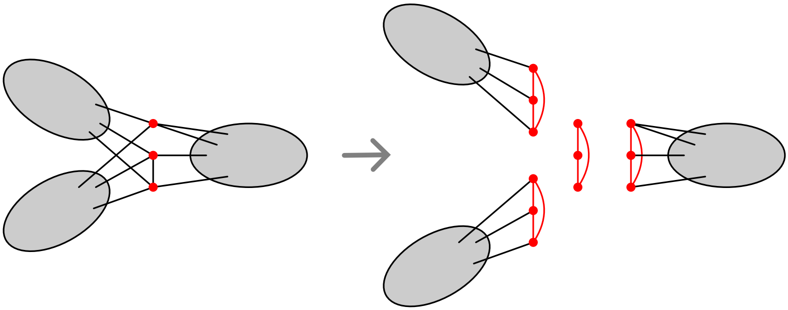


Joint work with Johannes Carmesin

University of Birmingham

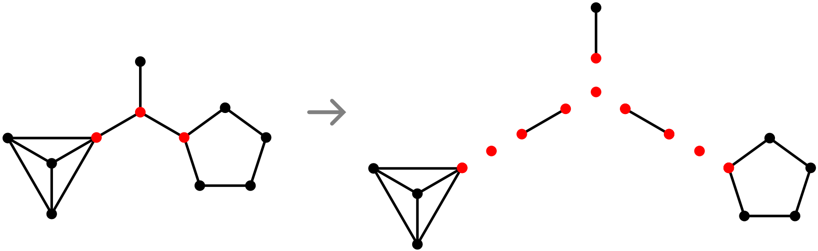
Problem: Decompose k-con'd G along k-separators
into pieces that are $(k + 1)$-con'd or 'basic'.

Decomposing G along a k-separator:



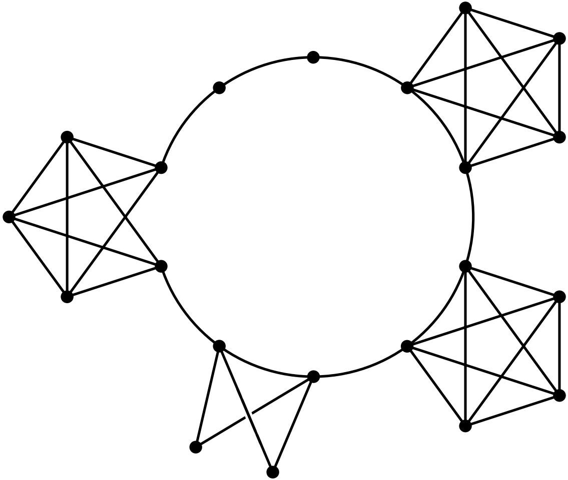
Problem: Decompose k-con'd G along k-separators into pieces that are $(k+1)$-con'd or 'basic'.

$k = 1$:



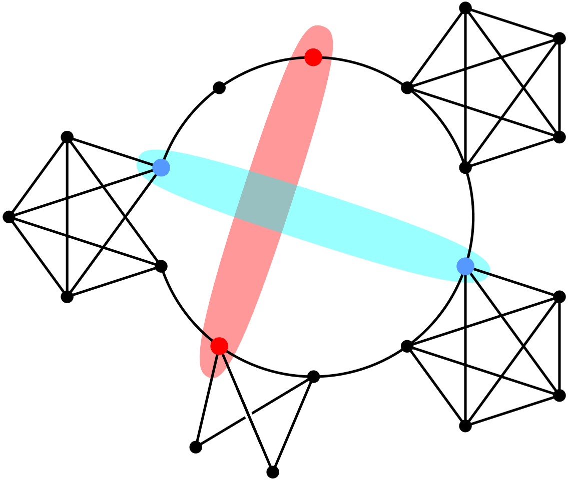
Problem: Decompose k-con'd G along k-separators
into pieces that are $(k + 1)$-con'd or 'basic'.

$k = 2$:



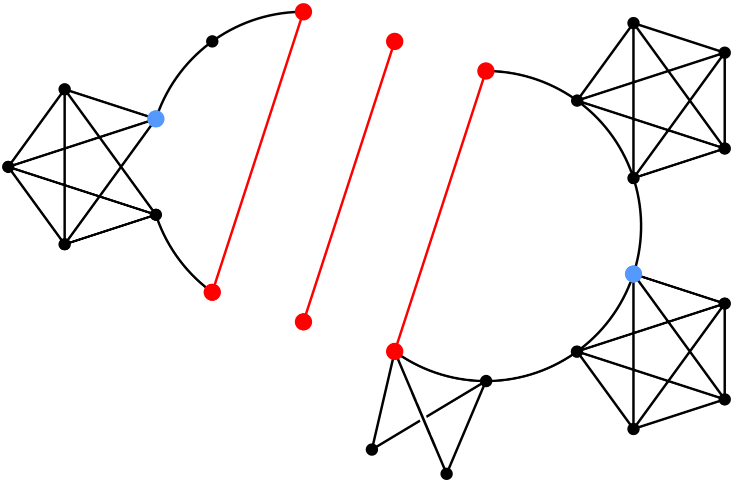
Problem: Decompose k-con'd G along k-separators
into pieces that are $(k+1)$-con'd or 'basic'.

$k = 2$:

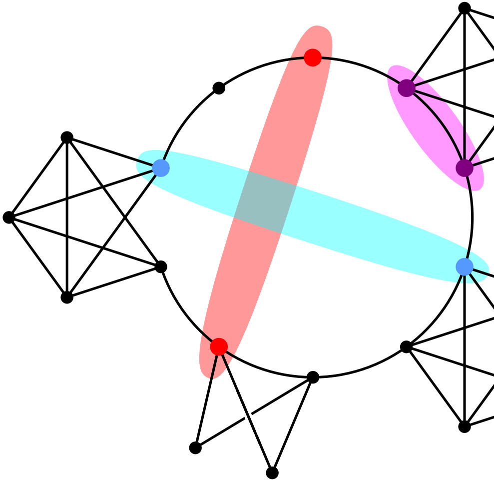


Problem: Decompose k-con'd G along k-separators into pieces that are $(k + 1)$-con'd or 'basic'.

$k = 2$:

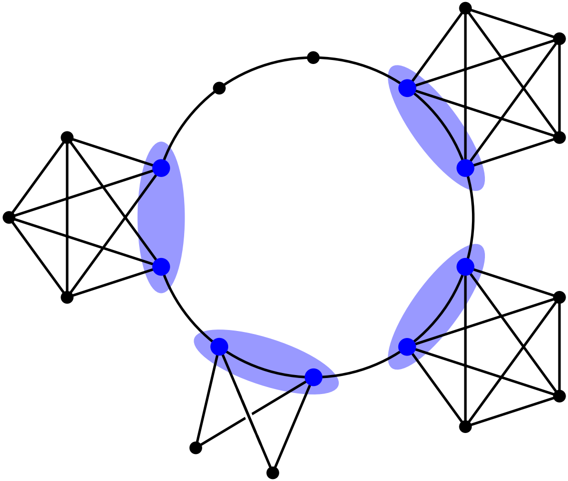


Two k-separators are *nested* if neither separates the other; otherwise they *cross*.



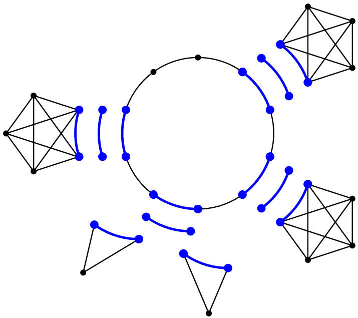
Two k-separators are *nested* if neither separates the other;
otherwise they *cross*.

A k-separator is *totally-nested* if it is nested with every k-separator.



Two k-separators are *nested* if neither separates the other;
otherwise they *cross*.

A k-separator is *totally-nested* if it is nested with every k-separator.

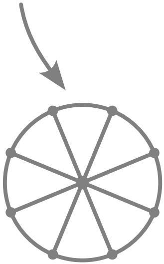


Theorem (Cunningham & Edmonds 80)

Every 2-con'd G decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and K_2's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



Theorem (Cunningham & Edmonds 80)

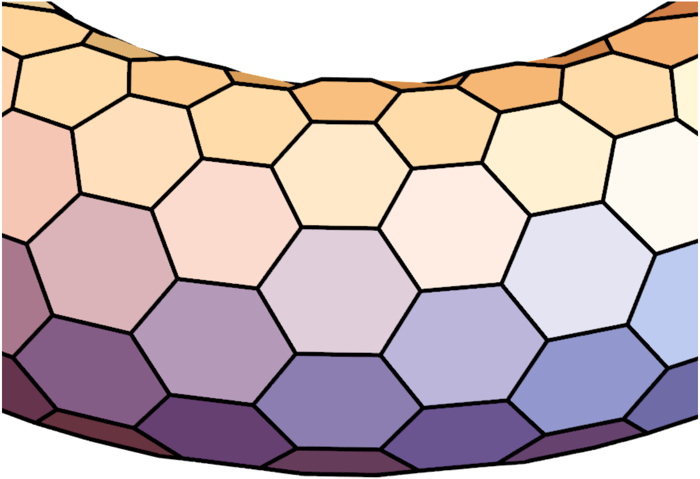
Every 2-con'd G decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and K_2's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.

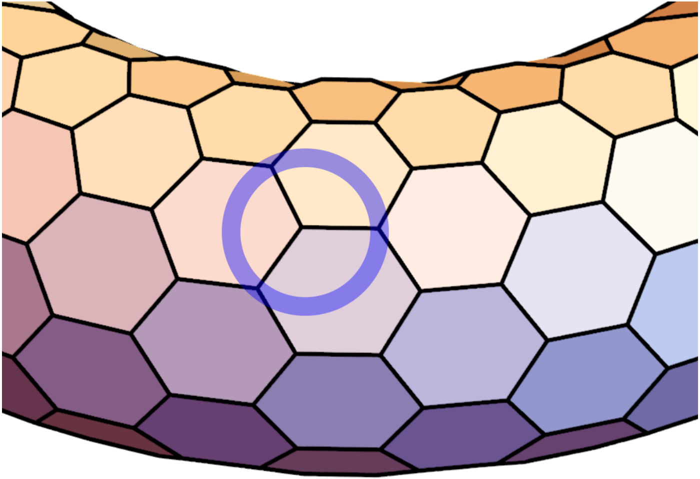
Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



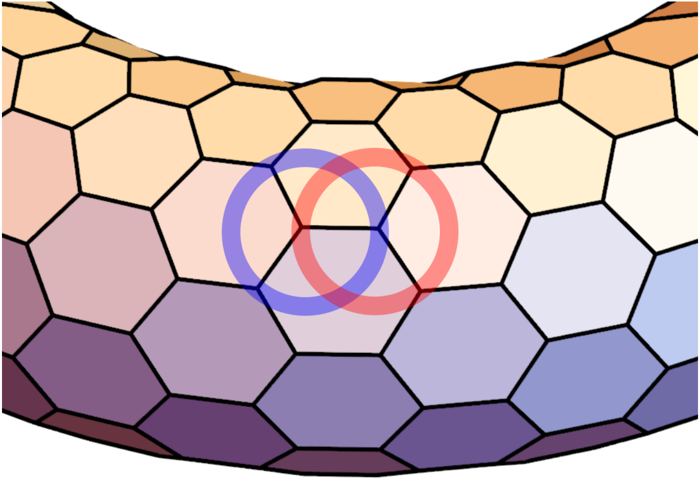
Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



Guess

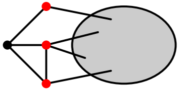
Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



Guess

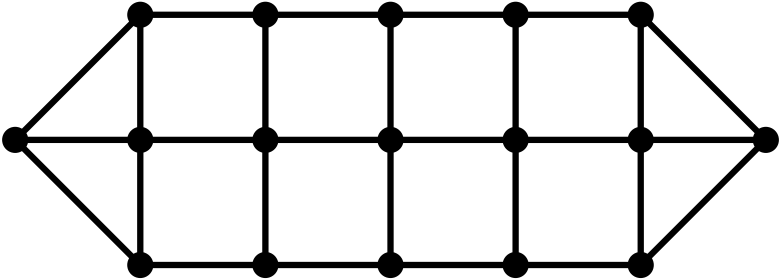
Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

↓

3-con'd, > 4 vertices, every 3-separator has form 

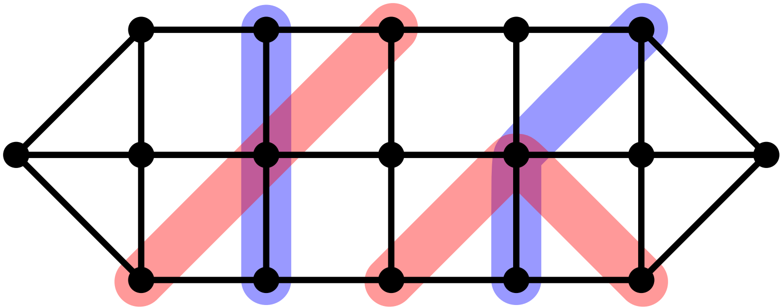
Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.



Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.



Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and $E(A \setminus B, B \setminus A) = \emptyset$

separator of $\{A, B\}$: $A \cap B$

mixed-separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and

$E(A \setminus B, B \setminus A) = \emptyset$; $A \not\subseteq B \not\subseteq A$

separator of $\{A, B\}$: $(A \cap B) \cup E(A \setminus B, B \setminus A)$

mixed-separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and $E(A \setminus B, B \setminus A) = \emptyset$, $A \not\subseteq B \not\subseteq A$

separator of $\{A, B\}$: $(A \cap B) \cup E(A \setminus B, B \setminus A)$

tri-separation of G: mixed-sep'n $\{A, B\}$ with $|\text{sep'n}| = 3$ and every vx in $A \cap B$ has two neighb's in $G[A]$ and in $G[B]$

mixed-separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and
$E(A \setminus B, B \setminus A) = \emptyset$, $A \not\subseteq B \not\subseteq A$

separator of $\{A, B\}$: $(A \cap B) \cup E(A \setminus B, B \setminus A)$

tri-separation of G: mixed-sep'n $\{A, B\}$ with $|\text{sep'n}| = 3$
and every vx in $A \cap B$ has two neighb's
in $G[A]$ and in $G[B]$

mixed-separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and

$E(A \setminus B, B \setminus A) = \emptyset$ $A \not\subseteq B \not\subseteq A$

separator of $\{A, B\}$: $(A \cap B) \cup E(A \setminus B, B \setminus A)$

tri-separation of G: mixed-sep'n $\{A, B\}$ with $|\text{sep'n}| = 3$

and every vx in $A \cap B$ has two neighb's

in $G[A]$ and in $G[B]$

trivial

mixed-separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and $E(A \setminus B, B \setminus A) = \emptyset$ $A \not\subseteq B \not\subseteq A$

separator of $\{A, B\}$: $(A \cap B) \cup E(A \setminus B, B \setminus A)$

tri-separation of G: mixed-sep'n $\{A, B\}$ with $|\text{sep'n}| = 3$ and every vx in $A \cap B$ has two neighb's in $G[A]$ and in $G[B]$

$\{A; B\}$ and $\{C; D\}$ are *nested* if $A \quad C$ and $B \quad D$ after possibly switching A with B or C with D; otherwise they *cross*.

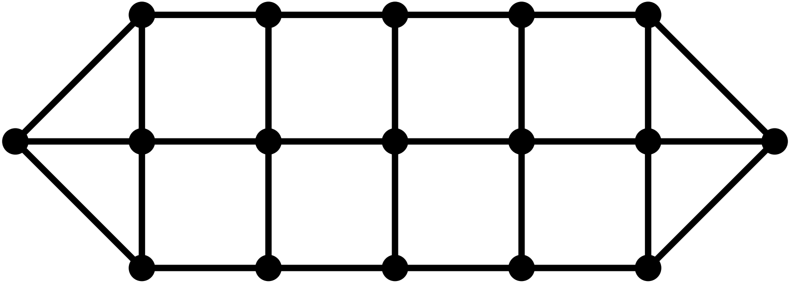
nested crossing

totally-nested nontrivial tri-separations

totally-nested nontrivial tri-separations

none

totally-nested nontrivial tri-separations

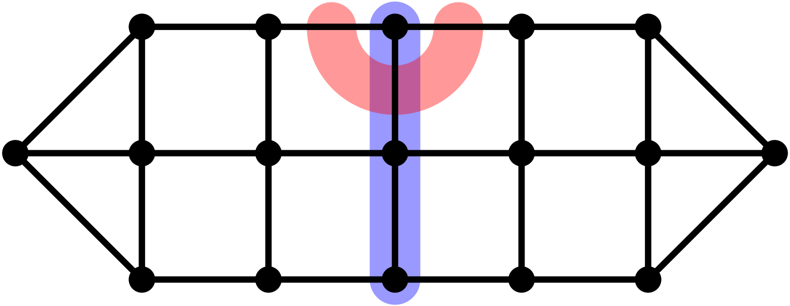


none

totally-nested nontrivial tri-separations

none

totally-nested nontrivial tri-separations

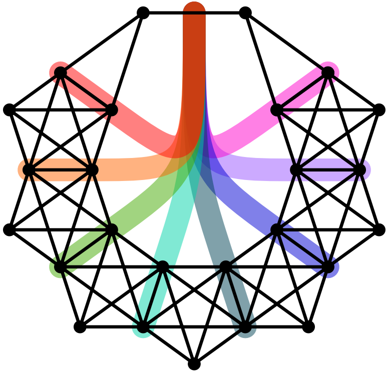


none

totally-nested nontrivial tri-separations

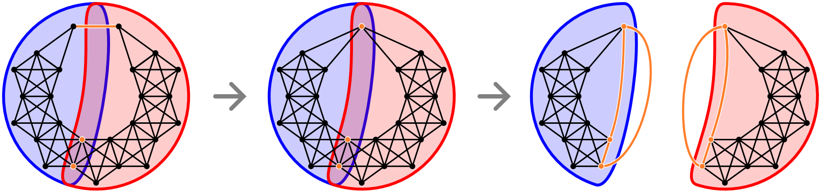
none

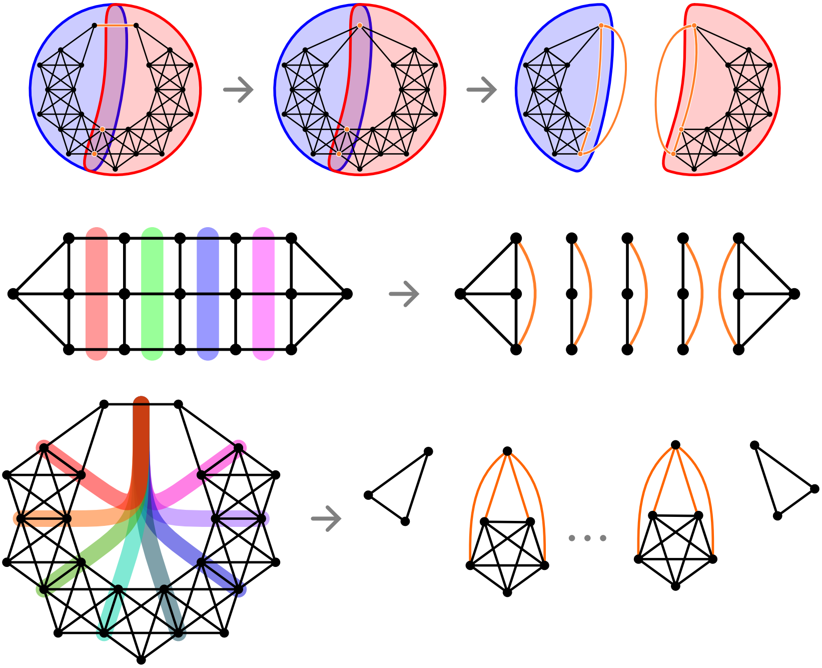
totally-nested nontrivial tri-separations



none

Decomposing along a tri-separation





Main result (Carmesin & K. 23)

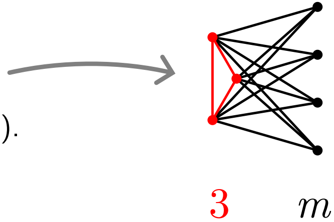
Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into minors of G that are

quasi 4-con'd

wheels

thickened $K_{3;m}$

or $G = K_{3;m}$ $(m > 0)$.

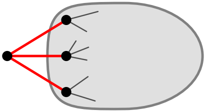


Application 1 (Carmesin & K. 23)

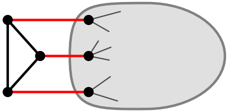
Every vertex-transitive finite con'd G is either

4-con'd

3-con'd and 3-regular and every tri-sep'n has form

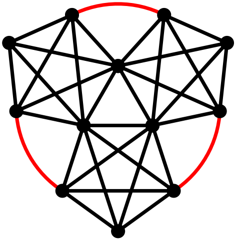


or



$K_1;::::;K_4$ or a cycle.

Application 2: Connectivity Augmentation to 4



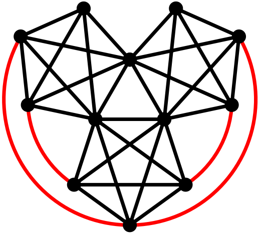
Theorem (Carmesin & Ramanujan 23+)

$\exists$ FPT-algorithm with runtime $C(k) \cdot \mathrm{Poly}(|V(G)|)$ and

Input: Graph G, $k \in \mathbb{N}$ and $F \subseteq E(\overline{G})$

Output: No, or $\leq k$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

Application 2: Connectivity Augmentation to 4



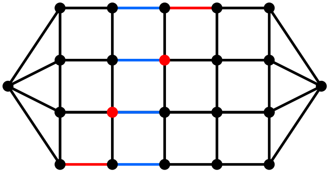
Theorem (Carmesin & Ramanujan 23+)

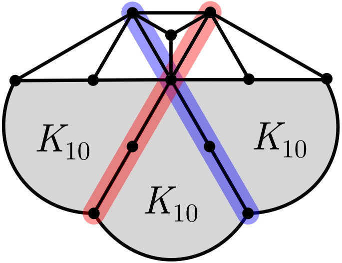
9 FPT-algorithm with runtime $C(k)$ Poly($jV(G)j$) and

Input: Graph G, k 2 N and F $E(\overline{G})$

Output: No, or 6 k-sized X F such that $G + X$ is 4-con'd

Open: Extend the main result to k-separations for $k > 4$.





Open: Tri-separations for matroids

$k = 2$:	X finite	Cunningham & Edmonds 80
	X infinite	Aigner-Horev, Diestel & Postle 16
$k = 3$:	???	Related: Oxley, Semple & Whittle 04

Tri-separation

Mixed-sep'n $\{A, B\}$ with $|$sep'r$| = 3$ such that every vx in $A \setminus B$ has two neighb's in $G[A]$ and in $G[B]$.

Main result (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into minors of G that are quasi 4-con'd, wheels, thickened $K_{3,m}$'s or $G = K_{3,m}$ ($m > 0$).

Open

Extend to k-separations for $k > 4$. Tri-separations for matroids.

arXiv: 2304.00945 Slides: web.mat.bham.ac.uk/J.Kurkofka/

Tri-separation

Mixed-sep'n $\{A, B\}$ with $|$sep'r$|$ = 3 such that every vx in $A \setminus B$ has two neighb's in $G[A]$ and in $G[B]$.

Main result (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into minors of G that are quasi 4-con'd, wheels, thickened $K_{3,m}$'s or $G = K_{3,m}$ ($m > 0$).

Open

Extend to k-separations for $k > 4$. Tri-separations for matroids.

arXiv: 2304.00945 Slides: web.mat.bham.ac.uk/J.Kurkofka/

Thank you!