

Counting Hamilton decompositions of oriented graphs

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Abstract

A Hamilton cycle in a directed graph G is a cycle that passes through every vertex of G . A Hamilton decomposition of G is a partition of its edge set into disjoint Hamilton cycles. In the late 60s Kelly conjectured that every regular tournament has a Hamilton decomposition. This conjecture was recently settled for large tournaments by Kühn and Osthus [15], who proved more generally that every r -regular n -vertex oriented graph G (without antiparallel edges) with $r = cn$ for some fixed $c > 3/8$ has a Hamilton decomposition, provided $n = n(c)$ is sufficiently large. In this paper we address the natural question of estimating the number of such decompositions of G and show that this number is $n^{(1-o(1))cn^2}$. In addition, we also obtain a new and much simpler proof for the approximate version of Kelly’s conjecture.

1 Introduction

A *Hamilton cycle* in a graph or a directed graph G is a cycle passing through every vertex of G exactly once, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity is one of the most central notions in graph theory, and has been intensively studied by numerous researchers in recent decades. The decision problem of whether a given graph contains a Hamilton cycle is known to be \mathcal{NP} -hard and in fact, already appears on Karp’s original list of 21 \mathcal{NP} -hard problems [10]. Therefore, it is important to find general sufficient conditions for Hamiltonicity (for a detailed discussion of this topic we refer the interested reader to two surveys of Kühn and Osthus [13, 14]).

In this paper we discuss Hamiltonicity problems for directed graphs. A *tournament* T_n on n vertices is an orientation of an n -vertex complete graph K_n . The tournament is *regular* if all in/outdegrees are the same and equal $(n - 1)/2$. It is an easy exercise to show that every tournament contains a Hamilton path (that is, a directed path passing through all the vertices). Moreover, one can further show that a regular tournament contains a Hamilton cycle.

A tournament is a special case of a more general family of directed graphs, so called *oriented* graphs. An oriented graph is a directed graph obtained by orienting the edges of a simple graph (that is, a graph without loops or multiple edges). Given an oriented graph G , let $\delta^+(G)$ be its minimum outdegree, $\delta^-(G)$ be its minimum indegree and let the *minimum semi-degree* $\delta^0(G)$ be the minimum of $\delta^+(G)$ and $\delta^-(G)$. A natural question, originally raised by Thomassen in the late 70s, asks to determine the minimum semi-degree which ensures Hamiltonicity in the oriented setting. Following a long line of research, Keevash, Kühn and Osthus [11] settled this problem, showing that $\delta^0(G) \geq \lceil \frac{3n-4}{8} \rceil$ is enough

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to obtain a Hamilton cycle in any n -vertex oriented graph. A construction showing that this is tight was obtained much earlier by Häggkvist [9].

Once Hamiltonicity of G has been established, it is natural to further ask whether G contains many edge-disjoint Hamilton cycles or even a Hamilton decomposition. A *Hamilton decomposition* is a collection of edge-disjoint Hamilton cycles covering all the edges of a graph. In the late 60s, Kelly conjectured (see [14, 13] and their references) that every regular tournament has a Hamilton decomposition. Kelly's Conjecture has been studied extensively in recent decades, and quite recently was settled for large tournaments in a remarkable tour de force by Kühn and Osthus [15]. In fact, Kühn and Osthus [15] proved the following stronger statement for dense r -regular oriented graphs (that is, oriented graphs with all in/outdegrees equal to r).

Theorem 1. *Let $\epsilon > 0$ and let n be a sufficiently large integer. Then, every r -regular oriented graph G on n vertices with $r \geq 3n/8 + \epsilon n$ has a Hamilton decomposition.*

The bound on r in this theorem is best possible up to the additive term of ϵn . Indeed, as we already mentioned above, if r is smaller than $3n/8$ then G may not even be Hamiltonian.

Counting various combinatorial objects has a long history in Discrete Mathematics and such problems have been extensively studied. Motivated by Theorem 1, in this paper we consider the number of distinct Hamilton decompositions of dense regular oriented graphs. One can obtain an upper bound for this question by using the famous Minc conjecture, established by Brégman [3], which provides an upper-bound on the permanent of a matrix A . Let S_n be the set of all permutations of the set $[n]$. The permanent of an $n \times n$ matrix A is defined as $\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n A_{i\sigma(i)}$. Note that every permutation $\sigma \in S_n$ has a cycle representation which is unique up to the order of cycles. When A is a $0-1$ adjacency matrix of an oriented graph (that is $A_{ij} = 1$ iff $\vec{ij} \in E(G)$), every non-zero summand in the permanent is 1 and it corresponds to a collection of disjoint cycles covering all the vertices. Hence, the permanent counts the number of such cycle factors and, in particular, gives an upper bound on the number of Hamilton cycles in the corresponding graph. For an r -regular oriented graph G with adjacency matrix A , where r is large, Brégman's Theorem asserts that

$$\text{per}(A) \leq (r!)^{n/r} = (1 - o(1))^n (r/e)^n.$$

Therefore, G has at most $(1 - o(1))^n (r/e)^n$ Hamilton cycles. Note that upon removing the edges of such a cycle from G , we are left with an $(r - 1)$ -regular oriented graph G' . Again by Brégman's Theorem, G' contains at most $(1 - o(1))^n ((r - 1)/e)^n$ distinct Hamilton cycles. Repeating this process and taking the product of all these estimates, we deduce that G has at most

$$\left((1 + o(1)) \frac{r}{e^2} \right)^{rn}$$

Hamilton decompositions. When r is linear in n this bound is of the form $n^{(1-o(1))rn}$.

Our first result gives a corresponding lower bound, which together with the above estimates determine asymptotically the number of Hamilton decompositions of dense regular oriented graphs. It is worth drawing attention to the fact that our result shows that all such graphs have roughly the same number of Hamilton decompositions.

Theorem 2. *Let $c > 3/8$ be a fixed constant, let $\epsilon > 0$ be an arbitrary small constant, and let n be a sufficiently large integer. Then, every cn -regular oriented graph G on n vertices contains at least $n^{(1-\epsilon)cn^2}$ distinct Hamilton decompositions.*

The main step in the proof of this theorem is to construct many almost Hamilton decompositions, each of which can be further completed to a full decomposition. This is done by extending some ideas from [6] and differs from the approach used in [15]. In particular, we obtain a new and much simpler proof for the approximate version of Kelly’s conjecture, originally established by Kühn, Osthus and Treglown in [17]. Furthermore, note that a Hamilton decomposition of a regular tournament also gives a Hamilton decomposition of the underlying complete (undirected) graph. Therefore Theorem 2 implies that, for odd n , the n -vertex complete graph has $n^{(1-o(1))n^2/2}$ Hamilton decompositions. This estimate, together with more general results concerning counting Hamilton decompositions of various dense regular graphs, was recently obtained in [8].

Another natural problem studied in this paper concerns how many edge-disjoint Hamilton cycles one can find in a given (not necessarily regular) oriented graph. Observe that if an oriented graph G contains r edge-disjoint Hamilton cycles, then their union gives a spanning, r -regular subgraph of G . We refer to such a subgraph as an r -factor of G . Given an oriented graph G , let $\text{reg}(G)$ be the maximal integer r for which G contains an r -factor. Clearly, G contain at most $\text{reg}(G)$ edge-disjoint Hamilton cycles. We propose the following conjecture which, if true, is best possible.

Conjecture 3. *Let $c > 3/8$ be a fixed constant and let n be sufficiently large. Let G be an oriented graph on n vertices with $\delta^0(G) \geq cn$. Then, G contains $\text{reg}(G)$ edge-disjoint Hamilton cycles.*

Our second result gives supporting evidence for this conjecture, proving that such oriented graphs G contain $(1 - o(1)) \text{reg}(G)$ edge-disjoint Hamilton cycles.

Theorem 4. *Let $c > 3/8$ and $\varepsilon > 0$ be fixed constants and let n be sufficiently large. Let G be an oriented graph on n vertices with $\delta^0(G) \geq cn$. Then, G contains a collection of $(1 - \varepsilon) \text{reg}(G)$ edge-disjoint Hamilton cycles.*

This theorem follows immediately from our proof of Theorem 2. For a regular tournament Theorem 4 implies an approximate version of Kelly’s Conjecture from [17].

Notation: Given an oriented graph G and $v \in V(G)$, let $N_G^+(v) = \{w \in V(G) : \vec{vw} \in E(G)\}$ denote the out-neighbourhood of v in G and $d_G^+(v) = |N_G^+(v)|$ denote the out-degree of v in G . Define $N_G^-(v)$ and $d_G^-(v)$ similarly. Given a set $W \subset V(G)$, let $N_G^+(v, W) = N_G^+(v) \cap W$ and let $d_G^+(v, W) = |N_G^+(v, W)|$. Similarly define $N_G^-(v, W)$ and $d_G^-(v, W)$. We omit the subscript G whenever there is no risk of confusion. We also define $\delta^+(G) := \min_v d^+(v)$, $\delta^-(G) := \min_v d^-(v)$, $\Delta^+(G) := \max_v d^+(v)$, $\Delta^-(G) := \max_v d^-(v)$, and set $\delta^0(G) = \min\{\delta^+(G), \delta^-(G)\}$ and $\Delta^0(G) = \max\{\Delta^+(G), \Delta^-(G)\}$. We also write $a \pm b$ to denote a value which lies in the interval $[a - b, a + b]$.

2 Proof outline

In this section we give a general overview of our proof strategy for Theorems 2 and 4. We only give a ‘high level’ description; the exact details will appear in later sections. The proof method is similar to that introduced in [6], although a number of new ideas are required in order extend from the pseudorandom setting to the general case.

Let G be an n -vertex digraph with $\delta^0(G) \geq \beta n$ and let D denote an arbitrary $d := \text{reg}(G)$ -factor of G . Let us first aim to find one collection of ‘many’ edge-disjoint Hamilton cycles, without attempting

to count the number of such collections. To do this, we will find edge-disjoint spanning subgraphs H_1, \dots, H_t of G with the following properties:

1. $t = \log^C(n)$, for some constant $C > 0$;
2. For each i , there is a partition $V(H_i) = V(G) = U_i \cup W_i$, with $|W_i| \ll |U_i|$;
3. For every i , the graph $D_i := H_i[U_i]$ is an almost regular subgraph of D , with degree roughly d/t ;
4. Each vertex $u \in U_i$ has many in and out-neighbours in the graph H_i to W_i ;
5. $\delta^0(H_i[W_i]) \geq (1 - o(1))\beta|W_i|$.

The existence of such a collection is proven in Section 6 (see Lemma 27).

Given such a partition, we will describe how to find many edge-disjoint Hamilton cycles in each H_i . This consists of two stages. In the first stage, contained in Section 5, we find $(1 - o(1))d/t$ edge-disjoint collections of paths from D_i , with the property that each collection has few paths, and covers all vertices of $V(D_i)$. These paths will be built from matchings. Concretely, we partition each set $V(D_i)$ into $b = \log^{C'}(n)$ sets $V(D_i) = V_1^i \cup \dots \cup V_b^i$, with $||V_{j_1}^i| - |V_{j_2}^i|| \leq 1$ for all $j_1, j_2 \in [b]$. By concentration inequalities, all in-/out-degrees in each $D_i[V_j^i, V_k^i]$ are roughly $d' \approx d/tb$. Now, any Hamilton path $v_{i_1} \dots v_{i_b}$ of K_b corresponds to a b -partite subgraph of D_i consisting of all the edges in $\vec{D}_i[V_j^i, V_{j+1}^i]$, $j \in [b-1]$. Furthermore, by a result of Tillson [19], one can partition each K_b into b edge-disjoint Hamilton paths, giving a partition of D_i into b edge-disjoint subgraphs. Each collection of paths will be taken from these subgraphs.

To see how this is achieved, fix a Hamilton path $v_1 v_2 \dots v_b$ in K_b . Observe that if we are able to find roughly d' edge-disjoint perfect matchings in $B_j := \vec{D}_i[V_j^i, V_{j+1}^i]$ for all $j \in [b-1]$, by combining a matching from each B_j we obtain roughly d' edge-disjoint collections of $|V_1^i|$ edge-disjoint paths, with each collection covering all vertices of D_i . Taking such collections for each of the b Hamilton paths above, we find $d'b \approx d/t$ edge-disjoint collections of roughly $|V_1^i| \approx n/b$ edge-disjoint paths covering $V(D_i)$.

Given such an idyllic situation, the second stage of the proof, contained in Section 4, aims to complete each collection of paths above to a Hamilton cycle using edges from H_i adjacent to vertices in W_i . As each collection consists of few paths, and as each vertex has ‘many’ neighbours in W_i in the graph H_i , we can (essentially greedily) extend each collection to a collection of $\approx n/b$ vertex disjoint paths that start and end in W_i , so that all paths are edge-disjoint. The final step completes each collection of paths to a Hamilton cycle using edges from W_i . Provided β is large and the number of paths in each collection ($\approx n/b$) is much smaller than $|W_i|$, this can be carried out using known results for dense oriented graphs (see Section 3.3). In this way, we complete each collection of paths to a Hamilton cycle. However care must be taken during this completion phase, so that $H_i[W_i]$ does not become ‘too sparse’, and there is some sensitivity in our choice of parameters (choices of b and t) as a result.

A difficulty, which was glossed over above, is that during the first stage as d' is not that large and the graphs B_j are not in general regular, it can be the case that B_j does not even contain a single perfect matching, let alone d' of them. To overcome this difficulty, we prove that each *almost* regular bipartite graph has the ‘correct number’ of large (not necessarily complete) edge-disjoint matchings. These slightly smaller matchings are sufficient to prove the theorem, as in this scenario we still have

collections of not too many paths covering all the vertices of $V(D_i)$ (perhaps some paths consist of a single vertex), although they present some extra technicalities, which are handled in Section 5. Note also that, as we show each almost regular bipartite graph contains the ‘correct number’ of collections of roughly d' edge-disjoint large matchings, we not only get existence, but also a counting result from our approach. In order to get a decomposition of G (when G is regular), we initially remove a carefully chosen regular subgraph from G to obtain G' , apply the above procedure on G' to obtain the ‘correct number’ of approximate decompositions and then complete the decomposition using the remaining edges and the graph we left outside. This is done using a celebrated result of Kühn and Osthus [15]. The rest of the details appear below.

3 Tools

In this section we have collected a number of tools to be used in proving our results.

3.1 Chernoff’s inequality

Throughout the paper we will make extensive use of the following well-known bound on the upper and lower tails of the Binomial distribution, due to Chernoff (see Appendix A in [1]).

Lemma 5 (Chernoff’s inequality). *Let $X \sim \text{Bin}(n, p)$ and let $\mathbb{E}(X) = \mu$. Then*

- $\mathbb{P}[X < (1 - a)\mu] < e^{-a^2\mu/2}$ for every $a > 0$;
- $\mathbb{P}[X > (1 + a)\mu] < e^{-a^2\mu/3}$ for every $0 < a < 3/2$.

Remark 6. *These bounds also hold when X is hypergeometrically distributed with mean μ .*

We also need the following easy proposition:

Proposition 7. *Let $s, K, N \in \mathbb{N}$ with $s \leq N$ and $p = s/N$. Let S be a set of size N . Suppose we select random subsets U_1, \dots, U_K from S of order s , all choices independent. Then $U = \cup_{i \in [K]} U_i$ satisfies $\mathbb{E}(|U|) = Np'$ where $p' = 1 - (1 - p)^K$, and for $t \leq Np'$ we have*

$$\mathbb{P}(|U| - \mathbb{E}(|U|) \geq t) \leq 2(N + 1)^K e^{-t^2/3Np'}.$$

Proof. Fixing an element $s \in S$, it appears in each U_i independently with probability p . Therefore, the probability that s appears in U is $p' = 1 - (1 - p)^K$ and by linearity of expectation we obtain $\mathbb{E}(|U|) = Np'$.

For the concentration bounds, select K random sets W_1, \dots, W_K by including each element of S in W_i with probability p , independently at random (that is, $|W_i|$ is not necessarily of size Np for all i). Setting $W = \cup_{i \in [K]} W_i$ we see that $|W|$ is binomially distributed according to $\text{Bin}(N, p')$, and that $\mathbb{E}(|W|) = \mathbb{E}(|U|)$. Therefore, by Chernoff’s inequality we have

$$\mathbb{P}(|W| - \mathbb{E}(|W|) > t) \leq 2e^{-t^2/3Np'}.$$

Let \mathcal{E} to denote the event “ $|W_i| = Np$ for all $i \in [K]$ ” and note that conditioned on \mathcal{E} , the random variable W has the same distribution as U . Therefore

$$\mathbb{P}(|U| - \mathbb{E}(|U|) > t) \cdot \mathbb{P}(\mathcal{E}) \leq \mathbb{P}(|W| - \mathbb{E}(|W|) > t) \leq 2e^{-t^2/3Np'}. \quad (1)$$

It easy to see that $\mathbb{P}(|W_i| = m)$ is maximized when $m = Np$ and therefore, by independence, we have

$$\mathbb{P}(\mathcal{E}) = \prod_{i \in [K]} \mathbb{P}(|W_i| = pn) \geq (N + 1)^{-K}.$$

Combined with (1) this completes the proof. □

3.2 Perfect matchings in a bipartite graph

Here we present a number of results related to perfect matchings in bipartite graphs. The first result is a criterion for the existence of r -factors (that is, spanning and r -regular subgraphs) in bipartite graphs, due to Gale and Ryser (see [7], [18]).

Theorem 8. *Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = m$, and let r be an integer. Then G contains an r -factor if and only if for all $X \subseteq A$ and $Y \subseteq B$*

$$e_G(X, Y) \geq r(|X| + |Y| - m).$$

Next we present Brégman’s Theorem which provides an upper bound for the number of perfect matchings in a bipartite graph based on its degrees (see e.g. [1] page 24).

Theorem 9. (*Brégman’s Theorem*) *Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B|$. Then the number of perfect matchings in G is at most*

$$\prod_{a \in A} (d_G(a))^{1/d_G(a)}.$$

Remark 10. *It will be useful for us to give an upper bound with respect to the maximum degree of G . Suppose that $|A| = |B| = m$ and let $\Delta := \Delta(G)$. Using Theorem 9 and Stirling’s approximation, one obtains that the number of perfect matchings in G is at most*

$$(\Delta!)^{m/\Delta} \leq (8\Delta)^{m/\Delta} \left(\frac{\Delta}{e}\right)^m.$$

Lastly, we require the following result which provides a lower bound for the number of perfect matchings in a regular bipartite graph. This result is known as the Van der Waerden Conjecture, and it was proven by Egorychev [4], and independently by Falikman [5].

Theorem 11. (*Van der Waerden’s Conjecture*) *Let $G = (A \cup B, E)$ be a d -regular bipartite graph with both parts of size m . Then the number of perfect matchings in G is at least*

$$d^m \frac{m!}{m^m} \geq \left(\frac{d}{e}\right)^m.$$

3.3 Hamilton paths, cycles and absorbers

We make use of the following theorem of Keevash, Kühn and Osthus [11].

Theorem 12. *Every n -vertex oriented graph G with $\delta^0(G) \geq (3n - 4)/8$ contains a Hamilton cycle, provided n is sufficiently large.*

We also make use of the following related result of Kelly, Kühn and Osthus, which follows immediately from the proof of the main theorem in [12].

Theorem 13. *Let $c > 3/8$ be a constant and n be sufficiently large. Suppose that G is an oriented graph on n vertices with $\delta^0(G) \geq cn$, and let $x, y \in V(G)$ be any two distinct vertices. Then there is a Hamilton path in G with x as its starting point and y as its final point.*

Before describing the next tool we need the following definition.

Definition 14. *Given an n -vertex oriented graph G , a subgraph $D \subseteq G$ is said to be a δ -absorber if, for any given d -regular spanning subgraph T which is edge-disjoint from D with $d \leq \delta n$, the oriented graph $D \cup T$ has a Hamilton decomposition.*

The following result is the main ingredient in the seminal paper of Kühn and Osthus in which they solved Kelly's conjecture [15]. Roughly speaking, the theorem states that there are δ -absorbers for arbitrarily small δ in any sufficiently large regular oriented graph. The result follows from Lemma 3.4 in [16].

Theorem 15. *Let $\varepsilon > 0$ and $c > 3/8$ be two constants. Then, there is $\delta > 0$ such that for sufficiently large n the following holds. Suppose that G is an n -vertex oriented graph with $\delta^0(G) \geq cn$. Then G contains a δ -absorber A as an oriented subgraph, where A is r -regular with $r \leq \varepsilon n$.*

4 Almost Hamilton decompositions of special oriented graphs

Our aim in this section is to show how certain special oriented graphs can be almost decomposed into Hamilton cycles.

4.1 Completing one Hamilton cycle

The following simple lemma will allow us to complete disjoint directed paths into a Hamilton cycle.

Lemma 16. *Let $c > 3/8$ and $a, N \in \mathbb{N}$ with $a \ll \frac{N}{\log N}$ and N sufficiently large. Let F be an oriented graph with $|V(F)| = N$ and $\delta^0(F) \geq cN$. Let $\{P_i\}_{i \in [a]}$ be a collection of vertex disjoint oriented paths contained in an oriented graph G , where $V(F) \cap V(G) = \emptyset$. Let x_i and y_i denote the first and last vertices of P_i , for each i , and assume that $d^-(x_i, V(F)), d^+(y_i, V(F)) \geq 2a$. Then there is a cycle C with the following properties:*

1. Each P_i appears as a segment of C ;
2. $V(F) \subseteq V(C)$.

Proof of Lemma 16. For each $i \in [a]$ select $t_i \in N^-(x_i, V(F))$ and $s_i \in N^+(y_i, V(F))$ such that all $2a$ vertices are distinct. Note that this is possible as $d^-(x_i, V(F)), d^+(y_i, V(F)) \geq 2a$. Let $S = \{s_i : i \in [a]\}$, $T = \{t_i : i \in [a]\}$ and $W = V(F)$.

Let us create a partition of W into a sets, W_1, \dots, W_a , by assigning s_i and t_{i+1} to W_i for all $i \in [a]$ (taking $a+1$ to be 1) and by randomly assigning each vertex $v \in W \setminus (S \cup T)$ to one of the sets uniformly and independently at random. Now, let $\varepsilon_0 = (c - 3/8)/4 > 0$ and consider the events:

$$A = \left| |W_i| \in (1 \pm \varepsilon_0) \frac{|W|}{a} \text{ for all } i \in [a] \right|$$

$$B = \left| d_{F[W_i]}^\pm(v) \geq (c - \varepsilon_0) \frac{|W|}{a} \text{ for all } v \in W \text{ and } i \in [a] \right|.$$

As $\mathbb{E}(|W_r|) = \frac{|W|}{a}$, using that $N \gg a \log a$ and Chernoff's inequality, we obtain

$$\mathbb{P}[A^c] \leq 2a \exp\left(-\frac{\varepsilon_0^2 |W|}{3a}\right) = o(1). \quad (2)$$

Also, as $\delta^0(F) \geq cN = c|W|$ and all but at most $2a$ vertices were assigned randomly, we have

$$\mathbb{E}(d^\pm(v, W_i)) \geq \frac{c|W| - 2a}{a} = c \frac{|W|}{a} - 2.$$

Again using that $|W| \gg a \log N$ together with Chernoff's inequality, we have

$$\mathbb{P}[B^c] \leq 2N \exp\left(-\Theta\left(\frac{\varepsilon_0^2 |W|}{a}\right)\right) = o(1). \quad (3)$$

Combining (2) with (3) we conclude $\mathbb{P}(A \cap B) > 0$. Fix a partition W_1, \dots, W_a such that $A \cap B$ holds.

To complete the proof, set $F_i := F[W_i]$ for each $i \in [a]$. As $A \cap B$ holds, we have

$$\delta^0(F_i) \geq (c - \varepsilon_0) \frac{|W|}{a} \geq (c - 3\varepsilon_0) |V(F_i)| = (3/8 + \varepsilon_0) |V(F_i)|.$$

Therefore, using that $|V(F_i)| \geq (1 - \varepsilon_0) |W|/a \geq N/2a \gg \log N$ and N is sufficiently large, it follows from Theorem 13 that F_i contains a Hamilton path I_i from s_i to t_{i+1} , for each i . All in all, the cycle $C = P_1 I_1 P_2 I_2 \dots P_a I_a P_1$ (with the connecting edges $y_i s_i$ and $t_{i+1} x_{i+1}$) gives the desired cycle. This completes the proof of the lemma. \square

4.2 Completing ‘many’ edge-disjoint Hamilton cycles

Next we will show how to repeatedly apply Lemma 16 to obtain ‘many’ edge-disjoint Hamilton cycles. Before stating this result we introduce the following definitions.

Definition 17. *Let G be an oriented graph.*

1. A path cover of G of size a is a collection of a vertex disjoint directed paths in G which cover all vertices in $V(G)$.
2. An $(a, t)_\mathcal{P}$ -family is a collection of t edge-disjoint paths covers of G , each of which is of size at most a .

3. Let $\mathcal{P}(G, a, t)$ denote the set of all (a, t) -families in G .
4. Given $\mathbf{P} \in \mathcal{P}(G, a, t)$, let $G_{\mathbf{P}}$ denote the oriented subgraph $G_{\mathbf{P}} = \bigcup_{P \in \mathbf{P}} E(P)$.

Remark: The above definitions include the possibility of paths of length 0, i.e. isolated vertices.

It may seem odd to refer to the oriented graph G in the definition of $G_{\mathbf{P}}$, as the oriented subgraph $G_{\mathbf{P}}$ only depends on the edges that appear in the paths from \mathbf{P} and not on G itself. Our notation is however intended to reflect a ‘choice’ of \mathbf{P} from G . This dependence will be relevant later in proving Theorem 2, as our eventual count on the number of Hamilton decompositions of G in Theorem 2 will follow from a lower bound on the number of choices of \mathbf{P}_i from certain subgraphs D_i of G .

One can think about a path cover P of small size as an ‘almost Hamilton cycle’, in the sense that by adjoining a small number of edges to P we can obtain a Hamilton cycle. Our aim in the following lemma is to show how, given ‘many’ edge-disjoint path covers, one can build ‘many’ edge-disjoint Hamilton cycles.

Lemma 18. *Let $c > 3/8$ and let $a, b, n, s, t \in \mathbb{N}$ with $t + a \log n \ll s \ll n$. Suppose that H is an n -vertex oriented graph with partition $V(H) = U \cup W$, where $|W| = s$, with the following properties:*

1. *There is $\mathbf{P} = \{\mathcal{P}_j | j \in [t]\} \in \mathcal{P}(H[U], a, t)$;*
2. *$\delta^0(H_{\mathbf{P}}[U]) \geq t - b$;*
3. *$d^{\pm}(u, W) > 2a + b$ for all $u \in U$;*
4. *The oriented subgraph $F = H[W]$ satisfies $\delta^0(F) \geq c|W|$;*

Then H contains a family $\mathcal{C} = \{C_1, \dots, C_t\}$ of t edge disjoint Hamilton cycles, where each cycle C_i contains all the paths in \mathcal{P}_i as segments.

Proof. For each $j \in [t]$, let $\mathcal{P}_j = \{P_{j,r}\}_{r \in [R_j]}$ denote the collection of all directed paths in the path cover \mathcal{P}_j . As \mathcal{P}_j has size at most a we have $R_j \leq a$.

Now we wish to turn each \mathcal{P}_j into a Hamilton cycle C_j of H in such a way that

- (i) all the paths in \mathcal{P}_j are segments of C_j , and
- (ii) C_i and C_j are edge-disjoint for all $i \neq j$.

This will be carried out over a sequence of steps where in step j we have already selected C_1, \dots, C_{j-1} , and the cycle C_j is chosen by showing that the oriented graph $H_j = H \setminus (\bigcup_{i \leq j-1} E(C_i))$ satisfies the requirements of Lemma 16. Let us fix $c > c' > 3/8$.

Suppose that we have already found C_1, \dots, C_{j-1} and we wish to find C_j . Let x_i and y_i denote the start and end vertices of $P_{j,i}$, for all $i \leq R_j$. First note that by property 3, each vertex $u \in \{x_i, y_i \mid i \leq R_j\}$ satisfies $d^{\pm}(u, W) > 2a + b$. By property 2, each vertex v appears as the first vertex of at most b paths and as the last vertex of at most b paths (otherwise v would have in-degree or out-degree less than $t - b$ in $H_{\mathbf{P}}(U)$). Therefore, for all $u \in U$ we have

$$d_{H_j}^{\pm}(u, W) \geq 2a.$$

Second, as the edges of less than j Hamilton cycles have been deleted from H , from property 4. we find that $F_j = H_j[W]$ satisfies $\delta^0(F_j) \geq c|W| - j + 1 \geq c'|W|$, using $|W| = s \gg t \geq j$. Lastly, we have $|W| = s \gg a \log n \gg a \log s$ by hypothesis.

All combined, we have shown that the graph H_j satisfies the conditions of Lemma 16 with $N = |W|$. Therefore Lemma 16 guarantees the cycle C_j exists. Thus we can find C_1, \dots, C_t , as required. \square

5 Path covers of oriented graphs

In the previous section we have shown how to extend edge-disjoint path covers to edge-disjoint Hamilton cycles in certain special oriented graphs. In this section we will show how to locate such path covers, using a number of well-known matching results. The main result of the section is the following:

Lemma 19. *Let $m, r \in \mathbb{N}$ with $r \geq m^{49/50}$ and m sufficiently large. Suppose that H is an m -vertex oriented graph with*

$$r - r^{3/5} \leq \delta^0(H) \leq \Delta^0(H) \leq r + r^{3/5}.$$

Then, taking $a = m/\log^4 m$ and $t = r - m^{24/25} \log m$, the following hold:

1. *There is a set $\mathbf{S} \subseteq \mathcal{P}(H, a, t)$ with $|\mathbf{S}| \geq r^{(1-o(1))rm}$;*
2. *For all $\mathbf{P} \in \mathbf{S}$ the oriented subgraph $H_{\mathbf{P}}$ satisfies $\delta^0(H_{\mathbf{P}}) \geq r - m/\log^4 m$.*

5.1 Finding r -factors in bipartite graphs

We show that given a dense bipartite graph $G = (A \cup B, E)$ which is ‘almost regular’, G contains a spanning r -regular subgraph (an r -factor), with r very close to $\delta(G)$.

Lemma 20. *Let $\alpha \geq 1/2$, $m, \xi \in \mathbb{N}$. Suppose $G = (A \cup B, E)$ is a bipartite graph with $|A| = |B| = m$ and $\alpha m + \xi \leq \delta(G) \leq \Delta(G) \leq \alpha m + \xi + \xi^2/m$. Then G contains an αm -factor.*

Proof. By Theorem 8, to prove the lemma it suffices to show that for all $X \subset A$ and $Y \subset B$ we have

$$e_G(X, Y) \geq \alpha m(|X| + |Y| - m). \quad (4)$$

Given such sets X and Y , let $x = |X|$ and $y = |Y|$. We may assume that $x \leq y$, as the case $y \leq x$ follows by symmetry. We will make use of the following two trivial estimates for $e_G(X, Y)$:

- (i) $e_G(X, Y) \geq x(\delta(G) + y - m)$;
- (ii) $e_G(X, Y) = e_G(X, B) - e_G(X, B \setminus Y) \geq \delta(G)x - \Delta(G)(m - y)$.

The required bound follows from the following cases.

Case 1: $x + y \leq m$. In this case (4) trivially holds.

Case 2: $x \leq y$ and $x \leq \delta(G)$. In this case, note that since $y - m \leq 0$ we obtain

$$x(\delta(G) + y - m) \geq \delta(G)(x + y - m).$$

which by (i) proves (4).

Case 3: $x \leq y$, and $x > \delta(G)$. Observe that in this case since $\alpha \geq 1/2$ we have

$$x + y - m \geq 2\delta(G) - m \geq 2\xi. \quad (5)$$

Also, from (ii), we have

$$e_G(X, Y) \geq \delta(G)x - \Delta(G)(m - y) \geq \alpha m(x + y - m) + \xi(x + y - m) - \frac{\xi^2}{m}(m - y). \quad (6)$$

Combining (5) with (6) and using that $x + y > m$, we conclude that

$$\alpha m(x + y - m) + \xi(x + y - m) - \frac{\xi^2}{m}(m - y) \geq \alpha m(x + y - m) + 2\xi^2 - \xi^2 \geq \alpha m(x + y - m),$$

which again proves (4). This completes the proof. \square

Using the previous lemma we obtain the following corollary, which shows that by adjoining a small number of edges to an almost regular bipartite graph, one can obtain a regular bipartite graph.

Corollary 21. *Let $d, m, \xi \in \mathbb{N}$, $d \leq m/2$. Suppose that $G = (A \cup B, E)$ is a bipartite graph with $|A| = |B| = m$ and that $d - \xi - \xi^2/m \leq \delta(G) \leq \Delta(G) \leq d - \xi$. Then there is a bipartite d -regular graph $H = (A \cup B, E')$ which contains G as a subgraph.*

Proof. Given G as in the lemma, consider the graph $G^c = (A \cup B, E^*)$ where $e \in E^*$ if and only if $e \notin E$. Clearly $m - d + \xi \leq \delta(G^c) \leq \Delta(G^c) \leq m - d + \xi + \xi^2/m$. Therefore, Lemma 20 guarantees an $(m - d)$ -regular subgraph $S \subseteq G^c$. Letting $H := S^c$ completes the proof. \square

5.2 Small subgraphs contribute many edges to few matchings

Lemma 22. *Let $m, r \in \mathbb{N}$ with $r \geq m^{24/25}$ and m sufficiently large. Suppose that $G = (A \cup B, E)$ is a bipartite graph with $|A| = |B| = m$ and that $E = E_1 \cup E_2$ is a partition of E . For $i \in \{1, 2\}$ let H_i be the spanning subgraph of G induced by the edges in E_i . Suppose also that:*

1. G is r -regular, and
2. $d_{H_2}(v) \leq 2m^{5/6}$ for all $v \in A \cup B$.

Then G contains at least $(1 - o(1)) \left(\frac{r}{e}\right)^m$ perfect matchings, each with at most $m^{7/8}$ edges from E_2 .

Proof. Set $s = 2m^{5/6}$ and $\ell = m^{7/8}$. First note that since G is r -regular, by Theorem 11, the number of perfect matchings in G is at least $\left(\frac{r}{e}\right)^m$. Therefore it is enough to show that at most $o(1)(r/e)^m$ matchings of G contain at least ℓ edges from E_2 .

Now given a matching $M \subseteq E_2$ of size ℓ , let G' be the subgraph of G obtained by deleting the vertices covered by M . Clearly $\Delta(G') \leq r$ and $|V(G')| = 2(m - \ell)$. By Remark 10 it follows that the number of ways to complete M into a perfect matching is at most

$$(8r)^{\frac{m-\ell}{r}} \left(\frac{r}{e}\right)^{m-\ell} \leq (8r)^{m^{1/25}} \left(\frac{e}{r}\right)^\ell \left(\frac{r}{e}\right)^m.$$

However, the number of matchings of size ℓ in H_2 is at most $\binom{m}{\ell} s^\ell \leq (ems/\ell)^\ell$. Therefore the number of perfect matchings of G with at least ℓ edges from E_2 is at most

$$(8r)^{m^{1/25}} \left(\frac{e^2 ms}{r\ell} \right)^\ell \left(\frac{r}{e} \right)^m \leq (8m)^{m^{1/25}} \left(\frac{2e^2 m^{1/25}}{m^{1/24}} \right)^{m^{7/8}} \left(\frac{r}{e} \right)^m = o(1) \left(\frac{r}{e} \right)^m.$$

This completes the proof of the lemma. \square

5.3 Decomposing almost regular bipartite graphs into large matchings

In order to prove Lemma 19 in the next subsection we will construct $(a, t)_\mathbf{P}$ -families $\mathbf{P} \in \mathcal{P}(H, a, t)$ by carefully combining collections of matchings from certain bipartite graphs. The following definition will be useful to refer to the key properties required from these matchings.

Definition 23. Let $G = (A \cup B, E)$ be a bipartite graph.

1. Given two integers a and t , we define an $(a, t)_\mathcal{M}$ -family in G to be a collection of t edge-disjoint matchings in G , each of which of size at least a .
2. Let $\mathcal{M}(G, a, t)$ denote the collection of all $(a, t)_\mathcal{M}$ -families in G .
3. Given $\mathbf{M} \in \mathcal{M}(G, a, t)$, we let $G_\mathbf{M}$ denote the spanning subgraph of G consisting of the edge set $\bigcup_{M \in \mathbf{M}} E(M)$.

Our main aim in the following lemma is to show that if $G = (A \cup B, E)$ is an almost r -regular bipartite graph with $|A| = |B|$, then for many elements $\mathbf{M} \in \mathcal{M}(G, a, t)$, where $a \approx |A|$ and $t \approx r$, the graph $G_\mathbf{M}$ is also almost regular.

Lemma 24. Let $\varepsilon > 0$ and $m, r \in \mathbb{N}$ with m sufficiently large and $2m^{24/25} \leq r \leq (1 - \varepsilon)m/2$. Suppose that $G = (A \cup B, E)$ is a bipartite graph with $|A| = |B| = m$ and $r \leq \delta(G) \leq \Delta(G) \leq r + r^{2/3}$. Then, taking $t = r - m^{24/25}$ and $a = m - m^{7/8}$, the following hold:

1. There is $\mathcal{M} \subset \mathcal{M}(G, a, t)$, with $|\mathcal{M}| = r^{(1-o(1))rm}$;
2. For each $\mathbf{M} \in \mathcal{M}$, the subgraph $G_\mathbf{M}$ has minimum degree at least $t - 2m^{5/6}$.

Proof. Set $\xi = m^{5/6}$ and $r' = r + \xi + \xi^2/m$. Then, using that $r^{2/3} \leq m^{2/3} = \xi^2/m$, combined with the hypothesis of the lemma, we have

$$r' - \xi - \xi^2/m = r \leq \delta(G) \leq \Delta(G) \leq r + r^{2/3} = r' - \xi.$$

Thus by Corollary 21 there is an r' -regular graph $H = (A \cup B, E')$ which contains G as a subgraph. Set $E_1 := E(G)$ and $E_2 := E(H) \setminus E_1$. By the above, we have

$$d_{E_2}(v) \leq r' - r = \xi + \frac{\xi^2}{2m} \leq 2m^{5/6} \tag{7}$$

for all $v \in A \cup B$.

We will now show, using Lemma 22, that there are many ways to build a sequence (M_1, \dots, M_t) of edge-disjoint perfect matchings in H , where each matching contains at least a edges from E_1 . To do

this, begin by setting $H_0 := H$. Having selected M_1, \dots, M_{i-1} , set $H_i := H \setminus (\cup_{j < i} E(M_j))$ and note that H_i is $(r' - i + 1)$ -regular. Since $r' - i \geq r - t \geq m^{24/25}$ and by (7), we can apply Lemma 22 to H_i to find at least $(1 - o(1))\left(\frac{r' - i + 1}{e}\right)^m$ perfect matchings of H_i with at least a edges in E_1 . Multiplying all this estimates gives at least

$$\prod_{i=1}^t (1 - o(1)) \left(\frac{r' - i + 1}{e}\right)^m = r^{(1-o(1))tm} = r^{(1-o(1))rm}$$

possible choices for (M_1, \dots, M_t) .

To complete the proof, simply note that each sequence (M_1, \dots, M_t) above gives rise to an $(a, t)_{\mathcal{M}}$ -family of G , given by $\mathbf{M} = \{M_i \cap E_1 : i \in [t]\}$. As each \mathbf{M} can occur at most $t!$ times in this way, these sequences give rise to $\mathcal{M} \subset \mathcal{M}(G, a, t)$ with

$$|\mathcal{M}| \geq \frac{1}{t!} \times r^{(1-o(1))rm} = r^{(1-o(1))rm}.$$

Lastly, for each such $(a, t)_{\mathcal{M}}$ -family \mathbf{M} , the minimum degree of $G_{\mathbf{M}}$ is at least $t - 2m^{5/6}$ by (7). This completes the proof of the lemma. \square

5.4 Path covers in almost regular oriented graphs

We are now ready to complete the proof of Lemma 19.

Proof of Lemma 19. Let $b = 2 \log^4 m$ and select a partition $V(H) = V_1 \cup \dots \cup V_b$ uniformly at random, where $|V_i| \in \{\lfloor m/b \rfloor, \lceil m/b \rceil\}$ holds for all $i \in [b]$. For convenience we will assume $|V_i| = m' := m/b$ for all $i \in [b]$, although this assumption is easily removed. By Chernoff's inequality, with probability $1 - o(1)$ we find that for all $v \in V(H)$ and $j \in [b]$ we have

$$d_H^\pm(v, V_j) = d_H^\pm(v)/b \pm 4\sqrt{m' \log m} = d \pm d^{2/3}/2, \quad (8)$$

where $d = r/b$. Fix a choice of partition such that (8) holds.

Now consider the complete directed graph on b vertices, denoted by D_b (this graph contains both directed edges (u, v) and (v, u) for all pairs of distinct vertices u, v). By a result of Tillson [19], the complete digraph D_b has an edge decomposition into b directed Hamilton paths Q_1, \dots, Q_b . Each such path $Q_i = v_{i_1} \dots v_{i_b}$ naturally corresponds to an oriented subgraph H_i of H consisting of all edges in $B_{ij} := \vec{H}[V_{i_j}, V_{i_{j+1}}]$ for $j \in [b-1]$. As the paths $\{Q_i\}_{i \in [b]}$ are edge-disjoint, so are the oriented subgraphs $\{H_i\}_{i \in [b]}$. Note that as B_{ij} only consists of edges oriented from V_{i_j} to $V_{i_{j+1}}$, we can view B_{ij} as a bipartite graph by ignoring the orientation of its edges.

Our aim now is to show that each oriented graph H_i has many path covers. Let us fix such a H_i and assume without loss of generality that H_i is given by the path $Q_i = v_1 \dots v_b$, so that $B_{ij} = \vec{H}[V_j, V_{j+1}]$ for all $j \in [b-1]$. The following observation is key:

Observation 25. *Suppose that M_j is a matching of size at least $m' - \ell$ in B_{ij} for all $j \in [b-1]$. Then $\cup M_j$ is a path cover of H_i (perhaps with some paths of length 0). Moreover, as $\cup M_j$ has at least $(m' - \ell)(b-1)$ edges and H_i has m vertices, such path covers are of size at most $m' + b\ell$.*

We now exploit this observation using Lemma 24. Note that $d - d^{2/3}/2 \geq 2m^{24/25}$. Secondly, by (8) for all $j \in [b-1]$ we have

$$d - d^{2/3}/2 \leq \delta(B_{ij}) \leq \Delta(B_{ij}) \leq d + d^{2/3}/2.$$

Therefore, we can apply Lemma 24 to B_{ij} , taking $a' = m' - (m')^{7/8}$ and $t' = d - d^{2/3}/2 - (m')^{24/25}$, to get

- (a) $\mathcal{M}_{ij} \subseteq \mathcal{M}(B_{ij}, a', t')$ with $|\mathcal{M}_{ij}| = d^{(1-o(1))dm'}$;
- (b) For all $\mathbf{M}_{ij} \in \mathcal{M}_{ij}$, letting $B := B_{ij}$, the graph $B_{\mathbf{M}_{ij}}$ has minimum degree at least $t' - 2(m')^{5/6}$.

Let us now fix $\mathbf{M}_{ij} \in \mathcal{M}_{ij}$ for all $j \in [b-1]$. As each \mathbf{M}_{ij} consists of t' edge-disjoint matchings, by Observation 25 we can use $\{\mathbf{M}_{ij}\}_{j \in [b-1]}$ to construct t' edge-disjoint path covers of H_i , each of size at most $m' + b(m')^{7/8} \leq n/\log^4 n = a$. Furthermore, it is easy to see that different choices of $\{\mathbf{M}_{ij}\}_{j \in [b-1]}$ give rise to a different collection of path covers. Combined with (a), this gives at least

$$\prod_{j \in [b-1]} |\mathcal{M}_{ij}| \geq d^{(1-o(1))(b-1)dm/b} = d^{(1-o(1))dm}$$

distinct (a, t') -families of H_i .

Now we have partitioned H into b edge-disjoint oriented graphs H_1, \dots, H_b , each of which consists of at least $d^{(1-o(1))dm}$ distinct (a, t') -families. Further, distinct choice of such families from each H_i yield distinct (a, bt') -families of H . Taking $t = bt' \geq r - 2b(m')^{24/25} \geq r - m^{24/25} \log m$, it follows that there is $\mathbf{S} \subset \mathcal{P}(H, a, t)$ with

$$|\mathbf{S}| \geq d^{(1-o(1))dmb} = d^{(1-o(1))rm} = r^{(1-o(1))rm}.$$

Here we have used that $b = 2 \log^4 m$, that $d = r/b$ and that $r \geq d/2b$, giving $b^{-rm} = r^{-o(rm)}$.

To complete the proof of the lemma, it only remains to prove the following:

Claim 26. *For each $\mathbf{P} \in \mathbf{S}$ we have $\delta^0(H_{\mathbf{P}}) \geq r - m/\log^4 m$.*

To see this, simply note that by construction

$$E(H_{\mathbf{P}}) = \bigcup_{i,j} E(B_{\mathbf{M}_{ij}})$$

for some choices of $\mathbf{M}_{ij} \in \mathcal{M}_{ij}$ where $i \in [b]$ and $j \in [b-1]$. Given $v \in V_k$ say, the out-edges of v in $H_{\mathbf{P}}$ are therefore those out-edges of v in $B_{\mathbf{M}_{ij}}$, where $i_j = k$. However, $i_j = k$ only occurs when an out-edge of v_k appears in Q_i , which happens exactly $b-1$ times, since Q_1, \dots, Q_b forms a Hamilton path decomposition of D_b . Combined with (b), $t' = d - d^{2/3}/2 - (m')^{24/25}$ and $d = r/b$, we find

$$d_{H_{\mathbf{P}}}^+(v) \geq (b-1)(t' - 2(m')^{5/6}) \geq bt' - t' - 2b(m')^{5/6} \geq r - t' - 4b(m')^{24/25} \geq r - 2m' = r - m/\log^4 m.$$

As an identical argument lower bounds the $d_{H_{\mathbf{P}}}^-(v)$, this completes the proof of the claim, and hence the proof of the lemma. \square

6 Partitions of oriented graphs

In this final section before the proof of Theorem 4 and Theorem 2 we prove a technical lemma which will allow us to decompose oriented graphs as given in Theorem 4 into smaller subgraphs, each of which satisfy the hypothesis of Lemma 18 and Lemma 19.

Lemma 27. *Let $\beta \geq \alpha > \varepsilon > 0$, let $K, d, n \in \mathbb{N}$, with n sufficiently large, $d = \alpha n$ and $K = \log n$. Suppose that G is an oriented graph on n vertices with $\delta^0(G) \geq \beta n$ and that D is a d -factor of G . Then there are K^3 edge-disjoint spanning subgraphs H_1, \dots, H_{K^3} of G with the following properties:*

1. For each H_i there is a partition $V(G) = U_i \cup W_i$ with $|W_i| = n/K^2 \pm 1$;
2. Letting $D_i = H_i[U_i]$ for all i , then $D_i \subseteq D$ and for some $r \geq (1 - 2\varepsilon)d/K^3$ we have
$$r - r^{3/5} \leq \delta^0(D_i) \leq \Delta^0(D_i) \leq r + r^{3/5};$$
3. Letting $E_i = H_i[U_i, W_i]$ we have $d_{E_i}^\pm(u, W_i) \geq \varepsilon|W_i|/4K$ for all $u \in U_i$;
4. Letting $F_i = H_i[W_i]$ we have $\delta^0(F_i) \geq (\beta - \varepsilon)|W_i|$.

Proof. To begin, select K partitions of $V(G)$ uniformly and independently at random where, for each $k \in [K]$, we partition $V(G)$ into K^2 sets, $V(G) = \bigcup_{\ell \in [K^2]} S_{k,\ell}$ with $|S_{k,\ell}| \in \{\lfloor n/K^2 \rfloor, \lceil n/K^2 \rceil\}$. Note that for each $k \in [K]$ and $v \in V(G)$ there exists a unique $\ell := \ell(k, v) \in [K^2]$ for which $v \in S_{k,\ell}$. In particular, every $v \in V(G)$ belongs to *exactly* K sets $S_{k,\ell}$.

Second, observe that by Chernoff's inequality for a hypergeometrical distribution (see Remark 6), letting $s = \lfloor n/K^2 \rfloor$, with probability $1 - nK^3 e^{-\omega(\log n)} = 1 - o(1)$ we have

$$d_D^\pm(v, S_{k,\ell}) = \alpha|S_{k,\ell}| \pm 4\sqrt{s \log n} \quad \text{and} \quad d_G^\pm(v, S_{k,\ell}) = d_G^\pm(v)|S_{k,\ell}|/n \pm 4\sqrt{s \log n} \quad (9)$$

for all $v \in V(G)$, $k \in [K]$ and $\ell \in [K^2]$. In particular, as $|S_{k,\ell}| = s \pm 1 > n/2K^2 \gg \log n$, for all k and ℓ we have

$$\delta^0(G[S_{k,\ell}]) \geq \beta|S_{k,\ell}| - 4\sqrt{s \log n} \geq (\beta - \varepsilon/2)|S_{k,\ell}|. \quad (10)$$

For each $v \in V(G)$ and $k \in [K]$, let $X^+(v, k)$ denote the random variable which counts the number of $w \in N_G^+(v)$ such that $w \in S_{k,\ell(k,v)} \cap S_{k',\ell(k',v)}$ for some $k' \neq k$. Define $X^-(v, k)$ similarly.

Note that for $\sigma \in \{+, -\}$ we have

$$\mathbb{E}[X^\sigma(v, k)] \leq K \left(\frac{n}{K^4} \right) = \frac{n}{K^3} = o(s).$$

By Chernoff's inequality, with probability $1 - Kne^{-\Theta(n/K^3)} = 1 - o(1)$, for all $k \in [K]$ and $v \in V(G)$ we have

$$X^\sigma(v, k) \leq \frac{2n}{K^3} = o(s). \quad (11)$$

Lastly, for $\sigma \in \{+, -\}$ and $v \in V(D)$ we define the random variable $Y^\sigma(v)$ to be the number of vertices $u \in N_D^\sigma(v)$ with $u \in S_{k,\ell(k,v)}$ for some k . For all $\sigma \in \{+, -\}$ and $v \in V(D)$ we have

$$b := \mathbb{E}[Y^\sigma(v)] \leq Ks.$$

Note that, since all the vertices of D have the same in/outdegrees, the value of $\mathbb{E}[Y^\sigma(v)]$ is indeed independent of v . By Proposition 7, with probability $1 - 2nKn^Ke^{-(2\sqrt{K^2s \log n})^2/3Ks} = 1 - o(1)$, for all $\sigma \in \{+, -\}$ and $v \in V(D)$ we have

$$Y^\sigma(v) = b \pm 2\sqrt{K^2s \log n}. \quad (12)$$

Thus, with positive probability a collection of partitions satisfy (9), (11) and (12). Fix such a collection. We relabel $\{S_{k,\ell} \mid k \in [K] \text{ and } \ell \in [K^2]\}$ as $\{W_1, \dots, W_{K^3}\}$ (arbitrarily). Also set $F_i = G[W_i] \setminus R_i$, where R_i is the set of all edges which appear in more than one W_i . From (10) and (11), for each $i \in [K^3]$ we obtain

$$\delta^0(F_i) \geq (\beta - \varepsilon)|W_i|.$$

Next, let $D' = D \setminus (\bigcup_i E(G[W_i]))$. As D is d -regular, by (12), we have that for all $\sigma \in \{+, -\}$ and $v \in V(D)$

$$d_{D'}^\sigma(v) = d_D^\sigma(v) - Y^\sigma(v) = d - b \pm 2\sqrt{K^2s \log n}.$$

To complete the proof we partition the edges of D' into further oriented subgraphs

$$\{D_i\}_{i \in [K^3]} \text{ and } \{E_i\}_{i \in [K^3]}.$$

Each D_i will be an oriented subgraph with $V(D_i) = V(D) \setminus W_i := U_i$, and each E_i will consist of some directed edges between U_i and W_i . To obtain these graphs we will partition the edges at random as follows: Suppose that $e = uv \in E(D')$, and let $I_u = \{i \in [K^3] \mid u \in W_i\}$. Similarly, define I_v . By construction, $|I_u| = |I_v| = K$ and $I_u \cap I_v = \emptyset$. Now, we randomly and independently assign each $e \in E(D')$ to a subgraph according to the following distribution:

- for $i \notin I_u \cup I_v$, we assign e to D_i with probability $\frac{1-\varepsilon}{K^3-2K}$;
- for $i \in I_u \cup I_v$, we assign e to E_i with probability $\frac{\varepsilon}{2K}$.

Note that the probability for e to be assigned to some subgraph is 1.

By Chernoff's inequality, with probability at least $1 - nK^3e^{-\Theta(\sqrt{n \log n})^2/n} - nK^3e^{-\Theta(\frac{\varepsilon}{K})} = 1 - o(1)$ the resulting oriented graphs satisfy

- $r - r^{3/5} \leq r - 4\sqrt{n} \log n \leq \delta^0(D_i) \leq \Delta^0(D_i) \leq r + 4\sqrt{n} \log n \leq r + r^{3/5}$, where $r := \frac{(1-\varepsilon)(d-b)}{K^3-2K} \geq \frac{(1-2\varepsilon)d}{K^3}$;
- $d_{E_i}^\pm(v, W_i) \geq \varepsilon|W_i|/4K$ for all $v \in U_i$.

Finally, taking $H_i = D_i \cup E_i \cup F_i$ for each $i \in [K^3]$, it is easy to check that these graphs satisfy the requirements. \square

7 Proof of Theorem 4

We are now ready to complete the proof of Theorem 4.

Proof of Theorem 4. Let G be an oriented graph as in the assumptions of the theorem. Let $d := \text{reg}(G) = \alpha n$ and let $D \subseteq G$ be a d -factor of G . From Theorem 12, we find that G contains $(c - 3/8)n$ edge-disjoint Hamilton cycles, and so $\alpha \geq c - 3/8 > 0$.

First, we apply Lemma 27 to G and D , with $\beta = c$, α and $\varepsilon/4$ in place of ε . Setting $K = \log n$, this gives edge-disjoint subgraphs H_1, \dots, H_{K^3} of G with the following properties:

1. For each H_i there is a partition $V(G) = U_i \cup W_i$ with $|W_i| = n/K^2 \pm 1$;
2. Letting $D_i = H_i[U_i]$, for some $r \geq (1 - \varepsilon/2)d/K^3$, we have

$$r - r^{3/5} \leq \delta^0(D_i) \leq \Delta^0(D_i) \leq r + r^{3/5};$$

3. Letting $E_i = H_i[U_i, W_i]$ we have $d_{E_i}^\pm(u, W_i) \geq \varepsilon|W_i|/4K$ for all $u \in U_i$;
4. Letting $F_i = H_i[W_i]$ we have $\delta^0(F_i) \geq (\beta - \varepsilon)|W_i|$;

Secondly, by property 2. above we can apply Lemma 19 to each oriented graph D_i . This gives $\mathbf{P}_i \in \mathcal{P}(D_i, n/\log^4 n, r - n/\log^4 n)$ which satisfies

$$\delta^0(D_{\mathbf{P}_i}) \geq r - n/\log^4 n. \tag{13}$$

Lastly, apply Lemma 18 to \mathbf{P}_i for each i . Taking $t = r - n/\log^4 n$ and $a = b = n/\log^4 n$ and $s = |W_i| = n/K^2 \pm 1$, it is easy to check that the conditions of Lemma 18 hold using (13) and properties 3. and 4. above. This gives a collection $\mathcal{C}_i := \{C_{i1}, \dots, C_{it}\}$ of edge-disjoint Hamilton cycles in H_i .

To complete the proof, set $\mathcal{C} := \bigcup_i \mathcal{C}_i$. Since the H_i are edge-disjoint, together with property 3., we find that \mathcal{C} consists of

$$K^3 t \geq (1 - \varepsilon/2)K^3 r \geq (1 - \varepsilon)d$$

edge-disjoint Hamilton cycles of G . This completes the proof. \square

8 Proof of Theorem 2

Before proving Theorem 2 let us introduce a final convenient definition.

Definition 28. *Given an oriented graph H , a collection of t edge-disjoint Hamilton cycles $\{C_1, \dots, C_t\}$ of G is called an $(H, t)_{\mathcal{C}}$ -family. Let $\mathcal{C}(H, t)$ denote the set of all $(H, t)_{\mathcal{C}}$ -families of H .*

We are now ready for the proof of Theorem 2.

Proof of Theorem 2. Let $c > 3/8$ be fixed and $d = cn$. We would like to show that given any $\varepsilon > 0$ and a large enough n , every d -regular oriented graph G on n vertices satisfies

$$|\mathcal{C}(G, d)| \geq n^{(1-\varepsilon)dn}.$$

Let $K = \log n$ and $\alpha = \varepsilon/4$. Our proof proceeds in five steps.

Step 1. Removing a δ -absorbing subgraph from G .

By Theorem 15, there exists $\delta > 0$ such that G contains a δ -absorber subgraph A , where A is a -regular, with $a \leq \alpha n$. Fix such a choice of A and let $G_0 := G \setminus A$.

Step 2. Partitioning G_0 .

Note that G_0 is $d' := cn - a$ regular with $\beta := d'/n > 3/8$. Therefore, taking $D = G$ and $\varepsilon_0 = \varepsilon/10$, applying Lemma 27, one can find K^3 edge-disjoint spanning subgraphs H_1, \dots, H_{K^3} of G_0 satisfying:

1. For each H_i there is a partition $V(G) = U_i \cup W_i$ with $|W_i| = n/K^2 \pm 1$;
2. Letting $D_i = H_i[U_i]$, with $r \geq (1 - 2\varepsilon_0)d'/K^3$, we have

$$r - r^{3/5} \leq \delta^0(D_i) \leq \Delta^0(D_i) \leq r + r^{3/5};$$

3. Letting $E_i = H_i[U_i, W_i]$ we have $d_{E_i}^\pm(u, W_i) \geq \varepsilon_0|W_i|/4K$ for all $u \in U_i$;
4. Letting $F_i = H_i[W_i]$ we have $\delta^0(F_i) \geq (\beta - \varepsilon_0)|W_i|$.

Step 3. Showing that for some $t = r - o(r)$ and for every $i \in [K^3]$ the set $\mathcal{C}(H_i, t)$ is large.

To this end, let us first apply Lemma 19 to each of the D_i s (note that by Property 2 above, the assumptions are fulfilled, and that $|U_i| = m = (1 - o(1))n$). It thus follows that for every i we have a collection

$$\mathcal{P}_i \subseteq \mathcal{P}(D_i, n/\log^4 n, r - n/\log^4 n)$$

which satisfies

$$|\mathcal{P}_i| \geq r^{(1-o(1))rn},$$

such that $\delta^0(D_{\mathbf{P}_i}) \geq r - n/\log^4 n$ for all $\mathbf{P}_i \in \mathcal{P}_i$.

Therefore, by Properties 3, 4 and the lower bound on $\delta^0(D_{\mathbf{P}_i})$, the hypothesis of Lemma 18 apply to H_i and \mathbf{P}_i , taking $a = b = n/\log^4 n$, $t = r - n/\log^4 n$ and $s = |W_i| = n/K^2 \pm 1$. This lemma allows us to turn \mathbf{P}_i into a collection of $t = r - n/\log^4 n$ edge-disjoint Hamilton cycles. Noting that we fix the W_i sets throughout the proof, we can trivially recover the path cover used to build each of the cycles. Therefore, for all $i \in [K^3]$ we have

$$|\mathcal{C}(H_i, t)| \geq |\mathcal{P}_i| \geq r^{(1-o(1))rn}. \quad (14)$$

Step 4. Showing that G_0 has $n^{(1-\varepsilon)dn}$ ‘almost Hamilton decompositions’.

To see this, note that if we pick $\mathbf{C}_i \in \mathcal{C}(H_i, t)$ for all i , then $\mathbf{C} = \bigcup_i \mathbf{C}_i \in \mathcal{C}(G_0, K^3 t)$. Therefore, by (14), for $t' = K^3 t$ we conclude that

$$|\mathcal{C}(G_0, t')| \geq r^{(1-o(1))rnK^3} \geq d^{(1-\varepsilon/5)d'n} \geq n^{(1-\varepsilon/4)(1-\alpha)dn} \geq n^{(1-\varepsilon/2)dn}. \quad (15)$$

Step 5. Completing every $\mathbf{C} \in \mathcal{C}(G_0, t')$ to a Hamilton decomposition of G .

Let $\mathbf{C} \in \mathcal{C}(G_0, t')$ and note that $G' = G_0 \setminus \mathbf{C}$ is a b -regular oriented graph with $b = o(n)$. Since $A := G \setminus G_0$ is a δ -absorber, and $b < \delta n$, it follows from Theorem 15 that $A \cup G'$ has a Hamilton decomposition \mathbf{C}' . But then $\mathbf{C} \cup \mathbf{C}'$ is a Hamilton decomposition of G . Lastly, note that although

different choices of $\mathbf{C} \in \mathcal{C}(G_0, t')$ may give rise to the same Hamilton decomposition in this way, it is easy to see that each such decomposition occurs at most $\binom{d}{t'} \leq 2^n$ times. By (15), this gives

$$|\mathcal{C}(G, d)| \geq |\mathcal{C}(G_0, t')|/2^n \geq n^{(1-\varepsilon)dn}.$$

This completes the proof. □

9 Concluding remarks

In this paper we have given bounds on the number of Hamilton decompositions of dense regular oriented graphs. Theorem 4 shows that if G is an r -regular n -vertex oriented graph, with $r = cn$ for some fixed $c > 3/8$, then it has $r^{(1+o(1))rn}$ Hamilton decompositions. As indicated in the Introduction this bound is tight for every such graph, up to the $o(1)$ -term in the exponent.

We believe that such oriented graphs should in fact have $\left((1 + o(1))\frac{r}{e^2}\right)^{rn}$ Hamilton decompositions. This would agree with the more precise upper bound obtained from the Minc conjecture in the Introduction. To prove this seems to require a version of Theorem 15 which can be applied to oriented graphs with sublinear density. In this respect, it would be very interesting to obtain an alternative proof of Kelly's conjecture that does not make use of regularity, as it seems likely to lead to such a theorem.

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References

- [1] N. Alon, J. Spencer, **The Probabilistic Method**, 3rd ed., John Wiley and Sons (2008).
- [2] B. Bollobás, **Modern Graph Theory**, Springer, New York (1998).
- [3] L. M. Brégman, *Some properties of non-negative matrices and their permanents*, Sov. Mat. Dokl. (1973) 14, 945–949.
- [4] G. Egorychev, *The solution of the Van der Waerden problem for permanents*, Dokl. Akad. Nauk SSSR (1981) 258, 1041–1044.
- [5] D. Falikman, *A proof of the Van der Waerden problem for permanents of a doubly stochastic matrix*, Mat. Zametki (1981) 29, 931–938.
- [6] A. Ferber, G. Kronenberg, and E. Long, *Packing, Counting and Covering Hamilton cycles in random directed graphs*, to appear in Israel J. Math., arXiv:1506.00618.
- [7] D. Gale, *A theorem on flows in networks*, Pacific J. Math. (1957) **7**, 1073–1082.
- [8] R. Gelbov, Z. Luria and B. Sudakov, *The number of Hamiltonian decompositions of regular graphs*, *Israel Journal of Mathematics*, to appear.

- [9] R. Häggkvist, *Hamilton cycles in oriented graphs*, *Combin. Probab. Comput.* (1993) **2**, 25–32.
- [10] R. M. Karp. *Reducibility among combinatorial problems*. Springer, 1972.
- [11] P. Keevash, D. Kühn and D. Osthus, *An exact minimum degree condition for Hamilton cycles in oriented graphs*, *J. Lond. Math. Soc.* (2009) **79**, 144–166.
- [12] L. Kelly, D. Kühn and D. Osthus, *A Dirac type result on Hamilton cycles in oriented graphs*, *Combin. Probab. Comput.* (2008) **17**, 689–709.
- [13] D. Kühn and D. Osthus, *A survey on Hamilton cycles in directed graphs*, *European J. Combinatorics* (2012) **33**, 750–766.
- [14] D. Kühn and D. Osthus, *Hamilton cycles in graphs and hypergraphs: an extremal perspective*, *Proceedings of the International Congress of Mathematicians 2014, Seoul, Korea, Vol 4*, 381–406.
- [15] D. Kühn and D. Osthus, *Hamilton decompositions of regular expanders: a proof of Kellys conjecture for large tournaments*, *Adv. Math.* (2013) **237**, 62–146.
- [16] D. Kühn and D. Osthus, *Hamilton decompositions of regular expanders: applications*, *J. Combin. Theory, Ser. B*, (2014) **104**, 1–27.
- [17] D. Kühn, D. Osthus and A. Treglown, *Hamilton decompositions of regular tournaments*, *Proc. Lond. Math. Soc.* (2010) **101**, 303–335.
- [18] H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, *Canad. J. Math.* (1957) **9**, 371–377.
- [19] T. W. Tillson, *A Hamiltonian decomposition of K_{2m}^* , $2m \geq 8$* , *J. Combin. Theory, Ser. B*, (1980) (1) **29**, 68–74.