Autumn 2012 **Extremal Combinatorics - Sheet 2** E.P. Long

There are also exercises in the notes; these are intended to assist in your understanding of the material and should all be attempted. The examples sheets are unassessed, but you are welcome to hand in your attempts the week after they are handed out for feedback.

1. The purpose of this exercise is to guide you through a proof of a slightly weakened version of Turán's theorem, starting with another proof of Mantel's theorem. So, complete the steps:
   (a) Given a graph G on vertex set $\{1, 2, \ldots, n\}$ with at least one edge, define the polynomial
   $$f_G(x_1, \ldots, x_n) = \sum_{ij \in E(G)} x_i x_j.$$
   This is sometimes known as the *Lagrangian* of G.
   (b) Let S denote the region $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ and } \sum_{i=1}^n x_i = 1\}$. Explain why the supremum $\sup_{\mathbf{x} \in S} f_G(\mathbf{x})$ is actually a maximum. Let us call this maximum λ.
   (c) Show that there is a tuple $\mathbf{a} \in S$ such that $f_G(\mathbf{a}) = \lambda$ and if $ij \notin E(G)$ then at least one of a_i and a_j is 0. (*Hint: consider a variational argument.*)
   (d) Supposing G is triangle-free, conclude that $\lambda = a_i a_j$ for some edge ij, and hence that $\lambda \leq 1/4$.
   (e) Finally evaluate $f_G\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ and use the fact that it is at most λ to give a bound for $e(G)$.

   Now extend this argument to show that $\mathrm{ex}(n, K_{r+1}) \leq \left(1 - \frac{1}{r}\right) n^2/2$.

2. Let A be the graph on 4 vertices comprising 2 triangles sharing an edge. Let B be the graph on 5 vertices comprising 2 triangles sharing a vertex.
   (a) Show by induction on n that if G is a graph on $n \geq 4$ vertices with $e(G) \geq \lfloor \frac{n^2}{4} \rfloor + 1$ then G contains a copy of A as a subgraph.
   (b) Show by induction on n that if G is a graph on $n \geq 5$ vertices with $e(G) \geq \lfloor \frac{n^2}{4} \rfloor + 2$ then G contains a copy of B as a subgraph.
   (c) Determine $\mathrm{ex}(n, A)$ and $\mathrm{ex}(n, B)$.

3. By modifying the proof of our upper bound for $\mathrm{ex}(n, C_4)$, show that
   $$\mathrm{ex}(n, K_{r,r}) \leq C n^{2-1/r}$$
   for some constant $C = C(r) > 0$ and all $r \geq 2$. (*Hint: you can use the inequality* $\sum_{i=1}^n \binom{a_i}{r} \geq n\binom{a}{r}$ *where* $a_i \in \mathbb{N}$ *and* $a = \frac{a_1 + \ldots + a_n}{n}$ *satisfies* $a \geq r$.) Deduce that $\pi(H) = 0$ for any bipartite graph H.

4. Let H be a graph on h vertices and let $\pi(H)$ be the Turán density of H. The purpose of the following exercise is to show that given $\epsilon > 0$, there exists $c = c(H, \epsilon) > 0$ so that for n sufficently large, any graph n-vertex graph G with $e(G) \geq (\pi(H) + \epsilon)\binom{n}{2}$ contains at least cn^h copies of H.
   (a) First show that for n_0 sufficiently large $\mathrm{ex}(n_0, H) < (\pi(H) + \frac{\epsilon}{2})\binom{n_0}{2}$.
   (b) Now show that for $n \geq n_0$, $V(G)$ must contain at least $c' n^{n_0}$ subsets S with $|S| = n_0$ such that $e(G[S]) \geq (\pi(H) + \frac{\epsilon}{2})\binom{n_0}{2}$, where $c' > 0$ and depends on n_0 and ϵ but not on n.
   (c) By choice of n_0, each of these subgraphs $G([S])$ contains a copy of H. By counting how many times a given copy of H can appear in one of these subgraphs $G[S]$, prove the bound claimed above.

Please mail me if you have any comments or corrections.

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