# A proof of Kelly's conjecture for large tournaments

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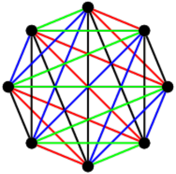
# Hamilton decompositions of graphs

Hamilton decomposition of $G$
$=$ edge-disjoint Hamilton cycles covering all edges of $G$

**Theorem (Walecki, 1892)**

*Complete graph $K_n$ has a Hamilton decomposition $\Leftrightarrow$ $n$ odd*

**Construction:** find Hamilton path decomposition for $K_{n-1}$



then add extra vertex and close paths into Hamilton cycles

**Theorem (Walecki, 1892)**

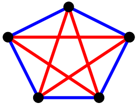
*Complete graph $K_n$ has a Hamilton decomposition $\Leftrightarrow$ $n$ odd*

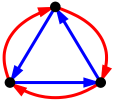
**Theorem (Tillson, 1980)**

*Complete digraph $K_n$ has a Hamilton decomposition $\Leftrightarrow$ $n \neq 4, 6$*

digraph: allow 1 edge in each direction between 2 vertices

oriented graph: allow at most 1 edge between 2 vertices



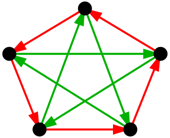


tournament: orientation of a complete graph

**Conjecture (Kelly, 1968)**

*Every regular tournament has a Hamilton decomposition.*

Decomposition of regular tournament into 2 Hamilton cycles



**Partial results on Kelly's conjecture**

*Thomassen (1979,1982), Jackson (1981), Alspach et al. (1990), Häggkvist(1993), Häggkvist & Thomason (1997), Bang-Jensen & Yeo (2004), Frieze & Krivelevich (2005), Keevash et al. (2009) ...*

Approximate solution to Kelly's conjecture:

**Theorem (Kühn, Osthus & Treglown, 2010)**

*Every regular tournament contains a set of edge-disjoint Hamilton cycles covering almost all the edges.*

Exact solution:

**Theorem (Kühn & Osthus $2012^+$)**

*Every large regular tournament has a Hamilton decomposition.*

Exact solution uses approximate one as a tool

# Finding Approximate Decompositions

**Theorem (Kühn, Osthus & Treglown, 2010)**

*Every regular tournament $G$ contains a set of edge-disjoint Hamilton cycles covering almost all the edges.*

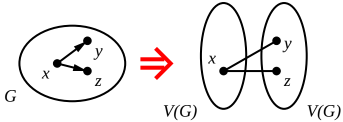
**Strategy:**

- Decompose almost all of $G$ into suitable 1-factors
- Transform 1-factors into Hamilton cycles using remaining edges

(1-factor: union of directed cycles spanning $V(G)$)

# Finding Approximate Decompositions

**Claim:** $G$ regular & oriented $\Rightarrow$ $G$ has 1-factor
**Proof:** consider (regular) auxiliary bipartite graph $H$




perfect matching in $H$ $\Leftrightarrow$ 1-factor in $G$
Use this successively to get almost decomposition into 1-factors.

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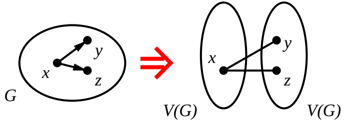


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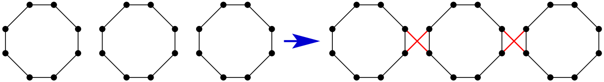
**Aim**

*Use remaining edges to 'merge' each 1-factor into Hamilton cycle by 'rotation-extension'*



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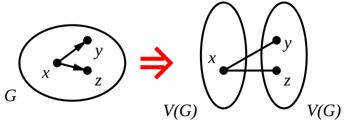


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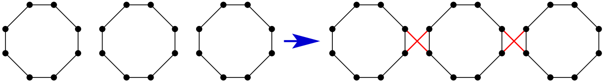


Hopeless as might need many such red edges for this.

**Theorem (Frieze & Krivelevich, 2005)**

*Choosing 1-factors randomly gives 1-factors with few cycles*

(applied to find approx. decompositions of quasi-random graphs)

**Aim still hopeless:**
leftover edges might not be the ones needed for merging

$\Rightarrow$ Need to find almost 1-factor decomposition with more structure
$\Rightarrow$ Apply regularity lemma and work with an almost 1-factor decomposition of the 'weighted reduced digraph'

This strategy finds an approx. decomposition but not a 'complete' decomposition – as the merging needs a reservoir of leftover unused edges

# Finding Hamilton decompositions

Will now sketch strategy of main result

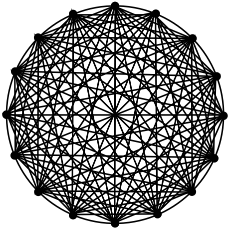
**Theorem (Kühn & Osthus $2012^+$)**

*Every large regular tournament has a Hamilton decomposition.*

**Crucial notion:** $H$ is robustly decomposable if:
for any $G$ which is regular and sparse compared to $H$
$H \cup G$ has a Hamilton decomposition

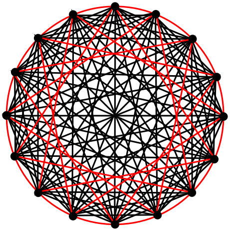
- Far from clear whether such $H$ exists!!
- Will use this in combination with approx. result

**Aim:** decompose regular tournament $G$ into Hamilton cycles



# Finding Hamilton decompositions

**Aim:** decompose regular tournament $G$ into Hamilton cycles



**Rough Strategy:**

- Find sparse $H$ inside $G$ which is 'robustly' Hamilton decomposable and let $G_1$ consist of remaining edges

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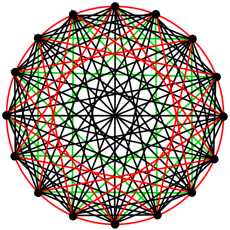


**Rough Strategy:**

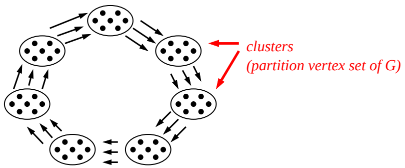
- Find sparse $H$ inside $G$ which is 'robustly' Hamilton decomposable and let $G_1$ consist of remaining edges
- Find 'almost' decomposition of $G_1$ using result of Kühn, Osthus & Treglown'10 to obtain very sparse leftover $G_2$
- Find Hamilton decomposition of $G_2 \cup H$ using robustness of $H$

Instead of one robustly decomposable graph $H$, will use two graphs in two successive steps:

- $CA$ (chord absorber)
- $CyA$ (cycle absorber)

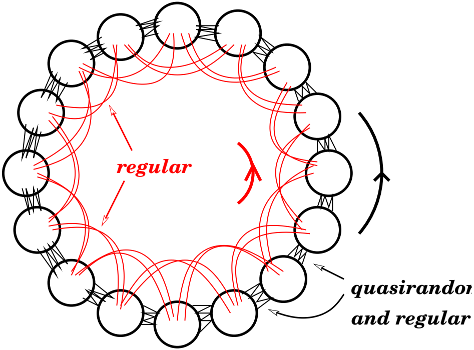
**Actual Strategy:**

- Remove sparse $CA$, $CyA$ from $G$ to obtain leftover $G_1$
- Find 'almost' decomposition of $G_1$ using result of KOT to obtain very sparse leftover $G_2$
- Find edge disjoint Hamilton cycles in $CA \cup G_2$ covering $G_2$ Leftover $G_3$ is sparse and is a blown-up Hamilton cycle




- Find a Hamilton decomposition of $G_3 \cup CyA$

**Definition of chord absorber** *CA***:**
*CA* blow up of a (directed) square of a Hamilton cycle

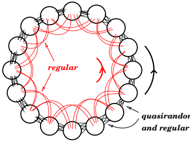



Can find this within a regular tournament

**Strategy for chord-absorbing step:**

- Partition leftover (from approx. decomposition step) $G_2$ into 1-factors $F_1, \ldots, F_r$
- Split each $F_i$ into small matchings $M_{i1}, \ldots, M_{is}$
- Extend each $M_{ij}$ into a Hamilton cycle using edges of chord absorber $CA$

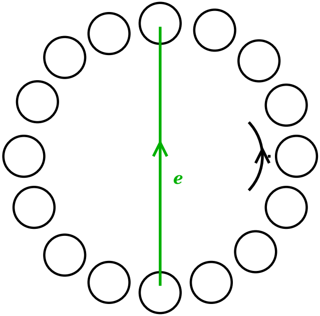
So leftover $G_3$ of chord-absorbing is a subgraph of $CA$




**Main challenge:** $G_3$ needs to be the blow-up of a cycle
i.e. Hamilton cycles need to use up all red edges

# Chord absorbing

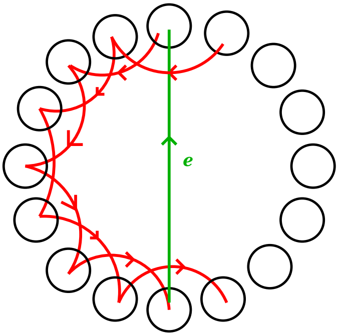
**Example:** Leftover matching $M_{ij}$ is a single edge $e$




**Cannot extend** $e$ to a Hamilton cycle using cyclic edges

# Chord absorbing

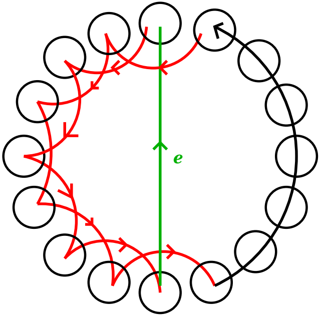
**Example:** $M_{ij}$ is a single edge $e$
Extend by adding suitable red edges




union of red and green edges is 'locally balanced':
for each edge entering a cluster there is one leaving predecessor

# Chord absorbing

**Example:** $M_{ij}$ is a single edge $e$



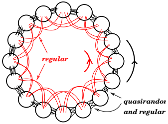

Let $W$ :=e+red+black path
Local balance $\Rightarrow$
Edges of $W$ enter and leave every cluster exactly once.

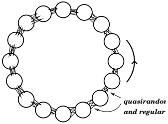
$\Rightarrow$ can extend $W$ to Hamilton cycle using cyclic edges:

Leftover is now subgraph of chord absorber $CA$
Following lemma implies desired stronger property:

**Lemma**

*After 'absorbing' entire (regular) green leftover $G_1$ into Hamilton cycles, have used* ***all*** *red edges at each cluster*

Chord absorber 

$\Rightarrow$ Leftover 

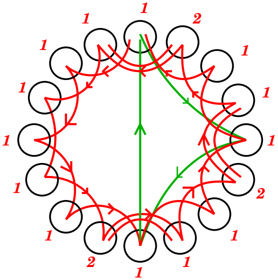

Will just verify weaker result (which is used in lemma proof)

**Claim**

*After 'absorbing' any green 1-factor $F_i$ into Hamilton cycles, have used* ***same number*** *of red edges at each cluster*

**Claim proof:**

Consider red edges used after absorbing a green triangle of $F_i$




'used' red outdegrees at clusters preceding green outedges $\mathbf{=2}$

'used' red outdegrees at other clusters $\mathbf{=1}$

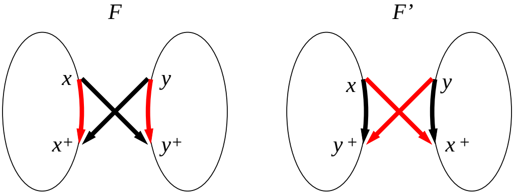
But # edges of 1-factor $F_i$ leaving each cluster is same

$\Rightarrow$ red outdegrees of clusters used for absorbing **entire** $F_i$ are equal!

# Cycle switching

**Aim:** Hamilton decomposition of $G_3 \cup CyA$, where
Cycle absorber $CyA$ is pre-chosen regular digraph
Leftover $G_3$ from chord-absorbing is regular blown-up cycle

**Rough idea:** Decompose $G_3 \cup CyA$ into 1-factors $F$
Switch pairs of edges between different 1-factors
$\Rightarrow$ successively reduce the total number of cycles




For simplicity, we consider undirected graphs in what follows.

Building blocks of cycle absorber $CyA$:

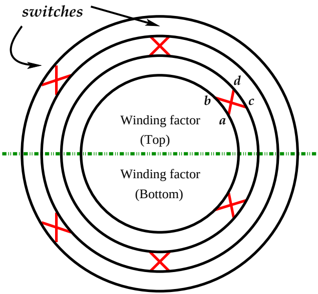
**Winding factors and switching cycles:**
Find 1-factor $WF$ (winding factor) & $SC$ (switching cycle) so that: for any leftover factor $F$ which winds around a blown-up cycle $C$ $WF \cup SC \cup F$ has a decomposition into 3 Hamilton cycles

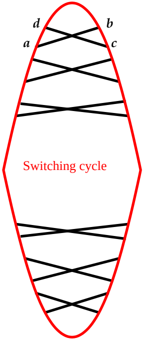
**The Hamilton decomposition:**
Let $r$ be degree of leftover $G_3$ from chord absorbing step.
The cycle absorber $CyA$ will consist of (edge-disjoint)

- winding factors $WF_1, \dots, WF_r$
- switching cycles $SC_1, \dots, SC_r$

Then decompose $G_3$ into leftover 1-factors $F_1, \dots, F_r$.
Finally $F_i \cup SC_i \cup WF_i$ has a Hamilton decomposition for each $i$.

# The winding factor $WF$ and the switching cycle $SC$



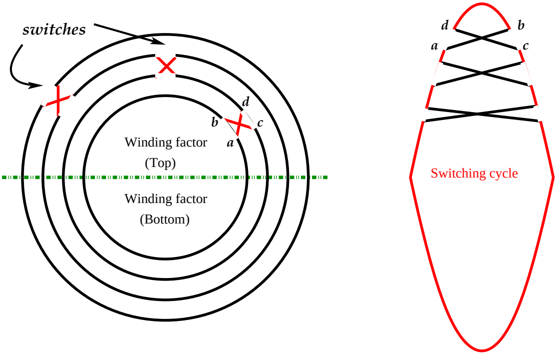

**Note:** switching cycle $SC$ remains a cycle after switching e.g. the edges at $abcd$

$\Rightarrow$ $WF \cup SC$ has a Hamilton decomposition

(carry out the three switches in the top half)

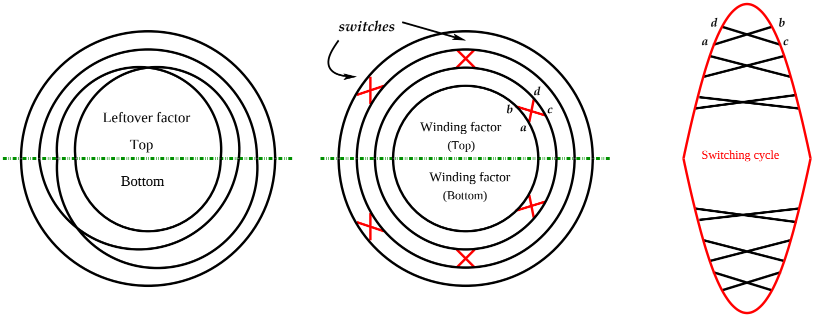
# The winding factor $WF$ and the switching cycle $SC$

A Hamilton decomposition of the winding factor $WF$ and the switching cycle $SC$




This also works if we replace the bottom half of the winding factor with the bottom half of the leftover factor!

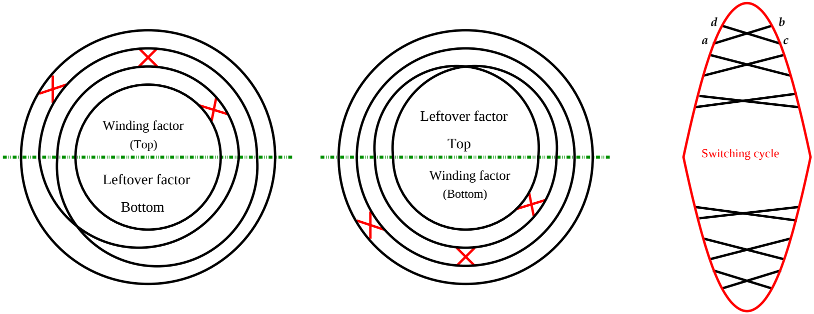
**Recall:** $WF \cup SC$ has Hamilton decomposition using switches
Now also consider a leftover factor $F$




**Key idea:** Switching approach also works if replace bottom half of the winding factor $WF$ with bottom half of the leftover factor $F$!

# The winding factor $WF$ and the switching cycle $SC$

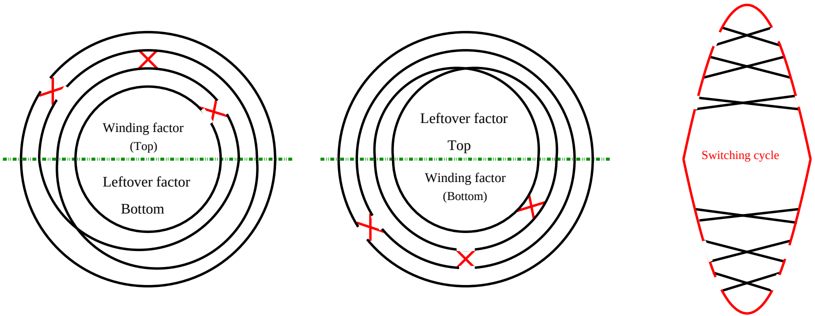
Recombining the leftover factor $F$ and the winding factor $WF$




Now just use those switches we need to turn the recombined factors into Hamilton cycles

**Note:** Recombination step used that $F$ is a blown-up cycle

A Hamilton decomposition of the union of:
leftover factor $F$, winding factor $WF$ and switching cycle $SC$




**Recall:** altogether this gives Hamilton decomposition of $G_3 \cup CyA$, and thus Hamilton decomposition of tournament $G$

# Concluding remarks on the proof

Many additional difficulties arise, e.g.

- switching more difficult for directed graphs
- there are exceptional vertices outside the clusters in all of the steps

**But** method of 'robust decompositions' to turn an approximate decomposition into a complete decomposition seems to be very general

# Robust expanders

Recall structural generalization of tournament decompositions

**Theorem (Kühn & Osthus, $2012^+$)**

*Every large regular robust outexpander of linear degree has a Hamilton decomposition*

proof uses approximate version as a tool
(i.e. edge-disjoint Hamilton cycles covering almost all edges)

**Theorem (Osthus & Staden, $2012^+$)**

*Every large regular robust outexpander of linear degree has an approximate Hamilton decomposition*

- both proofs are algorithmic
- in proof of approximate version cannot use trick (mentioned in tournament sketch) of using random 1-factorization

# TSP tour domination

**Asymmetric travelling salesman problem (ATSP)**

*Hamilton cycle of least weight in an edge-weighted complete digraph (opposite edges are allowed to have different weight).*

- $\nexists$ approximation algorithm for the ATSP whose approximation ratio is bounded unless $P = NP$.
- Note total number of possible solutions is $(n-1)!$

For any problem instance $I$ let $w(I)$ be the weight of the solution produced by algorithm $A$.

**Domination ratio of an algorithm $A$**

*$A$ has domination ratio $p(n)$ iff $\forall n$ and $\forall$ instances $I$ on $n$ vertices, there are at least $p(n)(n-1)!$ solutions to instance $I$ whose weight is also at least $w(I)$.*

i.e. fraction of solutions which are 'worse' is at least $p(n)$

$\exists$ TSP algorithms achieve a domination ratio of $\Omega(1/n)$ for ATSP

**Question (Glover & Punnen 1997, Alon, Gutin & Krivelevich 2004)**

*Is there a polynomial time algorithm which achieves a constant domination ratio for the ATSP?*

Gutin and Yeo (2001):
algorithmic proof of existence of Hamilton decompositions of sufficiently dense digraphs
$\Rightarrow \exists$ polynomial time algorithm with domination ratio $1/2 - \varepsilon$

**Theorem (Kühn & Osthus, $2012^+$)**

*For any $\varepsilon > 0$, there is a polynomial time algorithm for the ATSP whose domination ratio is $1/2 - \varepsilon$.*

# TSP tour domination

**Algorithm** (for finding TSP tour with large domination ratio):

- Find regular subgraph $H$ (of weighted complete digraph) with degree $n/2 + \varepsilon n$ and of minimum weight
- $H$ is robust outexpander $\Rightarrow$ $H$ has a Hamilton decomposition
- Let $C$ be Hamilton cycle of minimum weight in this decomposition
- $C$ has 'dominates' $(1/2 - \varepsilon)$ fraction of all tours

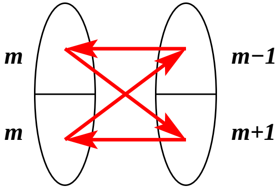
# Open problems I

Bipartite analogue of Kelly's conjecture

**Conjecture (Jackson)**

*Every regular bipartite tournament has a Hamilton decomposition.*

No analogue for almost regular bipartite tournaments:




Cannot even find a single Hamilton cycle in above example

# Open problems II

**Conjecture (Thomassen, 1982)**

*$\forall k\ \exists f(k)$ so that every strongly $f(k)$-connected tournament has $k$ edge-disjoint Hamilton cycles*

Generalization of Kelly's conjecture:

**Conjecture (Bang-Jensen & Yeo, 2004)**

*Every $k$-edge connected tournament has a decomposition into $k$ spanning strongly connected subgraphs*

- Kelly $\Leftrightarrow k = (n-1)/2$
- Bang-Jensen & Yeo: $k = 2$