HAMILTON ℓ-CYCLES IN k-GRAPHS

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Abstract. We say that a k-uniform hypergraph C is an *ℓ-cycle* if there exists a cyclic ordering of the vertices of C such that every edge of C consists of k consecutive vertices and such that every pair of adjacent edges (in the natural ordering of the edges) intersects in precisely ℓ vertices. We prove that if $1 \leq \ell < k$ and $k - \ell$ does not divide k then any k-uniform hypergraph on n vertices with minimum degree at least $\frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + o(n)$ contains a Hamilton ℓ-cycle. This confirms a conjecture of Hàn and Schacht. Together with results of Rödl, Ruciński and Szemerédi, our result asymptotically determines the minimum degree which forces an ℓ-cycle for any ℓ with $1 \leq \ell < k$.

1. Introduction

A *k-graph* $\mathcal{H}$ (also known as a k-uniform hypergraph), consists of a set of *vertices* $V(\mathcal{H})$ and a set of *edges* $E(\mathcal{H}) \subseteq \{X \subseteq V(\mathcal{H}) : |X| = k\}$, so that each edge of $\mathcal{H}$ consists of k vertices. Let $\mathcal{H}$ be a k-graph, and let A be a set of $k-1$ vertices of $\mathcal{H}$. Then the *degree* of A, denoted $d_{\mathcal{H}}(A)$, is the number of edges of $\mathcal{H}$ which contain A as a subset. The *minimum degree* $\delta(\mathcal{H})$ of $\mathcal{H}$ is then the minimum value of $d_{\mathcal{H}}(A)$ taken over all sets A of $k-1$ vertices of $\mathcal{H}$.

We say that a k-graph C is an *ℓ-cycle* if there exists a cyclic ordering of the vertices of C such that every edge of C consists of k consecutive vertices and such that every pair of adjacent edges (in the natural ordering of the edges) intersects in precisely ℓ vertices. We say that a k-graph $\mathcal{H}$ contains a *Hamilton ℓ-cycle* if it contains a spanning sub-k-graph which is an ℓ-cycle. Note that if a k-graph $\mathcal{H}$ on n vertices contains a Hamilton ℓ-cycle then $(k-\ell)|n$, since every edge contains exactly $k-\ell$ vertices which were not contained in the previous edge.

We shall give an asymptotic solution to the question of what minimum degree will guarantee that a k-graph $\mathcal{H}$ on n vertices contains a Hamilton ℓ-cycle. This can be viewed as a generalisation of Dirac's theorem [4], which states that any graph (i.e. 2-graph) with $n \geq 3$ vertices and of minimum degree at least $n/2$ contains a Hamilton cycle.

In [13] and [14], Rödl, Ruciński and Szemerédi proved the following theorem for $\ell = k-1$; the other cases follow, since if $(k-\ell)|n$ then any $(k-1)$-cycle of order n contains an ℓ-cycle on the same vertices.

Theorem 1.1. *For all $k \geq 3$, $1 \leq \ell \leq k-1$ and any $\eta > 0$ there exists n_0 so that if $n > n_0$ and $(k-\ell)|n$ then any k-graph $\mathcal{H}$ on n vertices with $\delta(\mathcal{H}) \geq \left(\frac{1}{2} + \eta\right) n$ contains a Hamilton ℓ-cycle.*

This proved a conjecture of Katona and Kierstead [7]. Proposition 2.2 shows that Theorem 1.1 is best possible up to the error term ηn if $(k-\ell)|k$. This then raises the natural question of what minimum degree guarantees a Hamilton ℓ-cycle if $(k-\ell) \nmid k$. In [11], Kühn and Osthus showed that any 3-graph $\mathcal{H}$ on n vertices with n even and $\delta(\mathcal{H}) \geq (\frac{1}{4} + o(1))n$ contains a Hamilton 1-cycle. Keevash, Kühn, Mycroft and Osthus [9] extended this result to k-graphs, showing that any k-graph $\mathcal{H}$ on n vertices with $(k-1)|n$ and $\delta(\mathcal{H}) \geq (\frac{1}{2k-2} + o(1))n$ contains a Hamilton 1-cycle. (The proof in [9] is based on a 'hypergraph blow-up lemma' due to Keevash [8].) This was also proved independently by Hàn and Schacht [6] using a different method. In fact, they showed that if $1 \leq \ell < k/2$, then any k-graph $\mathcal{H}$ on n vertices with $(k-\ell)|n$ and $\delta(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} + o(1))n$ contains a Hamilton ℓ-cycle. They raised the question

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of determining the correct minimum degree for those values of k and ℓ not covered by their result or by Theorem 1.1. Our main result confirms their conjecture and generalises their result.

Theorem 1.2. *For all $k \geq 3$, $1 \leq \ell \leq k-1$ such that $(k-\ell) \nmid k$ and any $\eta > 0$ there exists n_0 so that if $n > n_0$ and $(k-\ell)|n$ then any k-graph $\mathcal{H}$ on n vertices with $\delta(\mathcal{H}) \geq \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + \eta\right) n$ contains a Hamilton ℓ-cycle.*

This result is best possible up to the error term ηn, as shown by Proposition 2.1. Thus Theorem 1.1 and Theorem 1.2 together give asymptotically, for any k and ℓ, the minimum degree required to guarantee that a k-graph on n vertices contains a Hamilton ℓ-cycle.

Also, less restrictive notions of hypergraph cycles have been considered, e.g. in [1].

2. Extremal examples and outline of the proof of Theorem 1.2

The next proposition shows that Theorem 1.2 is best possible, up to the error term ηn.

Proposition 2.1. *For all $k \geq 3$, $1 \leq \ell \leq k-1$ and every n with $(k-\ell)|n$ there exists a k-graph $\mathcal{H}$ on n vertices with $\delta(\mathcal{H}) \geq \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} - 1$ which does not contain a Hamilton ℓ-cycle.*

Proof. Let $a := \lceil \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} \rceil - 1$ and let V_1 and V_2 be disjoint sets of size a and $n-a$ respectively. Let $\mathcal{H}$ be the k-graph on vertex set $V = V_1 \cup V_2$ whose edges are all those k-sets of vertices which contain at least one vertex from V_1. Then $\delta(\mathcal{H}) = a$. However, an ℓ-cycle on n vertices has $n/(k-\ell)$ edges and every vertex on such a cycle lies in at most $\lceil \frac{k}{k-\ell} \rceil$ edges. Since $\lceil \frac{k}{k-\ell} \rceil |V_1| < n/(k-\ell)$, $\mathcal{H}$ cannot contain a Hamilton ℓ-cycle. □

The next proposition shows that, if $(k-\ell)|k$, then Theorem 1.1 is asymptotically best possible. By a *perfect matching* in a k-graph $\mathcal{H}$, we mean a set of disjoint edges of $\mathcal{H}$ whose union contains every vertex of $\mathcal{H}$. The construction is well known, but we include it here for completeness.

Proposition 2.2. *For all $k \geq 3$, $1 \leq \ell \leq k-1$ and every $n \geq 3k$ such that $(k-\ell)|k$ and $k|n$ there exists a k-graph $\mathcal{H}$ on n vertices with $\delta(\mathcal{H}) \geq \frac{n}{2} - k$ which does not contain a Hamilton ℓ-cycle.*

Proof. Choose $\frac{n}{2} - 1 \leq a \leq \frac{n}{2} + 1$ so that a is odd. Let V_1 and V_2 be disjoint sets of size a and $n-a$ respectively, and let $\mathcal{H}$ be the k-graph on vertex set $V = V_1 \cup V_2$ and with all those k-element subsets S of V such that $|S \cap V_1|$ is even as edges. Then $\delta(\mathcal{H}) \geq \min(a, n-a) - k + 1 \geq \frac{n}{2} - k$. Now, any Hamilton ℓ-cycle C in $\mathcal{H}$ would contain a perfect matching, consisting of every $\frac{k}{k-\ell}$th edge of C. Every edge in this matching would contain an even number of vertices from V_1, and so $|V_1|$ would be even. Since $|V_1| = a$ is odd, $\mathcal{H}$ cannot contain a Hamilton ℓ-cycle. □

In our proof of Theorem 1.2 we construct the Hamilton ℓ-cycle by finding several ℓ-paths and joining them into a spanning ℓ-cycle. Here a k-graph P is an *ℓ-path* if its vertices can be given a linear ordering such that every edge of P consists of k consecutive vertices, and so that every pair of adjacent edges of P (in the natural ordering induced on the edges) intersect in precisely ℓ vertices. We say that an enumeration $v_1, v_2, \dots, v_r$ of the vertices of P is a *vertex sequence* of P if the edges of P are $\{v_{s(k-\ell)+1}, \dots, v_{s(k-\ell)+k}\}$ for each $0 \leq s \leq (r-k)/(k-\ell)$. We say that ordered sets A and B are *ordered ends* of P if $|A| = |B| = \ell$ and A and B are initial and final segments of a vertex sequence of P. This allows us to join up ℓ-paths in the following manner. Let P and Q be ℓ-paths, and let P^{beg} and P^{end} be ordered ends of P, and Q^{beg} and Q^{end} be ordered ends of Q. Suppose that $P^{end} = Q^{beg}$, and that $V(P) \cap V(Q) = P^{end}$. Then the k-graph with vertex set $V(P) \cup V(Q)$ and with all the edges of P and of Q is an ℓ-path with ordered ends P^{beg} and Q^{end}.

Our proof of Theorem 1.2 uses ideas of [6], which in turn were based on the 'absorbing path' method of [13] and [14]. Our proof contains further developments of the method, which may be of independent interest. Roughly speaking, the absorbing technique proceeds as follows. We shall prove an 'absorbing path lemma', which states that in any sufficiently large k-graph of large minimum degree there exists an ℓ-path P which can 'absorb' any small set X of vertices outside P. By this we mean that for any such small set X there is another ℓ-path Q with the same ordered ends as P and with $V(Q) = V(P) \cup X$. Then we can think of replacing P with Q as 'absorbing' the vertices of X into P. We shall also prove a 'path cover lemma', which states that any sufficiently large k-graph satisfying the minimum degree condition of Theorem 1.2 can be almost covered by a bounded number of disjoint ℓ-paths. We can then prove Theorem 1.2 by combining these lemmas as follows. Firstly, we find in $\mathcal{H}$ an absorbing ℓ-path, and then we almost cover the induced k-graph on the remaining vertices by disjoint ℓ-paths. We connect up all of these ℓ-paths to form an ℓ-cycle C which thus contains almost every vertex of $\mathcal{H}$. Finally, we absorb all vertices of $\mathcal{H}$ not contained in C into our absorbing path, thereby forming an ℓ-cycle containing every vertex of $\mathcal{H}$.

Beyond these similarities, we have had to make substantial changes to the method of Hàn and Schacht. For example, it is simple to 'connect up' ℓ-paths P and Q in a k-graph $\mathcal{H}$ of large minimum degree when $1 \leq \ell < k/2$. Indeed, we may add any $k-1-2\ell$ vertices from outside P and Q to the ordered ends of P and Q to obtain a set S of size $k-1$. Then we can apply the minimum degree condition of $\mathcal{H}$ to find a vertex $x \in V(\mathcal{H}) \setminus (V(P) \cup V(Q))$ such that $S \cup \{x\}$ is an edge of $\mathcal{H}$. Then $P, S \cup \{x\}$ and Q together form a single ℓ-path in $\mathcal{H}$. However, if $\ell \geq k/2$ then things are more difficult. So to allow us to connect ℓ-paths, in Section 5 we shall use strong hypergraph regularity to prove a 'diameter lemma', which states that if $1 \leq \ell \leq k-1$ is such that $(k-\ell) \nmid k$, and A and B are ordered sets of ℓ vertices of a k-graph $\mathcal{H}$ which has large minimum degree, then $\mathcal{H}$ contains an ℓ-path with ordered ends A and B with a bounded number of vertices (i.e. the number of vertices depends only on k).

In Section 6 we prove our absorbing path lemma. Actually, we will not be able to absorb arbitrary sets of vertices, but only 'good' ℓ-sets of vertices. We will use strong hypergraph regularity to show that most ℓ-sets of vertices are good, which will be sufficient for our purposes. This weaker notion of absorption may be useful for other problems. In Section 7 we shall prove the path cover lemma. A similar result was already proved in [14]. The main difference is that they used weak regularity, whereas we have used strong regularity, but this is simply to avoid having to introduce two different notions of regularity — weak regularity would have sufficed for this part of our proof. Finally, in Section 8 we complete the proof as outlined earlier.

3. Definitions and a preliminary result

We begin with some notation. By $[r]$ we denote the set of integers from 1 to r. For a set A, we use $\binom{A}{k}$ to denote the collection of subsets of A of size k. We write $x = y \pm z$ to denote that $y - z \leq x \leq y + z$. By $0 < \alpha \ll \beta$ we mean that there exists an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that the following argument is valid for any $0 < \alpha \leq f(\beta)$. We write $o(1)$ to denote a function which tends to zero as n tends to infinity, holding all other variables involved constant. We shall omit floors and ceilings throughout this paper whenever they do not affect the argument.

Let $\mathcal{H}$ be a k-graph on vertex set V, with edge set E. Then the *order* of $\mathcal{H}$, denoted $|\mathcal{H}|$, is the number of vertices of $\mathcal{H}$ (so $|\mathcal{H}| = |V|$). For $A \subseteq V$, the *neighbourhood* of A is $N_{\mathcal{H}}(A) := \{B \subseteq V : A \cup B \in E, A \cap B = \emptyset\}$. The *degree* of A, denoted $d_{\mathcal{H}}(A)$, is the number of edges of $\mathcal{H}$ which contain A as a subset, so $d_{\mathcal{H}}(A) = |N_{\mathcal{H}}(A)|$. This is consistent with our previous definition of degree for sets of $k-1$ vertices. For any $V' \subseteq V$, the *restriction of $\mathcal{H}$ to V'*, denoted $\mathcal{H}[V']$, is the k-graph with vertex set V' and edges all those edges of $\mathcal{H}$ which are subsets of V'.

Given two ordered ℓ-sets of vertices of $\mathcal{H}$, say S and T, an *ℓ-path from S to T* in $\mathcal{H}$ is an ℓ-path in $\mathcal{H}$ which has a vertex sequence beginning with the ordered ℓ-set S and ending with the ordered ℓ-set T (i.e. an ℓ-path with ordered ends S and T). We say that a k-graph $\mathcal{H}$ is *s-partite* if its vertex set V can be partitioned into s vertex classes $V_1, \dots, V_s$ such that no edge of $\mathcal{H}$ contains more than one vertex from any vertex class V_i. We denote by $\mathcal{K}[V_1, \dots, V_s]$ the *complete s-partite k-graph* with vertex classes $V_1, \dots, V_s$, that is, the k-graph with vertex set $V = V_1 \cup \dots \cup V_s$ and edges all sets $S \in \binom{V}{k}$ with $|S \cap V_i| \leq 1$ for all i.

The following proposition regarding the existence of ℓ-paths in complete k-partite k-graphs will be required in the proof of both the diameter lemma and the absorbing path lemma.

Proposition 3.1. *Suppose that $k \geq 3$, and that $1 \leq \ell \leq k-1$ is such that $(k-\ell) \nmid k$. Let V be a set of vertices partitioned into k vertex classes $V_1, ..., V_k$, with $|V_i| = k\ell(k-\ell)+1$ for each i, and let P^{beg} and P^{end} be disjoint ordered sets of ℓ vertices from V such that $|P^{beg} \cap V_i| \leq 1$ and $|P^{end} \cap V_i| \leq 1$ for each $1 \leq i \leq k$. Then $\mathcal{K}[V_1, ..., V_k]$ contains an ℓ-path P from P^{beg} to P^{end} containing every vertex of V (so $|V(P)| = k^2\ell(k-\ell)+k$).*

Proof. To prove this result, we consider strings (finite sequences of characters) on character set $[k]$. We denote the ith character of a string S by S_i. By an ordering of $[k]$ we mean a string of length k which contains each character precisely once. Let A and B be orderings of $[k]$. We say that A and B are adjacent if we can obtain B from A by swapping a single pair of adjacent characters in A. So for example, 12345 is adjacent to 12435.

Suppose that A and B are adjacent orderings of $[k]$, and let i and $i+1$ be the positions in A of the characters swapped to obtain B from A (so $1 \leq i \leq k-1$). Since $(k-\ell) \nmid k$ we may choose $p \in \{1, 2\}$ such that $(k-\ell) \nmid ((p-1)k+i)$. Then define the string $S(A, B)$ to consist of p consecutive copies of A followed by $(k-\ell+1)-p$ copies of B. Then $S(A, B)$ has length $(k-\ell+1)k$ and the property that $S(A, B)$ starts with A and ends with B. Note that the only consecutive subsequence of S of length k which contains some character more than once is $S' = S(A, B)_{(p-1)k+i+1} \dots S(A, B)_{pk+i}$. In other words, S' contains the final $k-i$ characters of A and the first i characters of B, and the first and final character of S' is A_{i+1}. Therefore, as $(k-\ell) \nmid ((p-1)k+i)$, we know that no character appears twice in $S(A, B)_{r(k-\ell)+1}, \dots, S(A, B)_{r(k-\ell)+k}$ for any $0 \leq r \leq k$. Furthermore, $S(A, B)$ contains the same number of copies of each character. Note that if $(k-\ell)|k$ it would not be possible to make such a choice of p as we have done here, and the lemma would not hold in this case.

Now, choose a string C to be any ordering of $[k]$ such that for $1 \leq i \leq \ell$, the ith vertex of the ordered set P^{beg} lies in vertex class V_{C_i}. Define a string D to be an ordering of $[k]$ such that for $1 \leq i \leq \ell$, the ith vertex of the ordered set P^{end} lies in vertex class $V_{D_{i+k-\ell}}$, and the characters D_i for $1 \leq i \leq k-\ell$ appear in the same order as they do in C. Then we may transform C into D through at most $k\ell$ swaps of pairs of consecutive vertices. So we may choose $A^0, \dots, A^{k\ell}$ to be orderings of $[k]$ such that $A^0 = C$, $A^{k\ell} = D$, and for any $0 \leq i \leq k\ell - 1$, A^i and A^{i+1} are either adjacent or identical.

Then for each $0 \leq i \leq k\ell - 1$ we may choose a string S^i of length $k(k-\ell+1)$ such that S^i starts with A^i and ends with A^{i+1}, each character appears an equal number of times in S^i and for each $0 \leq r \leq k$ no character appears more than once in $S^i_{r(k-\ell)+1}, \dots, S^i_{r(k-\ell)+k}$.

Indeed, if A^i and A^{i+1} are adjacent, take S^i to be $S(A^i, A^{i+1})$, and if $A^i = A^{i+1}$, take S^i to be the string consisting of $k-\ell+1$ consecutive copies of A^i. For each $0 \leq i \leq k\ell - 2$, let T^i be the string obtained by deleting the final k characters of S^i, and let $T^{k\ell-1} = S^{k\ell-1}$. Let S be the string formed by concatenating $T^0, \dots, T^{k\ell-1}$. Then S starts with C and ends with D and has the property that no character appears twice in $S_{r(k-\ell)+1}, \dots, S_{r(k-\ell)+k}$ for any $0 \leq r \leq k^2\ell$. Also $|S| = k^2\ell(k-\ell) + k$, and so since S contains each character the same number of times, each character appears $k\ell(k-\ell)+1$ times in S.

We can now construct the vertex sequence of our desired ℓ-path P. To do so, let P have vertex sequence beginning with P^{beg} and ending with P^{end}. In between, let the ith vertex of P be chosen from V_{S_i}, and make these choices without choosing the same vertex twice. Then P contains all $k\ell(k-\ell)+1$ vertices from each vertex class and is an ℓ-path. Indeed, the edges of an ℓ-path P consist of the vertices in positions $r(k-\ell)+1, \dots, r(k-\ell)+k$ for $0 \leq r \leq |E(P)| - 1$. So by construction these vertices are from different vertex classes, and so form an edge in $\mathcal{K}[V_1, \dots, V_k]$. □

4. The regularity lemma for k-graphs

4.1. **Regular complexes.** Before we can state the regularity lemma, we first have to say what we mean by a regular or 'quasi-random' hypergraph and, more generally, by a regular complex. A *hypergraph* $\mathcal{H}$ consists of a vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H})$, where every edge $e \in E(\mathcal{H})$ is a non-empty subset of $V(\mathcal{H})$. So a k-graph as defined earlier is a hypergraph in which every edge has size k. A hypergraph $\mathcal{H}$ is a *complex* if whenever $e \in E(\mathcal{H})$ and e' is a non-empty subset of e we have that $e' \in E(\mathcal{H})$. All the complexes considered in this paper have the property that every vertex forms an edge. A complex $\mathcal{H}$ is a *k-complex* if every edge of $\mathcal{H}$ consists of at most k vertices. The edges of size i are called *i-edges* of $\mathcal{H}$. We write $|\mathcal{H}| := |V(\mathcal{H})|$ for the *order* of $\mathcal{H}$. Given a k-complex $\mathcal{H}$, for each $i = 1, \dots, k$ we denote by $\mathcal{H}_i$ the *underlying i-graph of $\mathcal{H}$*. So the vertices of $\mathcal{H}_i$ are those of $\mathcal{H}$ and the edges of $\mathcal{H}_i$ are the i-edges of $\mathcal{H}$.

Note that a k-graph $\mathcal{H}$ can be turned into a k-complex, which we denote by $\mathcal{H}^{\leq}$, by making every edge into a complete i-graph $K_k^{(i)}$, for each $1 \leq i \leq k$. (In a more general k-complex we may have i-edges which do not lie within an $(i+1)$-edge.) Given $k \leq s$, a *(k,s)-complex* $\mathcal{H}$ is an s-partite k-complex, by which we mean that the vertex set of $\mathcal{H}$ can be partitioned into sets $V_1, \dots, V_s$ such that every edge of $\mathcal{H}$ meets each V_i in at most one vertex.

Given $i \geq 2$, an i-partite i-graph $\mathcal{H}_i$, and an i-partite $(i-1)$-graph $\mathcal{H}_{i-1}$ on the same vertex set, we write $\mathcal{K}_i(\mathcal{H}_{i-1})$ for the set of i-sets of vertices which form a copy of the complete $(i-1)$-graph $K_i^{(i-1)}$ on i vertices in $\mathcal{H}_{i-1}$. We define the *density of $\mathcal{H}_i$ with respect to $\mathcal{H}_{i-1}$* to be

$$d(\mathcal{H}_i|\mathcal{H}_{i-1}) := \frac{|\mathcal{K}_i(\mathcal{H}_{i-1}) \cap E(\mathcal{H}_i)|}{|\mathcal{K}_i(\mathcal{H}_{i-1})|}$$

if $|\mathcal{K}_i(\mathcal{H}_{i-1})| > 0$, and $d(\mathcal{H}_i|\mathcal{H}_{i-1}) := 0$ otherwise. More generally, if $\mathbf{Q} := (Q(1), Q(2), \dots, Q(r))$ is a collection of r subhypergraphs of $\mathcal{H}_{i-1}$, we define $\mathcal{K}_i(\mathbf{Q}) := \bigcup_{j=1}^r \mathcal{K}_i(Q(j))$ and

$$d(\mathcal{H}_i|\mathbf{Q}) := \frac{|\mathcal{K}_i(\mathbf{Q}) \cap E(\mathcal{H}_i)|}{|\mathcal{K}_i(\mathbf{Q})|}$$

if $|\mathcal{K}_i(\mathbf{Q})| > 0$, and $d(\mathcal{H}_i|\mathbf{Q}) := 0$ otherwise.

We say that $\mathcal{H}_i$ is *(d_i, δ, r)-regular with respect to $\mathcal{H}_{i-1}$* if every r-tuple $\mathbf{Q}$ with $|\mathcal{K}_i(\mathbf{Q})| > \delta|\mathcal{K}_i(\mathcal{H}_{i-1})|$ satisfies $d(\mathcal{H}_i|\mathbf{Q}) = d_i \pm \delta$. Instead of $(d_i, \delta, 1)$-regularity we sometimes refer to *(d_i, δ)-regularity.*

Given $3 \le k \le s$ and a (k,s)-complex $\mathcal{H}$, we say that $\mathcal{H}$ is $(d_k,\dots,d_2,\delta_k,\delta,r)$-*regular* if the following conditions hold:

- For every $i = 2,\dots,k-1$ and for every i-tuple K of vertex classes either $\mathcal{H}_i[K]$ is (d_i,δ)-regular with respect to $\mathcal{H}_{i-1}[K]$ or $d(\mathcal{H}_i[K]|\mathcal{H}_{i-1}[K]) = 0$.
- For every k-tuple K of vertex classes either $\mathcal{H}_k[K]$ is (d_k,δ_k,r)-regular with respect to $\mathcal{H}_{k-1}[K]$ or $d(\mathcal{H}_k[K]|\mathcal{H}_{k-1}[K]) = 0$.

Here we write $\mathcal{H}_i[K]$ for the restriction of $\mathcal{H}_i$ to the union of all vertex classes in K. We sometimes denote $(d_k,\dots,d_2)$ by $\mathbf{d}$ and refer to $(\mathbf{d},\delta_k,\delta,r)$-regularity.

We will need the following lemma which states that the restriction of regular complexes to a sufficiently large set of vertices is still regular.

Lemma 4.1. *Let k,s,r,m be positive integers and $\alpha,d_2,\dots,d_k,\delta,\delta_k$ be positive constants such that*
$$1/m \ll 1/r, \delta \le \min\{\delta_k,d_2,\dots,d_{k-1}\} \le \delta_k \ll \alpha \ll d_k, 1/s.$$
Let $\mathcal{H}$ be a $(\mathbf{d},\delta_k,\delta,r)$-regular (k,s)-complex with vertex classes $V_1,\dots,V_s$ of size m. For each i let $V_i' \subseteq V_i$ be a set of size at least αm. Then the restriction $\mathcal{H}' = \mathcal{H}[V_1' \cup \dots \cup V_s']$ of $\mathcal{H}$ to $V_1' \cup \dots \cup V_s'$ is $(\mathbf{d},\sqrt{\delta_k},\sqrt{\delta},r)$-regular.

It is easy to prove Lemma 4.1 by induction on i (where $2 \le i \le k$ is as in the definition of a regular complex). In the induction step, use the dense hypergraph counting lemma (Corollary 6.11 in [10]) to show that $\sqrt{\delta}|\mathcal{K}_i(\mathcal{H}'_{i-1})| \ge \delta|\mathcal{K}_i(\mathcal{H}_{i-1})|$ and likewise when $i = k$.

4.2. **Statement of the regularity lemma.** In this section we state the version of the regularity lemma for k-graphs due to Rödl and Schacht [15], which we will use several times in our proof. To prepare for this we will first need some more notation. Suppose that V is a finite set of vertices and $\mathcal{P}^{(1)}$ is a partition of V into sets $V_1,\dots,V_{a_1}$, which will be called *clusters*. Given $k \ge 3$ and any $j \in [k]$, we denote by $\mathrm{Cross}_j = \mathrm{Cross}_j(\mathcal{P}^{(1)})$ the set of all those j-subsets of V that meet each V_i in at most 1 vertex. For every set $A \subseteq [a_1]$ with $2 \le |A| \le k-1$ we write Cross_A for all those $|A|$-subsets of V that meet each V_i with $i \in A$. Let $\mathcal{P}_A$ be a partition of Cross_A. We refer to the partition classes of $\mathcal{P}_A$ as *cells*. For each $i = 2,\dots,k-1$ let $\mathcal{P}^{(i)}$ be the union of all the $\mathcal{P}_A$ with $|A| = i$. So $\mathcal{P}^{(i)}$ is a partition of Cross_i.

$\mathcal{P}(k-1) = \{\mathcal{P}^{(1)},\dots,\mathcal{P}^{(k-1)}\}$ is a *family of partitions on* V if the following condition holds. Recall that a_1 denotes the number of clusters in $\mathcal{P}^{(1)}$. Consider any $B \subseteq A \subseteq [a_1]$ such that $2 \le |B| < |A| \le k-1$ and suppose that $S,T \in \mathrm{Cross}_A$ lie in the same cell of $\mathcal{P}_A$. Let $S_B := S \cap \bigcup_{i\in B} V_i$ and define T_B similarly. Then S_B and T_B lie in the same cell of $\mathcal{P}_B$.

To illustrate this condition, suppose that $k = 4$ and $A = [3]$. Then $\mathcal{P}_{\{1,2\}}$, $\mathcal{P}_{\{2,3\}}$ and $\mathcal{P}_{\{1,3\}}$ partition the edges of the 3 complete bipartite graphs induced by the pairs V_1V_2, V_2V_3 and V_1V_3. These the partitions together naturally induce a partition $\mathcal{Q}$ of the set of triples induced by V_1, V_2 and V_3. The above condition says that $\mathcal{P}_{\{1,2,3\}}$ must be a refinement of $\mathcal{Q}$.

Given $1 \le i \le j \le k$ with $i < k$, $J \in \mathrm{Cross}_j$ and an i-set $Q \subseteq J$, we write C_Q for the set of all those i-sets in Cross_i that lie in the same cell of $\mathcal{P}^{(i)}$ as Q. (In particular, if $i = 1$ then C_Q is the cluster containing the unique element in Q.) The *polyad* $\hat{P}^{(i)}(J)$ *of* J is defined by $\hat{P}^{(i)}(J) := \bigcup_Q C_Q$, where the union is over all i-subsets Q of J. So we can view $\hat{P}^{(i)}(J)$ as an j-partite i-graph (whose vertex classes are the clusters intersecting J). We let $\hat{\mathcal{P}}^{(j-1)}$ be the set consisting of all the $\hat{P}^{(j-1)}(J)$ for all $J \in \mathrm{Cross}_j$. So for each $K \in \mathrm{Cross}_k$ we can view $\bigcup_{i=1}^{k-1} \hat{P}^{(i)}(K)$ as a $(k-1,k)$-complex.

We say that $\mathcal{P} = \mathcal{P}(k-1)$ is (η,δ,t)-*equitable* if

- there exists $\mathbf{d} = (d_{k-1},\dots,d_2)$ such that $d_i \ge 1/t$ and $1/d_i \in \mathbb{N}$ for all $i = 2,\dots,k-1$,

- $\mathcal{P}^{(1)}$ is a partition of V into a_1 clusters of equal size, where $1/\eta \leq a_1 \leq t$,
- for all $i = 2, \dots, k-1$, $\mathcal{P}^{(i)}$ is a partition of Cross_i into at most t cells,
- for every $K \in \mathrm{Cross}_k$, the $(k-1,k)$-complex $\bigcup_{i=1}^{k-1} \hat{P}^{(i)}(K)$ is $(\mathbf{d}, \delta, \delta, 1)$-regular.

Note that the final condition implies that for all $i = 2, \dots, k-1$ the cells of $\mathcal{P}^{(i)}$ have almost equal size.

Let $\delta_k > 0$ and $r \in \mathbb{N}$. Suppose that $\mathcal{H}$ is a k-graph on V and $\mathcal{P} = \mathcal{P}(k-1)$ is a family of partitions on V. Given a polyad $\hat{P}^{(k-1)} \in \hat{\mathcal{P}}^{(k-1)}$, we say that $\mathcal{H}$ is (δ_k, r)-*regular with respect to* $\hat{P}^{(k-1)}$ if $\mathcal{H}$ is (d, δ_k, r)-regular with respect to $\hat{P}^{(k-1)}$ for some d. We say that $\mathcal{H}$ is (δ_k, r)-*regular with respect to* $\mathcal{P}$ if

$$\left| \bigcup \{\mathcal{K}_k(\hat{P}^{(k-1)}) : \ \mathcal{H} \text{ is not } (\delta_k, r)\text{-regular with respect to } \hat{P}^{(k-1)} \in \hat{\mathcal{P}}^{(k-1)}\} \right| \leq \delta_k |V|^k.$$

This means that not much more than a δ_k-fraction of the k-subsets of V form a $K_k^{(k-1)}$ that lies within a polyad with respect to which $\mathcal{H}$ is not regular.

Now we are ready to state the regularity lemma.

Theorem 4.2 (Rödl and Schacht [15], Theorem 17)**.** *Let $k \geq 3$ be a fixed integer. For all positive constants η and δ_k and all functions $r : \mathbb{N} \to \mathbb{N}$ and $\delta : \mathbb{N} \to (0,1]$, there are integers t and n_0 such that the following holds for all $n \geq n_0$ which are divisible by $t!$. Suppose that $\mathcal{H}$ is a k-graph of order n. Then there exists a family of partitions $\mathcal{P} = \mathcal{P}(k-1)$ of the vertex set V of $\mathcal{H}$ such that*

(1) *$\mathcal{P}$ is $(\eta, \delta(t), t)$-equitable and*
(2) *$\mathcal{H}$ is $(\delta_k, r(t))$-regular with respect to $\mathcal{P}$.*

Similar results were proved earlier by Rödl and Skokan [17] and Gowers [5]. Note that the constants in Theorem 4.2 can be chosen so that they satisfy the following hierarchy:

$$\frac{1}{n_0} \ll \frac{1}{r} = \frac{1}{r(t)}, \delta = \delta(t) \ll \min\{\delta_k, 1/t\} \ll \eta.$$

4.3. **The reduced k-graph.** To prove the absorbing lemma and the path cover lemma, we will use the so-called reduced k-graph. Suppose that we have constants

$$\frac{1}{n_0} \ll \frac{1}{r}, \delta \ll \min\{\delta_k, 1/t\} \leq \delta_k, \eta \ll d \ll \theta \ll \mu, 1/k.$$

and a k-graph $\mathcal{H}$ on V of order $n \geq n_0$ with $\delta(\mathcal{H}) \geq (\mu + \theta)n$. We may apply the regularity lemma to $\mathcal{H}$ to obtain a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(k-1)}\}$ of V. Then the *reduced k-graph* $\mathcal{R} = \mathcal{R}(\mathcal{H}, \mathcal{P})$ is the k-graph whose vertices are the clusters of $\mathcal{H}$, i.e. the parts of $\mathcal{P}^{(1)}$. A k-tuple of clusters forms an edge of $\mathcal{R}$ if there is some polyad $\hat{P}^{(k-1)}$ induced on these k clusters such that $\mathcal{H}$ is (d', δ_k, r)-regular with respect to $\hat{P}^{(k-1)}$ for some $d' \geq d$. To make use of the reduced k-graph, we shall need to show that it almost inherits the minimum degree condition from $\mathcal{H}$.

Lemma 4.3. *All but at most $\theta |\mathcal{R}|^{k-1}$ sets $S \in \binom{V(\mathcal{R})}{k-1}$ satisfy $d_{\mathcal{R}}(S) \geq \mu |\mathcal{R}|$.*

Similar results have been proved in previous papers on hypergraph Hamilton cycles, but we include the short proof for completeness, in which we will need the following lemma. We say that an edge e of $\mathcal{H}$ is *useful* if it lies in $\mathcal{K}_k(\hat{P}^{(k-1)})$ for some $\hat{P}^{(k-1)} \in \hat{\mathcal{P}}^{(k-1)}$ such that $\mathcal{H}$ is (d', δ_k, r)-regular with respect to $\hat{P}^{(k-1)}$ for some $d' \geq d$. Note that if e lies in $\mathcal{K}_k(\hat{P}^{(k-1)})$ then $\hat{P}^{(k-1)} = \hat{P}^{(k-1)}(e)$ is the polyad of e. Moreover, if e is a useful edge of $\mathcal{H}$, and $V_{i_1}, \dots, V_{i_k}$ are the clusters containing the vertices of e, then these k clusters will form an edge of $\mathcal{R}$.

Lemma 4.4. *At most $2dn^k$ edges of $\mathcal{H}$ are not useful.*

Proof. There are three reasons why an edge of $\mathcal{H}$ may not be useful. Firstly, it may lie in $\binom{V}{k} \setminus \mathrm{Cross}_k$. Since $\mathcal{P}^{(1)}$ partitions V into a_1 clusters of equal size, there are at most $\frac{n}{a_1}n^{k-1} \leq \eta n^k$ edges of this type. Secondly, the edge may lie in a polyad $\hat{P}^{(k-1)} \in \hat{\mathcal{P}}^{(k-1)}$ such that $|E(\mathcal{H}) \cap \mathcal{K}_k(\hat{P}^{(k-1)})| \leq d|\mathcal{K}_k(\hat{P}^{(k-1)})|$. There are at most dn^k edges of this type. Finally, the edge may lie in a polyad $\hat{P}^{(k-1)} \in \hat{\mathcal{P}}^{(k-1)}$ such that $\mathcal{H}$ is not (δ_k, r)-regular with respect to $\hat{P}^{(k-1)}$. Since $\mathcal{H}$ is (δ_k, r)-regular with respect to $\mathcal{P}$, there are at most $\delta_k n^k$ edges of this type. So altogether, at most $(\delta_k + d + \eta)n^k \leq 2dn^k$ edges of $\mathcal{H}$ are not useful. □

Proof of Lemma 4.3. Let $m = |V_1| = \cdots = |V_{a_1}|$ be the size of the clusters. We say that a $(k-1)$-tuple of clusters of $\mathcal{H}$ is *poor* if there are at least $\theta m^{k-1}n$ edges of $\mathcal{H}$ which intersect each of the $k-1$ clusters in precisely one vertex and which are not useful. Then it follows from Lemma 4.4 that at most $\theta|\mathcal{R}|^{k-1}$ such $(k-1)$-tuples are poor. So it remains to show that any $(k-1)$-tuple which is not poor has many neighbours in $\mathcal{R}$. But if $V_{i_1}, \dots, V_{i_{k-1}}$ is a $(k-1)$-tuple which is not poor, then there are at least $m^{k-1}\delta(\mathcal{H}) - \theta m^{k-1}n \geq \mu m^{k-1}n$ useful edges of $\mathcal{H}$ which intersect each of $V_{i_1}, \dots, V_{i_{k-1}}$ in precisely one vertex. For any other cluster V_j at most m^k edges of $\mathcal{H}$ intersect each of $V_{i_1}, \dots, V_{i_{k-1}}, V_j$ in precisely one vertex, and so there are at least $\mu n/m = \mu|\mathcal{R}|$ choices of V_j such that there is at least one such useful edge. This useful edge indicates the existence of a polyad satisfying the conditions of an edge in the reduced k-graph $\mathcal{R}$. □

4.4. **The embedding and the extension lemmas.** In our proof we will also use an embedding lemma, which guarantees the existence of a copy of a complex $\mathcal{G}$ of bounded maximum degree inside a suitable regular complex $\mathcal{H}$, where the order of $\mathcal{G}$ is allowed to be linear in the order of $\mathcal{H}$. In order to state this lemma, we need some more definitions.

The degree of a vertex x in a complex $\mathcal{G}$ is the number of edges containing x. The *maximum vertex degree* of $\mathcal{G}$ is the largest degree of a vertex of $\mathcal{G}$.

Suppose that $\mathcal{H}$ is a (k, s)-complex with vertex classes $V_1, \dots, V_s$, which all have size m. Suppose also that $\mathcal{G}$ is a (k, s)-complex with vertex classes $X_1, \dots, X_s$ of size at most m. We say that $\mathcal{H}$ *respects the partition of* $\mathcal{G}$ if whenever $\mathcal{G}$ contains an i-edge with vertices in $X_{j_1}, \dots, X_{j_i}$, then there is an i-edge of $\mathcal{H}$ with vertices in $V_{j_1}, \dots, V_{j_i}$. On the other hand, we say that a labelled copy of $\mathcal{G}$ in $\mathcal{H}$ is *partition-respecting* if for each $i = 1, \dots, s$ the vertices corresponding to those in X_i lie within V_i.

Lemma 4.5 (Embedding lemma, [2], Theorem 3)**.** *Let Δ, k, s, r, m_0 be positive integers and let $c, d_2, \dots, d_k, \delta, \delta_k$ be positive constants such that $1/d_i \in \mathbb{N}$ for all $i < k$,*

$$1/m_0 \ll 1/r, \delta \ll \min\{\delta_k, d_2, \dots, d_{k-1}\} \leq \delta_k \ll d_k, 1/\Delta, 1/s$$

and

$$c \ll d_2, \dots, d_k.$$

Then the following holds for all integers $m \geq m_0$. Suppose that $\mathcal{G}$ is a (k, s)-complex of maximum vertex degree at most Δ with vertex classes $X_1, \dots, X_s$ such that $|X_i| \leq cm$ for all $i = 1, \dots, s$. Suppose also that $\mathcal{H}$ is a $(\mathbf{d}, \delta_k, \delta, r)$-regular (k, s)-complex with vertex classes $V_1, \dots, V_s$, all of size m, which respects the partition of $\mathcal{G}$. Then $\mathcal{H}$ contains a labelled partition-respecting copy of $\mathcal{G}$.

We will also use the following weak version of a lemma from [2]. Roughly speaking, it states that if $\mathcal{G}$ is an induced subcomplex of $\mathcal{G}'$, and $\mathcal{H}$ is suitably regular, then almost all copies of $\mathcal{G}$ in $\mathcal{H}$ can be extended to a large number of copies of $\mathcal{G}'$ in $\mathcal{H}$. We write $|\mathcal{G}|_{\mathcal{H}}$ for the number of labelled partition-respecting copies of $\mathcal{G}$ in $\mathcal{H}$.

Lemma 4.6 (Extension lemma, [2], Lemma 5)**.** *Let k, s, r, b', b'', m_0 be positive integers, where $b' < b''$, and let $c, \beta, d_2, \dots, d_k, \delta, \delta_k$ be positive constants such that $1/d_i \in \mathbb{N}$ for all $i < k$ and*

$$1/m_0 \ll 1/r, \delta \ll c \ll \min\{\delta_k, d_2, \dots, d_{k-1}\} \leq \delta_k \ll \beta, d_k, 1/s, 1/b''.$$

Then the following holds for all integers $m \geq m_0$. Suppose that $\mathcal{G}'$ is a (k, s)-complex on b'' vertices with vertex classes $X_1, \dots, X_s$ and let $\mathcal{G}$ be an induced subcomplex of $\mathcal{G}'$ on b' vertices. Suppose also that $\mathcal{H}$ is a $(\mathbf{d}, \delta_k, \delta, r)$-regular (k, s)-complex with vertex classes $V_1, \dots, V_s$, all of size m, which respects the partition of $\mathcal{G}'$. Then all but at most $\beta|\mathcal{G}|_{\mathcal{H}}$ labelled partition-respecting copies of $\mathcal{G}$ in $\mathcal{H}$ are extendible to at least $cn^{b''-b'}$ labelled partition-respecting copies of $\mathcal{G}'$ in $\mathcal{H}$.

The proofs of Lemmas 4.5 and 4.6 rely on the hypergraph counting lemma (Theorem 9 in [16]). In particular, the extension lemma is a straightforward consequence of the counting lemma. Actually both the embedding lemma and the extension lemma involved the additional condition that $1/d_k \in \mathbb{N}$. However, this can easily be achieved by working with a subcomplex $\mathcal{H}'$ of $\mathcal{H}$ which is $(d'', d_{k-1}, \dots, d_2, \delta_k, \delta, r)$-regular with respect to $\hat{P}^{(k-1)}$ for some $d'' \gg \delta_k$ with $1/d'' \in \mathbb{N}$. The existence of such a $\mathcal{H}'$ follows immediately from the slicing lemma ([15], Proposition 22), which is proved using a simple application of a Chernoff bound.

Now suppose that we have applied the regularity lemma (Theorem 4.2) to a k-graph $\mathcal{H}$ to obtain a reduced k-graph $\mathcal{R}$. An edge e of $\mathcal{R}$ indicates that we can apply the embedding lemma or the extension lemma to the subcomplex of $\mathcal{H}$ whose vertex classes are the clusters $V_1, \dots, V_k$ corresponding to the vertices of e. More precisely, since e is an edge of $\mathcal{R}$, there is some polyad $\hat{P}^{(k-1)} = \hat{P}^{(k-1)}(K)$ (where $K \in \mathrm{Cross}_k$) induced by $V_1, \dots, V_k$ such that $\mathcal{H}$ is (d', δ_k, r)-regular with respect to $\hat{P}^{(k-1)}$ for some $d' \geq d$. Let $\mathcal{H}^*$ be the (k, k)-complex obtained from the $(k-1, k)$ complex $\bigcup_{i=1}^{k-1} \hat{P}^{(i)}(K)$ by adding $E(\mathcal{H}) \cap \mathcal{K}(\hat{P}^{(k-1)})$ as the 'kth level'. Then $\mathcal{H}^*$ is a $(\mathbf{d}, \delta_k, \delta, r)$-regular subcomplex of $\mathcal{H}$, where $\mathbf{d} = (d', d_{k-1}, \dots, d_2)$, and $(d_{k-1}, \dots, d_2)$ is as in the definition of a (η, δ, t)-equitable family of partitions. Also $\mathcal{H}^*$ satisfies the conditions of the embedding (or extension) lemma. So in particular, the embedding lemma implies that if $m := |V_1|$ and $\mathcal{G}$ is a k-partite k-graph of bounded maximum vertex degree whose vertex classes have size at most cm, then $\mathcal{H}$ contains a copy of $\mathcal{G}$.

5. Diameter Lemma

In this section, we shall prove a diameter lemma, which will state that in any sufficiently large k-graph of large minimum degree, we can find an ℓ-path from any ordered ℓ-set of vertices to any other ordered ℓ-set of vertices. To prove this, we shall first consider a k-graph $\mathcal{W}(k, \ell)$, for which a similar statement is easier to prove (Proposition 5.1). For $k/2 \leq \ell \leq k-2$, the k-graph $\mathcal{W}(k, \ell)$ has $4\ell - k + 2$ vertices in three disjoint sets X, Y and Z, where $X = \{x_1, \dots, x_\ell\}, Y = \{y_1, \dots, y_\ell\}$ and $Z = \{z_1, \dots, z_{2\ell-k+2}\}$. $\mathcal{W}(k, \ell)$ has $2\ell - k + 2$ edges, where for $1 \leq i \leq 2\ell - k + 2$ the ith edge of $\mathcal{W}(k, \ell)$ is $\{x_1, \dots, x_{\ell+1-i}\} \cup \{y_1, \dots, y_{k-2-\ell+i}\} \cup \{z_i\}$. So each edge of $\mathcal{W}(k, \ell)$ intersects the following edge in precisely $k-2$ vertices. We shall sometimes view $\mathcal{W}(k, \ell)$ as a $(4\ell - k + 2)$-partite k-graph with a single vertex in each vertex class, and consider the $(k, 4\ell - k + 2)$-complex $\mathcal{W}(k, \ell)^{\leq}$. We refer to the ordered sets X and Y as the ordered ends of $\mathcal{W}(k, \ell)$.

The next lemma states that for most pairs of sets S and T of ℓ vertices in a k-graph $\mathcal{H}$ of large minimum degree, $\mathcal{H}$ contains many copies of $\mathcal{W}(k, \ell)$ with S and T as ordered ends.

Proposition 5.1. *Suppose that $k \geq 3$, that $k/2 \leq \ell \leq k-2$ and that $1/n \ll \gamma \ll \beta \ll \mu, 1/k$. Let $\mathcal{H}$ be a k-graph on n vertices such that $d(S) \geq \mu n$ for all but at most γn^{k-1} sets $S \in \binom{V(\mathcal{H})}{k-1}$.*

Then for all but at most $\beta n^{2\ell}$ pairs S, T of ordered ℓ-sets of vertices of $\mathcal{H}$ there are at least $\beta n^{2\ell-k+2}$ copies of $\mathcal{W}(k,\ell)$ in $\mathcal{H}$ with ordered ends S and T.

Proof. We refer to the at most γn^{k-1} sets S of $k-1$ vertices in $\mathcal{H}$ which do not satisfy $d(S) \geq \mu n$ as unfriendly $(k-1)$-sets. We say that a pair of ℓ-sets S and T is unfriendly if there exist $S' \subseteq S$, $T' \subseteq T$ such that $S' \cup T'$ is a unfriendly $(k-1)$-set. Then for any unfriendly $(k-1)$-set B, there are at most $2^{k-1}n^{2\ell-k+1}$ pairs of ℓ-sets S and T with $S' \cup T' = B$ for some $S' \subseteq S$ and $T' \subseteq T$, and so since there are at most γn^{k-1} unfriendly $(k-1)$-sets, and $\gamma \ll \beta \ll 1/k$, we know that there are at most $\beta n^{2\ell}$ unfriendly pairs of ℓ-sets.

To complete the proof, it is sufficient to show that if the pair S, T of ordered ℓ-sets is not unfriendly, then $\mathcal{H}$ contains at least $\beta n^{2\ell-k+2}$ copies of $\mathcal{W}(k,\ell)$ with ordered ends S and T. Let $S = \{x_1, \dots, x_\ell\}$, and let $T = \{y_1, \dots, y_\ell\}$. For each $1 \leq i \leq 2k-\ell+2$ we choose a vertex z_i such that $z_i \notin S \cup T$, $z_i \neq z_j$ for any $j < i$, and such that $\{x_1, \dots, x_{\ell+1-i}, y_1, \dots, y_{k-2-\ell+i}, z_i\}$ is an edge of $\mathcal{H}$. This is possible for each i as we know that S, T is not a unfriendly pair, and so $d(\{x_1, \dots, x_{\ell+1-i}, y_1, \dots, y_{k-2-\ell+i}\}) \geq \mu n$, and hence there are at least $\mu n - (4\ell - k + 2)$ vertices to choose from. Then S, T and the chosen vertices z_i together form a copy of $\mathcal{W}(k,\ell)$ in $\mathcal{H}$ with ordered ends S and T. Since $\beta \ll \mu$, by counting the choices we could have made for the z_i we find that $\mathcal{H}$ contains at least $\beta n^{2\ell-k+2}$ copies of $\mathcal{W}(k,\ell)$ with ordered ends S and T. □

The following proposition relates the k-graph $\mathcal{W}(k,\ell)$ to a k-graph $\mathcal{P}(k,\ell)$ which consists of several ℓ-paths from one ordered ℓ-set to another. We say that ℓ-paths P and Q with ordered ends P^{beg}, P^{end}, Q^{beg} and Q^{end} are *internally disjoint* if P and Q do not intersect other than in these ordered ends.

Proposition 5.2. *Suppose that $k \geq 3$ and that $k/2 \leq \ell \leq k-1$ is such that $(k-\ell) \nmid k$. Then there exists a $(4\ell-k+2)$-partite k-graph $\mathcal{P}(k,\ell)$ such that the following conditions hold.*

1. *$\mathcal{P}(k,\ell)$ is the union of $4\ell+1$ internally disjoint ℓ-paths, each containing between $k^2\ell$ and $2k^5$ vertices, with identical ordered ℓ-sets T_1 and T_2 as ordered ends (we refer to these as the ordered ends of $\mathcal{P}(k,\ell)$). In particular, $\mathcal{P}(k,\ell)$ contains at most $10k^6$ vertices.*
2. *The vertex classes of $\mathcal{P}(k,\ell)$ are disjoint sets V_w, one for each vertex w of $\mathcal{W}(k,\ell)$.*
3. *Whenever $v_1 \in V_{w_1}, v_2 \in V_{w_2}, \dots, v_k \in V_{w_k}$ are such that $\{v_1, v_2, \dots, v_k\}$ is an edge of $\mathcal{P}(k,\ell)$, $\{w_1, \dots, w_k\}$ is an edge of $\mathcal{W}(k,\ell)$. Furthermore, let X and Y be the ordered ends of $\mathcal{W}(k,\ell)$. Then the ordered ends of $\mathcal{P}(k,\ell)$ are contained in $\bigcup_{w\in X} V_w$ and $\bigcup_{w\in Y} V_w$ respectively.*

Proof. For every vertex w of $\mathcal{W}(k,\ell)$, take a large vertex set V_w. Define $\mathcal{W}^*$ to have vertex set $V = \bigcup_{w\in\mathcal{W}(k,\ell)} V_w$, and edges precisely those k-sets of vertices which lie in sets corresponding to an edge of $\mathcal{W}(k,\ell)$. We shall construct $\mathcal{P}(k,\ell)$ to be a sub-k-graph of $\mathcal{W}^*$, with the ordered ends of $\mathcal{P}(k,\ell)$ in the sets V_w corresponding to the ordered ends of $\mathcal{W}(k,\ell)$. Then $\mathcal{P}(k,\ell)$ will be a $(4\ell-k+2)$-partite k-graph which satisfies (2) and (3).

For each $1 \leq i \leq 2\ell-k+2$ let e_i be the ith edge of $\mathcal{W}(k,\ell)$ as in the definition of $\mathcal{W}(k,\ell)$. Then for each $1 \leq i \leq 2\ell-k+1$ we know that $|e_i \cap e_{i+1}| = k-2$, and so we may choose S_i to be an ordered set of ℓ vertices chosen from $\bigcup_{w\in e_i\cap e_{i+1}} V_w$. Also, let S_0 and $S_{2\ell-k+2}$ be ordered sets of ℓ vertices chosen from the V_w corresponding to the ordered ends of $\mathcal{W}(k,\ell)$. So S_0 and $S_{2\ell-k+2}$ are subsets of $\bigcup_{w\in e_1} V_w$, and $\bigcup_{w\in e_{2\ell-k+2}} V_w$ respectively. We choose these sets S_i to be disjoint and to contain at most one vertex from any one vertex class V_w. Then by Proposition 3.1, for each $1 \leq i \leq 2\ell-k+2$ we can find an ℓ-path from S_{i-1} to S_i in $\mathcal{K}[V_w : w \in e_i]$ which contains $k^2\ell(k-\ell)+k$ vertices. We do this so that the ℓ-paths chosen only intersect in the appropriate S_i. Then the union of all of these ℓ-paths is an ℓ-path P from S_0 to $S_{2\ell-k+2}$ with

$$k^2\ell \leq k^2\ell(k-\ell)+k \leq |P| \leq (2\ell-k+2)(k^2\ell(k-\ell)+k) \leq 2k^5.$$

In the same way we find another 4ℓ ℓ-paths from S_0 to $S_{2\ell-k+2}$, so that all $4\ell+1$ of the ℓ-paths obtained are internally disjoint. Then the union of all of these ℓ-paths is the $\mathcal{P}(k,\ell)$ we seek. □

Fix any such $\mathcal{P}(k,\ell)$, which we shall mean when we refer to $\mathcal{P}(k,\ell)$ in the rest of this paper. Also, let S_1 and S_2 be the ordered ends of $\mathcal{P}(k,\ell)$, so that S_1 and S_2 are disjoint ordered ℓ-sets. Let $\mathcal{S}(k,\ell)$ be the complex with vertex set $S_1 \cup S_2$ and with edges being all subsets of S_1 and all subsets of S_2. Then since each of the ℓ-paths which form $\mathcal{P}(k,\ell)$ contain at least $k^2\ell$ vertices, the complex $\mathcal{S}(k,\ell)$ is an induced subcomplex of the complex $\mathcal{P}(k,\ell)^{\leq}$ corresponding to $\mathcal{P}(k,\ell)$, so under appropriate circumstances we will be able to use the extension lemma (Lemma 4.6) to extend $\mathcal{S}(k,\ell)$ to $\mathcal{P}(k,\ell)$. This is the key to the following lemma, which states that for the values of k and ℓ considered, almost all pairs of ordered ℓ-sets of vertices of a sufficiently large k-graph of large minimum degree form the ordered ends of a copy of $\mathcal{P}(k,\ell)$.

Lemma 5.3. *Suppose that $k \geq 3$, that $k/2 \leq \ell \leq k-1$ is such that $(k-\ell) \nmid k$, and that $1/n \ll \beta \ll \mu, 1/k$. Let $\mathcal{H}$ be a k-graph of order n with $\delta(\mathcal{H}) \geq \mu n$. Then there are at most $\beta n^{2\ell}$ pairs of ordered ℓ-sets S_1 and S_2 of vertices of $\mathcal{H}$ for which $\mathcal{H}$ does not contain a copy of $\mathcal{P}(k,\ell)$ with ordered ends S_1 and S_2.*

Proof. To prove this, we use hypergraph regularity. So introduce new constants

$$\frac{1}{n} \ll \frac{1}{r}, \delta \ll c \ll \min\{\delta_k, 1/t\} \leq \delta_k, \eta \ll d \ll \gamma \ll \beta \ll \mu.$$

We may assume that $t!$ divides $|\mathcal{H}|$, so apply the regularity lemma to $\mathcal{H}$, and let $V_1,\dots,V_{a_1}$ be the clusters of the partition obtained. As in Section 4.3, we say that an edge of $\mathcal{H}$ is useful if it lies in $\mathcal{K}_k(\hat{P}^{(k-1)})$ such that $\mathcal{H}$ is (d',δ_k,r)-regular with respect to $\hat{P}^{(k-1)}$ for some $d' \geq d$. Let $\mathcal{H}'$ be the subgraph of $\mathcal{H}$ consisting of all useful edges. Note that no edge of $\mathcal{H}'$ contains 2 vertices from the same cluster. Then by Lemma 4.4, at most $2dn^k$ edges of $\mathcal{H}$ are not useful, and so $d_{\mathcal{H}'}(S) \geq \mu n/2$ for all but at most γn^{k-1} of the $(k-1)$-sets S of vertices of $\mathcal{H}'$.

Let C_1 and C_2 be cells of the partition $\mathcal{P}^{(\ell)}$ obtained from the regularity lemma. We say that C_1 and C_2 are *connected* if $\mathcal{H}'$ contains a copy $\mathcal{W}$ of $\mathcal{W}(k,\ell)$ with ordered ends A and B such that $A \in C_1$, $B \in C_2$, and such that no two vertices of $\mathcal{W}$ lie in the same cluster. We shall first show that there are at most $\beta n^{2\ell}/2$ pairs A and B of ordered ℓ-sets of vertices of $\mathcal{H}$ such that either

(i) at least one of A and B does not lie in a cell of $\mathcal{P}^{(\ell)}$, or
(ii) the cells C_A and C_B of $\mathcal{P}^{(\ell)}$ which contain A and B respectively are not connected.

Indeed, for (i) note that at most $\ell^2 \frac{n}{a_1} n^{\ell-1} \leq \ell^2\eta n^\ell$ ordered ℓ-sets of vertices of $\mathcal{H}$ do not lie in Cross_ℓ, and so there are at most $\ell^2\eta n^{2\ell}$ pairs A and B of ordered ℓ-sets of vertices of $\mathcal{H}$ such that at least one of A and B does not lie in a cell of $\mathcal{P}^{(\ell)}$. Similarly, for (ii) note that there are at most $\ell^2\eta n^{2\ell}$ pairs A and B of ordered ℓ-sets such that the cells C_A and C_B of $\mathcal{P}^{(\ell)}$ which contain A and B respectively share at least one cluster. Finally, if the cells C_A and C_B of $\mathcal{P}^{(\ell)}$ which contain A and B respectively do not share any clusters, but are not connected, then $\mathcal{H}'$ must contain fewer than $\binom{4\ell-k+2}{2}\eta n^{2\ell-k+2}$ copies of $\mathcal{W}(k,\ell)$ with ordered ends A and B. So by Proposition 5.1, there are at most $\beta n^{2\ell}/3$ pairs A and B of ordered ℓ-sets of vertices of $\mathcal{H}$ which lie in such pairs of cells of $\mathcal{P}^{(\ell)}$.

To prove the lemma, it is therefore sufficient to show that there are at most $\beta n^{2\ell}/2$ pairs S_1, S_2 of ordered ℓ-sets of vertices of $\mathcal{H}$ such that C_{S_1} and C_{S_2} are connected cells of $\mathcal{P}^{(\ell)}$ but S_1 and S_2 do not form the ordered ends of a copy of $\mathcal{P}(k,\ell)$ in $\mathcal{H}$. So suppose cells C_1 and C_2 of $\mathcal{P}(k,\ell)$ are connected. Then there is a copy $\mathcal{W}$ of $\mathcal{W}(k,\ell)$ in $\mathcal{H}'$ with ordered ends A and B such that $A \in C_1$, $B \in C_2$, and such that no two vertices of $\mathcal{W}$ lie in the same cluster. Since every edge of $\mathcal{H}'$ is a useful edge, for each edge $e \in E(\mathcal{W})$ the polyad $\hat{P}^{(k-1)}(e)$ of e is such that $\mathcal{H}$ is (d', δ_k, r)-regular with respect to $\hat{P}^{(k-1)}(e)$ for some $d' \geq d$. Then these polyads 'fit together'. By this we mean that if edges e and e' of $\mathcal{W}$ intersect in q vertices, then

$$\left(\bigcup_{i=1}^{k-1} \hat{P}^{(i)}(e)\right) \cap \left(\bigcup_{i=1}^{k-1} \hat{P}^{(i)}(e')\right) = \bigcup_{i=1}^{q} \hat{P}^{(i)}(e \cap e'),$$

i.e. the intersection of the $(k-1,k)$-complexes corresponding to e and e' is the (q,q)-complex corresponding to $e \cap e'$. Therefore we can define $\mathcal{H}^*$ to be the $(k, 4\ell-k+2)$-complex obtained from the $(k-1, 4\ell-k+2)$ complex $\bigcup_{e\in E(\mathcal{W})} \bigcup_{i=1}^{k-1} \hat{P}^{(i)}(e)$ by adding $E(\mathcal{H}) \cap \bigcup_{e\in E(\mathcal{W})} \mathcal{K}(\hat{P}^{(k-1)}(e))$ as the 'kth level'. Then $\mathcal{H}^*$ is a $(\mathbf{d}, \delta_k, \delta, r)$-regular $(k, 4\ell-k+2)$-complex, where $\mathbf{d} = (d', d_{k-1}, \dots, d_2)$ and $(d_{k-1}, \dots, d_2)$ is as in the definition of a (η, δ, t)-equitable family of partitions. (Here we may assume a common density d' for the kth level by applying the slicing lemma ([15], Proposition 22) if necessary.) Furthermore, by construction $\mathcal{H}^*$ respects the partition of the complex $\mathcal{W}(k,\ell)^{\leq}$ corresponding to $\mathcal{W}(k,\ell)$, and so property (3) of Proposition 5.2 implies that $\mathcal{H}^*$ also respects the partition of $\mathcal{P}(k,\ell)^{\leq}$. Let S_1 and S_2 be disjoint ordered ℓ-sets lying in the cells C_1 and C_2 of $\mathcal{P}^{(\ell)}$ respectively. Then $S_1 \cup S_2$ is the vertex set of a labelled copy $\mathcal{S}$ of $\mathcal{S}(k,\ell)$ in $\mathcal{H}^*$. So by Lemma 4.6, for all but at most $\beta|C_1||C_2|/2$ choices of $S_1 \in C_1$ and $S_2 \in C_2$ we can extend the labelled complex $\mathcal{S}$ to at least one labelled partition respecting copy of $\mathcal{P}(k,\ell)$ with ordered ends S_1 and S_2. Summing over all C_1 and C_2, we find that there are at most $\beta n^{2\ell}/2$ ordered ℓ-sets S_1 and S_2 of vertices of $\mathcal{H}$ which lie in connected cells of $\mathcal{P}^{(\ell)}$ and which cannot be extended to a labelled partition respecting copy of $\mathcal{P}(k,\ell)$, completing the proof. □

We can now prove the following corollary, the diameter lemma we were aiming for. The idea behind this is that if S and T are ordered ℓ-sets in a large k-graph $\mathcal{H}$ of large minimum degree, then there are many ordered ℓ-sets S' and T' such that $\mathcal{H}$ contains ℓ-paths from S to S' and T to T'. So by the previous lemma, at least one such pair S' and T' will form the ordered ends of a copy of $\mathcal{P}(k,\ell)$, and then combining these ℓ-paths we will obtain an ℓ-path from S to T.

Corollary 5.4 (Diameter lemma). *Suppose that $k \geq 3$, that $1 \leq \ell \leq k-1$ is such that that $(k-\ell) \nmid k$, and that $1/n \ll \mu, 1/k$. Let $\mathcal{H}$ be a k-graph of order n with $\delta(\mathcal{H}) \geq \mu n$. Then for any two disjoint ordered ℓ-sets S and T of vertices of $\mathcal{H}$, there exists an ℓ-path P in $\mathcal{H}$ from S to T such that P contains at most $8k^5$ vertices.*

Proof. Recall that if $\ell < k/2$ we can find such an ℓ-path consisting of just one single edge, so we may assume that $\ell \geq k/2$. Introduce constants β, β' such that $1/n \ll \beta' \ll \beta \ll \mu, 1/k$. Let A be an arbitrary ordered ℓ-set of vertices of $\mathcal{H}$, and let X be an arbitrary set of 3ℓ vertices which is disjoint from A. We begin by showing that there are many ordered ℓ-sets B such that $\mathcal{H}$ contains an ℓ-path P from A to B having at most 3ℓ vertices, none of which are from X. To show this, we shall demonstrate how a vertex sequence of P may be chosen, and then count the number of choices.

Since A will be an ordered end of P, we begin the vertex sequence of P with the ordered ℓ-set A. We then arbitrarily choose any $k-\ell-1$ vertices of $\mathcal{H}$ to add to the sequence. To finish the sequence, we repeatedly make use of the fact that $\delta(\mathcal{H}) \geq \mu n$. More precisely, we repeat the following step: let V be the set of the final $k-1$ vertices of the current vertex sequence. Then $d_{\mathcal{H}}(V) \geq \mu n$, and so there are at least $\mu n - 6\ell$ vertices which together with V form an edge of $\mathcal{H}$ and which are not in the vertex sequence constructed thus far or in X. Choose v to be one of these vertices, and append it to the vertex sequence. We stop as soon as the number r of vertices in the sequence satisfies $r > 2\ell$ and $r \equiv k$ (modulo $(k-\ell)$), so in particular $r \leq 3\ell$. Let B be the ordered set consisting of the last ℓ vertices of the sequence. Then $\mathcal{H}$ contains an ℓ-path P with this vertex sequence, and P is therefore an ℓ-path of order at most 3ℓ from A to B which does not contain any vertex of X. There are at least $(\mu n - 6\ell)^{r-\ell}$ vertex sequences we could have chosen, and hence there are at least $(\mu n - 6\ell)^{r-\ell}/n^{r-2\ell} > \beta n^{\ell}$ possibilities for an ordered ℓ-set B such that there is an ℓ-path from A to B in $\mathcal{H}$, not containing any vertex of X.

Now, let S and T be the two ordered ℓ-sets of vertices of $\mathcal{H}$ given in the statement of the corollary. Then there are at least βn^{ℓ} ordered ℓ-sets S' of vertices of $\mathcal{H}$ such that there exists an ℓ-path P_1 from S to S' in $\mathcal{H}$, which contains at most 3ℓ vertices and such that $V(P_1) \cap T = \emptyset$. Likewise for each such choice of S' and P_1, there are at least βn^{ℓ} ordered ℓ-sets T' of vertices of $\mathcal{H}$ such that there exists an ℓ-path P_2 from T to T' of order at most 3ℓ in $\mathcal{H}$ and such that $V(P_2) \cap V(P_1) = \emptyset$. By Lemma 5.3, at most $\beta' n^{2\ell}$ of these pairs S', T' do not form ordered ends of a copy of $\mathcal{P}(k,\ell)$ in $\mathcal{H}$. Since $\beta' \ll \beta$ we may therefore choose such a pair S', T' such that S' and T' are ordered ends of a copy of $\mathcal{P}(k,\ell)$ in $\mathcal{H}$. Then there are at least $4\ell+1$ internally disjoint ℓ-paths of order at most $2k^5$ from S' to T' in $\mathcal{H}$. At most 4ℓ of these ℓ-paths contain any vertex from $V(P_1) \setminus S'$ or $V(P_2) \setminus T'$, and so we may choose an ℓ-path Q from S' to T' in $\mathcal{P}(k,\ell) \subseteq \mathcal{H}$ of order at most $2k^5$ which contains no vertex from $V(P_1) \setminus S'$ or $V(P_2) \setminus T'$. Then $P_1 Q P_2$ is the ℓ-path from S to T of order at most $2k^5 + 6\ell \leq 8k^5$ we seek. □

6. Absorbing Path Lemma

Let $\mathcal{H}$ be a k-graph, and let S be a set of $k-\ell$ vertices of $\mathcal{H}$. Recall that an ℓ-path P in $\mathcal{H}$ with ordered ends P^{beg} and P^{end} is *absorbing* for S if P does not contain any vertex of S, and $\mathcal{H}$ contains an ℓ-path Q with the same ordered ends P^{beg} and P^{end}, where $V(Q) = V(P) \cup S$. This means that if P is a section of an ℓ-path P^* which does not contain any vertices of S, then we can 'absorb' the vertices of S into P^* by replacing P with Q. P^* will still be an ℓ-path after this change as P and Q have the same ordered ends. Similarly, we say that an ℓ-path P in $\mathcal{H}$ with ordered ends P^{beg} and P^{end} can *absorb* a collection $S_1, \dots, S_r$ of $(k-\ell)$-sets of vertices of $\mathcal{H}$ if P does not contain any vertex of $\bigcup_{i=1}^{r} S_i$, and $\mathcal{H}$ contains an ℓ-path Q with the same ordered ends P^{beg} and P^{end}, where $V(Q) = V(P) \cup \bigcup_{i=1}^{r} S_i$. The reason we absorb $(k-\ell)$-sets is that the order of an ℓ-path must be congruent to k, modulo $k-\ell$. The next result describes the absorbing path as a k-graph, which we shall use to absorb a set S.

Proposition 6.1. *Suppose that $k \geq 3$, and that $1 \leq \ell \leq k-1$ is such that $(k-\ell) \nmid k$. Then there is a k-partite k-graph $\mathcal{AP}(k,\ell)$ with the following properties.*

(1) $|\mathcal{AP}(k,\ell)| \leq k^4$.
(2) *The vertex set of $\mathcal{AP}(k,\ell)$ consists of two disjoint sets S and X with $|S| = k-\ell$.*
(3) *$\mathcal{AP}(k,\ell)$ contains an ℓ-path P with vertex set X and ordered ends P^{beg} and P^{end}.*
(4) *$\mathcal{AP}(k,\ell)$ contains an ℓ-path Q with vertex set $S \cup X$ and ordered ends P^{beg} and P^{end}.*
(5) *No edge of $\mathcal{AP}(k,\ell)$ contains more than one vertex of S.*
(6) *No vertex class of $\mathcal{AP}(k,\ell)$ contains more than one vertex of S.*

Proof. Let $V_1, \dots, V_k$ be disjoint vertex sets of size $k\ell(k-\ell)+1$. Let S be an ordered $(k-\ell)$-set such that for each $1 \le i \le k-\ell$, the ith vertex of S lies in $V_{\ell+i}$. Let P be an ℓ-path in $\mathcal{K}[V_1, \dots, V_k]$ with ordered ends P^{beg} and P^{end} such that both P^{beg} and P^{end} contain at most one vertex from each V_i and such that $V(P) = (V_1 \cup \dots \cup V_k) \setminus S$. (One can easily choose such a P if for all $j = 1, \dots, |P|$ one chooses the jth vertex of P in the V_i for which $j \equiv i$ modulo k.) Then $V(P) \cup S = V_1 \cup \dots \cup V_k$. Thus we can apply Proposition 3.1 to obtain an ℓ-path Q from P^{beg} to P^{end} in $\mathcal{K}[V_1, \dots, V_k]$ such that $V(Q) = V(P) \cup S$. By swapping some vertices in S with some vertices in $V(Q) \setminus S$ (lying in the same V_i) if necessary we can ensure that the vertices in S are distributed in such a way that in some vertex sequence of Q they have distance at least k from each other. (This ensures (5).) We can now take $\mathcal{AP}(k, \ell) := P \cup Q$. □

Fix an $\mathcal{AP}(k, \ell)$ satisfying Proposition 6.1, which we shall refer to simply as $\mathcal{AP}(k, \ell)$ for the rest of this section. Let $b(k, \ell) := |\mathcal{AP}(k, \ell)| - k + \ell$, so that $b(k, \ell)$ is the number of vertices of the ℓ-path P in the definition of $\mathcal{AP}(k, \ell)$.

Now, given a $(k-\ell)$-set S of vertices of $\mathcal{H}$, we can think of S as a labelled (k, k)-complex with no i-edges for any $i \ge 2$. We will apply the extension lemma (Lemma 4.6) to deduce that for most such $(k-\ell)$-sets S, there are many labelled copies of $\mathcal{AP}(k, \ell)^{\le}$ extending S in $\mathcal{H}$, which will imply that $\mathcal{H}$ contains many absorbing paths for these sets S.

Suppose that $\mathcal{H}$ is a k-graph on n vertices, and that c is a positive constant. We say that a $(k-\ell)$-set S of vertices of $\mathcal{H}$ is *c-good* if $\mathcal{H}$ contains at least $cn^{b(k,\ell)}$ absorbing paths for S, each on $b(k, \ell)$ vertices. S is *c-bad* if it is not c-good. The next lemma states that for the values of k and ℓ we are interested in, and any small c, if $\mathcal{H}$ is sufficiently large and has large minimum degree, then almost all $(k-\ell)$-sets S of vertices of $\mathcal{H}$ are c-good.

Lemma 6.2. *Suppose that $k \ge 3$, that $1 \le \ell \le k-1$ is such that $(k-\ell) \nmid k$, and that $1/n \ll c \ll \gamma \ll \mu, 1/k$. Let $\mathcal{H}$ be a k-graph on n vertices such that $\delta(\mathcal{H}) \ge \mu n$. Then at most $\gamma n^{k-\ell}$ sets S of $k-\ell$ vertices of $\mathcal{H}$ are c-bad.*

Proof. Let $b = b(k, \ell)$, and introduce new constants
$$\frac{1}{n} \ll \frac{1}{r}, \delta \ll c \ll \min\{\delta_k, 1/t\} \le \delta_k, \eta \ll d \ll \gamma.$$
We may assume that $t!$ divides $|\mathcal{H}|$, so apply the regularity lemma to $\mathcal{H}$, and let $V_1, \dots, V_{a_1}$ be the clusters of the partition obtained. Let $m = n/a_1$ be the size of each of these clusters. Form the reduced k-graph $\mathcal{R}$ on these clusters as defined in Section 4.3.

We begin by showing that almost all sets of $k-\ell$ vertices of $\mathcal{H}$ are contained in clusters which lie in a common edge of $\mathcal{R}$. More precisely, for all but at most $\gamma n^{k-\ell}/2$ sets $\{v_1, \dots, v_{k-\ell}\}$ of $k-\ell$ vertices of $\mathcal{H}$ we can choose clusters $V_{i_1}, \dots, V_{i_k}$ such that $v_j \in V_{i_j}$ for each $1 \le j \le k-\ell$ and such that $\{V_{i_1}, \dots, V_{i_k}\}$ forms an edge of $\mathcal{R}$. Indeed, by Lemma 4.3, $d_{\mathcal{R}}(S) \ge 1$ for all but at most $\gamma a_1^{k-\ell}/3$ 'neighbourless' sets S of $k-\ell$ clusters. At most $\eta n^{k-\ell} \ll \gamma n^{k-\ell}$ sets T of $k-\ell$ vertices of $\mathcal{H}$ do not lie in $\mathrm{Cross}_{k-\ell}$. But if $T \in \mathrm{Cross}_{k-\ell}$, then unless the set S of clusters containing the vertices of T is one of the at most $\gamma a_1^{k-\ell}/3$ 'neighbourless' sets of $k-\ell$ clusters (which will be the case for at most $\gamma n^{k-\ell}/3$ sets of $k-\ell$ vertices of $\mathcal{H}$), there is an edge of $\mathcal{R}$ containing S as required.

Now, suppose that $V_{i_1}, \dots, V_{i_k}$ are clusters which form an edge of $\mathcal{R}$. Note that there are $m^{k-\ell}$ sets $\{v_1, \dots, v_{k-\ell}\}$ such that $v_j \in V_{i_j}$ for each $1 \le j \le k-\ell$. Since $e = \{V_{i_1}, \dots, V_{i_k}\}$ is an edge of $\mathcal{R}$, we may define the complex $\mathcal{H}^*$ corresponding to e as in the paragraph after the statement of the extension lemma (Lemma 4.6). Then $\mathcal{H}^*$ satisfies the conditions of the extension lemma (with $\gamma/2$ and k playing the roles of β and s), and respects the partition of $\mathcal{AP}(k,\ell)$. Let S be an ordered set of size $k-\ell$, which we may view as a labelled (k,k)-complex with no j-edges for $j \geq 2$. Then by Lemma 4.6, all but at most $\gamma m^{k-\ell}/2$ ordered sets $S' = \{v_1, \dots, v_{k-\ell}\}$ such that $v_j \in V_{i_j}$ for each j (these are the labelled copies of S) are extendible to at least $cb(k,\ell)!n^{b(k,\ell)}$ labelled partition-respecting copies of $\mathcal{AP}(k,\ell)$ in $\mathcal{H}$. This is where we use property (5) of Proposition 6.1 – it ensures that the complex S is an induced subcomplex of $\mathcal{AP}(k,\ell)$. For each copy C of $\mathcal{AP}(k,\ell)$, $C - S'$ is an absorbing path for S' on $b(k,\ell)$ vertices, and so $\mathcal{H}^*$ (and therefore $\mathcal{H}$) contains at least $cn^{b(k,\ell)}$ absorbing paths on $b(k,\ell)$ vertices for S'. So at most $\gamma m^{k-\ell}/2$ such sets S' are c-bad.

Recall that the number of $(k-\ell)$-sets of vertices of $\mathcal{H}$ which do not lie in distinct clusters corresponding to an edge of $\mathcal{R}$ is at most $\gamma n^{k-\ell}/2$. Summing over all sets of $k-\ell$ clusters, we see that at most $\gamma n^{k-\ell}/2$ of the $(k-\ell)$-sets which do lie in distinct clusters corresponding to an edge of $\mathcal{R}$ are c-bad. Thus at most $\gamma n^{k-\ell}$ sets of $k-\ell$ vertices of $\mathcal{H}$ are c-bad, completing the proof. □

We are now in a position to prove the main lemma of this section. It states that for any positive c, if $\mathcal{H}$ is a sufficiently large k-graph of large minimum degree, then we can find an ℓ-path in $\mathcal{H}$ which contains a small proportion of the vertices of $\mathcal{H}$, includes all vertices of $\mathcal{H}$ which lie in many c-bad $(k-\ell)$-sets and can absorb any small collection of c-good $(k-\ell)$-sets of vertices of $\mathcal{H}$.

Lemma 6.3 (Absorbing path lemma)**.** *Suppose that $k \geq 3$, that $1 \leq \ell \leq k-1$ is such that $(k-\ell) \nmid k$, and that $1/n \ll \alpha \ll c \ll \gamma \ll \mu, 1/k$. Let $\mathcal{H}$ be a k-graph of order n with $\delta(\mathcal{H}) \geq \mu n$. Then $\mathcal{H}$ contains an ℓ-path P on at most μn vertices such that the following properties hold:*

1. *Every vertex of $\mathcal{H} - V(P)$ lies in at most $\gamma n^{k-\ell-1}$ c-bad $(k-\ell)$-sets.*
2. *P can absorb any collection of at most αn disjoint c-good $(k-\ell)$-sets of vertices of $\mathcal{H} - V(P)$.*

Proof. Let $b := b(k,\ell)$, and choose a family $\mathcal{T}$ of ordered b-sets of vertices of $\mathcal{H}$ at random by including each ordered b-set T into $\mathcal{T}$ with probability c^2n^{1-b}, independently of all other ordered b-sets. Now, for any c-good set S of $k-\ell$ vertices of $\mathcal{H}$, the expected number of $T \in \mathcal{T}$ for which $\mathcal{H}$ contains an absorbing path for S with T as a vertex sequence is at least c^3n, by the definition of a c-good set. So by a standard Chernoff bound, with probability $1-o(1)$, for every c-good $(k-\ell)$-set S of vertices of $\mathcal{H}$ the number of such ordered b-sets $T \in \mathcal{T}$ is at least $c^3n/2$. Furthermore, with probability $1-o(1)$ we have $|\mathcal{T}| \leq 2c^2n$. The expected number of ordered pairs T, T' in $\mathcal{T}$ which intersect (i.e. for which the corresponding unordered sets intersect) is at most $(c^2n^{1-b})^2b^2n^{2b-1} = c^4b^2n$. So with probability at least 1/2 the number of such pairs is at most $2c^4b^2n$. Thus we may fix an outcome of our random selection of $\mathcal{T}$ such that all of these events hold.

Delete from $\mathcal{T}$ every $T \in \mathcal{T}$ which intersects any other $T' \in \mathcal{T}$. Also delete from $\mathcal{T}$ every $T \in \mathcal{T}$ which is not a vertex sequence of an absorbing path for some c-good $(k-\ell)$-set S of vertices of $\mathcal{H}$. Let $T_1, \dots, T_q$ be the remaining members of $\mathcal{T}$. So $q \leq 2c^2n$, and for each $1 \leq i \leq q$ we can choose an ℓ-path P_i in $\mathcal{H}$ with vertex sequence T_i which is absorbing for some such S. Then all the ℓ-paths P_i are disjoint, and for every c-good $(k-\ell)$-set S of vertices of $\mathcal{H}$ at least $c^3n/2 - 2c^4b^2n \geq \alpha n$ of the ℓ-paths P_i are absorbing.

Let X be the set of vertices of $\mathcal{H}$ which are not contained in any P_i and which lie in more than $\gamma n^{k-\ell-1}$ c-bad $(k-\ell)$-sets. Then $|X| \leq \gamma n$, since by Lemma 6.2 there are at most $\gamma^2n^{k-\ell}/(k-\ell)$ c-bad $(k-\ell)$-sets in total. We shall use the minimum degree condition on $\mathcal{H}$ to greedily construct an ℓ-path P_0 containing all vertices in X and not intersecting the previous paths $P_i, 1 \leq i \leq q$. Then if we incorporate each of the P_i $(0 \leq i \leq q)$ into the

ℓ-path P we are constructing, conditions (1) and (2) of the lemma will be satisfied. So let X' be a set of $k-1$ vertices of X. Then $d_{\mathcal{H}}(X') \geq \mu n$ by the minimum degree condition on $\mathcal{H}$. Since $|\bigcup_{i=1}^q P_i| < \mu n$, we may choose a vertex $x \in V(\mathcal{H}) \setminus \bigcup_{i=1}^q V(P_i)$ which together with X' forms an edge of $\mathcal{H}$. Then $X' \cup \{x\}$ is the first edge of P_0. We then greedily extend P_0 as follows. Let X'' be the set of the final ℓ vertices of the vertex sequence of P_0. Add to X'' any $k-1-\ell \geq 1$ vertices from X not yet contained in P_0. Then $d_{\mathcal{H}}(X'') \geq \mu n$, and so we may choose a vertex y of $\mathcal{H}$ which is not in $\bigcup_{i=1}^q P_i$ nor already contained in P_0. We then extend P_0 by the edge $X'' \cup \{y\}$. At the end of this process we obtain an ℓ-path P_0 which is disjoint from all the P_i ($i = 1, \ldots q$), which contains every vertex of X, and which satisfies $|V(P_0)| \leq 2\gamma n$. Let P_i^{beg} and P_i^{end} be ordered ends of P_i for each $0 \leq i \leq q$.

To complete the proof, we now use the diameter lemma (Lemma 5.4) to greedily join each ordered ℓ-set P_i^{end} to the ordered ℓ-set P_{i+1}^{beg} by an ℓ-path P_i', such that P_i' intersects P_i and P_{i+1} only in the sets P_{i+1}^{beg} and P_i^{end} and does not intersect any other P_j or any previously chosen P_j'. More precisely, suppose we have chosen such $P_0', \ldots, P_{i-1}'$. Let $\mathcal{H}'$ be the k-graph obtained from $\mathcal{H}$ by removing all the vertices in $P_0, \ldots, P_q$ and all the vertices in $P_0', \ldots, P_{i-1}'$ and then adding back P_i^{end} and P_{i+1}^{beg}. Then $\delta(\mathcal{H}') \geq \mu n/2$, and so we may apply Lemma 5.4 to find an ℓ-path P_i' in $\mathcal{H}'$ from P_i^{end} to P_{i+1}^{beg} containing at most $8k^5$ vertices. Having found these ℓ-paths, the absorbing path P^* is the ℓ-path $P_0P_0'P_1P_1'P_2\ldots P_{q-1}P_{q-1}'P_q$. □

7. Path Cover Lemma

Lemma 7.1 (Path cover lemma)**.** *Suppose $k \geq 3$, that $1 \leq \ell \leq k-1$, and that $1/n \ll 1/D \ll \varepsilon \ll \mu, 1/k$. Let $\mathcal{H}$ be a k-graph of order n with $\delta(\mathcal{H}) \geq (\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + \mu)n$. Then $\mathcal{H}$ contains a set of at most D disjoint ℓ-paths covering all but at most εn vertices of $\mathcal{H}$.*

Let

$$a := \left\lceil \frac{k}{k-\ell} \right\rceil (k-\ell) \tag{1}$$

and let $\mathcal{F}_{k,\ell}$ be the k-graph whose vertex set is the disjoint union of sets $A_1, \ldots, A_{a-1}$ and B of size $k-1$ and whose edges are all the k-sets of the form $A_i \cup \{b\}$ (for all $i = 1, \ldots, a-1$ and all $b \in B$). An *$\mathcal{F}_{k,\ell}$-packing* in a k-graph $\mathcal{R}$ is a collection of pairwise vertex-disjoint copies of $\mathcal{F}_{k,\ell}$ in $\mathcal{R}$.

The idea of the proof of the path cover lemma is to apply the regularity lemma to $\mathcal{H}$ in order to obtain a reduced k-graph $\mathcal{R}$. Recall that by Lemma 4.3 the minimum degree of $\mathcal{H}$ is almost inherited by $\mathcal{R}$. So we can use the following lemma (Lemma 7.2) to obtain an almost perfect $\mathcal{F}_{k,\ell}$-packing in $\mathcal{R}$. Consider any copy $\mathcal{F}$ of $\mathcal{F}_{k,\ell}$ in this packing. We will repeatedly apply the embedding lemma (Lemma 4.5) to the sub-k-graph $\mathcal{H}(\mathcal{F})$ of $\mathcal{H}$ corresponding to $\mathcal{F}$ to obtain a bounded number of ℓ-paths which cover almost all vertices of $\mathcal{H}(\mathcal{F})$. Doing this for all the copies of $\mathcal{F}_{k,\ell}$ in the $\mathcal{F}_{k,\ell}$-packing of $\mathcal{R}$ will give a set of ℓ-paths as required in Lemma 7.1.

Lemma 7.2. *Suppose that $k \geq 3$, that $1 \leq \ell \leq k-1$, and that $1/n \ll \theta \ll \varepsilon \ll 1/k$. Let $\mathcal{H}$ be a k-graph of order n such that $d(S) \geq n/a$ for all but at most θn^{k-1} sets $S \in \binom{V(\mathcal{H})}{k-1}$, where a is as defined in (1). Then $\mathcal{H}$ contains a $\mathcal{F}_{k,\ell}$-packing covering all but at most $(1-\varepsilon)n$ vertices.*

We omit the proof of this lemma. It was first proved in [11] for the case when $k = 3$ and $\ell = 1$. A proof for the case when $\ell < k/2$ can be found in [6]. The general case can be proved similarly, see [12] for details.

Lemma 7.3. *Let P be an ℓ-path on n vertices and let a be as defined in (1). Then there is a k-colouring of P with colours $1, \dots, k$ such that colour k is used $n/a \pm 1$ times and the sizes of all other colour classes are as equal as possible.*

Proof. Let $x_1, \dots, x_n$ be a vertex sequence of P. Colour vertices $x_k, x_{k+a}, x_{k+2a}, \dots$ with colour k and remove these vertices from the sequence $x_1, \dots, x_n$. Colour the remaining vertices in turn with colours $1, \dots, k-1$ as follows. Colour the first vertex with colour 1. Suppose that we just coloured the ith vertex with some colour j. Then we colour the next vertex with colour $j+1$ if $j \le k-2$ and with colour 1 if $j = k-1$. To show that this yields a proper colouring, it suffices to show that every edge of P contains some vertex of colour k. Clearly this holds for the first edge e_1 of P and for all edges intersecting e_1 (since x_k lies in all those edges). Note that the first vertex of the ith edge e_i of P is $x_{f(i)}$, where $f(i) = (i-1)(k-\ell)+1$. Also note that $i^* := \lceil \frac{k}{k-\ell} \rceil + 1$ is the smallest integer so that $f(i^*) > k$. In other words, the i^*th edge e_{i^*} of P is the first edge which does not contain x_k. But the vertices of e_{i^*} are $x_{a+1}, \dots, x_{a+k}$. So e_{i^*} as well as all succeeding edges which intersect e_{i^*} contain a vertex of colour k (namely x_{a+k}). Continuing in this way gives the claim. □

Proof of Lemma 7.1. Choose new constants such that

$$\frac{1}{n} \ll \frac{1}{D} \ll \frac{1}{r}, \delta, c \ll \min\{\delta_k, 1/t\} \le \delta_k, \eta \ll d \ll \theta \ll \varepsilon.$$

We may assume that $t!|n$, so apply Lemma 4.2 (the regularity lemma) to $\mathcal{H}$, and let $V_1, \dots, V_{a_1}$ be the clusters of the partition obtained. Let $m = n/a_1$ be the size of each of these clusters. Form the reduced k-graph $\mathcal{R}$ on these clusters as discussed in Section 4.3. Lemmas 4.3 and 7.2 together imply that $\mathcal{R}$ has a $\mathcal{F}_{k,\ell}$-packing $\mathcal{A}$ covering all but at most $\varepsilon n/2$ vertices of $\mathcal{R}$. Consider any copy $\mathcal{F}$ of $\mathcal{F}_{k,\ell}$ in this packing. Our aim is to cover almost all vertices in the clusters belonging to $\mathcal{F}$ by a bounded number of disjoint ℓ-paths.

So let $A_1, \dots, A_{a-1}$ and B be $(k-1)$-element subsets of $V(\mathcal{F})$ as in the definition of $\mathcal{F}_{k,\ell}$. So the edges of $\mathcal{F}_{k,\ell}$ are all the k-tuples of the form $A_i \cup \{b\}$ for all $i = 1, \dots, a-1$ and all $b \in B$. Pick $b \in B$ and consider the edge $A_1 \cup \{b\} =: e$. Let $\mathcal{V}$ be the set of all clusters corresponding to vertices in A_1 and let V_b be the cluster corresponding to b. Define the complex $\mathcal{H}^*$ corresponding to the edge e as in the paragraph after the statement of the extension lemma (Lemma 4.6). Then Lemma 7.3 and the embedding lemma (Lemma 4.5 applied to $\mathcal{H}^*$) together imply that the sub-k-graph of $\mathcal{H}$ spanned by the vertices in $V_b \cup \bigcup_{V \in \mathcal{V}} V$ contains an ℓ-path P_1 on $acm/(a-1)$ vertices which intersects each cluster from $\mathcal{V}$ in $cm/(k-1) \pm 1$ vertices and V_b in $cm/(a-1) \pm 1$ vertices. Lemma 4.1 implies that the subcomplex of $\mathcal{H}^*$ obtained by deleting the vertices of P_1 is still $(\mathbf{d}, \sqrt{\delta_k}, \sqrt{\delta}, r)$-regular. So we can find another ℓ-path P_2 which is disjoint from P_1 and intersects each cluster from $\mathcal{V}$ in $cm/(k-1) \pm 1$ vertices and V_b in $cm/(a-1) \pm 1$ vertices. We do this until we have used about $m/(k-1)$ vertices in each cluster from $\mathcal{V}$. So we have found $1/c$ disjoint ℓ-paths. Now we pick $b' \in B \setminus \{b\}$ and argue as before to get $1/c$ disjoint ℓ-paths, such that each of them intersects (the remainder of) each cluster from $\mathcal{V}$ in $cm/(k-1) \pm 1$ vertices and $V_{b'}$ in $cm/(a-1) \pm 1$ vertices. We do this for all the $k-1$ vertices in B. However, when considering the last vertex b'' of B, we stop as soon as one of the subclusters from $\mathcal{V}$ has size less than $\varepsilon m/4a$ (and thus all the other subclusters from $\mathcal{V}$ have size at most $\varepsilon m/2a$) since we need to ensure that the subcomplex of $\mathcal{H}^*$ restricted to the remaining subclusters is still $(\mathbf{d}, \sqrt{\delta_k}, \sqrt{\delta}, r)$-regular. So in total we have chosen close to $(k-1)/c$ disjoint ℓ-paths covering all but at most $\varepsilon m/2a$ vertices in each cluster from $\mathcal{V}$ and covering between $m/(a-1) - \varepsilon m/2a$ and $m/(a-1)$ vertices in each cluster V_b with $b \in B$. We now repeat this process for each of $A_2, \dots, A_{a-1}$ in turn. When considering the final set A_{a-1}, we also stop choosing paths for some $b \in B$ if the subcluster

V_b has size less than $\varepsilon m/4a$. Altogether this gives us a collection of close to $(k-1)(a-1)/c$ disjoint ℓ-paths covering all but at most $\varepsilon m/2$ vertices in the clusters belonging to $\mathcal{F}$. Doing this for all the copies of $\mathcal{F}_{k,\ell}$ in the $\mathcal{F}_{k,\ell}$-packing $\mathcal{A}$ of $\mathcal{R}$ we obtain a collection of at most $|\mathcal{A}|(k-1)(a-1)/c \ll D$ disjoint ℓ-paths covering all but at most $\varepsilon m/2$ vertices from each cluster, and hence all but at most $\varepsilon|\mathcal{H}|$ vertices of $\mathcal{H}$, as required. □

8. Proof of Theorem 1.2

We shall use the following two results in our proof of Theorem 1.2. The first says that if $1 \leq s \leq k-1$ and $\mathcal{H}$ is a large k-graph in which all sets of s vertices have a large neighbourhood, then if we choose $R \subseteq V(\mathcal{H})$ uniformly at random, with high probability all sets of s vertices have a large neighbourhood in R.

Lemma 8.1 (Reservoir Lemma)**.** *Suppose that $k \geq 2$, that $1 \leq s \leq k-1$, and that $1/n \ll \alpha, \mu, 1/k$. Let $\mathcal{H}$ be a k-graph of order n with $d_{\mathcal{H}}(S) \geq \mu\binom{n}{k-s}$ for any set $S \in \binom{V(\mathcal{H})}{s}$, and let R be a subset of $V(\mathcal{H})$ of size αn chosen uniformly at random. Then the probability that $|N_{\mathcal{H}}(S) \cap \binom{R}{k-s}| \geq \mu\binom{\alpha n}{k-s} - n^{k-s-1/3}$ for every $S \in \binom{V(\mathcal{H})}{s}$ is $1-o(1)$.*

The proof of Lemma 8.1 is a standard probabilistic proof, which proceeds by applying Chernoff bounds to the size of the neighbourhood of each set S, and summing the probabilities of failure over all S. We omit the details.

The second result is the following theorem of Daykin and Häggkvist [3], giving an upper bound on the vertex degree needed to guarantee the existence of a perfect matching in a k-graph $\mathcal{H}$.

Theorem 8.2 ([3])**.** *Suppose that $k \geq 2$ and $k|n$. Let $\mathcal{H}$ be a k-graph of order n with minimum vertex degree at least $\frac{k-1}{k}\left(\binom{n-1}{k-1} - 1\right)$. Then $\mathcal{H}$ contains a perfect matching.*

Proof of Theorem 1.2. In our proof we will use constants that satisfy the hierarchy

$$1/n \ll 1/D \ll \varepsilon \ll \alpha \ll c \ll \gamma \ll \gamma' \ll \eta \ll \eta' \ll 1/k.$$

Apply Lemma 6.3 to find an absorbing ℓ-path P_0 in $\mathcal{H}$ which contains at most $\eta n/4$ vertices and which can absorb any set of at most αn c-good $(k-\ell)$-sets of vertices of $\mathcal{H}$. Define the $(k-\ell)$-graph $\mathcal{G}$ on the same vertex set as $\mathcal{H}$ to consist of all the $(k-\ell)$-sets of vertices of $\mathcal{H}$ which are c-good . Then by condition (1) of Lemma 6.3, $d_{\mathcal{G}}(v) \geq \binom{n-1}{k-\ell-1} - \gamma n^{k-\ell-1} \geq (1-\gamma')\binom{n}{k-\ell-1}$ for every vertex v in $V(\mathcal{G}) \setminus V(P_0)$.

Now, let R be a set of αn vertices of $\mathcal{H}$ chosen uniformly at random. Then by Lemma 8.1, with probability $1-o(1)$ we have that $|N_{\mathcal{G}}(v) \cap \binom{R}{k-\ell-1}| \geq (1-2\gamma')\binom{\alpha n}{k-\ell-1}$ for every vertex v in $V(\mathcal{G}) \setminus V(P_0)$. Likewise, with probability $1-o(1)$ we have that

$$|N_{\mathcal{H}}(S) \cap R| \geq \left(\frac{1}{\lceil\frac{k}{k-\ell}\rceil(k-\ell)} + \frac{\eta}{2}\right)\alpha n.$$

for any $(k-1)$-set S of vertices of $\mathcal{H}$. Finally, $\mathbb{E}[|R \cap V(P_0)|] = \alpha|P_0|$, and so with probability at least $1/2$ we have that $|R \cap V(P_0)| \leq \alpha\eta n/2$. Thus we may fix a choice of R such that each of these three properties hold. Let $R' = R \setminus V(P_0)$, so $|R'| \geq (1-\eta/2)\alpha n$. Then $|N_{\mathcal{G}}(v) \cap \binom{R'}{k-\ell-1}| \geq (1-\eta')\binom{\alpha n}{k-\ell-1}$ for every vertex v in $V(\mathcal{G}) \setminus V(P_0)$, and $|N_{\mathcal{H}}(S) \cap R'| \geq \frac{\alpha n}{\lceil\frac{k}{k-\ell}\rceil(k-\ell)}$ for any $(k-1)$-set S of vertices of $\mathcal{H}$.

Let $V' = V(\mathcal{H}) \setminus (V(P_0) \cup R)$, and let $\mathcal{H}' = \mathcal{H}[V']$ be the restriction of $\mathcal{H}$ to V'. Then as $|V(P_0) \cup R| \leq \eta n/2$, we must have

$$\delta(\mathcal{H}') \geq \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + \frac{\eta}{2} \right) n.$$

We may therefore apply Lemma 7.1 to $\mathcal{H}'$ to find a set of at most D disjoint ℓ-paths $P_1, \dots, P_q$ in $\mathcal{H}'$ which include all but at most εn vertices of $\mathcal{H}'$. Let X be the set of vertices not included in any of these ℓ-paths, so $|X| \leq \varepsilon n$.

For each $0 \leq i \leq q$, let P_i^{beg} and P_i^{end} be ordered ends of P_i. Next we shall find disjoint ℓ-paths P_i' for each $0 \leq i \leq q$, so that P_i' is an ℓ-path from P_i^{end} to P_{i+1}^{beg} (where subindices are taken modulo $q+1$). The ℓ-path P_i' will only contain vertices from $R' \cup P_i^{end} \cup P_{i+1}^{beg}$, and will contain at most $8k^5$ vertices in total. So, suppose that we have found such ℓ-paths $P_0', \dots, P_{i-1}'$. Let $R_i = (R' \cup P_i^{end} \cup P_{i+1}^{beg}) \setminus \bigcup_{j=0}^{i-1} V(P_j')$. Then $\delta(\mathcal{H}[R_i]) \geq \frac{\alpha n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} - 8k^5 D \geq \alpha n/2k$, and so by Corollary 5.4 we can choose such an ℓ-path P_i' in $\mathcal{H}[R_i]$.

Then $C = P_0 P_0' P_1 P_1' \dots P_q P_q'$ is an ℓ-cycle containing almost every vertex of $\mathcal{H}$. Indeed, C contains every vertex of $\mathcal{H}$ except for those in X and those in R' not contained in any P_i'. So let $R'' = V(\mathcal{H}) \setminus V(C)$. Then $(1-\eta)\alpha n \leq |R''| \leq (\alpha+\varepsilon)n$. Since $(k-\ell)|n$ and $(k-\ell)\big||C|$ (as C is an ℓ-cycle), we also have $(k-\ell)\big||R''|$. Furthermore, $N_{\mathcal{G}[R'']}(v) \geq (1-2\eta')\binom{\alpha n}{k-\ell-1}$ for every vertex $v \in R''$. Since $k-\ell \geq 2$, Theorem 8.2 tells us that $\mathcal{G}[R'']$ contains a perfect matching, and so we can partition R'' into at most αn c-good $(k-\ell)$-sets of vertices of $\mathcal{H}$. Since P_0 can absorb any collection of at most αn c-good $(k-\ell)$-sets, there exists an ℓ-path Q_0 with the same ordered ends as P_0 and such that $V(Q_0) = V(P_0) \cup R''$. Then $C' = Q_0 P_0' P_1 P_1' \dots P_q P_q'$ is a Hamilton ℓ-cycle in $\mathcal{H}$, completing the proof of Theorem 1.2.

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