EDGE CORRELATIONS IN RANDOM REGULAR HYPERGRAPHS AND APPLICATIONS TO SUBGRAPH TESTING

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Abstract. Compared to the classical binomial random (hyper)graph model, the study of random regular hypergraphs is made more challenging due to correlations between the occurrence of different edges. We develop an edge-switching technique for hypergraphs which allows us to show that these correlations are limited for a large range of densities. This extends some previous results of Kim, Sudakov and Vu for graphs. From our results we deduce several corollaries on subgraph counts in random d-regular hypergraphs. We also prove a conjecture of Dudek, Frieze, Ruciński and Šileikis on the threshold for the existence of an ℓ-overlapping Hamilton cycle in a random d-regular r-graph.

Moreover, we apply our results to prove bounds on the query complexity of testing subgraph-freeness. The problem of testing subgraph-freeness in the general graphs model was first studied by Alon, Kaufman, Krivelevich and Ron, who obtained several lower bounds on the query complexity of testing triangle-freeness. We extend some of these previous results beyond the triangle setting and to the hypergraph setting.

1. Introduction

1.1. **Random regular graphs.** While the consideration of random d-regular graphs is very natural and has a long history, this model is much more difficult to analyze than the seemingly similar $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$ models due to the dependencies between edges (here $\mathcal{G}(n,p)$ refers to the binomial n-vertex random graph model with edge probability p and $\mathcal{G}(n,m)$ refers to the uniform distribution on all n-vertex graphs with m edges). For small d, the configuration model (due to Bollobás [5]) has led to numerous results on random d-regular graphs. Moreover, the switching method introduced by McKay and Wormald [22] has led to results for a much larger range of d than can be handled by the configuration model. For example, Kim, Sudakov and Vu [18] used such ideas to show that the classical results on distributions of small subgraphs in $\mathcal{G}(n,p)$ carry over to random regular graphs.

In this paper we develop an edge switching technique for random regular r-uniform hypergraphs (also called r-graphs). More precisely, we show that correlations between the existence of edges in a random regular r-graph are small even if we condition on the (non-)existence of some further edges (see Corollary 2.3). This allows us to generalise results of Kim, Sudakov and Vu [18] on the appearance of fixed subgraphs in a random regular graph to the hypergraph setting (see Corollary 3.3). Moreover, even in the graph case, we can condition on the (non-)existence of a significantly larger edge set than in [18].

A general result of Dudek, Frieze, Ruciński and Šileikis [9] implies that one can transfer many statements from the binomial model to the random regular hypergraph model (see Theorem 3.5). This allows them to deduce (from the main result of Dudek and Frieze [7]) the following: if $2 \le \ell < r$ and $n^{\ell-1} \ll d \ll n^{r-1}$, then a random d-regular r-graph a.a.s. contains an ℓ-overlapping Hamilton cycle, that is, a Hamilton cycle in which consecutive edges overlap in precisely ℓ

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vertices (these cycles are defined formally in Section 1.4). They conjectured that the lower bound provides the correct threshold in the following sense:

$$\text{if } 2 \le \ell < r \text{ and } d \ll n^{\ell-1}, \text{ then a.a.s. a random } d\text{-regular } r\text{-graph contains no } \ell\text{-overlapping Hamilton cycle.} \tag{1.1}$$

Our correlation results from Section 2 allow us to confirm this conjecture (see Corollary 3.13). The threshold for a loose Hamilton cycle (i.e. a 1-overlapping Hamilton cycle) in a random d-regular r-graph was recently determined (via the configuration model) by Altman, Greenhill, Isaev and Ramadurai [3]. This improved earlier bounds by Dudek, Frieze, Ruciński and Šileikis [8]. Altman, Greenhill, Isaev and Ramadurai [3] also investigated the above conjecture and proved that (1.1) holds under the much stronger condition that $d \ll n$ if $r = 4$ and $d \ll n^{1/2}$ if $r = 3$. The graph case $r = 2$ where d is fixed is a classical result by Robinson and Wormald [25, 26]: if $d \ge 3$ is fixed, then a.a.s. a random d-regular graph has a Hamilton cycle. This was extended to larger d by Cooper, Frieze and Reed [6].

In a similar way, we can transfer several classical counting results for random graphs to the regular setting. We illustrate this for Hamilton cycles, where we extend the density range of a counting result of Krivelevich [21]: for $\log n \ll d \ll n$, a.a.s. the number of Hamilton cycles in a random d-regular n-vertex graph is fairly close to $n!(d/n)^n$ (see Corollary 3.8). The results by Krivelevich [21] imply the same behaviour for $d \gg e^{(\log n)^{1/2}}$. For constant d, this problem was studied by Janson [14]. Similarly, we transfer a general counting result for spanning subgraphs in $\mathcal{G}(n, m)$ due to Riordan [24] to the setting of random regular graphs.

1.2. **Property testing.** The running time of any "exact" algorithm that checks whether a given combinatorial object has a given property must be at least linear in the size of the input. Property testing algorithms have the potential to give much quicker answers, although at the cost of not knowing for certain if the desired property is satisfied by the object. A property testing algorithm is usually given oracle access to the combinatorial object, and answers whether the object satisfies the property or is "far" from satisfying it.

To be precise, following e.g. Goldreich, Goldwasser and Ron [11], we define testers as follows. Given a property $\mathcal{P}$, a *tester* for $\mathcal{P}$ is a (possibly randomized) algorithm that is given a distance parameter ε and oracle access to a structure S. If $S \in \mathcal{P}$, then the algorithm must accept with probability at least $2/3$. If S is ε-far from $\mathcal{P}$, then the algorithm should reject with probability at least $2/3$. If the algorithm is allowed to make an error in both cases, we say it is a *two-sided error tester*; if, on the contrary, the algorithm always gives the correct answer when S has the property, we say it is a *one-sided error tester*.

For graphs (and, more generally, r-graphs) there have been two classical models for testers: one of them is the dense model, and the other is the bounded-degree model. In the dense model, the density of the r-graph is assumed to bounded away from 0, and we say that an r-graph G is ε-far from having property $\mathcal{P}$ if at least εn^r edges have to be modified (added or deleted) to turn G into a graph that satisfies $\mathcal{P}$. Many results have been proved for the dense model. In particular, there exists a characterization of all properties which are testable with constant query complexity (by Alon, Fischer, Newman and Shapira [1] in the graph case and Joos, Kim, Kühn and Osthus [16] in the r-graph case). For the bounded degree graphs model (which assumes that the maximum degree of the input graphs is bounded by a fixed constant), several general results have also been obtained (see for example the results of Benjamini, Schramm and Shapira [4] as well as Newman and Sohler [23]).

Here, we consider the general graphs model and its generalization to r-graphs. In the general graphs model (introduced by Kaufman, Krivelevich and Ron [17]), a graph G with m edges is ε-far from having property $\mathcal{P}$ if at least εm edges have to be modified for the graph to satisfy $\mathcal{P}$. Furthermore, we also assume that the edges are labelled in the sense that for each vertex there is an ordering of its incident edges. It is natural to consider the following two types of queries. Firstly, we allow vertex-pair queries, where any algorithm may take two vertices and ask whether they are joined by an edge in the graph or not. Secondly, we allow neighbour queries, where any algorithm may take a vertex and ask which vertex is its i-th neighbour.

These notions generalise to hypergraphs in a straightforward way. More precisely, we will consider the following general hypergraphs model, where a hypergraph with m edges is ε-far from having property $\mathcal{P}$ if at least εm edges must be added or deleted to ensure the resulting hypergraph satisfies $\mathcal{P}$. As in the graph case, we will consider two types of queries:

- Vertex-set queries: Any algorithm may take a set of r vertices and ask whether they constitute an edge in the r-graph or not. The answer must be either yes or no.
- Neighbour queries: Any algorithm may take a vertex and ask for its i-th incident edge (according to the labelling of the edges). The answer is either a set of $r-1$ vertices or an error message if the degree of the queried vertex is smaller than i.

In this paper we consider the property $\mathcal{P}$ of being F-free for fixed r-graphs F. In the dense setting, the theory of hypergraph regularity (as developed by Rödl and Skokan [30], Rödl and Schacht [27, 28, 29] as well as Gowers [12]) implies the existence of testers with constant query complexity for this problem.

However, the problem is still wide open for general graphs and hypergraphs. Alon, Kaufman, Krivelevich and Ron [2] studied the problem of testing triangle-freeness. In Section 4, we provide lower and upper bounds for testing F-freeness which apply to large classes of hypergraphs F. In particular, we observe that testing F-freeness cannot be achieved in a constant number of queries whenever F is not a weak forest and the density of the graphs G to be tested is somewhat below the Turán threshold for F (see Proposition 4.1). Based on the results of Sections 2 and 3.1, we also provide a lower bound (see Theorem 4.5) which improves on Proposition 4.1 for a large range of parameters and r-graphs. Roughly speaking, Theorem 4.5 provides better bounds than Proposition 4.1 if the average degree d of the input r-graph G is not too small. On the other hand, the class of admissible F is more restricted. We also provide three upper bounds on the query complexity (see Section 4.3).

Kaufman, Krivelevich and Ron [17] also studied the problem of testing bipartiteness in general graphs. It would be interesting to obtain results for the general (hyper)graphs model covering further properties and to improve the lower and upper bounds we present for testing F-freeness.

1.3. **Outline of the paper.** The remainder of the paper is organised as follows. In Section 2 we develop a hypergraph generalisation of the edge-switching technique to prove a correlation result (Corollary 2.3) for the event that a given edge is present in a random d-regular r-graph even if we condition on the (non-)existence of some further edges.

Section 3 builds on this to obtain subgraph count results in random d-regular r-graphs. In particular, in Section 3.1 we consider the counting problem for small fixed graphs F, for which we prove a concentration result, thus also obtaining the threshold for their appearance, which generalises a result of Kim, Sudakov and Vu [18] for graphs. We also derive bounds on the number of edge-disjoint copies of fixed subgraphs F in a random d-regular r-graph, which we use in Section 4.2. In Section 3.2, we combine the results from Section 2 with known results for $\mathcal{G}^{(r)}(n,p)$ and $\mathcal{G}^{(r)}(n,m)$ to count the number of suitable spanning subgraphs (such as Hamilton cycles) in random d-regular r-graphs.

Finally, Section 4 provides lower and upper bounds on the query complexity for testing subgraph freeness for small, fixed r-graphs F. The proof of the main lower bound relies on Corollary 2.3 and the counting results derived in Section 3.1.

1.4. **Definitions and notation.** An *r-graph* (or r-uniform hypergraph) $H=(V,E)$ is an ordered pair where V is a set of vertices, and $E \subseteq \binom{V}{r}$ is a set of r-subsets of V, called edges. When $r=2$, we will simply refer to these as graphs and omit the presence of r in any notation. To indicate the vertex set and the edge set of a certain r-graph H we will use the notation $V(H)$ and $E(H)$, respectively. We will often abuse notation and write $e \in H$ to mean $e \in E(H)$, or use $E(H)$ instead of H to denote the r-graph. In particular, we write $|H|$ for $|E(H)|$. The *order* of an r-graph H is $|V(H)|$ and the *size* of H is $|E(H)|$. For a fixed r-graph H, we sometimes denote its number of vertices by v_H, while e_H will denote the number of edges.

Given a vertex $v \in V(H)$, the *degree* of v in H is $\deg_H(v) := |\{e \in H : v \in e\}|$. When H is clear from the context, it may be dropped from the notation. We will use $\Delta(H)$ to denote the maximum (vertex) degree of H, $\delta(H)$ to denote the minimum (vertex) degree of H and $d(H)$ to denote its average (vertex) degree. We say that H is *d-regular* if $\deg_H(v) = d$ for all $v \in V(H)$. The set of vertices lying in a common edge with v is called its *neighbourhood* and denoted by $N_H(v)$.

The *complete* r-graph of order n is denoted by $K_n^{(r)}$. If its vertex set V is given, we denote this by $K_V^{(r)}$. We say that an r-graph H is *k-partite* if there exists a partition of $V(H)$ into k sets such that every edge $e \in H$ contains at most one vertex in each of the sets. A *path* P between vertices u and v, also called a *(u,v)-path*, is an r-graph whose vertices admit a labelling $u, v_1, \ldots, v_k, v$ such that any two consecutive vertices lie in an edge of P and each edge consists of consecutive vertices. An r-graph H is *connected* if there exists a path joining any two vertices in $V(H)$. The *distance* between vertices u and v in H is defined by $\mathrm{dist}_H(u,u) \coloneqq 0$ and $\mathrm{dist}_H(u,v) \coloneqq \min\{|P| : P \text{ is an } (u,v)\text{-path}\}$ whenever $u \neq v$. If there is no such path, the distance is said to be infinite. The distance between sets of vertices S and T is $\mathrm{dist}_H(S,T) \coloneqq \min\{\mathrm{dist}_H(s,t) : s \in S, t \in T\}$. The *diameter* of an r-graph H is $D(H) \coloneqq \max_{(u,v)\in V(H)^2} \mathrm{dist}_H(u,v)$. An r-graph C is a *k-overlapping cycle* of length ℓ if $|C| = \ell$ and the vertices of C admit a cyclic labelling such that each edge in C consists of r consecutive vertices and any two consecutive edges have exactly k vertices in common (in the natural cyclic order induced on the edges of C). When $k = 1$, we refer to C as a *loose cycle*. When $k = r-1$, C is called a *tight cycle*. A k-overlapping cycle C is said to be *Hamiltonian* for an r-graph H if $E(C) \subseteq E(H)$ and $V(C) = V(H)$. We will write C_n^k for a k-overlapping cycle of order n. We say that a connected r-graph H is a *weak tree* if $|e \cap f| \leq 1$ for all $e, f \in H$ with $e \neq f$, and H contains no loose cycles. We say that an r-graph is a *weak forest* if it is the union of vertex-disjoint weak trees. Note that, for graphs, this is the usual definition of a forest. Given any r-graph H, its complement is denoted as $\overline{H}$.

The Erdős-Rényi random r-graph, also called the binomial model, is denoted by $\mathcal{G}^{(r)}(n,p)$, for $n \in \mathbb{N}$ and $p \in [0,1]$. An r-graph $G^{(r)}(n,p)$ on vertex set V with $|V| = n$ chosen according to this model is obtained by including each $e \in \binom{V}{r}$ with probability p independently from the other edges. For $n \in \mathbb{N}$ and $m \in [\binom{n}{r}] \cup \{0\}$, we denote by $\mathcal{G}^{(r)}(n,m)$ the set of all r-graphs on n vertices that have exactly m edges, and denote by $G^{(r)}(n,m)$ an r-graph chosen uniformly at random from this set. We denote the set of all d-regular r-graphs on vertex set V with $|V| = n$ by $\mathcal{G}_{n,d}^{(r)}$, for $n \in \mathbb{N}$ and $d \in [\binom{n-1}{r-1}] \cup \{0\}$, and denote by $G_{n,d}^{(r)}$ an r-graph chosen uniformly at random from $\mathcal{G}_{n,d}^{(r)}$. Note that d-regular r-graphs on n vertices do not always exist; however, if $d = o(n^{r-1})$, then they always exist as long as $r \mid nd$, which we will always implicitly assume. If H and H' are two r-graphs on vertex set V, we define $\mathcal{G}_{n,d,H,H'}^{(r)}$ as the set of all r-graphs $G \in \mathcal{G}_{n,d}^{(r)}$ such that $H \subseteq G$ and $H' \subseteq \overline{G}$. With a slight abuse of notation, we sometimes also treat $\mathcal{G}_{n,d,H,H'}^{(r)}$ as the event that $G_{n,d}^{(r)} \in \mathcal{G}_{n,d,H,H'}^{(r)}$. Given a sequence of events $\{\mathcal{A}_n\}_{n\geq 1}$, we will say that $\mathcal{A}_n$ holds *asymptotically almost surely*, and write a.a.s., if $\lim_{n\to\infty} \mathbb{P}[\mathcal{A}_n] = 1$.

2. Edge-correlation in random regular r-graphs

This section is devoted to estimating the probability that any fixed r-set of vertices forms an edge in a random d-regular r-graph, even if we require certain edges to be (not) present. More precisely, we obtain accurate bounds on $\mathbb{P}[e \in G_{n,d}^{(r)} \mid \mathcal{G}_{n,d,H,H'}^{(r)}]$ for a large range of d as long as H, H' are sparse (see Corollary 2.3). This result is the core ingredient for all the results in Section 3 and it will be used in the proof of our lower bound on the query complexity for testing F-freeness, for a fixed r-graph F, in Section 4.2.

Corollary 2.3 follows immediately from Lemma 2.1 (which provides the upper bound) and Lemma 2.2 (which provides the lower bound). To prove Lemmas 2.1 and 2.2 we develop a hypergraph generalization of the method of edge-switchings, which was introduced for graphs by McKay and Wormald [22]. The switchings we consider in the proof of Lemma 2.1 are similar to those used by Dudek, Frieze, Ruciński and Šileikis [9]. The switchings we use in Lemma 2.2 are more complex however. Moreover, to bound the number of certain 'bad' configurations, the proof of Lemma 2.2 relies on Lemma 2.1. The special case of Lemmas 2.1 and 2.2 when $r = 2$ and H, H' have bounded size (which is much simpler to prove) was obtained by Kim, Sudakov and Vu [18].

Lemma 2.1. *Let $r \geq 2$ be a fixed integer. Assume that $d = o(n^{r-1})$. Suppose $H, H' \subseteq \binom{V}{r}$ are two edge-disjoint r-graphs such that $|H| = o(nd)$ and $\Delta(H') = o(n^{r-1})$. Then, for all $e \in \binom{V}{r} \setminus (H \cup H')$, we have*

$$\mathbb{P}\left[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}\right] \leq (r-1)!\frac{d}{n^{r-1}}\left(1 + O\left(\frac{1}{n} + \frac{d}{n^{r-1}} + \frac{|H|}{nd} + \frac{\Delta(H')}{n^{r-1}}\right)\right).$$

Proof. Write $e = \{v_1, \dots, v_r\}$ and fix this labelling of the vertices in e. Let $e_1 \coloneqq e$ and let $e_2, \dots, e_r \in \binom{V}{r}$ be pairwise disjoint and also disjoint from e_1. Let $f_1, \dots, f_r \in \binom{V}{r}$ be pairwise disjoint and such that $f_i \cap e_1 = \{v_i\}$ for all $i \in [r]$. We say that $\Lambda_e \coloneqq (e_1, \dots, e_r)$ is an *out-switching configuration* and that $\Lambda_{\overline{e}} \coloneqq (f_1, \dots, f_r)$ is an *in-switching configuration*. If, furthermore, $|e_i \cap f_j| = 1$ for all $i, j \in [r]$, we say that Λ_e and $\Lambda_{\overline{e}}$ are *related*.

Given $\Lambda_e = (e_1, \dots, e_r)$, we denote the number of in-switching configurations related to Λ_e by $\lambda_{\text{in}} = \lambda_{\text{in}}(\Lambda_e)$; we claim that

$$\lambda_{\text{in}} = (r!)^{r-1}. \tag{2.1}$$

Indeed, for each $i \in [r] \setminus \{1\}$, write $e_i = \{v^i_1, \dots, v^i_r\}$ and let $\pi_i \colon [r] \to [r]$ be a permutation. For each $i \in [r]$, let $f_i \coloneqq \{v_i, v^2_{\pi_2(i)}, \dots, v^r_{\pi_r(i)}\}$. Then, $\Lambda_{\overline{e}} \coloneqq (f_1, \dots, f_r)$ is related to Λ_e. In this way, each (ordered) $(r-1)$-tuple of permutations $(\pi_2, \dots, \pi_r)$ defines a unique in-switching configuration. On the other hand, each $\Lambda_{\overline{e}} = (f_1, \dots, f_r)$ related to Λ_e gives rise to a different $(r-1)$-tuple of permutations $(\pi_2, \dots, \pi_r)$ by setting, for each $i \in [r] \setminus \{1\}$ and $j \in [r]$, $\pi_i(j)$ to be the subscript of the vertex in $e_i \cap f_j$. There are $(r!)^{r-1}$ such tuples of permutations, so (2.1) follows.

Similarly, given $\Lambda_{\overline{e}} = (f_1, \dots, f_r)$, we denote the number of out-switching configurations related to $\Lambda_{\overline{e}}$ by $\lambda_{\text{out}} = \lambda_{\text{out}}(\Lambda_{\overline{e}})$. We claim that

$$\lambda_{\text{out}} = ((r-1)!)^r. \tag{2.2}$$

Indeed, for each $i \in [r]$, write $f_i = \{v_i, v^i_2, \dots, v^i_r\}$ and let $\sigma_i \colon [r] \setminus \{1\} \to [r] \setminus \{1\}$ be a permutation. For each $i \in [r] \setminus \{1\}$, let $e_i \coloneqq \{v^1_{\sigma_1(i)}, \dots, v^r_{\sigma_r(i)}\}$. Then, $\Lambda_e \coloneqq (e_1, \dots, e_r)$ is related to $\Lambda_{\overline{e}}$. Each r-tuple of permutations $(\sigma_1, \dots, \sigma_r)$ defines a unique Λ_e. On the other hand, each $\Lambda_e = (e_1, \dots, e_r)$ related to $\Lambda_{\overline{e}}$ gives rise to a unique r-tuple of permutations $(\sigma_1, \dots, \sigma_r)$. Thus (2.2) holds.

Let $\Omega_1, \Omega_2 \subseteq \binom{V}{r}$. We define a function ψ on the set of all r-graphs G on V by $\psi(G, \Omega_1, \Omega_2) \coloneqq (G \setminus \Omega_1) \cup \Omega_2$. Now let G be an r-graph on V. Let Λ_e and $\Lambda_{\overline{e}}$ be related out- and in-switching configurations, respectively, such that $\Lambda_e \subseteq G$ and $\Lambda_{\overline{e}} \subseteq \overline{G}$. An *out-switching* on G from Λ_e to $\Lambda_{\overline{e}}$ is obtained by applying the operation $\psi(G, \Lambda_e, \Lambda_{\overline{e}})$ (here Λ_e and $\Lambda_{\overline{e}}$ are viewed as (unordered) sets of edges). We denote this out-switching by the triple $(G, \Lambda_e, \Lambda_{\overline{e}})$. Similarly, if Λ_e and $\Lambda_{\overline{e}}$ are related out- and in-switching configurations, respectively, such that $\Lambda_e \subseteq \overline{G}$ and $\Lambda_{\overline{e}} \subseteq G$, an *in-switching* on G from $\Lambda_{\overline{e}}$ to Λ_e is the operation $\psi(G, \Lambda_{\overline{e}}, \Lambda_e)$, and is denoted by $(G, \Lambda_{\overline{e}}, \Lambda_e)$. Note that $\psi(\psi(G, \Lambda_{\overline{e}}, \Lambda_e), \Lambda_e, \Lambda_{\overline{e}}) = G$, that is, switchings are involutions. Furthermore, both types of switchings preserve the vertex degrees of the r-graph G on which they act.

Let $\mathcal{F}_e \subseteq \mathcal{G}^{(r)}_{n,d,H,H'}$ be the set of all r-graphs $G \in \mathcal{G}^{(r)}_{n,d,H,H'}$ such that $e \in G$, and let $\mathcal{F}_{\overline{e}} \coloneqq \mathcal{G}^{(r)}_{n,d,H,H'} \setminus \mathcal{F}_e$. We define an auxiliary bipartite multigraph Γ with bipartition $(\mathcal{F}_e, \mathcal{F}_{\overline{e}})$ as follows. For each $G \in \mathcal{F}_e$, consider all possible out-switchings on G whose image is in $\mathcal{G}^{(r)}_{n,d,H,H'}$ (that is, all triples $(G, \Lambda_e, \Lambda_{\overline{e}})$ such that $\Lambda_e \subseteq G \setminus H$ and $\Lambda_{\overline{e}} \subseteq \overline{G} \setminus H'$ are related) and add an edge between G and $\psi(G, \Lambda_e, \Lambda_{\overline{e}})$ for each such triple $(G, \Lambda_e, \Lambda_{\overline{e}})$. Similarly, one could consider each $G \in \mathcal{F}_{\overline{e}}$ and every possible in-switching $(G, \Lambda_{\overline{e}}, \Lambda_e)$ on G with $\psi(G, \Lambda_{\overline{e}}, \Lambda_e) \in \mathcal{G}^{(r)}_{n,d,H,H'}$, and add an edge between G and $\psi(G, \Lambda_{\overline{e}}, \Lambda_e)$. Both constructions result in the same multigraph Γ.

We will use switchings to bound $\mathbb{P}[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}] = |\mathcal{F}_e|/|\mathcal{G}^{(r)}_{n,d,H,H'}|$ from above in terms of $\mathbb{P}[e \notin G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}]$. In order to obtain this bound, we will use a double-counting argument involving the edges of Γ.

Assume first that $G \in \mathcal{F}_{\overline{e}}$. Let $S_{\text{in}}(G)$ be the number of in-switchings $(G, \Lambda_{\overline{e}}, \Lambda_e)$ on G, thus $\deg_\Gamma(G) \leq S_{\text{in}}(G)$. We claim that

$$S_{\text{in}}(G) \leq ((r-1)!)^r d^r. \tag{2.3}$$

Clearly, $S_{\text{in}}(G)$ is at most the number of in-switching configurations $\Lambda_{\overline{e}} \subseteq G$ multiplied by λ_{out}. As G is d-regular and $\Lambda_{\overline{e}}$ must contain an edge incident to each $v_i \in e$, there are at most d^r such in-switching configurations. This, together with (2.2), yields (2.3).

Assume now that $G \in \mathcal{F}_e$. Let $\ell := |H|$ and $k' := \Delta(H')$, and let $\eta := \eta(n, d, \ell, k') = \frac{1}{n} + \frac{d}{n^{r-1}} + \frac{\ell}{nd} + \frac{k'}{n^{r-1}}$. Let $S_{\text{out}}(G)$ be the number of possible out-switchings $(G, \Lambda_e, \Lambda_{\overline{e}})$ on G with $\psi(G, \Lambda_e, \Lambda_{\overline{e}}) \in \mathcal{G}^{(r)}_{n,d,H,H'}$; thus, $\deg_\Gamma(G) = S_{\text{out}}(G)$. We claim that

$$S_{\text{out}}(G) \geq ((r-1)!)^{r-1} (nd)^{r-1} \left(1 - O\left(\eta\right)\right). \tag{2.4}$$

In order to have $\psi(G, \Lambda_e, \Lambda_{\overline{e}}) \in \mathcal{G}^{(r)}_{n,d,H,H'}$ we must have $\Lambda_e \subseteq G \setminus H$ and $\Lambda_{\overline{e}} \subseteq \overline{G} \setminus H'$. Let $\lambda_e(G)$ be the number of out-switching configurations Λ_e with $\Lambda_e \subseteq G \setminus H$. We first give a lower bound on $\lambda_e(G)$.

Choose $\Lambda_e = (e_1, \ldots, e_r)$ by sequentially choosing $e_2, \ldots, e_r \in G \setminus H$ in such a way that e_i is disjoint from $e_1, \ldots, e_{i-1}$, for $i \in [r] \setminus \{1\}$. As each vertex is incident to exactly d edges, the number of choices for e_i is at least $(nd/r - \ell - (r-1)rd)$. Thus,

$$\lambda_e(G) \geq \left(\frac{nd}{r} - \ell - (r-1)rd\right)^{r-1}. \tag{2.5}$$

We say that an out-switching configuration $\Lambda_e \subseteq G \setminus H$ is *good* (for G) if there are λ_{in} in-switching configurations $\Lambda_{\overline{e}} \subseteq \overline{G} \setminus H'$ related to Λ_e, and *bad* (for G) otherwise. Let $\lambda_{e,\text{bad}}(G)$ denote the number of bad out-switching configurations $\Lambda_e \subseteq G \setminus H$. We now provide an upper bound on this quantity. An out-switching configuration $\Lambda_e \subseteq G \setminus H$ can only be bad if

(a) one of the edges in some $\Lambda_{\overline{e}}$ related to Λ_e, say g, lies in G, or
(b) one of the edges in some $\Lambda_{\overline{e}}$ related to Λ_e, say h, lies in H'.

In case (a), the edge g has to intersect e, so there are at most rd possible such edges g. Furthermore, $g \setminus e$ must intersect every edge in $\Lambda_e \setminus \{e\}$, so each edge g can make at most $(r-1)!d^{r-1}$ out-switching configurations bad. Thus, there are at most $r!d^r$ out-switching configurations which are bad because of (a). In case (b), the edge h has to intersect e, so there are at most rk' such edges. As above, it follows that there are at most $r!k'd^{r-1}$ out-switching configurations which are bad because of (b). Overall,

$$\lambda_{e,\text{bad}}(G) \leq r!d^r + r!k'd^{r-1}. \tag{2.6}$$

By combining (2.1), (2.5) and (2.6), we have that

$$\begin{aligned} S_{\text{out}}(G) &\geq (r!)^{r-1} \left(\left(\frac{nd}{r} - \ell - (r-1)rd\right)^{r-1} - r!d^r - r!k'd^{r-1} \right) \\ &= ((r-1)!)^{r-1} (nd)^{r-1} \left(1 - O\left(\eta\right)\right). \end{aligned}$$

As (2.3) and (2.4) hold for every $G \in \mathcal{F}_{\overline{e}}$ and $G \in \mathcal{F}_e$, respectively, we can use these expressions to estimate the number $|\Gamma|$ of edges in Γ. We conclude that

$$((r-1)!)^{r-1} (nd)^{r-1} \left(1 - O\left(\eta\right)\right) |\mathcal{F}_e| \leq |\Gamma| \leq ((r-1)!)^r d^r |\mathcal{F}_{\overline{e}}|.$$

Noting that $|\mathcal{F}_{\overline{e}}| \leq |\mathcal{G}^{(r)}_{n,d,H,H'}|$ and dividing this by $|\mathcal{G}^{(r)}_{n,d,H,H'}|$ implies that

$$((r-1)!)^{r-1} (nd)^{r-1} \left(1 - O\left(\eta\right)\right) \cdot \mathbb{P}\left[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}\right] \leq ((r-1)!)^r d^r.$$

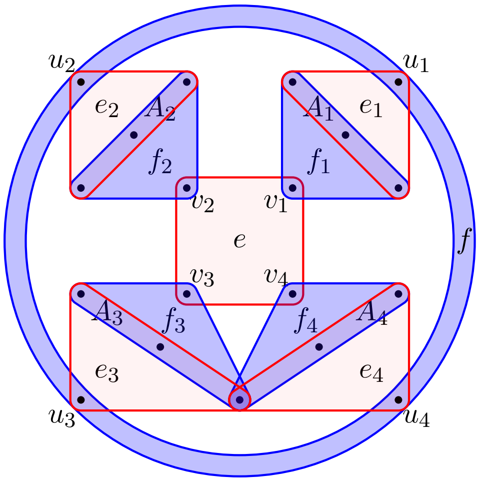

FIGURE 1. Representation of a switching for Lemma 2.2 in the case $r = 4$. Shaded (blue) edges represent an in-switching configuration, while clear (red) ones represent an out-switching configuration.

Thus, we conclude that

$$\mathbb{P}\left[e \in G_{n,d}^{(r)} \mid \mathcal{G}_{n,d,H,H'}^{(r)}\right] \le (r-1)!\frac{d}{n^{r-1}}\left(1 + O\left(\eta\right)\right). \qquad \square$$

Lemma 2.2. *Let $r \ge 2$ be a fixed integer. Suppose that $d = \omega(1)$ and $d = o(n^{r-1})$. Let $H, H' \subseteq \binom{V}{r}$ be two edge-disjoint r-graphs such that $\Delta(H), \Delta(H') = o(d)$. Then, for all $e \in \binom{V}{r} \setminus (H \cup H')$,*

$$\mathbb{P}\left[e \in G_{n,d}^{(r)} \mid \mathcal{G}_{n,d,H,H'}^{(r)}\right] \ge (r-1)!\frac{d}{n^{r-1}}\left(1 - O\left(\frac{1}{n} + \frac{1}{d} + \frac{d}{n^{r-1}} + \frac{\Delta(H)}{d} + \frac{\Delta(H')}{d}\right)\right).$$

Proof. Our strategy is similar as in Lemma 2.1, but we change the definition of a switching configuration. Write $e = \{v_1, \dots, v_r\}$. Let $e_1, \dots, e_r \in \binom{V}{r}$ be such that, for each $i \in [r]$, $v_i \notin e_i$ and there is a vertex $u_i \in e_i \setminus e$ such that $u_i \notin e_j$ for all $j \in [r] \setminus \{i\}$. Let $f_1, \dots, f_r \in \binom{V}{r} \setminus \{e\}$ be distinct such that $v_i \in f_i$, and let $f \in \binom{V}{r}$ be disjoint from $f_1, \dots, f_r$. We say that $\Lambda_e := (e, e_1, \dots, e_r)$ is an *out-switching configuration* and that $\Lambda_{\overline{e}} := (f_1, \dots, f_r, f)$ is an *in-switching configuration*. We say that Λ_e and $\Lambda_{\overline{e}}$ are *related* if, for each $i \in [r]$, one can find a set $A_i \in \binom{V}{r-1}$ such that $e_i \cap f_i = A_i$, and $f = (e_1 \setminus A_1) \cup \ldots \cup (e_r \setminus A_r)$ (note that in this case we must have $A_i = f_i \setminus \{v_i\}$). See Figure 1 for an illustration. Given related out- and in-switching configurations $\Lambda_e = (e, e_1, \dots, e_r)$ and $\Lambda_{\overline{e}} = (f_1, \dots, f_r, f)$, we will always write $A_i := e_i \cap f_i$ and $\{u_i\} := e_i \setminus f_i$ for $i \in [r]$. It is easy to check that this definition of u_i implies that $\{u_i\} = e_i \cap f$ and $u_i \notin e_j$ for all $j \in [r] \setminus \{i\}$. So u_i is indeed as required in the definition of an out-switching configuration.

Given $\Lambda_e = (e, e_1, \dots, e_r)$, we denote the number of in-switching configurations related to Λ_e by $\lambda_{\text{in}}(\Lambda_e)$. We claim that

$$\lambda_{\text{in}}(\Lambda_e) \le r^r. \tag{2.7}$$

Indeed, in order to obtain an in-switching configuration $\Lambda_{\overline{e}} = (f_1, \dots, f_r, f)$ related to Λ_e one has to choose $u_i \in e_i$ for each $i \in [r]$. There are at most r choices for each u_i. Each (admissible) choice of u_i uniquely determines f_i, and thus they determine f.

Similarly, given $\Lambda_{\overline{e}} = (f_1, \dots, f_r, f)$, we denote the number of out-switching configurations related to $\Lambda_{\overline{e}}$ by $\lambda_{\text{out}} = \lambda_{\text{out}}(\Lambda_{\overline{e}})$. We claim that

$$\lambda_{\text{out}} = r!. \tag{2.8}$$

This holds because, for each $i \in [r]$, the edge e_i must contain $f_i \setminus \{v_i\} = A_i$ and one vertex $u_i \in f$, hence each permutation of the labels of the vertices in f results in a different Λ_e.

We define $\psi(G,\Lambda_e,\Lambda_{\overline{e}})$, $\mathcal{F}_e$, $\mathcal{F}_{\overline{e}}$ and Γ as in the proof of Lemma 2.1. As before, neither out- nor in-switchings on an r-graph G change the vertex degrees.

Assume first that $G \in \mathcal{F}_e$. Let $S_{\mathrm{out}}(G)$ be the number of possible out-switchings $(G,\Lambda_e,\Lambda_{\overline{e}})$ on G satisfying that $\psi(G,\Lambda_e,\Lambda_{\overline{e}}) \in \mathcal{G}^{(r)}_{n,d,H,H'}$. Thus $\deg_\Gamma(G) = S_{\mathrm{out}}(G)$. Let $S_{\mathrm{out}} \coloneqq \sum_{G\in\mathcal{F}_e} S_{\mathrm{out}}(G)$ be the number of edges incident to $\mathcal{F}_e$ in Γ. We claim that

$$S_{\mathrm{out}}(G) \leq (nd)^r. \tag{2.9}$$

Indeed, (2.7) implies that $S_{\mathrm{out}}(G)$ is at most the number of out-switching configurations $\Lambda_e \subseteq G$ multiplied by r^r. The number of such out-switching configurations is given by the choice of $(e_1,\ldots,e_r)$, so there are at most $(nd/r)^r$ such configurations. This yields (2.9). As this is true for every G,

$$S_{\mathrm{out}} \leq |\mathcal{F}_e|(nd)^r. \tag{2.10}$$

Consider now any r-graph $G \in \mathcal{F}_{\overline{e}}$. Let $S_{\mathrm{in}}(G)$ be the number of possible in-switchings $(G,\Lambda_{\overline{e}},\Lambda_e)$ on G satisfying that $\psi(G,\Lambda_{\overline{e}},\Lambda_e) \in \mathcal{G}^{(r)}_{n,d,H,H'}$. Thus $\deg_\Gamma(G) = S_{\mathrm{in}}(G)$. Let $S_{\mathrm{in}} \coloneqq \sum_{G\in\mathcal{F}_{\overline{e}}} S_{\mathrm{in}}(G)$ be the number of edges incident to $\mathcal{F}_{\overline{e}}$ in Γ. Let $T_{\mathrm{in}}(G)$ denote the number of in-switching configurations $\Lambda_{\overline{e}} \subseteq G$. As an in-switching configuration is given by r edges, one incident to each of the vertices of e, and one more edge which is disjoint from the previous ones, by choosing each edge in turn and taking into consideration that G is d-regular, we conclude that

$$T_{\mathrm{in}}(G) \leq \frac{nd^{r+1}}{r}. \tag{2.11}$$

For a lower bound on $T_{\mathrm{in}}(G)$, observe that there are exactly d choices for f_1. Then, f_2 can be chosen in at least $d-1$ ways. More generally, there are at least $(d-r)^r$ choices for $(f_1,\ldots,f_r)$. Finally, f must be chosen disjoint from $f_1,\ldots,f_r$, so there are at least $nd/r - r^2d$ choices. Overall,

$$T_{\mathrm{in}}(G) \geq (d-r)^r\left(\frac{nd}{r} - r^2d\right) = \frac{nd^{r+1}}{r}\left(1 - O\left(\frac{1}{d}+\frac{1}{n}\right)\right). \tag{2.12}$$

We say that an in-switching configuration $\Lambda_{\overline{e}} \subseteq G$ is *good* (for G) if there are λ_{out} out-switching configurations $\Lambda_e \subseteq \overline{G}$ related to $\Lambda_{\overline{e}}$ which satisfy $\psi(G,\Lambda_{\overline{e}},\Lambda_e) \in \mathcal{G}^{(r)}_{n,d,H,H'}$. We say that $\Lambda_{\overline{e}}$ is *bad* (for G) otherwise. An in-switching configuration $\Lambda_{\overline{e}} = (f_1,\ldots,f_r,f)$ is bad for G if and only if any of the following occur:

(a) $(f_i \setminus \{v_i\}) \cup \{v\} \in H$ for some $i \in [r]$ and $v \in f$.
(b) $(f_i \setminus \{v_i\}) \cup \{v\} \in H'$ for some $i \in [r]$ and $v \in f$.
(c) $f_i \in H$ for some $i \in [r]$ or $f \in H$.
(d) Neither (a) nor (b) hold, but $(f_i \setminus \{v_i\}) \cup \{v\} \in G$ for some $i \in [r]$ and $v \in f$.

For each $G \in \mathcal{F}_{\overline{e}}$, let $\mathcal{L}(G)$ denote the set of in-switching configurations $\Lambda_{\overline{e}}$ with $\Lambda_{\overline{e}} \subseteq G$. Consider the set $\Omega \coloneqq \{(G,\Lambda_{\overline{e}}) \mid G \in \mathcal{F}_{\overline{e}}, \Lambda_{\overline{e}} \in \mathcal{L}(G)\}$. We say that a pair $(G,\Lambda_{\overline{e}})$ is *bad* if $\Lambda_{\overline{e}}$ is bad for G.

Let $k \coloneqq \Delta(H)$, $k' \coloneqq \Delta(H')$. We first count the number of in-switching configurations in $\mathcal{L}(G)$ which are bad because of (a)–(c). For this, fix an r-graph $G \in \mathcal{F}_{\overline{e}}$. Let $T_{\mathrm{a}}(G)$ be the number of in-switching configurations which are bad because of (a). Fix $e^* \in H$ and $i \in [r]$. To count the number of in-switching configurations $\Lambda_{\overline{e}} = (f_1,\ldots,f_r,f) \in \mathcal{L}(G)$ with $(f_i \setminus \{v_i\}) \cup \{v\} = e^*$ for some $v \in f$, note that there are at most r choices for v, and then at most d choices for f (since $v \in f$). Then we must have $f_i = (e^* \setminus \{v\}) \cup \{v_i\}$. Finally, there are at most d choices for each f_j with $j \in [r] \setminus \{i\}$ (since $v_j \in f_j$). Therefore, $T_{\mathrm{a}}(G) \leq |H| \cdot r \cdot r \cdot d \cdot d^{r-1} \leq rnkd^r$. Let $T_{\mathrm{a}} \coloneqq \sum_{G\in\mathcal{F}_{\overline{e}}} T_{\mathrm{a}}(G)$ be the number of pairs $(G,\Lambda_{\overline{e}})$ which are bad because of (a). Then,

$$T_{\mathrm{a}} \leq |\mathcal{F}_{\overline{e}}|rnkd^r. \tag{2.13}$$

Similarly, for $G \in \mathcal{F}_{\overline{e}}$, let $T_{\mathrm{b}}(G)$ be the number of in-switching configurations which are bad because of (b). As above, one can show that $T_{\mathrm{b}}(G) \leq |H'| \cdot r \cdot r \cdot d \cdot d^{r-1} \leq rnk'd^r$. Let $T_{\mathrm{b}} \coloneqq \sum_{G\in\mathcal{F}_{\overline{e}}} T_{\mathrm{b}}(G)$ be the number of pairs $(G,\Lambda_{\overline{e}})$ which are bad because of (b). Then,

$$T_{\mathrm{b}} \leq |\mathcal{F}_{\overline{e}}|rnk'd^r. \tag{2.14}$$

Next, for $G \in \mathcal{F}_{\overline{e}}$, let $T_{\mathrm{c}}(G)$ be the number of in-switching configurations which are bad because of (c). Given $i \in [r]$, there are at most k choices for $f_i \in H$ (as $v_i \in f_i$), and the remaining edges in the in-switching configuration can be chosen in at most $d^{r-1}nd/r$ ways. Similarly, if $f \in H$, then the remaining edges in the in-switching configuration can be chosen in at most d^r ways. Therefore, $T_{\mathrm{c}}(G) \le r \cdot k \cdot d^{r-1}nd/r + |H| \cdot d^r \le (r+1)nkd^r/r$. Let $T_{\mathrm{c}} \coloneqq \sum_{G\in\mathcal{F}_{\overline{e}}} T_{\mathrm{c}}(G)$ be the number of pairs $(G, \Lambda_{\overline{e}})$ which are bad because of (c). Then,

$$T_{\mathrm{c}} \le |\mathcal{F}_{\overline{e}}| \frac{(r+1)nkd^r}{r}. \tag{2.15}$$

Finally, we count the number of in-switching configurations which are bad because of (d). For this, fix $\Lambda_{\overline{e}} = (f_1, \dots, f_r, f) \in \bigcup_{G\in\mathcal{F}_{\overline{e}}} \mathcal{L}(G)$. Note that this implies that $\Lambda_{\overline{e}} \cap H' = \varnothing$. We now apply Lemma 2.1 with $H \cup \Lambda_{\overline{e}}$ playing the role of H to bound the number of pairs $(G, \Lambda_{\overline{e}})$ that are bad because of (d). We denote this number by T_{d}. Lemma 2.1 implies that, for any $\hat{e} \in \binom{V}{r} \setminus (H \cup H' \cup \Lambda_{\overline{e}})$,

$$\mathbb{P}\left[\hat{e} \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H\cup\Lambda_{\overline{e}},H'}\right] \le 2(r-1)!\frac{d}{n^{r-1}}.$$

In particular, this holds for all r-sets of the form $(f_i \setminus \{v_i\}) \cup \{v\}$ for some $i \in [r]$ and $v \in f$ (as long as they are not in H or H', which is guaranteed for condition (d)). Therefore, a union bound yields an upper bound on the probability that $\Lambda_{\overline{e}}$ is bad for G because of (d). Indeed, let $\mathcal{B}(G, \Lambda_{\overline{e}})$ denote the event that the pair $(G, \Lambda_{\overline{e}})$ is bad because of (d). Then,

$$\mathbb{P}\left[\mathcal{B}(G^{(r)}_{n,d}, \Lambda_{\overline{e}}) \mid \mathcal{G}^{(r)}_{n,d,H\cup\Lambda_{\overline{e}},H'}\right] \le 2r^2(r-1)!\frac{d}{n^{r-1}}.$$

The same approach works for all $\Lambda_{\overline{e}}$. By (2.11) we have that $|\Omega| \le |\mathcal{F}_{\overline{e}}|nd^{r+1}/r$. Hence, for the number T_{d} of pairs that are bad because of (d), it follows that

$$T_{\mathrm{d}} \le |\mathcal{F}_{\overline{e}}| 2r! \frac{d^{r+2}}{n^{r-2}}. \tag{2.16}$$

By (2.12) we have that $|\Omega| \ge |\mathcal{F}_{\overline{e}}|\frac{nd^{r+1}}{r}\left(1 - O\left(\frac{1}{d} + \frac{1}{n}\right)\right)$. Let $\varepsilon \coloneqq \varepsilon(n, d, k, k') = \frac{1}{n} + \frac{1}{d} + \frac{d}{n^{r-1}} + \frac{k}{d} + \frac{k'}{d}$. By (2.8) and (2.13)–(2.16), we conclude that

$$S_{\mathrm{in}} \ge \lambda_{\mathrm{out}}(|\Omega| - T_{\mathrm{a}} - T_{\mathrm{b}} - T_{\mathrm{c}} - T_{\mathrm{d}}) = |\mathcal{F}_{\overline{e}}|(r-1)!nd^{r+1}\left(1 - O\left(\varepsilon\right)\right). \tag{2.17}$$

Combining (2.10) and (2.17), we conclude that

$$|\mathcal{F}_{\overline{e}}|(r-1)!nd^{r+1}\left(1 - O\left(\varepsilon\right)\right) \le S_{\mathrm{in}} = S_{\mathrm{out}} \le |\mathcal{F}_e|(nd)^r.$$

Dividing this by $|\mathcal{G}^{(r)}_{n,d,H,H'}|$ implies that

$$(r-1)!nd^{r+1}\left(1 - O\left(\varepsilon\right)\right)\mathbb{P}\left[e \notin G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}\right] \le (nd)^r\mathbb{P}\left[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}\right].$$

Taking into account that $\mathbb{P}[e \notin G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}] = 1 - \mathbb{P}[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}]$, we conclude that

$$\mathbb{P}\left[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}\right] \ge (r-1)!\frac{d}{n^{r-1}}\left(1 - O\left(\varepsilon\right)\right). \qquad \square$$

Together, Lemma 2.1 and Lemma 2.2 imply the following result.

Corollary 2.3. *Let $r \ge 2$ be a fixed integer. Suppose that $d = \omega(1)$ and $d = o(n^{r-1})$. Let $H, H' \subseteq \binom{V}{r}$ be two edge-disjoint r-graphs such that $\Delta(H), \Delta(H') = o(d)$. Then, for all $e \in \binom{V}{r} \setminus (H \cup H')$ we have*

$$\mathbb{P}\left[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}\right] = (r-1)!\frac{d}{n^{r-1}}\left(1 \pm O\left(\frac{1}{n} + \frac{1}{d} + \frac{d}{n^{r-1}} + \frac{\Delta(H)}{d} + \frac{\Delta(H')}{d}\right)\right).$$

3. Counting subgraphs of random regular r-graphs

In this section we use the results of Section 2 to count the number of copies of certain r-graphs F inside a random d-regular r-graph. In Section 3.1 we consider the case when F is fixed. In particular, we will derive results on the number of edge-disjoint copies of F, which will be used in Section 4.2. In Section 3.2 we apply our results to count the number of copies of sparse but possibly spanning r-graphs such as Hamilton cycles.

3.1. **Counting small subgraphs.** For an r-graph F, let $\mathrm{aut}(F)$ denote the number of automorphisms of F. Let $X_F(G)$ denote the number of (unlabelled) copies of F in an r-graph G. We will often just write X_F whenever G is clear from the context. Observe that X_F is a random variable whenever G is randomly chosen from some set $\mathcal{G}$. We will consider the uniform distribution on the set $\mathcal{G}^{(r)}_{n,d}$. Furthermore, we define

$$p := (r-1)!\frac{d}{n^{r-1}} \qquad \text{and} \qquad \varepsilon_{n,d} := \frac{1}{n} + \frac{1}{d} + \frac{d}{n^{r-1}}.$$

Corollary 3.1. *Let $r \geq 2$ and $t \geq 1$ be fixed integers, and let F be a fixed r-graph. Suppose that $d = \omega(1)$ and $d = o(n^{r-1})$. Then,*

(i) for any set $\mathcal{E} \subseteq \binom{V}{r}$ of size t, $\mathbb{P}[\mathcal{E} \subseteq G^{(r)}_{n,d}] = p^t\,(1 \pm O\,(\varepsilon_{n,d}))$,

(ii) $\mathbb{E}[X_F] = \binom{n}{v_F}\dfrac{v_F!}{\mathrm{aut}(F)}p^{e_F}\,(1 \pm O\,(\varepsilon_{n,d}))$.

Proof. Enumerate the edges in $\mathcal{E}$ as $e_1, \dots, e_t$. (i) follows by applying Corollary 2.3 repeatedly. This in turn implies (ii). □

The next lemma implies that X_F is concentrated around $\mathbb{E}[X_F]$ whenever $\Phi_F = \omega(1)$, where

$$\Phi_F := \min\{\mathbb{E}[X_K] : K \subseteq F, e_K > 0\}.$$

Lemma 3.2. *Let $r \geq 2$ be a fixed integer. Suppose that $d = \omega(1)$ and $d = o(n^{r-1})$. Then, for any fixed r-graph F with $e_F \geq 1$, we have that $\mathrm{Var}[X_F] = O(\varepsilon_{n,d} + \Phi_F^{-1})\mathbb{E}[X_F]^2$.*

The proof follows a straightforward second moment approach (based on Corollary 3.1), so we omit the details. Corollary 3.1, Lemma 3.2 and Chebyshev's inequality imply the following result. In particular, this determines the threshold for the appearance of a copy of a fixed F in $\mathcal{G}^{(r)}_{n,d}$.

Corollary 3.3. *Let $r \geq 2$ be a fixed integer. Suppose that $d = \omega(1)$ and $d = o(n^{r-1})$. Then, for any fixed r-graph F with $\Phi_F = \omega(1)$, we a.a.s. have*

$$X_F = (1 \pm o(1))\binom{n}{v_F}\frac{v_F!}{\mathrm{aut}(F)}p^{e_F}.$$

The next result adresses the problem of counting edge-disjoint copies of an r-graph F in $G^{(r)}_{n,d}$. Its proof builds on an idea of Kreuter [20] for counting vertex-disjoint copies in the binomial random graph model (see also [15, Theorem 3.29]). The approach is to consider an auxiliary graph whose vertex set consists of the copies of F in $G^{(r)}_{n,d}$ and where an independent set corresponds to a set of edge-disjoint copies of F. To estimate the number of vertices and edges of this graph (with a view to apply Turán's theorem), one makes use of Corollary 3.1, Lemma 3.2 and Corollary 3.3. For the sake of completeness, we include the details in Appendix A.

Lemma 3.4. *Let F be a fixed r-graph. Assume that $d = \omega(1)$ and $d = o(n^{r-1})$. Let D_F be the maximum number of edge-disjoint copies of F in an r-graph chosen uniformly from $\mathcal{G}^{(r)}_{n,d}$. If $\Phi_F = \omega(1)$, then $D_F = \Theta(\Phi_F)$ a.a.s.*

3.2. **Counting spanning graphs.** Let $H = \{H_i\}_{i\geq 1}$ be a sequence of r-graphs with $|V(H_i)|$ strictly increasing. When we say that H is a subgraph of G, for some G of order n, we mean that the corresponding H_i of order n is a subgraph of G. This only makes sense when $n = |V(H_i)|$ for some i; we will implicitly assume this is the case, and study the asymptotic behaviour as i tends to infinity.

Our main tool for this section is the following result of Dudek, Frieze, Ruciński and Šileikis [9], which allows to translate results on the $\mathcal{G}^{(r)}(n,p)$ and $\mathcal{G}^{(r)}(n,m)$ random graph models to $\mathcal{G}^{(r)}_{n,d}$. Roughly speaking, their result asserts that $G^{(r)}(n,p) \subseteq G^{(r)}_{n,d}$ a.a.s. provided that p is at least a little smaller than $d/\binom{n-1}{r-1}$. For the graph case, a similar result was proved by Kim and Vu [19] (for a more restricted range of d).

Theorem 3.5 ([9])**.** *For every $r \geq 2$ there exists a constant $C > 0$ such that if for some positive integer $d = d(n)$,*

$$\delta_{n,d} \coloneqq C\left(\left(\frac{d}{n^{r-1}} + \frac{\log n}{d}\right)^{1/3} + \frac{1}{n}\right) < 1, \tag{3.1}$$

then there is a joint distribution of $G^{(r)}(n,p_d)$ and $G^{(r)}_{n,d}$ such that

$$\lim_{n\to\infty} \mathbb{P}\left[G^{(r)}(n,p_d) \subseteq G^{(r)}_{n,d}\right] = 1,$$

where $p_d \coloneqq (1-\delta_{n,d})d/\binom{n-1}{r-1}$. The analogous statement also holds with $G^{(r)}(n,p_d)$ replaced by $G^{(r)}(n,m_d)$ for $m_d \coloneqq (1-\delta_{n,d})nd/r$.

In order to be able to apply Theorem 3.5, from now on we always assume that $d = o(n^{r-1})$ and $d = \omega(\log n)$. We now combine Theorem 3.5 with our results from Section 2 to obtain a general result relating subgraph counts in $\mathcal{G}^{(r)}_{n,d}$ to those in $G^{(r)}(n,p_d)$ and $G^{(r)}(n,m_d)$.

Theorem 3.6. *Let $r \geq 2$ be a fixed integer and V be a set of n vertices. Assume that $d = \omega(\log n)$ and $d = o(n^{r-1})$. Let H be an r-graph on V with $\Delta(H) = O(1)$. Suppose that $\eta = \eta(n) = o(1)$ is such that*

$$\varepsilon_{n,d} = o(\eta), \qquad \delta_{n,d} = o(\eta), \qquad \eta = \omega(1/n), \tag{3.2}$$

and $X_H(G^{(r)}(n,p_d)) = (1\pm\eta)^{|H|}\mathbb{E}[X_H(G^{(r)}(n,p_d))]$ a.a.s. Then a.a.s.

$$X_H(G^{(r)}_{n,d}) = (1\pm 3\eta)^{|H|}\mathbb{E}[X_H(G^{(r)}(n,p_d))] \tag{3.3}$$

Similarly, if (3.2) holds and $X_H(G^{(r)}(n,m_d)) = (1\pm\eta)^{|H|}\mathbb{E}[X_H(G^{(r)}(n,m_d))]$ a.a.s., then a.a.s.

$$X_H(G^{(r)}_{n,d}) = (1\pm 3\eta)^{|H|}\mathbb{E}[X_H(G^{(r)}(n,m_d))]. \tag{3.4}$$

Proof. Observe first that, by Corollary 2.3, for any fixed copy H' of H we have

$$\mathbb{P}\left[H' \subseteq G^{(r)}_{n,d}\right] = ((1\pm O(\varepsilon_{n,d}))(r-1)!d/n^{r-1})^{|H|}. \tag{3.5}$$

Therefore,

$$\frac{\mathbb{E}[X_H(G^{(r)}_{n,d})]}{\mathbb{E}[X_H(G^{(r)}(n,p_d))]} = (1\pm O(\varepsilon_{n,d}+\delta_{n,d}))^{|H|} \leq (1+\eta)^{|H|}. \tag{3.6}$$

By using Markov's inequality and (3.6) we conclude that

$$\begin{aligned}&\mathbb{P}\left[X_H(G^{(r)}_{n,d}) \geq (1+3\eta)^{|H|}\mathbb{E}\left[X_H(G^{(r)}(n,p_d))\right]\right]\\ \leq\,&\mathbb{P}\left[X_H(G^{(r)}_{n,d}) \geq (1+\eta)^{|H|}\mathbb{E}\left[X_H(G^{(r)}_{n,d})\right]\right] \leq 1/(1+\eta)^{|H|} = o(1).\end{aligned} \tag{3.7}$$

Note that, as $G^{(r)}(n,p_d) \subseteq G^{(r)}_{n,d}$ a.a.s. by Theorem 3.5, then $X_H(G^{(r)}_{n,d}) \geq X_H(G^{(r)}(n,p_d))$ a.a.s. Thus, by assumption,

$$\mathbb{P}\left[X_H(G^{(r)}_{n,d}) \leq (1-\eta)^{|H|}\mathbb{E}\left[X_H(G^{(r)}(n,p_d))\right]\right]$$

$$\leq \mathbb{P}\left[X_H(G^{(r)}(n,p_d)) \leq (1-\eta)^{|H|}\mathbb{E}\left[X_H(G^{(r)}(n,p_d))\right]\right] + o(1) = o(1). \tag{3.8}$$

Combining equations (3.7) and (3.8) yields (3.3).

Finally, one can prove (3.4) in a very similar way. □

We may apply Theorem 3.6 to obtain estimates on the number of copies of certain spanning subgraphs. This requires concentration results in the $\mathcal{G}^{(r)}(n,p)$ model or the $\mathcal{G}^{(r)}(n,m)$ model in order to obtain results for $\mathcal{G}^{(r)}_{n,d}$.

We start with the following result of Glebov and Krivelevich [10] on counting Hamilton cycles in $\mathcal{G}(n,p)$. For a more restricted range of densities, Janson [13] proved more precise results in $\mathcal{G}(n,m)$.

Theorem 3.7 ([10])**.** *Let V be a set of n vertices. Let H be a Hamilton cycle on V. If $p \geq \frac{\ln n + \ln\ln n + \omega(1)}{n}$, then a.a.s.*

$$X_H(G(n,p)) = (1 \pm o(1))^n n! p^n.$$

Together with Theorem 3.6 this implies the following result.

Corollary 3.8. *Let V be a set of n vertices. Let H be a Hamilton cycle on V. Assume $d = \omega(\log n)$ and $d = o(n)$, then a.a.s.*

$$X_H(G_{n,d}) = (1 \pm o(1))^n n! \left(\frac{d}{n-1}\right)^n.$$

Corollary 3.8 improves a previous result of Krivelevich [21] by increasing the range of d in which the number of Hamilton cycles is estimated from $d = \omega(e^{(\log n)^{1/2}})$ to $d = \omega(\log n)$. Note that, on the other hand, the results of Krivelevich [21] also cover pseudo-random d-regular graphs.

A very general result due to Riordan [24] allows us to count the number of copies of H as a spanning subgraph of $G(n,m)$ for a large class of graphs H. We only state a special case of this result here. Let $\alpha_1(H) := |H|/\binom{n}{2}$, $\alpha_2(H) = X_{P_2}(H)/(3\binom{n}{3})$ (where P_2 stands for a path of length 2), $e_H(k) := \max\{|F| : F \subseteq H, |V(F)| = k\}$, $\gamma_1(H) := \max_{3\leq k\leq n}\{e_H(k)/(k-2)\}$ and $\gamma_2(H) := \max_{5\leq k\leq n}\{(e_H(k)-4)/(k-4)\}$.

Theorem 3.9 ([24])**.** *Let V be a set of n vertices. Let $p = \omega(\max\{1/n^{1/2}, 1/n^{1/\gamma_1}, 1/n^{1/\gamma_2}\})$, $p = o(1/\log n)$, $m := p\binom{n}{2}$, and let H be a triangle-free spanning graph on V with $|H| \geq n$, $\Delta(H) = O(1)$ and $|\alpha_2(H) - \alpha_1(H)^2| = \Omega(1/n^2)$. Then, $X_H(G(n,m))$ follows a normal distribution such that* $\mathrm{Var}[X_H(G(n,m))]/\mathbb{E}[X_H(G(n,m))]^2 = o(1)$.

Together with Theorem 3.6, we can deduce the following.

Corollary 3.10. *Let V be a set of n vertices. Assume that $d = \omega(\max\{n^{1/2}, n^{1-1/\gamma_1}, n^{1-1/\gamma_2}\})$, $d = o(n/\log n)$, and let H be a triangle-free spanning graph on V with $|H| \geq n$, $\Delta(H) = O(1)$ and $|\alpha_2(H) - \alpha_1(H)^2| = \Omega(1/n^2)$. Then, $X_H(G_{n,d}) = (1 \pm o(1))^n \mathbb{E}[X_H(G(n,m_d))]$ a.a.s., where $m_d = (1-o(1))dn/2$ is defined as in Theorem 3.6.*

As a particular case of this, we can estimate the number of spanning square lattices in a random d-regular graph. A square lattice L_k is defined by setting $V(L_k) = [k] \times [k]$ and $L_k = \{\{(x,y),(u,v)\} : u,v,x,y \in [k], \|(x,y)-(u,v)\| = 1\}$.

Corollary 3.11. *Let $n = k^2$. Let $d = \omega(1)$, $d = o(n/\log n)$ and $p := d/(n-1)$.*

(i) If $d = o(n^{1/2})$, then $\mathbb{P}[X_{L_k}(G_{n,d}) > 0] = o(1)$.
(ii) If $d = \omega(n^{1/2})$, then, $X_{L_k}(G_{n,d}) = (1 \pm o(1))^n n! p^{|L_k|}$ a.a.s.

In particular, as $|L_k| = 2n \pm O(n^{1/2})$, this determines the threshold for the existence of a spanning square lattice L_k in $G_{n,d}$. Corollary 3.11(i) follows from Corollary 2.3 and Markov's inequality, while Corollary 3.11(ii) follows from Corollary 3.10.

Much less is known for r-graphs when $r \geq 3$. For Hamilton cycles, we can apply the following result of Dudek and Frieze [7] on ℓ-overlapping Hamilton cycles.

Theorem 3.12 ([7], Section 2)**.** *Let $r > \ell \geq 2$ and assume that $(r-\ell) \mid n$. Assume $p = \omega(1/n^{r-\ell})$. Then, a.a.s.*

$$X_{C_n^\ell}(G^{(r)}(n,p)) = (1 \pm o(1))^n n! p^{n/(r-\ell)}.$$

Together with Theorem 3.6, Corollary 2.3 and Markov's inequality, this implies the following result.

Corollary 3.13. *Let $r > \ell \geq 2$ and assume that $(r-\ell) \mid n$. Let $p := d/\binom{n-1}{r-1}$.*

(i) If $d = o(n^{\ell-1})$ then $\mathbb{P}[X_{C_n^\ell}(G_{n,d}^{(r)}) > 0] = o(1)$.
(ii) If $d = \omega(n^{\ell-1})$ and $d = o(n^{r-1})$, then a.a.s. $X_{C_n^\ell}(G_{n,d}^{(r)}) = (1 \pm o(1))^n n! p^{n/(r-\ell)}$.

In particular, this determines the threshold for the existence of C_n^ℓ in $\mathcal{G}_{n,d}^{(r)}$ for $\ell \in [r-1] \setminus \{1\}$, solving a conjecture of Dudek, Frieze, Ruciński and Šileikis [9]. We note that Altman, Greenhill, Isaev and Ramadurai [3] recently determined the threshold for the appearance of loose Hamilton cycles in random regular r-graphs. Their results imply that for every $r \geq 3$ there exists a value d_0 (which is calculated explicitly in [3]) such that if $d \geq d_0$, then $G_{n,d}^{(r)}$ a.a.s. has a loose Hamilton cycle. For $\ell \in [r-1] \setminus \{1\}$, they also proved that $\mathbb{P}[X_{C_n^\ell}(G_{n,d}^{(r)}) > 0] = o(1)$ holds under the much stronger condition that $d = o(n)$ if $r = 4$ and $d = o(n^{1/2})$ if $r = 3$ (but to deduce Corollary 3.13(i) we do rely on their result when d is constant; we rely on Corollary 2.3 when $d = \omega(1)$).

4. Testing F-freeness in general r-graphs

We now give lower and upper bounds on the query complexity of testing F-freeness in the general r-graphs model, where F is a fixed r-graph. In the special case when F is a triangle, these (and other) bounds were already obtained by Alon, Kaufman, Krivelevich and Ron [2]. Our proofs develop ideas from their paper.

In Section 4.1, we observe a simple lower bound for the query complexity of any F-freeness tester. In Section 4.2, we use our results from Sections 2 and 3 to improve this bound for input r-graphs whose density is larger than a certain threshold. The bound that we obtain, however, only holds for one-sided error testers; extending it to two-sided error testers, as Alon, Kaufman, Krivelevich and Ron [2] do with their triangle-freeness tester, would be an interesting problem. Finally, Section 4.3 is devoted to upper bounds on the query complexity.

4.1. A lower bound for sparser r-graphs.

In this section we provide a lower bound on the query complexity of testing F-freeness which is stronger than that in Section 4.2 when the r-graphs that are being tested are sparser (the range of the average degree d for which this holds depends on the particular r-graph F). For a fixed r-graph F, let $ex(n, F)$ denote the maximum number of edges of an F-free r-graph G on n vertices.

Proposition 4.1. *Let $r \geq 2$ and F be an r-graph. Let $c, a > 0$ be fixed constants such that $c \cdot n^a \leq ex(n, F)$ and suppose that $d = \Omega(1)$ and $d = o(n^{a-1})$. Then, any F-freeness tester in r-graphs must perform $\Omega\left(n^{1-1/a} d^{-1/a}\right)$ queries, when restricted to input r-graphs on n vertices of average degree $d \pm o(d)$.*

Proof. It suffices to construct two families of r-graphs on n vertices $\mathcal{F}_1$ and $\mathcal{F}_2$ such that the following hold:

(i) All r-graphs in $\mathcal{F}_1$ are F-free.
(ii) All r-graphs in $\mathcal{F}_2$ are $\Theta(1)$-far from F-free.
(iii) All r-graphs in both families have average degree $d \pm o(d)$.
(iv) Consider an r-graph G chosen from $\mathcal{F}_1 \cup \mathcal{F}_2$ according to the following rule. First choose $i \in [2]$ uniformly at random. Then choose $G \in \mathcal{F}_i$ uniformly at random. Then any algorithm that determines with probability at least $2/3$ whether $G \in \mathcal{F}_1$ or $G \in \mathcal{F}_2$ must perform at least $\Omega(n^{1-1/a} d^{-1/a})$ queries.

Let H be an F-free r-graph on $(nd/(cr))^{1/a}$ vertices with nd/r edges. Let $\mathcal{F}_1$ be the family of all labelled r-graphs consisting of the disjoint union of H on $(nd/(cr))^{1/a}$ vertices and $n-(nd/(cr))^{1/a}$ isolated vertices. Let $\mathcal{F}_2$ be the family of all labelled r-graphs consisting of the disjoint union of a complete r-graph on a set of $(nd(r-1)!)^{1/r}$ vertices and $n-(nd(r-1)!)^{1/r}$ isolated vertices.

A simple computation shows that all r-graphs in both families have average degree $d \pm o(d)$. All r-graphs in $\mathcal{F}_1$ are F-free by definition. Since the number of distinct $K_{v_F}^{(r)}$ in $K_k^{(r)}$ is $\Theta(k^{v_F})$, it is easy to check that all r-graphs in $\mathcal{F}_2$ are $\Theta(1)$-far from being $K_{v_F}^{(r)}$-free, and hence $\Theta(1)$-far from being F-free. Thus, conditions (i), (ii) and (iii) hold.

Now consider any algorithm ALG that, given an r-graph G chosen at random from either $\mathcal{F}_1$ or $\mathcal{F}_2$ as in (iv), tries to determine with probability at least $2/3$ whether $G \in \mathcal{F}_1$ or $G \in \mathcal{F}_2$. If $G \in \mathcal{F}_1$, then the probability of finding a vertex with positive degree with any given query is $O(n^{1/a-1}d^{1/a})$. Similarly, if $G \in \mathcal{F}_2$, the probability of finding a vertex with positive degree with any given query is $O(n^{1/a-1}d^{1/a})$. Hence, if the number of queries is $Q = o(n^{1-1/a}d^{-1/a})$, by the union bound, one has that the probability of finding any such vertex is $o(1)$. So a.a.s. ALG only finds a set of isolated vertices, of size $O(Q)$, after the first Q queries.

Note that the number of isolated vertices in any graph in both $\mathcal{F}_1$ and $\mathcal{F}_2$ is $(1-o(1))n$. Thus we conclude that for $i \in [2]$, $\mathbb{P}[G \in \mathcal{F}_i \mid \text{ALG finds only isolated vertices}] = 1/2 \pm o(1)$. Therefore, the algorithm cannot distinguish between r-graphs in $\mathcal{F}_1$ and $\mathcal{F}_2$ with sufficiently high probability with only Q queries. □

If F is a non-r-partite r-graph, then $ex(n,F) = \Theta(n^r)$. Using this, Proposition 4.1 asserts that, for any non-r-partite graph F, testing F-freeness needs $\Omega((n^{r-1}/d)^{1/r})$ queries. This implies that for all non-r-partite r-graphs F there is no constant time F-freeness tester for input r-graphs G on n vertices with $d = o(n^{r-1})$ and $d = \Omega(1)$, as opposed to the constant time algorithms existing for dense r-graphs.

In more generality, Proposition 4.1 shows that there can be no F-freeness tester that requires a constant number of queries whenever the input r-graph G has average degree $d = o(ex(n,F)/n)$ and $d = \Omega(1)$. On the other hand, if the number of edges of the input r-graph is larger than the Turán number of F, then there is a trivial F-freeness tester: an algorithm that rejects every input, which has constant query complexity. As another example, it is well-known that $ex(n, C_4) = \Theta(n^{3/2})$. With this, we conclude that any algorithm testing C_4-freeness in graphs with average degree d, when $d = o(n^{1/2})$ and $d = \Omega(1)$, must perform at least $\Omega((n/d^2)^{1/3})$ queries.

The asymptotic growth of $ex(n,F)$ is not known for every F. Let $\beta(F) := \frac{v_F - r}{e_F - 1}$. An easy probabilistic argument shows that $ex(n,F) = \Omega\left(n^{r-\beta(F)}\right)$. This bound is superlinear in n as long as $\beta(F) < r-1$, which holds for every connected F that is not a weak tree. Using this bound on $ex(n,F)$, Proposition 4.1 asserts that for any connected r-graph F other than a weak tree the number of queries performed by any F-freeness tester on input r-graphs on at least $\Omega(n)$ and at most $o\left(n^{r-\beta(F)}\right)$ edges is $\Omega((n^{r-1-\beta(F)}/d)^{1/(r-\beta(F))})$.

4.2. **A lower bound for denser r-graphs.** The lower bound on the query complexity of F-freeness testers we present here improves the bound in Section 4.1 when d is large enough and either $r = 2$ or $r \geq 3$ and F is non-r-partite. However, this approach only works for one-sided error algorithms. The answer given by one-sided error algorithms must always be correct when the input r-graph is F-free, so any algorithm we consider must accept if it cannot rule out the possibility of G being F-free. Thus, in order to prove that the query complexity is at least Q, say, (roughly speaking) the idea is to find a family $\mathcal{F}$ of r-graphs which are far from being F-free and such that any algorithm, given an r-graph chosen uniformly at random from $\mathcal{F}$ as an input, must perform at least Q queries in order to find a copy of F (with high probability). As we will prove, the family $\mathcal{F}_{n,d(n)}^{(r)}$ described below has the required properties.

Let F be an r-graph other than a weak forest. Recall that $X_F(G)$ denotes the number of copies of F in G. Let $\Phi_{F,n,d} := \min\{\mathbb{E}[X_K(G^{(r)}_{n,d})] : K \subseteq F, e_K > 0\}$. Taking K to be an edge shows that $\Phi_{F,n,d'} \leq nd'/r$ for any d'.

Assume now that $d(n) = \omega(1)$ and $d(n) = o(n^{r-1})$. Choose $\eta(n)$ such that $\eta(n) = o(1)$. Let

$$n_* := \max\{n_0 \leq n : \Phi_{F,n_0,d(n)} \geq (1-\eta(n))n_0 d(n)/r\}.$$

We claim that n_* always exists. Indeed, let $n_1 \leq n$ be such that every r-graph G^* on n_1 vertices with average degree $d(n)$ has at least $(1-\eta(n)^2)\binom{n_1}{r}$ edges, and such that there exists such an r-graph G^*. Thus $n_1 = (1 \pm o(1))((r-1)!d(n))^{1/(r-1)}$ and, since $d = \omega(1)$, we have $n_1 = \omega(1)$. Consider any G^* as above. Given any $K \subseteq F$, the number of copies of K in G^* is given by $(1 \pm \eta(n))\binom{n_1}{v_K}\frac{v_K!}{\mathrm{aut}(K)}$ which, among all $K \subseteq F$ with $e_K \geq 1$, achieves its minimum for a single edge. Hence $\Phi_{F,n_1,d(n)} \geq (1-\eta(n))n_1 d(n)/r$ and $n_* \geq n_1$ must exist.

Lemma 4.2. *Let F be a fixed r-graph other than a weak forest and let $d(n)$ be such that $d(n) = \omega(1)$ and $d(n) = o(n^{r-1})$. Then $d(n) = o(n_*^{r-1})$.*

Proof. For any fixed r-graph K with $e_K > 1$, let $d^*(n,K)$ be the smallest integer such that $\mathbb{E}[X_K(G^{(r)}_{n,d^*(n,K)})] \geq nd^*(n,K)/r$. Let $d^*_F(n) := \max_{K \subseteq F: e_K > 1}\{d^*(n,K)\}$. We claim that $d^*_F(n) = o(n^{r-1})$. To prove the claim, note that by Corollary 3.1(ii), for any $K \subseteq F$ with $e_K > 1$ we have that

$$d^*(n,K) = \Theta\left(\left\lceil n^{\frac{(r-1)e_K - v_K + 1}{e_K - 1}}\right\rceil\right).$$

In particular, $d^*(n,K) = o(n^{r-1})$ as $v_K > r$. The claim follows by taking the maximum over all K.

Returning to the main proof, we now consider two cases. If $n_* = n$, then $d(n) = o(n_*^{r-1})$ by assumption. So suppose $n_* < n$. Let $n_+ > n_*$ be the smallest integer such that there exists a $d(n)$-regular r-graph on n_+ vertices. So $n_+ \leq 2n_*$ and $n_+ \leq n$. Then $\Phi_{F,n_+,d(n)} < (1-\eta(n))n_+ d(n)/r$. This in turn implies that $d(n) < d^*_F(n_+)$. But $d^*_F(n_+) = o(n_+^{r-1})$ by the above claim, and thus $d(n) = o(n_*^{r-1})$. □

Let $t := \lfloor n/n_* \rfloor$. Define $\mathcal{F}^{(r)}_{n,d(n)}$ by considering all possible partitions of V into sets $V_1, \ldots, V_t$ of size

$$\tilde{n} := n/t \tag{4.1}$$

and, for each of them, all possible labelled $d(n)$-regular r-graphs G_i on each of the sets V_i. By Lemma 4.2, $d(n) = o(\tilde{n}^{r-1})$ and so the G_i are well-defined. With these definitions, all the results in Sections 2 and 3.1 can be applied to each family $\mathcal{F}^{(r)}_{n,d(n)}[V_i]$ consisting of the subgraphs of each $G \in \mathcal{F}^{(r)}_{n,d(n)}$ restricted to vertex set V_i, and hence to $\mathcal{F}^{(r)}_{n,d(n)}$ by summing over all $i \in [t]$.

Lemma 4.3. *Let F be a fixed, connected r-graph other than a weak tree and let $d(n)$ be such that $d(n) = \omega(1)$ and $d(n) = o(n^{r-1})$. Let $\tilde{n}$ and $\mathcal{F}^{(r)}_{n,d(n)}$ be as defined above. Then an r-graph $G \in \mathcal{F}^{(r)}_{n,d(n)}$ chosen uniformly at random contains $\Theta(nd(n))$ edge-disjoint copies of F a.a.s.*

Note that this immediately implies that a.a.s. a graph $G \in \mathcal{F}^{(r)}_{n,d(n)}$ chosen uniformly at random is ε-far from being F-free for some fixed $\varepsilon > 0$.

Proof. Let D_L and $D_L^{n'}$ denote the maximum number of edge-disjoint copies of L in an r-graph chosen uniformly at random from $\mathcal{F}^{(r)}_{n,d(n)}$ and in an r-graph chosen uniformly at random from $\mathcal{G}^{(r)}_{n',d(n)}$ for any admissible value of n', respectively. Recall that $\mathcal{F}^{(r)}_{n,d(n)}$ is obtained by partitioning the set of vertices into sets $V_1, \ldots, V_t$ of size $\tilde{n}$, where $t = n/\tilde{n}$, and considering $d(n)$-regular r-graphs G_i on each of the V_i, where each G_i is chosen uniformly at random from $\mathcal{G}^{(r)}_{\tilde{n},d(n)}$. Note that $n_* \leq \tilde{n} \leq 2n_*$. Together with the definition of n_* this implies that the value of $\Phi_{F,\tilde{n},d(n)}$ in each G_i satisfies $\Phi_{F,\tilde{n},d(n)} = \Theta(\tilde{n}d(n))$. Then by Lemma 3.4 the maximum number of edge-disjoint copies of F in each of the G_i is $D_F^{\tilde{n}} = \Theta(\tilde{n}d(n))$ a.a.s. Summing over all $i \in [t]$, we conclude that $D_F = \Theta(nd(n))$ a.a.s. $\square$

We now provide a proof for the lower bound on the complexity of any algorithm that tests F-freeness in r-graphs (for graphs and non-r-partite r-graphs F with $r \geq 3$). In order to do so, consider any algorithm ALG that performs Q queries given an input r-graph G on n vertices with average degree $d(n) \pm o(d(n))$. ALG will retrieve some information about G from the queries it performs, namely a set of r-sets $E_1 \subseteq E(G)$, a set of r-sets $E_2 \subseteq E(\overline{G})$ and (potentially) some vertex degrees of G, i.e. a set $\mathcal{D} \subseteq \{(v, d_v) : v \in V(G), d_v = \deg_G(v)\}$. We call the information retrieved by ALG after Q queries the *history* of G seen by ALG, and denote it as $(E_1, E_2, \mathcal{D})$. We say that the history of G seen by ALG is *simple* if E_1 forms a weak forest and for all $(v, d_v) \in \mathcal{D}$ we have that $d_v = O(d(n))$.

For $d(n) = \omega(1)$ it is easy to see that any sufficiently large r-graph G with average degree $d(n) \pm o(d(n))$ contains any fixed weak forest F. Thus we assume that F is not a weak forest, that is, F contains at least two edges whose intersection has size at least 2 or a loose cycle. In order to prove our bound we first show the following result.

Lemma 4.4. *Let F be an r-graph which is not a weak forest and define $\tilde{n}$ as in (4.1). Assume that $d(n) = \omega(1)$ and $d(n) = o(n^{r-1})$. Suppose ALG is an algorithm whose input is an r-graph $G \in \mathcal{F}^{(r)}_{n,d(n)}$ and which for at least $1/3$ of the r-graphs $G \in \mathcal{F}^{(r)}_{n,d(n)}$ sees with probability at least $1/3$ a history which is not simple. Then, ALG must perform $\Omega(\min\{d(n), \tilde{n}^{r-1}/d(n), \tilde{n}^{1/2}\})$ queries.*

To prove Lemma 4.4, we will show that an algorithm that performs only $o(\min\{d(n), \tilde{n}^{r-1}/d(n), \tilde{n}^{1/2}\})$ queries will usually not succeed with the desired probability. For this, we consider a suitable randomised process P that answers the queries of the algorithm.

Proof. Suppose $Q = o(\min\{d(n), \tilde{n}^{r-1}/d(n), \tilde{n}^{1/2}\})$. Let ALG be a (possibly adaptive and randomised) algorithm that performs Q queries and searches for some history of the input $G \in \mathcal{F}^{(r)}_{n,d(n)}$ which is not simple. Since we have, for any history $(E_1, E_2, \mathcal{D})$ seen by any algorithm on any $G \in \mathcal{F}^{(r)}_{n,d(n)}$, that any pair $(v, d_v) \in \mathcal{D}$ satisfies $d_v = d(n)$, the only condition for $(E_1, E_2, \mathcal{D})$ being simple is that E_1 forms a weak forest. Therefore, ALG tries to find a set $E_1 \subseteq E(G)$ which forms an r-graph which is not a weak forest.

The queries performed by ALG are answered by a randomised process P. We denote the queries asked by ALG as $q_1, q_2, \ldots$, and the answers given by P as $a_1, a_2, \ldots$. After t queries, we refer to all the previous queries from ALG and all the answers provided by P as the *query-answer history*. The process P uses the query-answer history to build what we call the *history book*, defined for each $t \geq 0$ and denoted by $H^t = (V^t, E^t_*, \bar{E}^t)$, where $V^t \subseteq V$, $\bar{E}^t \subseteq \binom{V}{r}$ and E^t_* is a set of labelled r-sets in $\binom{V}{r}$ such that each r-set $e \in E^t_*$ has r labels $i_1, \ldots, i_r$, one for each vertex in e. We denote by E^t the set of edges consisting of the r-sets in E^t_*. Given an edge $e = \{v_1, \ldots, v_r\} \in E^t$, its labels in E^t_* indicate, for each vertex $v_j \in e$, that e is the i_j-th edge in the incidence list of v_j.

Initially, V^0, E^0_* and $\bar{E}^0$ are set to be empty. Note that we may always assume that in the t-th step ALG never asks a query whose answer can be deduced from the history book H^{t-1}. Given two r-graphs H and H', define $\mathcal{F}^{(r)}_{n,d(n),H,H'} := \{G \in \mathcal{F}^{(r)}_{n,d(n)} : H \subseteq G, H' \subseteq \overline{G}\}$. We abuse notation to write $\mathcal{F}^{(r)}_{n,d(n),H,H'}$ as the event that $G \in \mathcal{F}^{(r)}_{n,d(n),H,H'}$. The process P answers ALG's queries and builds the history book as follows.

If $q_t = \{v_1, \ldots, v_r\}$ is a vertex-set query, then P answers "yes" with probability $\mathbb{P}[q_t \in G \mid \mathcal{F}^{(r)}_{n,d(n),E^{t-1},\bar{E}^{t-1}}]$, and "no" otherwise. If the answer is "yes", then the history book is updated by setting $V^t := V^{t-1} \cup q_t$, $\bar{E}^t := \bar{E}^{t-1}$ and adding q_t together with its labels $j_1, \ldots, j_r$ to E^{t-1}_* to obtain E^t_*, where the labels $j_1, \ldots, j_r$ are chosen uniformly at random among all possible labellings which are consistent with the labels in E^{t-1}_*. In this case, the labels are also given to

ALG as part of the answer. Otherwise, the history book is updated by setting $V^t := V^{t-1} \cup q_t$, $E^t_* := E^{t-1}_*$ and $\bar{E}^t := \bar{E}^{t-1} \cup \{q_t\}$.

If $q_t = (u, i)$ is a neighbour query, P replies with $a_t := (v_1, \dots, v_{r-1}, j_1, \dots, j_{r-1})$, where a_t is chosen such that $e := \{u, v_1, \dots, v_{r-1}\}$ is an edge and for each $k \in [r-1]$, the number j_k is the position of e in the incidence list of v_k (we may assume that, as the r-graphs are $d(n)$-regular, the algorithm never queries $i > d(n)$). To determine its answer a_t, the process P will first choose an r-graph $G_t \in \mathcal{F}^{(r)}_{n,d(n),E^{t-1},\bar{E}^{t-1}}$ uniformly at random, and then choose a labelling of the edges of G_t which is consistent with H^{t-1} uniformly at random. The edge $e = \{u, v_1, \dots, v_{r-1}\}$ will be the i-th edge at u in G_t (in the chosen labelling) and j_s will be the label of e in the incidence list of v_s (for each $s \in [r-1]$). Note that the random labelling ensures that, given G_t, e is chosen uniformly at random from a set of edges of size at least $d(n) - t$ (namely from the set of those edges of G_t incident to u which have no label at u in H^{t-1}). This in turn means that for all $f \in G_t$ with $u \in f$, the probability that the label of u in f is i is at most $1/(d(n) - t)$. The history book is updated by setting $V^t := V^{t-1} \cup e$, $\bar{E}^t := \bar{E}^{t-1}$ and adding e together with the labels $i, j_1, \dots, j_{r-1}$ to E^{t-1}_* to obtain E^t_*.

Once P has answered all Q queries, it chooses an r-graph $G^* \in \mathcal{F}^{(r)}_{n,d(n),E^Q,\bar{E}^Q}$ uniformly at random. Note that P gives extra information to the algorithm in the form of labels that have not been queried. This extra information can only benefit the algorithm, so any lower bound on the query complexity in this setting will also be a lower bound in the general setting.

We claim that G^* is chosen uniformly at random in $\mathcal{F}^{(r)}_{n,d(n)}$. Indeed, let $s_0 := |\mathcal{F}^{(r)}_{n,d(n)}|$. Given a query-answer history $\mathcal{H} = (q_1, a_1, \dots, q_Q, a_Q)$, for each $t \in [Q] \cup \{0\}$, write $\mathcal{F}^t(\mathcal{H})$ for the set of all those graphs $G \in \mathcal{F}^{(r)}_{n,d(n)}$ which are "consistent" with $\mathcal{H}$ for at least the first t steps, i.e. all $G \in \mathcal{F}^{(r)}_{n,d(n),E^t,\bar{E}^t}$, where $(V^t, E^t_*, \bar{E}^t)$ is the history book associated with the first t steps of $\mathcal{H}$. Thus $\mathcal{F}^t$ is a random variable and $\mathcal{F}^0(\mathcal{H}) = \mathcal{F}^{(r)}_{n,d(n)}$ for each $\mathcal{H}$. Now consider any sequence $\mathcal{S} = (s_1, \dots, s_Q)$ such that $s_t \in \mathbb{N}$ and $\mathbb{P}[|\mathcal{F}^t| = s_t] > 0$ for all $t \in [Q]$. Write $\mathbb{P}_\mathcal{S}$ for the probability space consisting of all those query-answer histories $\mathcal{H} = (q_1, a_1, \dots, q_Q, a_Q)$ which satisfy $|\mathcal{F}^t| = s_t$ for all $t \in [Q]$. Take any fixed r-graph $G \in \mathcal{F}^{(r)}_{n,d(n)}$. Note that our choice of the t-th answer a_t given by P implies that $\mathbb{P}_\mathcal{S}\left[G \in \mathcal{F}^t \mid G \in \mathcal{F}^{t-1}\right] = s_t/s_{t-1}$ for all $t \in [Q]$. Thus,

$$
\begin{aligned}
\mathbb{P}_\mathcal{S}[G^* = G] &= \mathbb{P}_\mathcal{S}\left[G \in \mathcal{F}^Q\right]/s_Q = \frac{1}{s_Q}\left(\prod_{t=1}^{Q} \mathbb{P}_\mathcal{S}\left[G \in \mathcal{F}^t \mid G \in \mathcal{F}^{t-1}\right]\right)\mathbb{P}_\mathcal{S}\left[G \in \mathcal{F}^0\right] \\
&= \frac{1}{s_Q}\prod_{t=1}^{Q}\frac{s_t}{s_{t-1}} = \frac{1}{s_0}.
\end{aligned}
$$

Thus $\mathbb{P}[G^* = G] = 1/|\mathcal{F}^{(r)}_{n,d(n)}|$ by the law of total probability.

Now let us prove that ALG will a.a.s. only see a simple history $(E_1, E_2, \mathcal{D})$. Note that $E_1 = E^Q$ and $E_2 = \bar{E}^Q$. Hence it suffices to show that E^Q is a weak forest a.a.s. Recall that we can write each $G \in \mathcal{F}^{(r)}_{n,d(n)}$ as the disjoint union of $G_1, \dots, G_s$, where $s = n/\tilde{n}$, each G_j is uniformly distributed in $\mathcal{G}^{(r)}_{\tilde{n},d(n)}$ and G_j has vertex set V_j.

Assume q_t is a vertex-set query. The probability that P answers "yes" is given by Corollary 2.3 as $O(d(n)/\tilde{n}^{r-1})$, as long as $t = o(d(n))$. Thus, because the number of queries is $Q = o(\tilde{n}^{r-1}/d(n))$, then, by a union bound, the probability that any edge is found with vertex-set queries is $o(1)$.

Assume now that $q_t = (u, i)$ is a neighbour query, where $u \in V_j$, $j \in [s]$. We will bound the probability that some vertex returned by P in the t-th answer a_t lies in V^{t-1}. Note that such a vertex will always lie in V_j. To bound this probability, for any given vertex $v \in V^{t-1} \cap V_j$, let $S_v := \{f \in \binom{V_j}{r} : u, v \in f, f \notin E^{t-1} \cup \bar{E}^{t-1}\}$. Note that $|S_v| = O(\tilde{n}^{r-2})$. By Corollary 2.3, the probability that a given r-set in S_v is an edge of G_t is $\Theta(d(n)/\tilde{n}^{r-1})$. Furthermore, if we condition on $e \in S_v$ being an edge of G_t, recall that the probability that its label belonging to u

equals i is at most $1/(d(n)-t)$. Thus, by a union bound over all elements of S_v, the probability that some r-set in S_v is the i-th edge in the incidence list of u is $O(d(n)/(\tilde{n}(d(n)-t))) = O(1/\tilde{n})$. Note that $|V^{t-1}| \leq rt = O(Q)$. Thus, by a union bound over all $v \in V^{t-1}$, the probability that the answer to the t-th query results in E^t being not a weak forest is $O(Q/\tilde{n})$. By a union bound over all queries, the probability that any of the at most Q neighbour queries finds any vertex in the current history book is $O\left(Q^2/\tilde{n}\right) = o(1)$. This in turn implies that the probability that a neighbour query detects anything else than a weak forest is $o(1)$.

Combining the conditions and lower bounds for both types of queries, we have that the probability (taken over all queries of ALG and choices of P in the above process) that E^Q is not a weak forest is $o(1)$. The statement follows since we have shown that the r-graph G^* returned by P is chosen uniformly at random from $\mathcal{F}^{(r)}_{n,d(n)}$. □

Theorem 4.5. *The following statements hold:*

(i) Let F be a connected graph which is not a tree. Assume that $d(n) = \omega(1)$ and $d(n) = o(n)$. Assume, furthermore, that $nd(n)/2 \leq ex(n,F)$. Then, any one-sided error F-freeness tester must perform $\Omega(\min\{d(n), \tilde{n}/d(n), \tilde{n}^{1/2}\})$ queries when restricted to n-vertex inputs of average degree $d(n) - o(d(n))$, where $\tilde{n}$ is as defined in (4.1).

(ii) Let $r \geq 3$. Let F be a connected non-r-partite r-graph. Assume that $d(n) = \omega(1)$ and $d(n) = o(n^{r-1})$. Then, any one-sided error F-freeness tester in r-graphs must perform $\Omega(\min\{d(n), \tilde{n}^{r-1}/d(n), \tilde{n}^{1/2}\})$ queries when restricted to n-vertex inputs of average degree $d(n) \pm o(d(n))$, where $\tilde{n}$ is as defined in (4.1).

Proof. We first prove (ii) and later discuss which modifications are needed to prove (i). Let $Q = o(\min\{d(n), \tilde{n}^{r-1}/d(n), \tilde{n}^{1/2}\})$. Consider any algorithm ALG that performs Q queries given an input r-graph G on n vertices with average degree $d(n) \pm o(d(n))$. Assume that ALG is given an r-graph $G \in \mathcal{F}^{(r)}_{n,d(n)}$ as an input. By Lemma 4.4 we know that any algorithm that performs at most Q queries will see a simple history $(E_1, E_2, \mathcal{D})$ of G with probability at least $2/3$ for at least $2/3$ of the graphs $G \in \mathcal{F}^{(r)}_{n,d(n)}$. Note that any such simple history $(E_1, E_2, \mathcal{D})$ is such that $|E_1 \cup E_2| \leq Q$, $|\mathcal{D}| \leq Q$ and for every $(v, d_v) \in \mathcal{D}$, $d_v = d(n)$. We now show that there is a family $\mathcal{F}_2$ of F-free r-graphs that, for every such simple history, contains at least one r-graph for which ALG will see the same history with positive probability.

- For each simple history $(E_1, E_2, \mathcal{D})$, let H be the r-graph that has vertex set $\bigcup_{e \in E_1 \cup E_2} e$ and edge set E_1. Note that H is a weak forest with (possibly) some isolated vertices and $|V(H)| \leq rQ$. Consider a partition of $V(H)$ into $V_1, \ldots, V_r$ such that for every $e \in E(H)$ and $i \in [r]$, we have $|e \cap V_i| = 1$. Consider pairwise disjoint sets of vertices $W_1, \ldots, W_r$ of size $d(n)^{1/(r-1)}$ which are disjoint from $V(H)$.
- Define an r-graph K with vertex set $V(H) \cup W_1 \cup \ldots \cup W_r$. Note that for each $v \in V_i$ there are $d(n)$ r-sets f such that $v \in f$ and $|f \cap W_j| = 1$ for all $j \in [r] \setminus \{i\}$. Define $E(K)$ by including $E(H)$ and adding $d(n) - \deg_H(v)$ of these r-sets incident to each vertex $v \in V(H)$. Note that K is r-partite and, thus, F-free.
- Finally, for any such K, consider the r-graph G obtained as the vertex-disjoint union of K and any F-free r-graph on $n - |V(K)|$ vertices with average degree $d(n) - o(d(n))$ (to see that this is possible, note that $|V(K)| = o(n)$ and $ex(n,F) = \Theta(n^r)$ since F is non-r-partite).

We define $\mathcal{F}_2$ as the family that consists of all r-graphs G that can be constructed as above and all possible relabellings of their vertices. Note that each $G \in \mathcal{F}_2$ has n vertices, average degree $d(n) \pm o(d(n))$ and is F-free. Moreover, for every $G \in \mathcal{F}^{(r)}_{n,d(n)}$ and any simple history $(E_1, E_2, \mathcal{D})$ seen by ALG on G, there is some r-graph $G \in \mathcal{F}_2$ such that ALG would have seen $(E_1, E_2, \mathcal{D})$ on G.

Now suppose ALG is a one-sided error F-freeness tester for r-graphs of average degree $d(n) \pm o(d(n))$ that performs Q queries. Assume that ALG is given inputs as follows. With probability $99/100$, the input is an r-graph $G \in \mathcal{F}^{(r)}_{n,d}$ chosen uniformly at random. With probability $1/100$, the input is an r-graph $G \in \mathcal{F}_2$ chosen uniformly at random. By Lemma 4.4, the proportion of r-graphs $G \in \mathcal{F}^{(r)}_{n,d(n)}$ for which with probability at least $2/3$ ALG only sees a simple history is least $2/3$. Moreover, since ALG is a one-sided error tester, it can only reject an input G if ALG can guarantee the existence of a copy of F in G. Thus, if after Q queries ALG has seen a simple history $(E_1, E_2, \mathcal{D})$, then it cannot reject the input, as there are r-graphs $G \in \mathcal{F}_2$ which are F-free and for which ALG may see the same history with positive probability. So given a random input as described above, the probability that ALG accepts is at least $(99/100)(2/3)^2 > 2/5$.

On the other hand, by Lemma 4.3, the proportion of r-graphs in $\mathcal{F}^{(r)}_{n,d(n)}$ that are ε-far from being F-free is at least $99/100$. Since ALG is a one-sided error F-freeness tester, it must reject these inputs with probability at least $2/3$. Therefore, given a random input G, the probability that ALG rejects G must be at least $(99/100)^2(2/3) > 3/5$. This is a contradiction to the previous statement, so ALG cannot be a one-sided error F-freeness tester.

In order to prove (i), let $Q = o(\min\{d(n), \tilde{n}/d(n), \tilde{n}^{1/2}\})$. If F is not bipartite, then (ii) already shows the desired statement. In order to deal with bipartite graphs F, define a new family $\mathcal{F}_1$ (which also works for non-bipartite F) as follows. Given a simple history $(E_1, E_2, \mathcal{D})$, define H as above. For each $v \in V(H)$, consider $d(n) - \deg_H(v)$ new vertices and add an edge between v and each of them. Denote the resulting graph by K. Finally, consider the graph G obtained as the disjoint union of K and any F-free graph on $n - |V(K)|$ vertices with average degree $d(n) - o(d(n))$. We define $\mathcal{F}_1$ as the family that consists of all graphs G that can be constructed as above and all possible relabellings of their vertices. The remainder of the proof works in the same way. □

Note that if, for instance, $d(n) = 2ex(n, F)/n$ and $F = C_4$, then Theorem 4.5(i) (together with Corollary 3.1(ii)) implies a lower bound of $\Omega(n^{1/2})$. The bound on the number of queries in Theorem 4.5 is stronger than in Proposition 4.1 as long as d is not too small.

4.3. **Upper bounds.** Here, we present several upper bounds on the query complexity for testing F-freeness. Note that there is always the trivial bound of $O(nd)$ queries; the forthcoming results are only relevant whenever the presented bound is smaller than this. Proposition 4.6 provides a bound on the query complexity which applies to input r-graphs G in which the maximum degree does not differ too much from the average degree. Proposition 4.7 improves Proposition 4.6 for special r-graphs F. Finally, Theorem 4.8 provides a bound which works for arbitrary F and G. Propositions 4.6 and 4.7 give stronger bounds for very sparse r-graphs G, whereas Theorem 4.8 gives stronger bounds for denser r-graphs.

The techniques of our algorithms are based on two strategies: random sampling and local exploration. We will always write V for the vertex set of the input r-graph G and d for its average degree. Given any $S \subseteq V$, we denote by $G[S] \coloneqq \{e \in G : e \subseteq S\}$ the subgraph of G spanned by S. Thus $V(G[S]) = S$. We denote by $G\{S, \rho\} \coloneqq \{e \in G : \exists\, v \in e : \mathrm{dist}(S, v) < \rho\}$ the graph obtained from G by performing a breadth-first search of depth ρ from S. Throughout this section, the hidden constants in the O notation will be independent of both ε and n. When the constants depend on ε, we will denote this by writing O_ε.

Proposition 4.6. *Let F be a fixed, connected r-graph and let D be its diameter. For the class consisting of all input r-graphs G on n vertices with average degree d and maximum degree $\Delta(G) = O(d)$, it is possible to test F-freeness with $O_\varepsilon(d^{D+1})$ queries.*

Proof. We consider a one-sided error F-freeness tester. The procedure is as follows. First choose a set $S \subseteq V(G)$ of size $\Theta(1/\varepsilon)$ uniformly at random. For each $v \in S$, find $G\{v, D+1\}$ by performing neighbour queries. If any of the graphs $G\{v, D+1\}$ contains a copy of F, the algorithm rejects G. Otherwise, it accepts it. Clearly, the complexity is $O(d^{D+1}/\varepsilon)$ and the procedure will always accept G if it is F-free.

Assume now that the input is ε-far from being F-free. Then, it contains at least $\varepsilon nd/r$ edges that belong to copies of F. It follows that the number of vertices that belong to some copy of F is $\Omega(\varepsilon nd/\Delta(G)) = \Omega(\varepsilon n)$. Therefore, if the implicit constant in the bound on $|S|$ is large enough, the algorithm will choose one of the vertices that belong to a copy of F with probability at least $2/3$. If it chooses such a vertex, then, as F has diameter D, it rejects the input. $\square$

We can improve the bound in Proposition 4.6 for a certain class of r-graphs F. Given any r-graph F, let D_F be its diameter. Consider the partition of its vertices given by choosing an edge $e \in F$, taking $V_0(e) := e$ and $V_i(e) := \{u \in V(F) : \mathrm{dist}(e,u) = i\}$ for $i \in [D_F]$. We let $\mathcal{F}_E := \{F : |F[V_{D_F}(e)]| = 0 \ \ \forall\, e \in F\}$. The class $\mathcal{F}_E$ contains, for instance, complete r-partite r-graphs, loose cycles and tight cycles. If $r = 2$ then $\mathcal{F}_E$ also contains hypercubes, for example.

Proposition 4.7. *Let $F \in \mathcal{F}_E$ be an r-graph and let D be its diameter. For the class consisting of all input r-graphs G with average degree d and maximum degree $\Delta(G) = O(d)$, it is possible to test F-freeness with $O_\varepsilon(d^D)$ queries.*

Proof. We consider a one-sided error tester, which works in a very similar way as in the proof of Proposition 4.6. The F-freeness tester chooses a set $S \subseteq V$ of size $\Theta(1/\varepsilon)$ uniformly at random. It then chooses an edge e incident to each $v \in S$ uniformly at random and finds $G\{e, D\}$ by performing neighbour queries; then, it searches for a copy of F. If any copy of F is found, the algorithm rejects the input; otherwise, it accepts. The query complexity is clearly $O(d^D/\varepsilon)$. The analysis of the algorithm is similar to that of Proposition 4.6, so we omit the details. $\square$

We conclude with the following bound, which works for arbitrary G and any F without isolated vertices. Given an r-graph F, we define its *vertex-overlap index* $\ell(F)$ as the minimum integer ℓ such that two graphs isomorphic to F sharing ℓ vertices must share at least one edge; if this does not hold for any $\ell \in [v_F]$, we then set $\ell = v_F + 1$. For instance, $\ell(K_k^{(r)}) = r$, and for a matching M we have $\ell(M) = |V(M)| + 1$ if $|V(M)| \geq 2r$.

Theorem 4.8. *Let $r \geq 2$ and let F be an r-graph without isolated vertices. Let $\ell := \ell(F)$. For the class consisting of all input r-graphs G on n vertices with average degree d and maximum degree Δ, it is possible to test F-freeness with $O_\varepsilon(\max\{(n/(nd)^{1/v_F})^r, (n^{\ell-2}\Delta/d)^{r/(\ell-1)}\})$ queries.*

In the case when $F = K_k^{(r)}$ and the input r-graph G satisfies $\Delta(G) = O(d)$, the bound in Theorem 4.8 becomes $O_\varepsilon((n/(nd)^{1/k})^r)$ whenever $d = o(n^{k/(r-1)-1})$, and $O_\varepsilon(n^{r(r-2)/(r-1)})$ otherwise.

Proof. Choose a constant c which is large enough compared to v_F and e_F. We present a one-sided error tester in Algorithm 1. In this proof, the constants in the O notation are independent of c.

Algorithm 1 An F-freeness tester for r-graphs.

```
procedure CANONICAL F TESTER
    Let s = c max{n/(εnd)^{1/v_F}, (n^{ℓ-2}Δ/εd)^{1/(ℓ-1)}}.
    Choose a set S ⊆ V of size s uniformly at random.
    Find G[S] by performing all vertex-set queries.
    if G[S] contains a copy of F, then reject.
    otherwise, accept.
end procedure
```

It is easy to see that we may assume s is large compared to v_F. If G is F-free, the algorithm will never find a copy of F and will always accept the input. Assume now that G is ε-far from being F-free. Then, G must contain a set $\mathcal{F}$ of $\varepsilon nd/e_F$ edge-disjoint copies of F. For each $W \subseteq V$, we define $\deg_{\mathcal{F}}(W) := |\{F' \in \mathcal{F} : W \subseteq V(F')\}|$. It is clear that

$$\deg_{\mathcal{F}}(W) \leq \min_{v \in W} \deg(v) \leq \Delta. \tag{4.2}$$

For any fixed $F' \in \mathcal{F}$, we have $\mathbb{P}[F' \in G[S]] = (1 \pm 1/2)(s/n)^{v_F}$. We denote by X the number of $F' \in \mathcal{F}$ such that $F' \in G[S]$. We conclude that

$$\mathbb{E}[X] = (1 \pm 1/2)|\mathcal{F}|\left(\frac{s}{n}\right)^{v_F} = \Theta\left(\frac{\varepsilon d s^{v_F}}{n^{v_F - 1}}\right). \tag{4.3}$$

The variance of X can be estimated by observing that we only need to consider r-graphs $F', F'' \in \mathcal{F}$ whose vertex sets intersect, as otherwise the events are negatively correlated. Hence,

$$\mathrm{Var}[X] \leq \sum_{\substack{(F',F'')\in\mathcal{F}\times\mathcal{F}\\ V(F')\cap V(F'')\neq\varnothing}} \mathbb{P}[F'\cup F''\subseteq G[S]] = \sum_{i=1}^{v_F} \sum_{\substack{(F',F'')\in\mathcal{F}\times\mathcal{F}\\ |V(F')\cap V(F'')|=i}} \mathbb{P}[F'\cup F''\subseteq G[S]]. \tag{4.4}$$

Let us estimate this quantity for each $i \in [v_F]$. For $i \in [v_F - 1]$ we can apply a double counting argument to see that

$$|\{(F',F'')\in\mathcal{F}\times\mathcal{F} : |V(F')\cap V(F'')| = i\}| \leq 2 \sum_{W\in\binom{V}{i}} \binom{\deg_{\mathcal{F}}(W)}{2}, \tag{4.5}$$

while for $i = v_F$ we have that

$$|\{(F',F'')\in\mathcal{F}\times\mathcal{F} : |V(F')\cap V(F'')| = v_F\}| \leq |\mathcal{F}| + 2 \sum_{W\in\binom{V}{v_F}} \binom{\deg_{\mathcal{F}}(W)}{2}. \tag{4.6}$$

Note that

$$\sum_{W\in\binom{V}{i}} \deg_{\mathcal{F}}(W) = O(|\mathcal{F}|) = O(\varepsilon n d). \tag{4.7}$$

By assumption on F, we have that $\deg_{\mathcal{F}}(W) \leq 1$ for all W such that $|W| \geq \ell$, which implies that $\binom{\deg_{\mathcal{F}}(W)}{2} = 0$. Moreover, by (4.2) and (4.7), for each $i \in [\ell - 1]$ we obtain

$$\sum_{W\in\binom{V}{i}} \binom{\deg_{\mathcal{F}}(W)}{2} \leq \Delta \sum_{W\in\binom{V}{i}} \deg_{\mathcal{F}}(W) = O(\varepsilon n d \Delta). \tag{4.8}$$

Combining (4.5)–(4.8), the estimation in (4.4) yields

$$\mathrm{Var}[X] = O\left(\varepsilon n d \left(\frac{s}{n}\right)^{v_F}\right) + \sum_{i=1}^{\ell-1} O\left(\varepsilon n d \Delta \left(\frac{s}{n}\right)^{2v_F - i}\right) = \varepsilon n d \cdot O\left(\left(\frac{s}{n}\right)^{v_F} + \Delta\left(\frac{s}{n}\right)^{2v_F-\ell+1}\right). \tag{4.9}$$

By Chebyshev's inequality, $\mathbb{P}[X = 0] \leq \mathrm{Var}[X]/\mathbb{E}[X]^2$. Using (4.3), (4.9) and the fact that c is large compared to v_F and e_F, one can check that $\mathrm{Var}[X]/\mathbb{E}[X]^2 < 1/3$. Thus $G[S]$ contains a copy of F with probability at least $2/3$. Therefore, G will be rejected with probability at least $2/3$, which shows that Algorithm 1 is an F-freeness tester.

The query complexity of the algorithm is given by performing all $\binom{s}{r}$ vertex-set queries. This yields the stated complexity. □

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Appendix A. Proof of Lemma 3.4

Proof of Lemma 3.4. Clearly, $D_K \geq D_F$ for each $K \subseteq F$, and $D_K \leq X_K$. Thus, $D_F \leq \min\{X_K : K \subseteq F, e_K > 0\}$. Corollary 3.3 applied to each K implies that $\min\{X_K : K \subseteq F, e_K > 0\} = \Theta(\Phi_F)$ a.a.s., so $D_F = O(\Phi_F)$ a.a.s.

It now suffices to show that $D_F = \Omega(\Phi_F)$ a.a.s. To do so, for each $G \in \mathcal{G}^{(r)}_{n,d}$ we define an auxiliary graph $\Gamma = \Gamma(G)$ whose vertices are all the copies of F in G, and in which $F_1, F_2 \in V(\Gamma)$ are adjacent if and only if F_1 and F_2 share at least one edge. Let us denote by $\sum^*$ the sum over all graphs $\tilde{F}$ which can be written as $\tilde{F} = F' \cup F''$, where $F', F'' \subseteq K^{(r)}_V$, $F', F'' \cong F$ and $E(F') \cap E(F'') \neq \varnothing$. This means that $v_\Gamma = X_F$ and

$$e_\Gamma = O\left(\sum\nolimits^* X_{\tilde{F}}\right).$$

Note that the size of the largest independent set in Γ equals D_F. By Turán's theorem we have that

$$D_F \geq \frac{X_F^2}{X_F + O\left(\sum^* X_{\tilde{F}}\right)}.$$

Using Corollary 3.3, one can check that, a.a.s.,

$$\frac{X_F^2}{X_F + O\left(\sum^* X_{\tilde{F}}\right)} = \Omega(\Phi_F) \iff X_{\tilde{F}} = O\left(n^{2v_F} p^{2e_F} \Phi_F^{-1}\right) \tag{A.1}$$

for all $\tilde{F} = F' \cup F''$ such that $e_{F' \cap F''} > 0$. So it suffices to prove the final bound in (A.1).

For any fixed r-graph K, let $\Psi_K \coloneqq n^{v_K} p^{e_K}$. Note that if $K \subseteq F$, then $\Psi_K = \Theta(\mathbb{E}[X_K])$ (by Corollary 3.1(ii)). Furthermore, for any two r-graphs K and L on a vertex set V,

$$\Psi_L \Psi_K = \Psi_{L \cup K} \Psi_{L \cap K}. \tag{A.2}$$

Consider two copies F' and F'' of F whose intersection has at least one edge. Let $\tilde{F} \coloneqq F' \cup F''$ and $K \coloneqq F' \cap F''$, so $e_K > 0$. Thus, by Corollary 3.1(ii),

$$\mathbb{E}[X_{\tilde{F}}] = \Theta\left(n^{2v_F - v_K} p^{2e_F - e_K}\right) = \Theta\left(\frac{\Psi_F^2}{\Psi_K}\right) = O\left(\frac{\Psi_F^2}{\Phi_F}\right). \tag{A.3}$$

We now claim that

$$\Phi_{\tilde{F}} = \min\{\mathbb{E}[X_L] : L \subseteq \tilde{F}, e_L > 0\} = \Omega\left(\min\left\{\frac{\Phi_F^3}{\Psi_K^2}, \frac{n\Phi_F^2}{\Psi_K^2}\right\}\right). \tag{A.4}$$

Indeed, consider any r-graph $L \subseteq \tilde{F}$ with $e_L > 0$ and let $L' \coloneqq L \cap F'$ and $L'' \coloneqq L \cap F''$. Note that $L \cup K = (L' \cup K) \cup (L'' \cup K)$ and $K = (L' \cup K) \cap (L'' \cup K)$, so two applications of (A.2) yield

$$\Psi_L = \frac{\Psi_{L \cup K} \Psi_{L \cap K}}{\Psi_K} = \frac{\Psi_{L' \cup K} \Psi_{L'' \cup K} \Psi_{L \cap K}}{\Psi_K^2}. \tag{A.5}$$

If $e_{L \cap K} > 0$, then the values of $\Psi_{L' \cup K}$, $\Psi_{L'' \cup K}$ and $\Psi_{L \cap K}$ can be lower bounded by $\Omega(\Phi_F)$ (as $L' \cup K$, $L'' \cup K$ and $L \cap K$ are all subgraphs of F). Thus $\Psi_L = \Omega(\Phi_F^3/\Psi_K^2)$. So suppose that $e_{L \cap K} = 0$. Then L' and L'' are edge-disjoint, and at least one of them has at least one edge. We may assume that $e_{L'} > 0$ without loss of generality. Consider three cases. If $e_{L''} = 0$, then $\Psi_{L''} = n^{v_{L''}} \geq \Psi_{L' \cap L''}$ and, by (A.2), $\Psi_L = \Psi_{L'} \Psi_{L''} / \Psi_{L' \cap L''} = \Omega(\Psi_{L'}) = \Omega(\Phi_F) = \Omega(\Phi_F^3/\Psi_K^2)$, where the final equality holds since $\Psi_K = \Omega(\Phi_F)$. If $e_{L''} > 0$ but $L' \cap L'' = \varnothing$ then $\Psi_L = \Psi_{L'} \Psi_{L''} = \Omega(\Psi_{L'}) = \Omega(\Phi_F^3/\Psi_K^2)$. Otherwise, we have that $e_{L''} > 0$ and $L' \cap L'' \neq \varnothing$. We use (A.5) taking into account that $\Psi_{L \cap K} = n^{v_{L \cap K}} = \Omega(n)$ to conclude that $\Psi_L = \Omega(n\Phi_F^2/\Psi_K^2)$. This proves the claim.

By Lemma 3.2 we have that $\mathrm{Var}(X_{\tilde{F}}) = \mathbb{E}[X_{\tilde{F}}]^2 O(\varepsilon_{n,d} + \Phi_{\tilde{F}}^{-1})$. As $\varepsilon_{n,d} = o(1)$ by assumption, by (A.3) Chebyshev's inequality implies that the final bound in (A.1) holds a.a.s. if $\Phi_{\tilde{F}}^{-1} = O(\varepsilon_{n,d})$. Therefore, we may assume that $\mathrm{Var}(X_{\tilde{F}}) = O(\mathbb{E}[X_{\tilde{F}}]^2/\Phi_{\tilde{F}}) = O(\Psi_{\tilde{F}}^2/\Phi_{\tilde{F}})$. Consequently, by (A.4) and (A.2) we have

$$\mathrm{Var}(X_{\tilde{F}}) = O\left(\frac{\Psi_{\tilde{F}}^2 \Psi_K^2}{\Phi_F^3} + \frac{\Psi_{\tilde{F}}^2 \Psi_K^2}{n\Phi_F^2}\right) = O\left(\frac{\Psi_F^4}{\Phi_F^3} + \frac{\Psi_F^4}{n\Phi_F^2}\right).$$

Thus, Chebyshev's inequality gives

$$\mathbb{P}\left[X_{\tilde{F}} \geq \mathbb{E}[X_{\tilde{F}}] + \frac{\Psi_F^2}{\Phi_F}\right] = O\left(\Phi_F^{-1} + 1/n\right) = o(1)$$

by assumption. Hence $X_{\tilde{F}} = O\left(\Psi_F^2/\Phi_F\right)$ a.a.s., as required. □