Hamilton cycles in graphs and hypergraphs: an extremal perspective

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Joint work with
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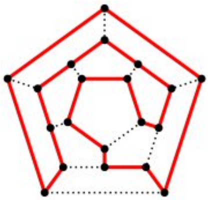
Hamilton cycles in graphs

Hamilton cycle contains every **vertex** exactly once



related to the Traveling salesman problem

Question

Which graphs contain a Hamilton cycle?

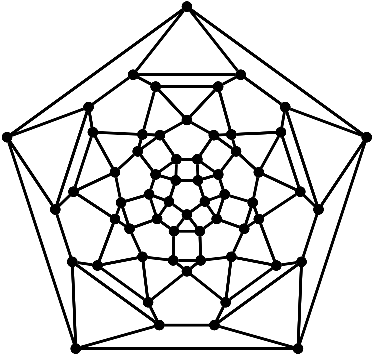
Hamilton cycle problem is difficult:

- no characterization of Hamiltonian graphs known
- Decision problem NP-complete
  ($\Rightarrow$ algorithm for checking Hamiltonicity would imply P=NP)

Hamilton cycles in graphs

Question

Can you decide whether this graph has a Hamilton cycle?



Hamilton cycles in graphs

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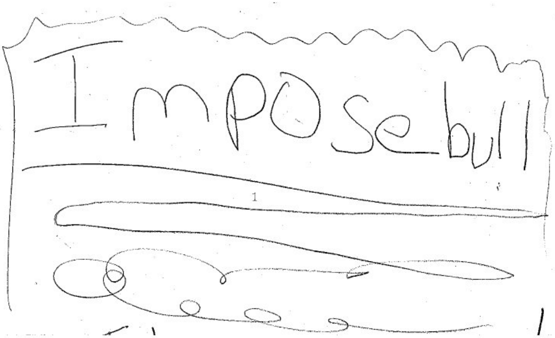

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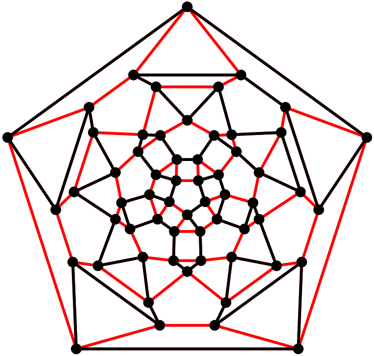

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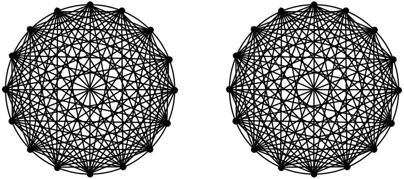
Aim

Simple and general sufficient conditions which guarantee a Hamilton cycle degree $d(x)$ of a vertex x = number of edges incident to x.

Theorem (Dirac 1952)

Suppose that G is a graph on $n \geq 3$ vertices with minimum vertex degree $\delta \geq n/2$. Then G has a Hamilton cycle.

Minimum degree bound cannot be improved



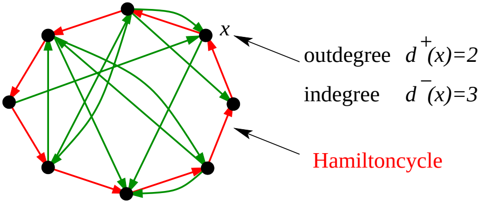
Disjoint union of two complete graphs is not even connected but each vertex has degree $n/2 - 1$

Theorem (Dirac, 1952)

Suppose that G is a graph on $n \geq 3$ vertices with minimum vertex degree $\delta \geq n/2$. Then G has a Hamilton cycle.

directed graph=digraph

Minimum outdegree $\delta^+ = 1$
Minimum indegree $\delta^- = 1$



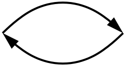
Theorem (Ghouila-Houri, 1960)

Suppose that D is a digraph with $\delta^+, \delta^- \geq n/2$ then D has a Hamilton cycle

As before, bound on δ^+, δ^- is best possible.

Question (Thomassen, 1979)

Analogue for oriented graphs?

i.e. digraphs without double edges 

Hamilton cycles in oriented graphs

Question (Thomassen, 1979)

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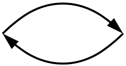
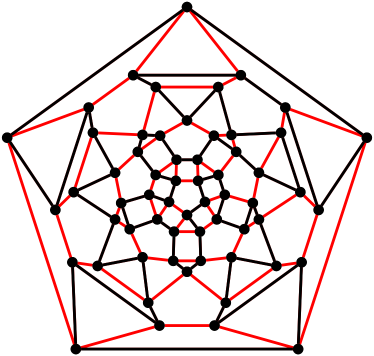
i.e. digraphs without double edges 

Theorem (Keevash, Kühn & Osthus, 2009)

Every large oriented graph with $\delta^+, \delta^- \geq \frac{3n-4}{8}$ has a Hamilton cycle.

- gives exact solution of Thomassen's problem for large graphs
- improves previous results e.g. by Jackson (1981), Thomassen (1982), Häggkvist (1993), Häggkvist & Thomason (1997), Kelly, Kühn & Osthus (2008)

Hamilton decomposition of G
= set of edge-disjoint Hamilton cycles covering all edges of G

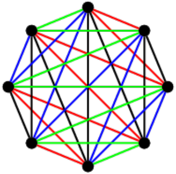


Which graphs/digraphs have Hamilton decompositions?
Very few general conditions known

Theorem (Walecki, 1892)

Complete graph K_n has a Hamilton decomposition $\Leftrightarrow$ n odd

Construction: find Hamilton path decomposition for K_{n-1}



then add extra vertex and close paths into Hamilton cycles

Theorem (Tillson, 1980)

Complete digraph K_n has a Hamilton decomposition $\Leftrightarrow$ $n \neq 4, 6$

Hamilton decompositions of tournaments

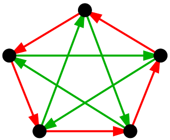
tournament: orientation of a complete graph

regular tournament: every vertex has same in- and outdegree

Conjecture (Kelly, 1968)

Every regular tournament has a Hamilton decomposition.

Decomposition of regular tournament into 2 Hamilton cycles



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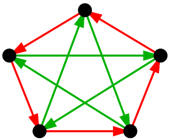

- even finding 2 edge-disjoint Hamilton cycles is not easy (Jackson 1981, Zhang 1980)
- further partial results due to Thomassen (1979, 1982), Alspach et al. (1990), Häggkvist (1993), Häggkvist & Thomason (1997), Bang-Jensen & Yeo (2004) ...

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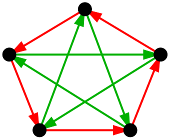
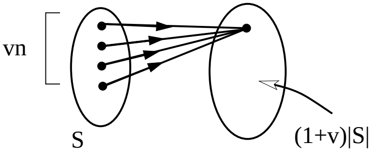


Theorem (Kühn & Osthus, 2013)

Every large regular tournament has a Hamilton decomposition.

Robust outexpanders

digraph G on n vertices is a (ν, τ)-robust outexpander
$\Leftrightarrow$ for every vertex set S with $\tau n \leq |S| \leq (1-\tau)n$ there are $(1+\nu)|S|$ vertices with νn inneighbours in S



Main result:

Theorem (Kühn & Osthus 2013)

Suppose $1/n \ll \nu, \tau \ll \alpha$. Then every αn-regular (ν, τ)-robust outexpander has a Hamilton decomposition.

Robust outexpanders: main result

Theorem (Kühn & Osthus 2013)

Suppose $1/n \ll \nu, \tau \ll \alpha$. Then every αn-regular (ν, τ)-robust outexpander on n vertices has a Hamilton decomposition.

Digraph classes which are robust outexpanders:

(1) oriented graphs G with $\delta^+(G), \delta^-(G) \geq 3n/8 + o(n)$

(2) digraphs G with $\delta^+(G), \delta^-(G) \geq n/2 + o(n)$

(3) dense quasi-random digraphs

$\Rightarrow$ such digraphs have Hamilton decompositions if they are regular

So (3) generalizes result of Alspach, Bryant & Dyer (2012) on Hamilton decompositions of Paley graphs (for large n)

Applications of main result on robust outexpanders

Theorem (Kühn & Osthus 2013)

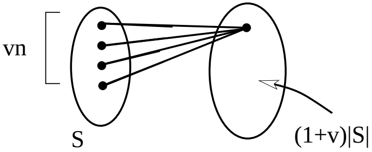
Suppose $1/n \ll \nu, \tau \ll \alpha$. Then every αn-regular (ν, τ)-robust outexpander on n vertices has a Hamilton decomposition.

Many applications:

- conjecture of Erdős from 1981 on random tournaments
- crucial ingredient for proof of the 1-factorization conjecture (1950's) and Hamilton decomposition conjecture (1970)
- conjecture of Nash-Williams from 1970's
- solves a problem on domination ratio for TSP tours by Glover & Punnen as well as Alon, Gutin & Krivelevich
- solves dense case of a conjecture of Frieze and Krivelevich on packing Hamilton cycles in random graphs
- ...

Can deduce a version of main result for undirected graphs:

graph G on n vertices is a (ν, τ)-robust expander
$\Leftrightarrow$ for every vertex set S with $\tau n \leq |S| \leq (1-\tau)n$ there are $(1+\nu)|S|$ vertices with νn neighbours in S



Theorem (Kühn & Osthus 2013)

Every large even-regular robust expander G of linear degree has a Hamilton decomposition.

Undirected robust expanders

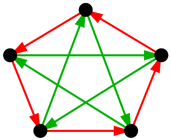
Theorem (Kühn & Osthus 2013)

Every large even-regular robust expander G of linear degree has a Hamilton decomposition.

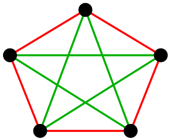
Proof strategy:

• Find orientation G_{orient} of G so that G_{orient} is a regular robust outexpander. Do this by choosing a random orientation and change directions of some edges to make it regular.

• Can apply main result on digraphs to obtain (directed) Hamilton decomposition of G_{orient}, which corresponds to (undirected) Hamilton decomposition of G.

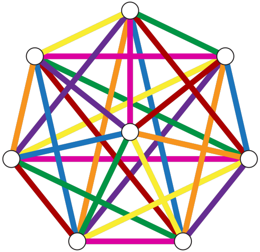


$\Rightarrow$



1-factor (or perfect matching) of G
$=$ set of disjoint edges covering all vertices of G
1-factorization of G
$=$ set of edge-disjoint 1-factors covering all edges of G

D-regular graph G has 1-factorization
$\iff$ G has an edge-colouring with D colours



A 1-factorization of the complete graph K_8 on 8 vertices

1-factorization conjecture

1-factorization conjecture (1950's)

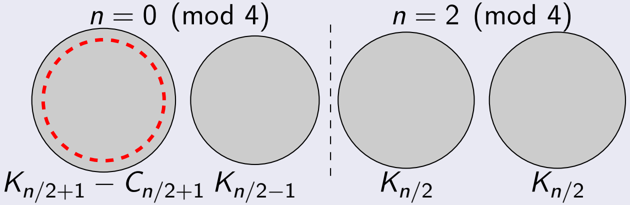
Every D-regular graph G on an even number n of vertices with $D \geq 2\lceil n/4 \rceil - 1$ has a 1-factorization.

Explicitly,

$$D \geq \begin{cases} n/2 - 1 & \text{if } n = 0 \pmod 4, \\ n/2 & \text{if } n = 2 \pmod 4. \end{cases}$$

Extremal examples

Odd component contains no 1-factor.



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Every D-regular graph G on an even number n of vertices with $D \geq 2\lceil n/4 \rceil - 1$ has a 1-factorization.

- True for $D = n - 1$, i.e. complete graphs.
- Chetwynd and Hilton (1989), and independently Niessen and Volkmann (1990), for $D \geq (\sqrt{7} - 1)n/2 \approx 0.82n$.
- Perkovic and Reed (1997) for $D \geq (1/2 + \varepsilon)n$ with $\varepsilon > 0$.
- Vaughan (2013) : an approximate multigraph version.

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Theorem (Csaba, Kühn, Lo, Osthus, Treglown 2013^+)

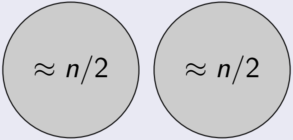
1-factorization conjecture holds for sufficiently large n.

Hamilton decomposition conjecture (Nash-Williams 1970)

Every D-regular graph on n vertices with $D \geq \lfloor n/2 \rfloor$ has a decomposition into Hamilton cycles and at most one perfect matching.

Extremal examples

No disconnected graph contains a Hamilton cycle.



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- Nash-Williams (1969), $D \geq \lfloor n/2 \rfloor$ guarantees Hamilton cycle.
- Jackson (1979), $D/2 - n/6$ edge-disjoint Hamilton cycles
- Christofides, Kühn and Osthus (2012), $D \geq n/2 + \varepsilon n$ guarantees an almost Hamilton decomposition.
- Kühn and Osthus (2013), $D \geq n/2 + \varepsilon n$ guarantees Hamilton decomposition.

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Theorem (Csaba, Kühn, Lo, Osthus, Treglown 2013$^+$)

Hamilton decomposition conjecture holds for sufficiently large n.

G is ε-close to H if G can be transformed to H by adding/removing at most εn^2 edges.

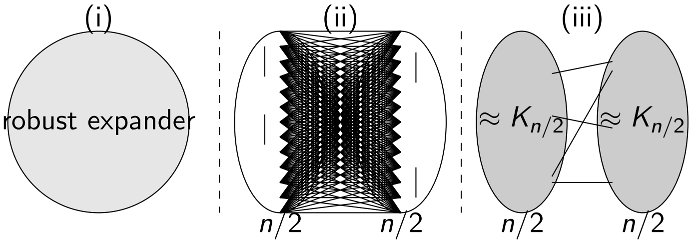
Lemma

Let G be a D-regular graph on n vertices and $D \geq n/2 - 1$. Then either

(i) *G is a robust expander;* ✓

(ii) *G is ε-close to complete bipartite graph $K_{n/2,n/2}$;*

(iii) *G is ε-close to union of two complete graphs $K_{n/2}$.*



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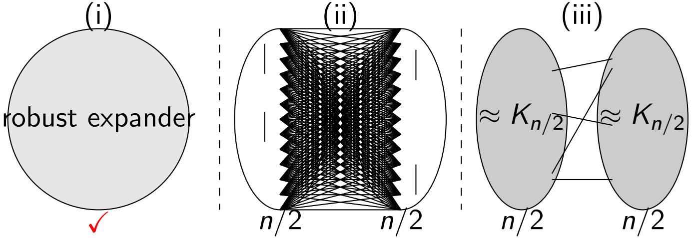

Packing Hamilton cycles in graphs of large mindegree

Theorem (Dirac, 1952)

Suppose that G is a graph on $n \geq 3$ vertices with minimum degree $\geq n/2$. Then G has a Hamilton cycle.

Minimum degree condition is best possible.

Theorem (Nash-Williams, 1971)

Suppose that G is a graph on $n \geq 3$ vertices with minimum degree $\geq n/2$. Then G has a at least $5n/224$ edge-disjoint Hamilton cycles.

Conjecture (Nash-Williams):
can improve this to $n/4$ (clearly best possible)

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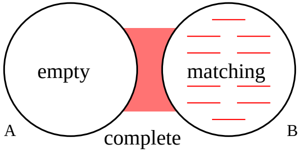
Babai: can't do better than $\approx n/8$

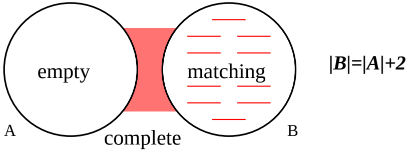
Babai's construction:



$|B|=|A|+2$

Every Hamilton cycle contains at least 2 edges from B
$\Rightarrow$ G has at most $|B|/4$ edge-disjoint Hamilton cycles
$\Rightarrow$ G has $\leq (n+2)/8$ edge-disjoint Hamilton cycles

Babai's construction:



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Theorem (Csaba, Kühn, Lapinskas, Lo, Osthus, Treglown 2013$^+$)

If n is sufficiently large then this is the correct bound.

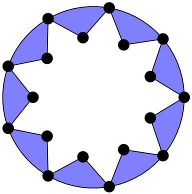
More generally:
determined the number of edge-disjoint Hamilton cycles which are guaranteed in a graph G of given minimum degree

Theorem (Walecki, 1892)

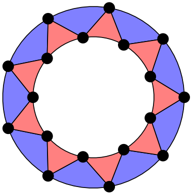
Complete graph K_n has a Hamilton decomposition $\Leftrightarrow$ n odd

Problem

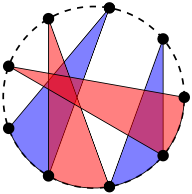
Prove a hypergraph version of Walecki's theorem.



loose Hamilton cycle

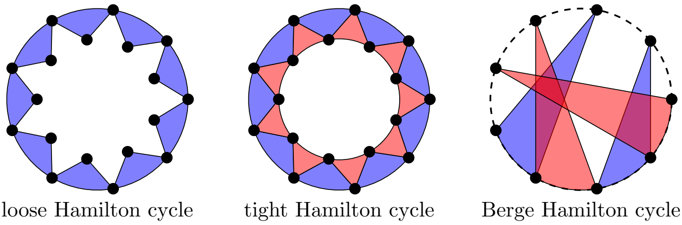


tight Hamilton cycle



Berge Hamilton cycle

Open problems: Hamilton decompositions of hypergraphs

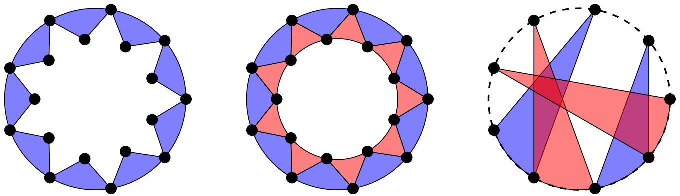


Conjecture (Kühn, Osthus, 2014)

If n is sufficiently large such that $k-1$ divides n and $n/(k-1)$ divides $\binom{n}{k}$ then $K_n^{(k)}$ has a decomposition into loose Hamilton cycles.

divisibility conditions are necessary:
k-uniform loose Hamilton cycles have $n/(k-1)$ edges and only exist if $k-1$ divides n

Open problems: Hamilton decompositions of hypergraphs



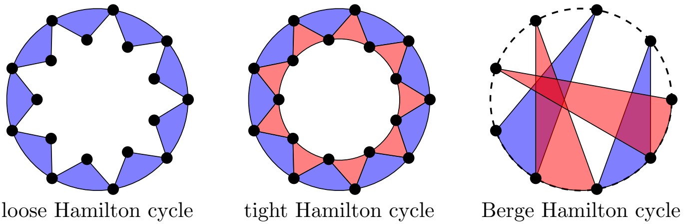
loose Hamilton cycle   tight Hamilton cycle   Berge Hamilton cycle

Conjecture (Bailey, Stevens, 2010)

If n is sufficiently large and n divides $\binom{n}{k}$ then $K_n^{(k)}$ has a decomposition into tight Hamilton cycles.

divisibility condition is necessary:
k-uniform tight Hamilton cycles have n edges

Open problems: Hamilton decompositions of hypergraphs



Theorem (Kühn, Osthus, 2014)

If $n \geq 30$ and n divides $\binom{n}{k}$ then $K_n^{(k)}$ has a decomposition into Hamilton Berge cycles.

- divisibility condition is necessary
- proves conjecture of Bermond, Germa, Heydemann, Sotteau from 1973
- case $k = 3$ already due to Verrall, building on results of Bermond