On the structure of oriented graphs and digraphs with forbidden tournaments or cycles

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Joint work with
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Structure of triangle-free graphs

What can we say about the structure of triangle-free graphs?

Theorem (Mantel 1907)

Every triangle-free graph on n vertices has at most $n^2/4$ edges, i.e. $\mathrm{ex}(n, K_3) \leq n^2/4$.

Complete bipartite graphs show that the bound is best possible.

Instead of the extremal viewpoint, we consider the 'average case':

- What does a typical triangle-free graph on n vertices look like?
- How many triangle-free graphs on n vertices are there?

Typical structure of triangle-free graphs

Theorem (Erdős, Kleitman, Rothschild, 1976)

Almost all triangle-free graphs are bipartite. i.e.

$$\lim_{n\to\infty} \frac{\#\text{ triangle-free graphs on } n \text{ vertices}}{\#\text{ bipartite graphs on } n \text{ vertices}} \to 1.$$

Corollary

The number of triangle-free graphs on n vertices is

$$2^{(1+o(1))\frac{n^2}{4}} = 2^{(1+o(1))\mathrm{ex}(n,K_3)}.$$

Trivial lower bound:

$K_{\frac{n}{2},\frac{n}{2}}$ has $\frac{n^2}{4} = \mathrm{ex}(n, K_3)$ edges and $2^{\frac{n^2}{4}}$ subgraphs.

Typical structure of H-free graphs

Related work:

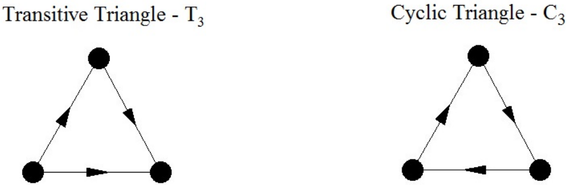
- Kolaitis, Prömel, Rothschild (K_r-free graphs)
- Prömel, Steger (H-free graphs)
- Bollobás, Thomason (Hereditary properties)
- Balogh, Bollobás, Simonovits (fine structure of H-free graphs)
- Osthus, Prömel, Taraz (sparse K_3-free graphs)
- Balogh, Morris, Samotij, Warnke (sparse K_r-free graphs)
- ...

Oriented graphs and digraphs

oriented graph:
edges directed, between any two vertices there is at most one edge

digraph:
edges directed, between any two vertices there are at most two edges – at most one in each direction

Typical structure of triangle-free oriented graphs

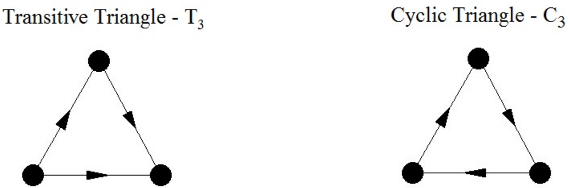


Conjecture (Cherlin, 1998)

(a) Almost all T_3-free oriented graphs are tripartite.

(b) Almost all C_3-free oriented graphs are acyclic.

Typical structure of triangle-free oriented graphs



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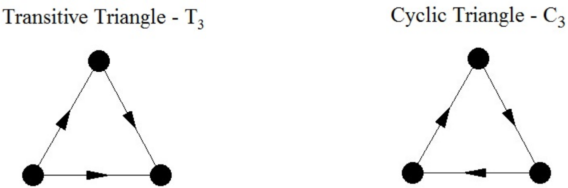

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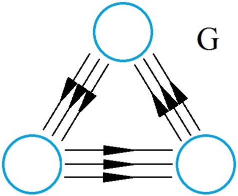
(a) is formally correct but morally wrong.
(b) is formally wrong but morally almost correct.

Forbidding transitive triangles

Note that $\mathrm{ex}(n, T_3) = \frac{n^2}{3}$.

$\#$ oriented subgraphs of G is $2^{\frac{n^2}{3}}$.



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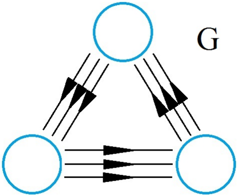
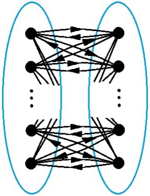


But the complete bipartite digraph has $\frac{n^2}{4}$ double edges, so has $3^{\frac{n^2}{4}}$ oriented subgraphs.



Since $3^{\frac{n^2}{4}} \gg 2^{\frac{n^2}{3}}$, this 'morally' disproves Cherlin's conjecture.

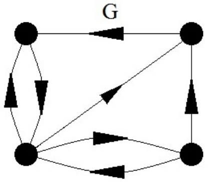
A general lower bound

Given a digraph G, let

$$e_1(G) = \#\text{ single edges,}$$

$$e_2(G) = \#\text{ double edges.}$$

G has $2^{e_1(G)}3^{e_2(G)} = 2^{e_1(G)+a\cdot e_2(G)}$ oriented subgraphs, where $a = \log_2 3$.



$e_1(G) = 3$

$e_2(G) = 2$

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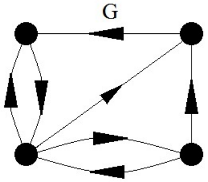

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So the number of oriented T_3-free graphs is at least $2^{\mathrm{ex}_a(n,T_3)}$, where

$$\mathrm{ex}_a(n, T_3) = \max\{e_1(G)+a\cdot e_2(G) : T_3\text{-free digraphs } G \text{ on } n \text{ vertices}\}.$$

Forbidding transitive triangles

$\mathrm{ex}_{\log_2 3}(n, T_3) = \log_2 3 \cdot \frac{n^2}{4}$
extremal graph: balanced complete bipartite digraph

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extremal graph: balanced complete bipartite digraph

Theorem (Kühn, Osthus, Townsend, Zhao, 2014+)

(i) Almost all T_3-free oriented graphs are bipartite (so # T_3-free oriented graphs on n vertices is $\approx 2^{\mathrm{ex}_{\log_2 3}(n, T_3)}$).

(ii) Almost all T_3-free digraphs are bipartite (so # T_3-free digraphs on n vertices is $\approx 2^{\mathrm{ex}_2(n, T_3)}$).

This generalises to T_k-free oriented graphs and digraphs.

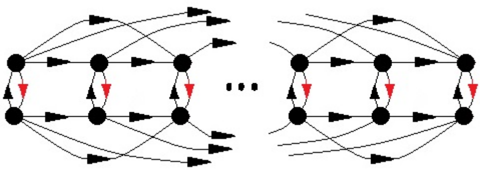
Forbidding cyclic triangles - oriented graphs

Recall:

Conjecture (Cherlin, 1998)

Almost all C_3-free oriented graphs are acyclic,
i.e. subgraphs of the transitive tournament T_n.

But $\mathrm{ex}_{\log_2 3}(n, C_3) = \binom{n}{2} + (\log_2 3 - 1)\frac{n}{2} > \binom{n}{2}$.
Extremal number attained by T_n + perfect matching.

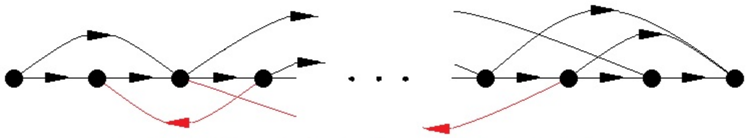


Forbidding cyclic triangles - oriented graphs

This motivates:

Theorem (Kühn, Osthus, Townsend, Zhao, 2014+)

Almost all C_3-free oriented graphs are almost - but not quite - acyclic, i.e.
$\frac{cn}{\log n} \leq$ typical $\#$ backwards edges in an optimal ordering $\ll n^2$.



Number of backwards edges is subquadratic but at least cn / log n

Conjecture (Kühn, Osthus, Townsend, Zhao, 2014+)

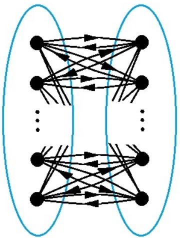
For almost all C_3-free oriented graphs, $\#$ backwards edges in an optimal ordering is $\Theta(n)$.

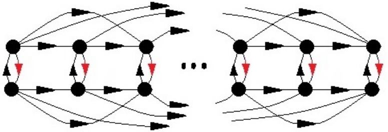
Given G, # subdigraphs of G is $2^{e_1(G)}4^{e_2(G)} = 2^{e_1(G)+2e_2(G)}$.
So # of C_3-free digraphs on n vertices is at least $2^{\mathrm{ex}_2(n,C_3)}$.

Theorem (Häggkvist, Thomassen, 1976)

$$\mathrm{ex}_2(n, C_3) = \frac{n^2}{2}.$$

Extremal graphs:



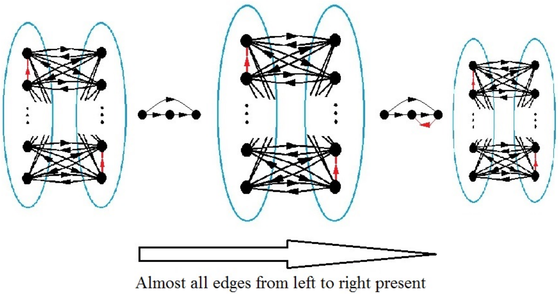


Forbidding cyclic triangles - digraphs

Theorem (Kühn, Osthus, Townsend, Zhao, 2014+)

Almost all C_3-free digraphs are close to being a subgraph of a transitive-bipartite blow up.

So # of C_3-free digraphs on n vertices is $2^{(1+o(1))\mathrm{ex}_2(n,C_3)}$.



Theorem generalises to longer forbidden cycles.
($DK_{\frac{n}{2},\frac{n}{2}}$ is only an extremal graph for odd cycles.)

Summary of results

- T_3-free oriented graphs: typically bipartite
- T_3-free digraphs: typically bipartite
- C_3-free oriented graphs: typically close to acyclic
- C_3-free digraphs: typically close to subgraph of transitive-bipartite blow up

Proofs

Step 1: Use container arguments + stability results for rough structure.
Step 2: Use this rough structure together with e.g. induction to obtain fine structure.

Proofs

Step 1: Use container arguments + stability results for rough structure.

Theorem - Containers
(Balogh, Morris, Samotij, 2014+; Saxton, Thomason, 2014+)

For all oriented graphs H and all sufficiently large n, there exists a collection of digraphs $\mathcal{C}$ such that

(a) For every H-free digraph G on $[n]$ there exists $C \in \mathcal{C}$ such that $G \subseteq C$,

(b) for every $C \in \mathcal{C}$, C contains almost no copies of H,

(c) $|\mathcal{C}| = 2^{o(n^2)}$.

Note that (b) together with supersaturation $\implies$

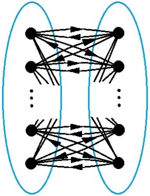
$$e_a(C) := e_1(C) + a \cdot e_2(C) \leq (1 + o(1))\mathrm{ex}_a(n, H).$$

Important ingredient: weighted stability results

Theorem (Kühn, Osthus, Townsend, Zhao, 2014+)

(i) If G is a T_3-free digraph with $e_{\log_2 3}(G) \approx \mathrm{ex}_{\log_2 3}(n, T_3)$ then G is close to the complete bipartite digraph.

(ii) If G is a T_3-free digraph with $e(G) \approx \mathrm{ex}(n, T_3)$ then G is close to the complete bipartite digraph.



Stability results II

Theorem (Kühn, Osthus, Townsend, Zhao, 2014+)

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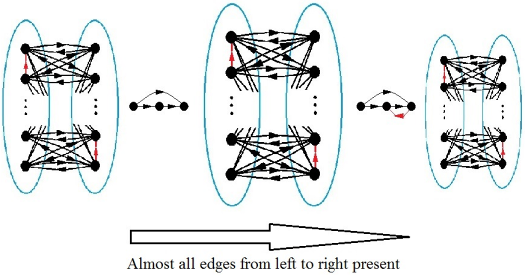
(ii) If G is a C_3-free digraph with $e(G) \approx \mathrm{ex}(n, C_3) = \frac{n^2}{2}$ then G is close to a transitive-bipartite blow up.



Summary

Oriented graphs and digraphs have a much ‘richer’ and more unexpected behaviour than (undirected) graphs regarding

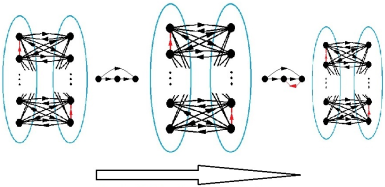
- extremal graphs
- stability results
- typical structure.

Open problems

Pin down the typical structure of C_3-free digraphs. Recall:

Theorem (Kühn, Osthus, Townsend, Zhao, 2014+)

Almost all C_3-free digraphs are close to being a subgraph of a transitive-bipartite blow up.



Can one strengthen this to prove that they are typically almost transitive?

Open problems II

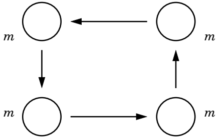
- What about forbidding other tournaments (i.e. non-transitive ones)?
- Sparse setting (i.e. H-free digraphs on n vertices and m edges)?
- Forbidding arbitrary digraphs? (container results do not hold for every digraph)

Open problems III: short cycles in oriented graphs

Conjecture

Suppose that G is oriented with $\delta^+, \delta^- \geq \lfloor n/4 \rfloor + 1$. Then G contains a 6-cycle.

Conjectured extremal example



Special case of a more general conjecture on arbitrary cycle lengths

More general conjecture

Conjecture (Kelly, Kühn, Osthus)

Let $\ell \geq 4$ and let $k \geq 3$ be minimal such that k does not divide ℓ. Then there exists $n_0 = n_0(\ell)$ such that every oriented graph G on $n \geq n_0$ vertices with $\delta^+(G), \delta^-(G) \geq \lfloor n/k \rfloor + 1$ contains an ℓ-cycle.

Previous partial results:

- Kelly, Kühn & Osthus (2010):
- Kühn, Osthus & Piguet (2013):