The Regularity Lemma and applications to packings in graphs

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Abstract

In this thesis we investigate applications of Szemerédi's Regularity Lemma [20]. This result was originally formulated to solve number-theoretical problems. However, we consider its applications in graph theory, in particular to packing results. We begin by introducing some of the basic notions that are needed to understand and use the Regularity Lemma. From this we give an outline of some tools which are useful in applications of the Regularity Lemma. For example, given a graph H we will see that the Key Lemma (see Section 3.2) can be applied to find almost perfect H-packings in graphs, whereas the Blow-up Lemma (see Section 7.2) is useful for finding perfect H-packings in graphs. Furthermore we give several examples which use these results. For instance, in Chapter 4 we include proofs of the Erdős-Stone Theorem [6] and the Alon-Yuster Theorem on almost perfect packings [1].

We give an account of several results concerning H-packings in large dense graphs. For example, when considering graphs G with large minimum degree Komlós' Theorem [11] tells us that the critical chromatic number of H is the parameter which governs whether G has a perfect H-packing. The Alon-Yuster Theorem on perfect packings [2] and a result by Kühn and Osthus [16] determine the corresponding parameter for perfect H-packings. We prove this result in Chapter 7.

We also investigate similar results involving so-called Ore-type degree conditions. In Chapter 5 we establish an analogue of Komlós' Theorem for such degree conditions. Further, in Chapter 8 we prove an analogue of the Alon-Yuster Theorem on perfect packings. However, we also see that the characterisation of the parameter that governs whether a graph has a perfect H-packing is not the same when considering minimum and Ore-type degree conditions. Indeed, we provide an example that shows that an Ore-type analogue of the result by Kühn and Osthus does not exist. This leads into the interesting question for which graphs we can improve the degree condition in the Ore-type analogue of the Alon-Yuster Theorem on perfect packings.

We should make the preliminary remark that throughout this thesis floors and ceilings are ignored whenever this does not affect the argument given.

Contents

Chapter 1

Introduction to ϵ-regularity

1.1 Density and ϵ-regular pairs

The majority of this thesis is concerned with the applications of Szemerédi's Regularity Lemma [20]. Before we discuss this we must be familiar with some of the most central and basic concepts concerning this topic. Thus, this section aims to introduce notions of density and regularity, as well as bringing together some of the simple results surrounding them.

Definition 1.1 *(Density) Let G be a graph, and let $X, Y \subseteq V(G)$ be disjoint. In particular G could be bipartite with vertex classes X and Y. We define the density, $d(X,Y)$, of the pair (X,Y) as: $d(X,Y) := \frac{e(X,Y)}{|X||Y|}$.*

Notice that the maximum possible number of X-Y edges is $|X||Y|$. Hence, the density of such a pair is a real number between 0 and 1: 0 when there are no X-Y edges, 1 when there are all possible X-Y edges. Thus, the density of (X,Y) gives the proportion of the pairs $(x,y) \subseteq X \times Y$ that form an edge xy in G. So a bipartite graph with density close to 0 is, in some sense sparse, whereas if it had density close to 1 the graph is dense.

With this definition we can now begin to consider the idea of ϵ-regularity.

Definition 1.2 *(ϵ-regularity) Let $\epsilon > 0$. Given a graph G and disjoint vertex sets $A, B \subseteq V(G)$ we say the pair (A,B) is ϵ-regular if for every $X \subseteq A$ and $Y \subseteq B$ such that $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have $|d(X,Y) - d(A,B)| < \epsilon$.*

The definition formalises the concept of a pair of vertex classes having the edges between them distributed fairly uniformly. That is, we are not in a situation where one (not too small) section of the pair yields a high concentration of edges compared to another. Thus, given an ϵ-regular pair (A,B), if we consider $X \subseteq A$ and $Y \subseteq B$ such that X and Y are not too small, the density of (X,Y) will be close to the density of (A,B). Notice the smaller we make ϵ, the closer the density of (X,Y) must be to $d(A,B)$, and that we consider smaller vertex classes X and Y. Thus, the smaller ϵ is, the more uniform the pair (A,B) will be.

We should also note that the condition that $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ in the definition is important. If we dropped this condition, then we could consider

$X \subseteq A$ and $Y \subseteq B$ such that $|X| = |Y| = 1$. Then $d(X, Y) = e(X, Y)$, but as there is only one vertex in both vertex classes there is either one edge between X and Y or none. That is, $d(X, Y) = 1$ or 0. So for a pair (A, B) to be ϵ-regular we would either have

1) $1 - d(A, B) < \epsilon$ or

2) $d(A, B) < \epsilon$.

Thus, the definition would then be about whether a pair (A, B) is very dense or very sparse, not whether the edges between them are uniformly distributed.

We are now in a position to introduce some simple results about ϵ-regularity.

Lemma 1.3 *Let (A, B) be an ϵ-regular pair in G of density d. Then (A, B) is also an ϵ-regular pair with density $1 - d$ in $\overline{G}$*

Proof. Consider any $X \subseteq A$, $Y \subseteq B$ such that $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$. Then by hypothesis we have $|d_G(X, Y) - d| < \epsilon$. Now $d_G(X, Y) := \frac{e_G(X,Y)}{|X||Y|}$ and $e_{\overline{G}}(X, Y) = |X||Y| - e_G(X, Y)$, therefore

$$d_{\overline{G}}(X, Y) = \frac{|X||Y| - e_G(X, Y)}{|X|Y|} = 1 - d_G(X, Y).$$

In particular this shows $d_{\overline{G}}(A, B) = 1 - d_G(A, B) = 1 - d$. Thus,

$$|d_{\overline{G}}(X, Y) - (1 - d)| = |(1 - d_G(X, Y)) - (1 - d)| = |d_G(X, Y) - d| < \epsilon,$$

as required. □

Lemma 1.4 *Let (A, B) be an ϵ-regular pair of density d and $Y \subseteq B$ such that $|Y| > \epsilon|B|$. Then,*

$$|\{x \in A|\ d_Y(x) \leq (d - \epsilon)|Y|\}| \leq \epsilon|A|,$$

i.e. all but at most $\epsilon|A|$ vertices in A have more than $(d - \epsilon)|Y|$ neighbours in Y each.

Proof. Let $X := \{x \in A|\ d_Y(x) \leq (d - \epsilon)|Y|\} \subseteq A$. Assume $|X| > \epsilon|A|$. Then as (A, B) is an ϵ-regular pair and $|Y| > \epsilon|B|$, we have that $d(X, Y) > d - \epsilon$.

However, $e(X, Y) \leq |X|(d - \epsilon)|Y|$ by definition of X. Thus, $d(X, Y) \leq d - \epsilon$, a contradiction. So our assumption that $|X| > \epsilon|A|$ was false. □

The previous two results followed straight from the definition of ϵ-regularity. The last of these two results just tells us that few vertices in an ϵ-regular pair have few neighbours (compared to the expected number of neighbours) in any relatively large set of vertices. This result is what one should expect to be a natural consequence of the definition of an ϵ-regular pair: If the distribution of edges in a bipartite graph is uniform, we should not have lots of vertices with fewer than the expected number of neighbours. Else, restricting our attention to this section of the graph, the density will be significantly less than in the whole graph, a contradiction to ϵ-regularity.

This idea can be extended so that if $Y \subseteq B$ is not too small then we cannot have many l-tuples of vertices that have few common neighbours in Y. Indeed, Lemma 1.4 is just the base case for the following result.

Lemma 1.5 *Let $l \in \mathbb{N}$. If (A, B) is an ϵ-regular pair with density d and $Y \subseteq B$, such that $(d-\epsilon)^{l-1}|Y| > \epsilon|B|$ then,*

$$\left| \left\{ (x_1, x_2, \ldots, x_l) : x_i \in A, \left| \bigcap_{i=1}^{l} N_Y(x_i) \right| \leq (d-\epsilon)^l |Y| \right\} \right| \leq l\epsilon|A|^l.$$

Proof. Throughout the proof when we refer to a set Y we mean $Y \subseteq B$. Also we can assume $d > \epsilon$ else the result is trivial.

Let $Z_l^Y := \{(x_1, x_2, \ldots, x_l) : x_i \in A, |\bigcap_{i=1}^{l} N_Y(x_i)| \leq (d-\epsilon)^l |Y|\}$, where $(d-\epsilon)^{l-1}|Y| > \epsilon|B|$. We will prove the claim by induction on l.

If $l = 1$ then by Lemma 1.4, the result holds. So now let $k > 1$ and assume the result holds for $l = k-1$. Thus, $|Z_{k-1}^Y| \leq (k-1)\epsilon|A|^{k-1}$ for all Y such that $(d-\epsilon)^{k-2}|Y| > \epsilon|B|$.

Now suppose $(d-\epsilon)^{k-1}|Y| > \epsilon|B|$. We wish to count the members of Z_k^Y. Notice $(d-\epsilon)^{k-2}|Y| > (d-\epsilon)^{k-1}|Y| > \epsilon|B|$ as $1 > d-\epsilon > 0$. So we can consider Z_{k-1}^Y. Given some k-tuple $(x_1, x_2, \ldots, x_{k-1}, x_k) \in Z_k^Y$, we have two possibilities: Firstly, we may have that $(x_1, x_2, \ldots, x_{k-1}) \in Z_{k-1}^Y$. There are at most $(k-1)\epsilon|A|^{k-1}$ members of Z_{k-1}^Y, so in this case there are at most this many possibilities for $(x_1, x_2, \ldots, x_{k-1})$. There are $|A|$ possibilities for x_k. So in total there are at most $(k-1)\epsilon|A|^{k-1}|A| = (k-1)\epsilon|A|^k$ such members of Z_k^Y.

Otherwise we must have that $(x_1, x_2, \ldots, x_{k-1}) \notin Z_{k-1}^Y$. There are up to $|A|^{k-1}$ such $(k-1)$-tuples since there are $|A|$ possibilities for each x_i. If $(x_1, x_2, \ldots, x_{k-1}) \notin Z_{k-1}^Y$ then these x_i have more than $(d-\epsilon)^{k-1}|Y|$ common neighbours in $|Y|$. Consider this set N of neighbours.

Now x_k must have at most $(d-\epsilon)^k|Y| < (d-\epsilon)|N|$ neighbours in $N \subseteq Y$. Since $|N| > (d-\epsilon)^{k-1}|Y| > \epsilon|B|$ our base case shows that there are only at most $\epsilon|A|$ such x_k. Thus, in total there can only be at most $|A|^{k-1} \cdot \epsilon|A| = \epsilon|A|^k$ such members of Z_k^Y.

In total therefore, we must have $|Z_k^Y| \leq (k-1)\epsilon|A|^k + \epsilon|A|^k = k\epsilon|A|^k$ as required. So we have proved the claim by induction. □

The next lemma tells us that reasonable size subgraphs of regular pairs are also regular.

Lemma 1.6 *(Slicing Lemma) Let $\alpha > \epsilon > 0$ and $\epsilon' := max\{\epsilon/\alpha, 2\epsilon\}$. Let (A, B) be an ϵ-regular pair with density d. Suppose $A' \subseteq A$ such that $|A'| \geq \alpha|A|$, and $B' \subseteq B$ such that $|B'| \geq \alpha|B|$. Then (A', B') is an ϵ'-regular pair with density d' where $|d' - d| < \epsilon$.*

Proof. Firstly, $|A'| \geq \alpha|A| > \epsilon|A|$ and $|B'| \geq \alpha|B| > \epsilon|B|$ since $\alpha > \epsilon$. Thus, as (A, B) is ϵ-regular we know $|d' - d| < \epsilon$. Consider $X \subseteq A'$ and $Y \subseteq B'$ such that $|X| > \epsilon'|A'|$ and $|Y| > \epsilon'|B'|$. Then as $\epsilon' \geq \epsilon/\alpha$ and $|A'| \geq \alpha|A|$ we have

$$|X| > \epsilon'|A'| \geq \frac{\epsilon}{\alpha}|A'| \geq \epsilon|A|.$$

Similarly we obtain $|Y| > \epsilon|B|$.

Therefore, as (A, B) is ϵ-regular, we have $|d(X,Y) - d| < \epsilon$. Thus, as $\epsilon \leq \epsilon'/2$ and by the triangle inequality,

$$|d(X,Y) - d'| = |(d(X,Y) - d) + (d - d')| \leq |d(X,Y) - d| + |d' - d| < \epsilon + \epsilon \leq \epsilon'.$$

So, by definition, (A', B') is an ϵ'-regular pair. $\square$

The Slicing Lemma tells us that not too small subgraphs of an ϵ-regular pair are also regular with density close to that of the original pair. To get an idea as to why it is useful suppose that we are in a situation, where, for whatever reason, we only consider some of the vertices in an ϵ-regular pair. Then it seems good to know that all the properties of the original pair do not just disappear. We will see that knowing this is useful in the proof of the Alon-Yuster Theorem (Theorem 2.8 in Chapter 2). But for now we will be content with an application of the Slicing Lemma which links the notion of regularity to that of super-regularity.

Definition 1.7 *(Super-regularity) Given a graph G and disjoint vertex sets $A, B \subseteq V(G)$, we say the pair (A, B) is (ϵ, δ)-super-regular if for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have*

$$e(X,Y) > \delta|X||Y|$$

and futhermore, $d_B(a) > \delta|B|$ for all $a \in A$, and $d_A(b) > \delta|A|$ for all $b \in B$.

Next we see that given a regular pair we can approximate it by a super-regular pair.

Lemma 1.8 *If (A, B) is an ϵ-regular pair with density d in a graph G (where $0 < \epsilon < 1/3$), then there exists $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq (1-\epsilon)|A|$ and $|B'| \geq (1-\epsilon)|B|$, such (A', B') is a $(2\epsilon, d - 3\epsilon)$-super-regular pair.*

Proof. Let A' be the set of all vertices $x \in A$ such that $d_B(x) \geq (d-\epsilon)|B|$. Notice Lemma 1.4 implies $|A'| \geq (1-\epsilon)|A|$. Similarly, let B' be the set of all vertices $y \in B$ such that $d_A(y) \geq (d-\epsilon)|A|$. So again, $|B'| \geq (1-\epsilon)|B|$. Now let $\alpha := 1/2 > \epsilon$. We have $|A'| \geq (1-\epsilon)|A| > \alpha|A|$ and $|B'| \geq (1-\epsilon)|B| > \alpha|B|$. So, by the Slicing Lemma, (A', B') is a 2ϵ-regular pair with density d', where $d' > d - \epsilon$. In particular this means for all $X \subseteq A'$, $Y \subseteq B'$, such that $|X| > 2\epsilon|A'|$ and $|Y| > 2\epsilon|B'|$ we have $|d(X,Y) - d'| < 2\epsilon$. So $d(X,Y) > d' - 2\epsilon > d - 3\epsilon$. Hence,

$$e(X,Y) > (d - 3\epsilon)|X||Y|.$$

Further, if $x \in A'$, $d_B(x) > (d-\epsilon)|B|$ and if $y \in B'$, $d_A(y) > (d-\epsilon)|A|$. Since $|A'| \geq (1-\epsilon)|A|$ and $|B'| \geq (1-\epsilon)|B|$ this tells us

$$d_{B'}(x) > (d-\epsilon)|B| - \epsilon|B| > (d - 3\epsilon)|B'|$$

and

$$d_{A'}(y) > (d-\epsilon)|A| - \epsilon|A| > (d - 3\epsilon)|A'|.$$

Thus, the result holds. $\square$

All we did in the proof above was discard the vertices of small degree in the ϵ-regular pair, and apply two of our lemmas. We could have been more precise and noticed that our pair (A', B') was $\frac{\epsilon}{1-\epsilon}$-regular for example. However, what is more important is to see the overlap in the notions of regularity and super-regularity. Notice though that super-regularity is a one-sided version of regularity: If (A, B) is (ϵ, δ)-super-regular then for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have $e(X, Y) > \delta|X||Y|$. However, if (A, B) is ϵ-regular with density d, then for X and Y as above, we have $(d + \epsilon)|X||Y| > e(X, Y) > (d - \epsilon)|X||Y|$.

1.2 The Regularity Lemma

Next we introduce the stimulus for this thesis, Szemerédi's Regularity Lemma. We will not state the original version of the lemma, but a 'cleaner' form of it. Later we will see that the theorem below is just one of a number of forms of the lemma.

Theorem 1.9 *(Szemerédi's Regularity Lemma [20]) For every $\epsilon > 0$ and every $m \in \mathbb{N}$ there exists an integer $M(\epsilon, m)$ such that every graph G of order $n \geq m$ admits a partition $\{V_0, V_1, \ldots, V_k\}$ of $V(G)$ such that:*

(i) $m \leq k \leq M$,

(ii) $0 \leq |V_0| \leq \epsilon|G|$,

(iii) $|V_1| = \cdots = |V_k|$,

(iv) all but at most ϵk^2 of the pairs (V_i, V_j) with $1 \leq i < j \leq k$ are ϵ-regular.

We call the set V_0 an exceptional set as, unlike usual partitions, V_0 may be empty. Further, the classes V_i are known as *clusters* and the partition described is called an *ϵ-regular partition.*

What the result essentially says is that all sufficiently large and dense graphs can be approximated by a 'random' graph. We can disregard the vertices in V_0 and the edges that lie inside some V_i or in a pair (V_i, V_j) which is not ϵ-regular. Then we have a subgraph such that all edges between two of our clusters are distributed fairly uniformly, as we would expect in a random graph. So V_0 acts like a bin, one which cannot get too full by condition (ii). We can choose m to be large so that the clusters are not too large and hence, most edges go between different clusters. Thus, together with condition (iv) this ensures not too many edges are disregarded when considering the random-like approximation we mentioned. The upper bound M, however, means that large graphs will have clusters that are large too.

Chapter 2

Embeddings and packings of graphs

2.1 Introduction to embedding results

In the previous chapter we began to get familiar with the Regularity Lemma. This chapter provides a breather before we venture into the applications of the lemma. We will give an account of some of the classical embedding and packing results in graph theory. That is, the chapter contains results concerning conditions that guarantee some H as a subgraph of other graphs. So for example, which conditions guarantee a copy of K_r in another graph G? As Diestel [4] puts it, 'we study how global parameters of a graph, such as its edge density or chromatic number, can influence its local substructures'. Questions such as whether a lower bound on the minimum degree of a graph G forces certain subgraphs in G are addressed.

There are several reasons for the importance of this chapter. Firstly, this subject, known as extremal graph theory, is an interesting area in its own right. Furthermore, many of the results given are applied in proofs involving the use of the Regularity Lemma. But, perhaps most importantly, we will later see that the Regularity Lemma is used in several of the proofs of these theorems.

We begin by introducing a definition.

Definition 2.1 *Let $n \in \mathbb{N}$ and H be a graph. Then*

$$ex(n, H) := \max\{e(G) : |G| = n \text{ and } H \nsubseteq G\}.$$

So $ex(n, H)$ is simply the maximum number of edges a graph on n vertices can have without containing a copy of H. A graph on n vertices with $ex(n, H)$ edges is called *extremal* for n and H. Notice that if a graph G on n vertices satisfies $e(G) > ex(n, H)$ then $H \subseteq G$. By looking at a special type of complete $(r-1)$-partite graphs, Turán [21] determined $ex(n, K_r)$. More precisely, the *Turán graph* $T_{r-1}(n)$ is the complete $(r-1)$-partite graph on n vertices such that the vertex classes are as equal as possible. So in particular $T_{r-1}(n) = K_n$

if $n < r$. We let $t_{r-1}(n) := e(T_{r-1}(n))$. It is not hard to see that

$$\lim_{n\to\infty} \frac{t_{r-1}(n)}{\binom{n}{2}} = 1 - \frac{1}{r-1} \tag{2.1}$$

and for $n \geq r$

$$t_{r-1}(n) \leq \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}. \tag{2.2}$$

Moreover, note that $t_{r-1}(n) = (1 - \frac{1}{r-1})\frac{n^2}{2}$ precisely when $(r-1)|n$. We have the following result about Turán graphs.

Theorem 2.2 *(Turán [21]) For $n, r \in \mathbb{N}$ with $r > 1$, we have $ex(n, K_r) = t_{r-1}(n)$. In particular if $|G| = n \geq r$ and $e(G) > (1 - \frac{1}{r-1})\frac{n^2}{2}$ then $K_r \subseteq G$.*

Notice that r is the chromatic number of K_r. Theorem 2.4 will show that this is important. Also, from our remarks before Theorem 2.2, we observe that the lower bound on the number of edges is asymptotically best possible.

Corollary 2.3 *For $r \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \frac{ex(n, K_r)}{\binom{n}{2}} = 1 - \frac{1}{r-1}.$$

These results by Turán were the first in extremal graph theory. The next result is an extension to this, giving a lower bound on the number of edges in a graph that guarantees some graph H as its subgraph.

Theorem 2.4 *(Erdős, Stone [6] and Erdős, Simonovits [5]) Given any $\epsilon > 0$ and any graph H there is an $N(H, \epsilon)$ such that if $n \geq N$ and G is a graph on n vertices with*

$$e(G) > \left(1 - \tfrac{1}{\chi(H)-1} + \epsilon\right)\tfrac{n^2}{2}$$

then $H \subseteq G$.

In Section 4.1 we will use the Regularity Lemma to prove this result, though it can also be proven directly. From Theorem 2.4 we can deduce the following.

Theorem 2.5 *(Fundamental Theorem of Extremal Graph Theory)*

$$\lim_{n\to\infty} \frac{ex(n, H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H) - 1}$$

What this tells us is that the chromatic number of H is the important value when it comes to forcing a copy H into another graph. All the results so far in this section have hinged on this: We have not been concerned with the minimum degree, number of edges, or any other property of H. In the next section we will see that $\chi(H)$ is also important when forcing multiple disjoint copies of H into another graph.

2.2 Packings

Definition 2.6 *(H-packing, H-factor) Given two graphs H and G an H-packing in G is a collection of vertex-disjoint copies of H. We call an H-packing in G perfect or an H-factor if all but at most $|H|-1$ vertices in G are covered by this H-packing. That is, there are $\lfloor \frac{|G|}{|H|} \rfloor$ copies of H in G.*

A natural question is whether we can get similar results to those in the previous section for H-packings. That is, if G satisfies certain conditions can we guarantee a perfect H-packing or at least an almost perfect H-packing in G. This obviously seems harder than guaranteeing just one copy of H in G. Thus, maybe we need to know more than the number of edges in a large graph G to determine whether it contains a perfect H-packing or not. Indeed, if we stop and think for a moment it seems clear that a large number of edges in a graph by no means forces a perfect H-packing: a large graph G could have lots of edges but in one section could not have enough to contain a copy of H. Thus, G may not have a perfect H-packing. For example, consider the disjoint union of K_n and $\overline{K}_r$. We can make n as large as we like, thus making the density of our graph very close to 1, yet it will not contain a perfect K_r-packing.

Our example highlights a key point; we must ensure that there are enough edges everywhere in a graph in order to ensure a perfect K_r-packing. Therefore, we should hope that a minimum degree condition on G (involving H somehow) should allow us to say something about perfect H-packings in G. We see this in terms of complete graphs in the next result.

Theorem 2.7 *(Hajnal, Szemerédi [8]) Let G be a graph on n vertices. If*

$$\delta(G) \geq \left(1 - \frac{1}{r}\right) n$$

then G has a perfect K_r-packing.

This looks very similar to Turán's Theorem. Our degree condition involves r, which is the chromatic number of K_r. As in Turán's Theorem this is the only piece of information about K_r in this condition. If we have a graph G such that $\delta(G) \geq (1-\frac{1}{r})n$ then we know $e(G) \geq (1-\frac{1}{r})\frac{n^2}{2}$. This is a stronger edge condition than the one which guaranteed just one copy of K_r in G. Thus, this reiterates that finding a perfect K_r-packing is harder than finding just one copy of K_r.

The Hajnal-Szemerédi Theorem is the 'equivalent' of Turán's Theorem for perfect K_r-packings. It would be nice to find a similar result for perfect H-packings, i.e. an analogous result to Theorem 2.4. The first result in this direction was proved by Alon and Yuster.

Theorem 2.8 *(Alon, Yuster [1]) For every graph H and $\epsilon > 0$ there exists a $n_0(H, \epsilon)$ such that, if G has $n \geq n_0$ vertices and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n$$

then G contains an H-packing covering all but at most ϵn vertices.

It is important that we are only considering almost perfect packings in Theorem 2.8 as under the current degree condition, the equivalent result for perfect packings is false (see Section 7.1 for more details). However, if we introduce an error term in the degree condition we can guarantee a perfect H-packing.

Theorem 2.9 *(Alon, Yuster [2]) For every H and $\epsilon > 0$ there exists a $n_0(H, \epsilon)$ such that, if G has $n \geq n_0$ vertices and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \epsilon\right) n$$

then G contains a perfect H-packing.

Theorem 2.8 can be proven quite easily using the Regularity Lemma and the Key Lemma. We will see this proof in Chapter 4. Chapter 5 will be concerned with whether we can improve the minimum degree condition in this result. In particular, we will see that we can replace $\chi(H)$ in the minimum degree condition of Theorem 2.8 with the so-called critical chromatic number. In Chapter 7 we will prove a result which strengthens Theorem 2.9 when considering H-packings for a special type of graphs H. We will then use this result to prove Theorem 2.9 in Section 7.4.

Alon and Yuster [2] conjectured that the slack ϵn could be replaced with a constant in the degree condition. This has been proved by Komlós, Sárközy and Szemerédi [13].

Theorem 2.10 *(Komlós, Sárközy and Szemerédi [13]) Given any graph H there is a constant $C(H)$ such that if a graph G on n vertices satisfies*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n + C$$

then it has a perfect H-packing.

Chapter 3

Proof of the Regularity Lemma and the Key Lemma

3.1 Proof of the Regularity Lemma

In this section we will prove Theorem 1.9. The proof given is based on the one found in [4]. The idea of the proof is the following: Given a graph G and a partition $\mathcal{P}$ of $V(G)$ we will introduce a quantity $q(\mathcal{P})$. If $\mathcal{P}$ is not an ϵ-regular partition then we can obtain a partition $\mathcal{P}'$ of $V(G)$ where $q(\mathcal{P}')$ is substantially larger than $q(\mathcal{P})$. This is made precise in Lemma 3.1. If $\mathcal{P}'$ is an ϵ-regular partition we are done. If not we can repeat the argument above to find another partition of $V(G)$. We will see that the value of q for any partition is bounded above by 1. So it will be the case that we can only apply our argument above a constant number of times. That is, eventually we will obtain an ϵ-regular partition of $V(G)$ consisting of a bounded number of clusters.

We now give an inequality which will be used to prove Theorem 1.9. For reals $a_1, \dots a_n, b_1, \dots, b_n$ we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.$$

This is known as the Cauchy-Schwarz inequality. Thus, for reals $\alpha_1, \dots, \alpha_n > 0$ and $\beta_1, \dots, \beta_n \geq 0$ we obtain

$$\frac{(\sum_{i=1}^{n} \beta_i)^2}{\sum_{i=1}^{n} \alpha_i} \leq \sum_{i=1}^{n} \frac{\beta_i^2}{\alpha_i} \tag{3.1}$$

by taking $a_i := \sqrt{\alpha_i}$ and $b_i := \frac{\beta_i}{\sqrt{\alpha_i}}$ in the Cauchy-Schwarz inequality.

We must also introduce some notation. Given a graph G on n vertices, and disjoint sets $A, B \subseteq V(G)$ we define

$$q(A, B) := \frac{|A||B|}{n^2} d(A, B)^2 = \frac{e(A, B)^2}{|A||B|n^2}.$$

Further, for partitions $\mathcal{A}$ of A and $\mathcal{B}$ of B we let

$$q(\mathcal{A}, \mathcal{B}) := \sum_{A' \in \mathcal{A}, B' \in \mathcal{B}} q(A', B'),$$

and given a partition $\mathcal{P} = \{C_1, \dots, C_k\}$ we define

$$q(\mathcal{P}) := \sum_{i<j} q(C_i, C_j).$$

If $\mathcal{P} = \{C_0, C_1, \dots, C_k\}$ is a partition of $V(G)$ with exceptional set C_0 we define

$$q(\mathcal{P}) := q(\mathcal{P}^*)$$

where $\mathcal{P}^* := \{C_1, \dots, C_k\} \cup \{\{v\} \mid v \in C_0\}$.

Given a partition $\mathcal{P} = \{C_1, \dots, C_k\}$ of a set V a *refinement* $\mathcal{P}'$ of $\mathcal{P}$ is a partition of V obtained from $\mathcal{P}$ by replacing each C_i in $\mathcal{P}$ by a partition $\mathcal{C}_i$ of C_i.

Recall we mentioned that given a partition $\mathcal{P}$ of $V(G)$ we have that $q(\mathcal{P}) \leq 1$. Indeed, if $\mathcal{P} = \{C_1, \dots, C_k\}$ we have that

$$q(\mathcal{P}) = \sum_{i<j} q(C_i, C_j) = \sum_{i<j} \frac{|C_i||C_j|}{n^2} d(C_i, C_j)^2 \leq \frac{1}{n^2} \sum_{i<j} |C_i||C_j| \leq 1. \quad (3.2)$$

As indicated above, the following lemma is crucial in the proof of Theorem 1.9. It shows that if a partition $\mathcal{P}$ of $V(G)$ fails to be ϵ-regular by virtue of containing too many irregular pairs of partition sets then we can subpartition these such sets, obtaining a partition $\mathcal{P}'$ of $V(G)$ where $q(\mathcal{P}') > q(\mathcal{P})$.

Lemma 3.1 *Let G be a graph on n vertices. Let $0 < \epsilon \leq \frac{1}{4}$ and $\mathcal{P} = \{C_0, C_1, \dots, C_k\}$ be a partition of $V(G)$, with exceptional set C_0 where $|C_0| \leq \epsilon n$ and $|C_1| = \dots = |C_k| =: c$. Suppose that $\mathcal{P}$ is not ϵ-regular. Then there is a partition $\mathcal{P}' = \{C'_0, C'_1, \dots, C'_l\}$ of $V(G)$ where $k \leq l \leq k4^k$. Further $\mathcal{P}'$ has exceptional set C'_0 where $|C'_0| \leq |C_0| + n/2^k$ and all other sets C'_i have equal size, and*

$$q(\mathcal{P}') \geq q(\mathcal{P}) + \epsilon^5/2.$$

In order to prove Lemma 3.1 we introduce the next two results. The first of these shows that the value of q for a refinement of a partition $\mathcal{P}$ is at least the value of $q(\mathcal{P})$.

Lemma 3.2 *Let G be a graph on n vertices.*

(i) Let $A, B \subseteq V(G)$ be disjoint. If $\mathcal{A}$ is a partition of A and $\mathcal{B}$ is a partition of B, then $q(\mathcal{A}, \mathcal{B}) \geq q(A, B)$.

(ii) If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $V(G)$ such that $\mathcal{P}'$ refines $\mathcal{P}$, then $q(\mathcal{P}') \geq q(\mathcal{P})$.

Proof. (i) Let $\mathcal{A} =: \{A_1, \dots, A_k\}$ and $\mathcal{B} =: \{B_1, \dots, B_l\}$. Then

$$\begin{aligned} q(\mathcal{A}, \mathcal{B}) = \sum_{i,j} q(A_i, B_j) = \frac{1}{n^2} \sum_{i,j} \frac{e(A_i, B_j)^2}{|A_i||B_j|} &\overset{(3.1)}{\geq} \frac{1}{n^2} \frac{(\sum_{i,j} e(A_i, B_j))^2}{\sum_{i,j} |A_i||B_j|} \\ = \frac{1}{n^2} \frac{e(A, B)^2}{(\sum_i |A_i|)(\sum_j |B_j|)} &= q(A, B). \end{aligned}$$

(ii) Let $\mathcal{P} =: \{C_1, \dots, C_k\}$, and for all $i \in [k]$ let $\mathcal{C}_i$ be the partition of C_i induced by $\mathcal{P}'$. Then

$$q(\mathcal{P}) = \sum_{i<j} q(C_i, C_j) \overset{(i)}{\leq} \sum_{i<j} q(\mathcal{C}_i, \mathcal{C}_j) \leq q(\mathcal{P}'),$$

as $q(\mathcal{P}') = \sum_i q(\mathcal{C}_i) + \sum_{i<j} q(\mathcal{C}_i, \mathcal{C}_j)$. □

The next lemma tells us that we can partition a pair of vertex classes which is not ϵ-regular to increase the value of q. The result will be used in the proof of Lemma 3.1 when we wish to refine a partition: we will subpartition pairs that are not ϵ-regular in the way described in Lemma 3.3.

Lemma 3.3 *Let G be a graph on n vertices. Let $\epsilon > 0$ and $A, B \subseteq V(G)$ be disjoint. If (A, B) is not ϵ-regular, then there are partitions $\mathcal{A} = \{A_1, A_2\}$ of A and $\mathcal{B} = \{B_1, B_2\}$ of B such that*

$$q(\mathcal{A}, \mathcal{B}) > q(A, B) + \epsilon^4 \frac{|A||B|}{n^2}.$$

Proof. Suppose (A, B) is not ϵ-regular. Then by definition there exists $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $|A_1| > \epsilon|A|$, $|B_1| > \epsilon|B|$ and

$$|\eta| > \epsilon \tag{3.3}$$

where $\eta := d(A_1, B_1) - d(A, B)$. Let $A_2 := A \backslash A_1$ and $B_2 := B \backslash B_1$. Thus we obtain partitions $\mathcal{A} := \{A_1, A_2\}$ and $\mathcal{B} := \{B_1, B_2\}$ of A and B respectively. Now,

$$\begin{aligned} q(\mathcal{A}, \mathcal{B}) &= \frac{1}{n^2} \sum_{i,j} \frac{e(A_i, B_j)^2}{|A_i||B_j|} = \frac{1}{n^2} \left(\frac{e(A_1, B_1)^2}{|A_1||B_1|} + \sum_{i+j>2} \frac{e(A_i, B_j)^2}{|A_i||B_j|} \right) \\ &\overset{(3.1)}{\geq} \frac{1}{n^2} \left(\frac{e(A_1, B_1)^2}{|A_1||B_1|} + \frac{(e(A, B) - e(A_1, B_1))^2}{|A||B| - |A_1||B_1|} \right). \end{aligned}$$

By definition of η we have $e(A_1, B_1) = |A_1||B_1|e(A, B)/|A||B| + \eta|A_1||B_1|$, hence,

$$\begin{aligned} n^2 q(\mathcal{A}, \mathcal{B}) \geq{}& \frac{1}{|A_1||B_1|} \left(\frac{|A_1||B_1|e(A, B)}{|A||B|} + \eta|A_1||B_1| \right)^2 \\ &+ \frac{1}{|A||B| - |A_1||B_1|} \left(\frac{|A||B| - |A_1||B_1|}{|A||B|} e(A, B) - \eta|A_1||B_1| \right)^2 \\ ={}& \frac{|A_1||B_1|e(A, B)^2}{|A|^2|B|^2} + \frac{2\eta e(A, B)|A_1||B_1|}{|A||B|} + \eta^2|A_1||B_1| \\ &+ \frac{|A||B| - |A_1||B_1|}{|A|^2|B|^2} e(A, B)^2 - \frac{2\eta e(A, B)|A_1||B_1|}{|A||B|} + \frac{\eta^2|A_1|^2|B_1|^2}{|A||B| - |A_1||B_1|} \\ \geq{}& \frac{e(A, B)^2}{|A||B|} + \eta^2|A_1||B_1| \overset{(3.3)}{>} \frac{e(A, B)^2}{|A||B|} + \epsilon^4|A||B| \end{aligned}$$

since $|A_1| > \epsilon|A|$ and $|B_1| > \epsilon|B|$. □

We are now in a position to prove Lemma 3.1.

Proof of Lemma 3.1. For all $1 \leq i < j \leq k$ we define a partition $\mathcal{C}_{ij}$ of C_i as follows: Given an ϵ-regular pair (C_i, C_j) we let $\mathcal{C}_{ij} := \{C_i\}$ and $\mathcal{C}_{ji} := \{C_j\}$. Otherwise, we choose the partitions as in Lemma 3.3. That is $\mathcal{C}_{ij}$ and $\mathcal{C}_{ji}$ are partitions of C_i and C_j respectively such that $|\mathcal{C}_{ij}| = |\mathcal{C}_{ji}| = 2$ and

$$q(\mathcal{C}_{ij}, \mathcal{C}_{ji}) \geq q(C_i, C_j) + \frac{\epsilon^4 c^2}{n^2}. \tag{3.4}$$

Given a fixed $1 \leq i \leq k$ we say two elements of C_i are equivalent if they lie in the same partition set of $\mathcal{C}_{ij}$ for every $j \neq i$. We thus define $\mathcal{C}_i$ to be the partition of C_i whose partition sets are precisely the sets of equivalence classes of the equivalence relation just described. Since, for all i, j, $|\mathcal{C}_{ij}| \leq 2$, we have $|\mathcal{C}_i| \leq 2^{k-1}$. Consider the partition

$$\mathcal{C} := \{C_0\} \cup \bigcup_{i=1}^{k} \mathcal{C}_i$$

of $V(G)$, with exceptional set C_0. Hence, $\mathcal{C}$ refines $\mathcal{P}$, and

$$k \leq |\mathcal{C}| \leq k2^k. \tag{3.5}$$

Let $\mathcal{C}_0 := \{\{v\} \,|\, v \in C_0\}$. Since $\mathcal{P}$ is not ϵ-regular there are more than ϵk^2 pairs (C_i, C_j), with $i < j$, and non-trivial partition $\mathcal{C}_{ij}$. Note when considering $q(\mathcal{C})$ we must consider the definition of q concerning partitions with an exceptional set. Thus, by part (i) of Lemma 3.2,

$$\begin{aligned}
q(\mathcal{C}) &= \sum_{1 \leq i < j} q(\mathcal{C}_i, \mathcal{C}_j) + \sum_{1 \leq i} q(\mathcal{C}_0, \mathcal{C}_i) + \sum_{0 \leq i} q(\mathcal{C}_i) \\
&\geq \sum_{1 \leq i < j} q(\mathcal{C}_{ij}, \mathcal{C}_{ji}) + \sum_{1 \leq i} q(\mathcal{C}_0, \{C_i\}) + q(\mathcal{C}_0) \\
&\overset{(3.4)}{\geq} \sum_{1 \leq i < j} q(C_i, C_j) + \epsilon k^2 \frac{\epsilon^4 c^2}{n^2} + \sum_{1 \leq i} q(\mathcal{C}_0, \{C_i\}) + q(\mathcal{C}_0) \\
&= q(\mathcal{P}) + \epsilon^5 \left(\frac{kc}{n}\right)^2 \geq q(\mathcal{P}) + \epsilon^5/2.
\end{aligned}$$

Note the last inequality holds since $|C_0| \leq \epsilon n \leq \frac{1}{4}n$, so $kc \geq \frac{3}{4}n$.

To complete the proof the idea will be to 'chop-up' the sets in $\mathcal{C}$ to obtain small sets of equal size thus giving us our desired partition $\mathcal{P}'$. Let $d := \lfloor c/4^k \rfloor$. Given each $C \in \mathcal{C} \backslash \{C_0\}$ we partition C into as many sets as possible of size d (so some vertices in C may not belong to one of these sets). We define $C'_1, \dots, C'_l$ to be the collection of disjoint sets of size d thus obtained. So $C'_i \subseteq C$ for some $C \in \mathcal{C} \backslash \{C_0\}$. Let $C'_0 := V(G) \backslash \bigcup C'_i$. Thus, $\mathcal{P}' := \{C'_0, C'_1, \dots, C'_l\}$ is a partition of $V(G)$. Notice that $(\mathcal{P}')^*$ refines $\mathcal{C}^*$, so Lemma 3.2(ii) implies

$$q(\mathcal{P}') \geq q(\mathcal{C}) \geq q(\mathcal{P}) + \epsilon^5/2.$$

Each set $C'_i \neq C'_0$ is contained in one of the sets $C_1, \ldots, C_k$, but by our choice of d only at most 4^k such sets can lie inside the same C_j. Thus, $k \leq l \leq k4^k$. At most d vertices from each set $C \neq C_0$ in $\mathcal{C}$ lie in C'_0. Hence,

$$|C'_0| \leq |C_0| + d|\mathcal{C}| \overset{(3.5)}{\leq} |C_0| + \frac{c}{4^k} k 2^k = |C_0| + ck/2^k \leq |C_0| + n/2^k.$$

Thus, we have obtained the desired partition $\mathcal{P}'$ of $V(G)$. □

We are now able to prove Theorem 1.9 by repeated use of Lemma 3.1.

Proof of Theorem 1.9. The idea of the proof is to repeatedly refine a partition $\mathcal{P}$ of $V(G)$ until after a bounded number of refinements we obtain an ϵ-regular partition of $V(G)$.

Let $\epsilon > 0$ and $m \geq 1$ be given. We may assume that $\epsilon \leq 1/4$. Now (3.2) gives us an upper bound $s := 2/\epsilon^5$ on the number of iterations of Lemma 3.1 that can be applied to $\mathcal{P}$ before we obtain an ϵ-regular partition. So if we are able to apply Lemma 3.1 s times then we obtain an ϵ-regular partition of G. It is just left to show this.

Given a partition $\{C_0, C_1, \ldots, C_k\}$ of $V(G)$ with $|C_1| = \cdots = |C_k|$, in order to (re)-apply Lemma 3.1 the exceptional set C_0 must have size at most ϵn. With each iteration of the lemma the size of the exceptional set increases by at most $n/2^k$. So let $k \geq m$ be large enough so that $2^{k-1} \geq s/\epsilon$. Then $s/2^k \leq \epsilon/2$, and hence for all $n' \geq 2k/\epsilon$ we have

$$k + \frac{s}{2^k} n' \leq \epsilon n'. \tag{3.6}$$

We now wish to find an upper bound M on the number of non-exceptional sets in our partition after up to s iterations. Given one iteration of Lemma 3.1, if we have r such sets then this will grow to at most $r4^r$. So given the function $f : x \mapsto x4^x$, we define $M := \max\{f^s(k), 2k/\epsilon\}$.

Now, if $n \leq M$, we can partition $V(G)$ into $k := n$ singleton sets with $V_0 := \emptyset$. This partition is ϵ-regular. If $n > M$, we define $\mathcal{P} := \{C_0, C_1, \ldots, C_k\}$ as follows: $C_0 \subseteq V(G)$ is chosen to be of smallest size so that k is a factor of $|V(G) \backslash C_0|$, and $\{C_1, \ldots, C_k\}$ is any partition of $V(G) \backslash C_0$ into sets of equal size. Then $|C_0| < k \leq \epsilon n$ by (3.6). Starting with $\mathcal{P}$ we repeatedly apply Lemma 3.1 until we obtain an ϵ-regular partition of G: this will happen as by (3.6) the size of the exceptional set is bounded from above by ϵn and so we can apply the lemma up to the s times required to obtain an ϵ-regular partition. □

3.2 The degree form of the Regularity Lemma and the Key Lemma

We know that an ϵ-regular partition of G gives us the structure that most pairs of clusters are ϵ-regular. We would like to be able to use this uniformity property. Hence, the rest of this chapter aims to give results that will be essential in applying the Regularity Lemma. We also want to begin to see in what circumstances we can apply our results.

It is clear that we can get rid of the exceptional set V_0 in an ϵ-regular partition of a graph G by distributing its vertices as evenly as possible throughout the other clusters, whilst ϵ-regularity among pairs of clusters is preserved with a slightly larger ϵ. So we obtain the following alternative form of the Regularity Lemma.

Theorem 3.4 *(Regularity Lemma – alternative form) For every $\epsilon > 0$ there exists $M(\epsilon)$ such that for any graph G on n vertices, $V(G)$ can be partitioned into k sets $V_1, \dots, V_k$, for some $k \leq M$ such that:*

(i) $|V_i| \leq \lceil \epsilon n \rceil$ *for every* i,

(ii) $||V_i| - |V_j|| \leq 1$ *for all* i, j,

(iii) *all but at most* ϵk^2 *of the pairs* (V_i, V_j) *are* ϵ*-regular.*

The next result, the so-called degree form of the Regularity Lemma, is a very useful and applicable form of the Regularity Lemma.

Theorem 3.5 *(Degree form of the Regularity Lemma) For every $\epsilon > 0$ there is an $M = M(\epsilon)$ such that if G is any graph and $d \in [0, 1]$, then there exists a partition of $V(G)$ into $k + 1$ clusters $V_0, V_1, ..., V_k$, and there is a spanning subgraph $G' \subseteq G$ with the following properties:*

- $k \leq M$,
- $|V_0| \leq \epsilon|G|$,
- *all clusters* V_i *for* $i \geq 1$ *are of the same size* $m \leq \lceil \epsilon|G| \rceil$,
- $d_{G'}(v) > d_G(v) - (d + \epsilon)|G|$ *for all* $v \in V(G)$,
- $e(G'[V_i]) = 0$ *for all* $i \geq 1$,
- *all pairs* $G'[V_i, V_j]$ $(1 \leq i < j \leq k)$ *are* ϵ*-regular, each with density either* 0 *or greater than* d.

What does this theorem tell us? Given a graph G, $\epsilon > 0$ and $d \in [0, 1]$ we can obtain a subgraph $G' \subseteq G$ which is a good approximation of G and has some structure. More precisely, what we mean by G' being a good approximation of G is that $V(G') = V(G)$ and $d_{G'}(v) > d_G(v) - (d + \epsilon)$ for all $v \in V(G)$. So for small ϵ and d this tells us that G' can be obtained from G by removing not too many edges at each vertex $v \in V(G)$. Another difference between Theorem 1.9 and Theorem 3.5 is that in G' all the pairs of clusters which do not form an ϵ-regular pair with density greater than d are empty. So G' can be thought of as obtained from G by tidying up. The main point of Theorem 3.5 is that this can be done in such a way that the degree of each vertex is only reduced slightly. In fact, Theorem 3.5 can be deduced from Theorem 1.9 by tidying up the ϵ-regular partition obtained there.

We can 'clean up' G' further by defining the *pure graph* of G to be $G'' = G' - V_0$. So a pure graph of G has all the nice structure of G' but we get rid of the exceptional set V_0 which tarnishes the neatness of G' somewhat. A convention we will adopt when considering a pure graph G'' or indeed G' is that these graphs contain the maximal number of edges possible for the partition of $V(G)$ associated with it. That is, if G^* is a spanning subgraph of G satisfying all the properties in Theorem 3.5 that G' satisfies, then $e(G^*) \leq e(G)$. So we do not remove edges from G to obtain G' or G'' if it is unnecessary to do so.

Since $|V_0| \leq \epsilon|G|$ we have that

$$d_{G''}(v) > d_G(v) - (d + 2\epsilon)|G|$$

for all $v \in V(G'')$. Thus,

$$\begin{aligned} e(G'') = \frac{1}{2} \sum_{v \in V(G'')} d_{G''}(v) &> \frac{1}{2} \left(\sum_{v \in V(G'')} d_G(v) \right) - \frac{|G''|}{2}(d + 2\epsilon)|G| \\ &\geq \frac{1}{2} \left(\sum_{v \in V(G'')} d_G(v) \right) - (d + 2\epsilon)\frac{|G|^2}{2}. \end{aligned}$$

Moreover,

$$\begin{aligned} e(G) = \frac{1}{2} \sum_{v \in V(G)} d_G(v) &= \frac{1}{2} \left(\sum_{v \in V(G - V_0)} d_G(v) + \sum_{v \in V_0} d_G(v) \right) \\ &\leq \frac{1}{2} \left(\sum_{v \in V(G'')} d_G(v) \right) + \frac{\epsilon|G|^2}{2} \end{aligned}$$

since $|V_0| \leq \epsilon|G|$. Hence,

$$e(G'') > \left(e(G) - \frac{\epsilon|G|^2}{2} \right) - (d + 2\epsilon)\frac{|G|^2}{2} = e(G) - (d + 3\epsilon)\frac{|G|^2}{2}.$$

Notice that Theorem 3.5 seems to lack an explicit lower bound on the number of clusters in the partition of G, unlike Theorem 1.9. However, we can use the bound on the vertex degrees and the bound on the size of the clusters to obtain a lower bound on the number of clusters. Intuitively it is clear that such a bound exists: If we had relatively few clusters in our partition of G then a large proportion of edges in G would lie inside a cluster. But then we would not have that G' contains 'most' of the edges of G. More precisely, we know from Theorem 3.5 that $|V_0| \leq \epsilon|G|$ and all other cluster have size $m \leq \lceil \epsilon|G| \rceil$. This gives us that $|G| \leq \epsilon|G| + k\lceil \epsilon|G| \rceil \leq \epsilon|G| + k(\epsilon|G| + 1)$ and so rearranging we get $\frac{(1-\epsilon)}{\epsilon + \frac{1}{|G|}} \leq k$. Thus, making ϵ small and $|G|$ large forces k to be large. Similarly we obtain a lower bound for m since $\epsilon|G| + mM \geq |G|$ and so $m \geq \frac{1-\epsilon}{M}|G|$.

We now introduce a very important type of graph which is vital in applying the Regularity Lemma.

Definition 3.6 *(Reduced graph) Let G be a graph and $\{V_1, \ldots, V_k\}$ a partition of $V(G)$. Given two parameters $\epsilon > 0$ and $d \in [0, 1)$ we define the reduced graph R of G as follows: its vertices are the clusters $V_1, \ldots, V_k$ and there exists an edge between V_i and V_j precisely when (V_i, V_j) is ϵ-regular with density more than d.*

Most proofs involving the Regularity Lemma use reduced graphs. Often we will look at graph G and apply Theorem 3.5 to get the pure graph G''. We then take the reduced graph R of G''. In this case we also say that R is the reduced graph of G. Notice R provides an overview of the layout of the graph G'': it shows us when there exists a reasonable number of edges between two given clusters in G''. We can also define a reduced graph from the ϵ-regular partition obtained when applying Theorem 1.9. If we have a property of G such as an edge or degree condition, this will often give us a similar condition for R. An example of this is the lemma below.

Lemma 3.7 *Let G be a graph such that $\delta(G) \geq c|G|$ where c is a constant. Suppose we have applied the degree form of the Regularity Lemma to G and have defined from this the reduced graph R with parameters ϵ and d such that $2\epsilon \leq d$. If $d < c/2$ then $\delta(R) \geq (c - 2d)|R|$.*

Proof. Suppose not. Then there exists some $V_i \in V(R)$ such that $d(V_i) < (c - 2d)|R|$. (Here we let $\{V_0, V_1, \ldots, V_k\}$ be the ϵ-regular partition of $V(G)$, such that V_0 is the exceptional set and $|V_j| = m$ for all $j \in \{1, \ldots, k\}$.) Let G'' be the pure graph of G.

By assumption, fewer than $(c - 2d)|R|$ of our clusters V_j form an ϵ-regular pair of density more than d with V_i in G''. At most m^2 edges go between such a pair. So the total number of edges coming out of V_i in G'' is less than

$$(c - 2d)|R|m^2 \leq (c - 2d)|G|m$$

since $|R|m \leq |G|$.

Given any vertex $x \in V(G'')$ we know that $d_{G''}(x) > d_G(x) - (d + 2\epsilon)|G|$. So for all $x \in V_i$, we have $d_{G''}(x) > (c - d - 2\epsilon)|G|$ since $\delta(G) \geq c|G|$. G'' contains no edges between vertices in V_i. So the number of edges coming out of V_i in G'' is just $\sum_{x \in V_i} d_{G''}(x)$. Thus,

$$(c - d - 2\epsilon)|G|m < \sum_{x \in V_i} d_{G''}(x) < (c - 2d)|G|m.$$

This implies $d < 2\epsilon$, a contradiction to our hypothesis. So our assumption is false, proving the claim. □

The lemma tells us that if we have a reduced graph R with parameters ϵ and d sufficiently small, then the minimum fraction of vertices a vertex in R is adjacent to is close to the corresponding fraction for the vertices in G. The fact that we can tell something about the minimum degree of the reduced graph of G from the minimum degree of G is very useful.

On the other hand, we will see that under certain conditions, properties of a reduced graph R of G are inherited by G. This is realised in the Key Lemma below. Before we state the Key Lemma we need some more notation. Given a graph R and positive integer t, let $R(t)$ be the graph obtained from R by replacing every vertex $x \in V(R)$ by a set U_x of t independent vertices, and joining $u \in U_x$ to $v \in U_y$ precisely when xy is an edge in R. That is we replace the edges of R by copies of $K_{t,t}$. We will refer to $R(t)$ as a 'blown-up' copy of R. Further, given graphs H and G, $||H \to G||$ denotes the number of labelled copies of H in G.

Theorem 3.8 *(Key Lemma) Given $d > \epsilon > 0$, a graph R, and a positive integer m, let us construct a graph G by replacing every vertex of R by m vertices, and replacing the edges of R with ϵ-regular pairs of density at least d. Let H be a subgraph of $R(t)$ with h vertices and maximum degree $\Delta > 0$, and let $\delta := d - \epsilon$ and $\epsilon_0 := \frac{\delta^\Delta}{2+\Delta}$. If $\epsilon \leq \epsilon_0$ and $t - 1 \leq \epsilon_0 m$, then $H \subseteq G$. In fact,*

$$||H \to G|| > (\epsilon_0 m)^h.$$

Proof. We will prove the Key Lemma by proving the following more general result.

$$\text{If } t - 1 \leq (\delta^\Delta - \Delta\epsilon)m \text{ then } ||H \to G|| > [(\delta^\Delta - \Delta\epsilon)m - (t-1)]^h.$$

Indeed, this is more general since $\epsilon_0 = \frac{\delta^\Delta}{\Delta+2}$ and $\epsilon \leq \epsilon_0$ imply that $2\epsilon_0 \leq \delta^\Delta - \Delta\epsilon$. This in turn implies that if $t-1 \leq \epsilon_0 m$ then $t - 1 \leq (\delta^\Delta - \Delta\epsilon)m$ and further assuming that $||H \to G|| > [(\delta^\Delta - \Delta\epsilon)m - (t-1)]^h$ we then have $||H \to G|| > (\epsilon_0 m)^h$.

We let $V(R) =: \{V_1, \dots, V_k\}$. By definition of G we will also use the convention that V_i is the vertex set in G that corresponds to the vertex $V_i \in V(R)$. We let U_i^t denote the vertex set of size t in $R(t)$ corresponding to V_i. Now H is a subgraph of $R(t)$ with vertices $u_1, \dots, u_h$ say. Each vertex u_i lies in one of the sets U_j^t. This defines a map $\sigma : [h] \mapsto [k]$. Our aim is to embed many labelled copies of H in G. We will define embeddings of the form $u_i \mapsto v_i \in V_{\sigma(i)}$. Thus, $v_1, \dots, v_h$ will be distinct.

We will describe an algorithm below such that in the ith step we define v_i. We will need that there is a sufficient number of choices for v_i for each i in order to obtain the required number of labelled copies of H in G. Given some u_i we will have at each step a candidate set for v_i. At the jth step this will be called Y_i^j. Initially (i.e. in 'step' 0) we have $Y_i^0 := V_{\sigma(i)}$ for all i. In particular, $|Y_i^0| = m$ for all i. After every application of the algorithm we will want to update the candidate sets for all vertices v_j yet to be defined. That is, if $u_i u_j \in E(H)$ and v_i is defined, we can only consider vertices in V_j as candidates for v_j if they are adjacent to v_i in G. The algorithm at step $i \geq 1$ consists of two steps.

Step 1: Picking v_i. We pick $v_i \in Y_i^{i-1}$ such that

$$d_G(v_i, Y_j^{i-1}) > \delta |Y_j^{i-1}| \tag{3.7}$$

for all $j > i$ such that $v_i v_j \in E(H)$.

Step 2: Updating the candidate sets. We set $Y_j^i := Y_j^{i-1} \cap N(v_i)$ if $u_iu_j \in E(H)$, or $Y_j^i := Y_j^{i-1}$ otherwise (for $j > i$).

For $i < j$ we define $d_{ij} := |\{l \in [i] : u_lu_j \in E(H)\}|$. We claim that if $d_{ij} > 0$ then $|Y_j^i| > \delta^{d_{ij}}m$: Given u_j the initial candidate set is $V_{\sigma(j)} = Y_j^0$ which has size m. To obtain the candidate set Y_j^i, by definition of d_{ij} we have had to shrink Y_j^0 d_{ij} times. But given some $l \leq i$ such that $u_lu_j \in E(H)$ we have $|Y_j^l| = |Y_j^{l-1} \cap N(v_j)| > \delta|Y_j^{l-1}|$. So $|Y_j^i| > \delta^{d_{ij}}m$.

Notice that if $d_{ij} = 0$ then $|Y_j^i| = m$. Now for all $i < j$, $|Y_j^i| > \delta^\Delta \geq \epsilon m$ and $Y_j^i \subseteq V_{\sigma(j)}$. If u_j is a neighbour of u_i in H then since $u_i \in U_{\sigma(i)}^t$ and $u_j \in U_{\sigma(j)}^t$, we have that $V_{\sigma(i)}$ and $V_{\sigma(i)}$ are adjacent in R. So by definition of G we have that $(V_{\sigma(i)}, V_{\sigma(j)})$ is an ϵ-regular pair in G with density at least d. Thus, by Lemma 1.4 all but at most ϵm vertices of Y_{i-1}^i satisfy (3.7) for our specific j. Now u_i has at most Δ neighbours in H. So all but at most $\Delta\epsilon m$ vertices of Y_{i-1}^i satisfy (3.7) for all $j > i$. At most $t-1$ vertices before u_i were given an image in $V_{\sigma(i)}$. Thus, we have at least

$$|Y_i^{i-1}| - \Delta\epsilon m - (t-1) > (\delta^\Delta - \Delta\epsilon)m - (t-1) \geq 0$$

free choices for each v_i. This shows we can apply the algorithm at each step, thus we obtain vertices $v_1, ..., v_h$ in G, and by the construction of our algorithm if $u_iu_j \in E(H)$ then $v_iv_j \in E(G)$. So $H \subseteq G$ and with the number of possible choices for each v_i, we obtain $||H \to G|| > [(\delta^\Delta - \Delta\epsilon)m - (t-1)]^h$. $\square$

The proof of the Key Lemma given here is based on the proof in [14]. Although the Key Lemma does not specify this, we usually think of R being a reduced graph of a graph G, and the way 'G' is specified in the Key Lemma, this refers to the pure graph G'' of G. Thus, the message one should get from the Key Lemma is the following: Given a graph G and the reduced graph R, G will contain a copy of any relatively sparse graph H for which we know that $H \subseteq R$ or $H \subseteq R(t)$. The denser H is, the larger Δ is in the theorem, and so the conditions ensuring a copy of H in G are stricter. That is, it is 'harder' to embed a dense graph into G.

If we are embedding a graph H into G where we know $H \subseteq R(t)$ for some $t > 1$, we have to ensure that $t - 1 \leq \epsilon_0 m$. Usually we will be dealing with a graph G such that R is the reduced graph of G with parameters ϵ and d, obtained by applying Theorem 3.5. Thus, m refers to the size of the (non-exceptional) clusters in an ϵ-regular partition of G. But recall $m \geq \frac{(1-\epsilon)}{M}|G|$ where M is the constant obtained from Theorem 3.5 on input ϵ. So if $|G|$ is sufficiently large compared to H then the condition involving m in the Key Lemma will be satisfied. Hence, in some of our proofs we will just state that $|G|$ is chosen sufficiently large to apply the Key Lemma.

Notice the only result about ϵ-regularity we needed to prove Theorem 3.8 was Lemma 1.4. The proof of this lemma required the property that for an ϵ-regular pair (A, B) of density d, if $X \subseteq A$, $Y \subseteq B$ such that $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ then $e(X, Y) > (d - \epsilon)|X||Y|$. Thus to prove the Key Lemma we did not need the full strength of ϵ-regularity, just this one-sided property.

Finally, note that all the copies of H in G which we found in the proof of the Key Lemma had the following property: If u_i is a vertex in a copy of H in $R(t)$ then it was embedded into $V_{\sigma(i)}$ where $u_i \in U^t_{\sigma(i)}$. In particular, if $t = 1$ then v_i lies in the cluster of G that corresponds to the vertex u_i in R. This is very important: Given a graph G with reduced graph R, if H is a subgraph of R and we embed H into G then we use at most one vertex in each cluster of G. Thus, in general removing these vertices will, by the Slicing lemma, maintain ϵ-regularity of pairs whose density remains about the same (where ϵ will be larger than before). Thus, we may be able to find another, disjoint, copy of H in G by applying the Key Lemma again. So this property can be useful (but not essential) in applying the Key Lemma to find H-packings in G. This subject is discussed in more detail in the next section.

3.3 Applying the Regularity Lemma

In the last section we noted that the Key Lemma can be used as a way of embedding some graph H into a graph G. Thus, many applications of the Regularity Lemma are concerned with embedding and packing problems. In this section we give the framework of a typical application of the Regularity Lemma in these situations. Throughout this section H is a graph that we are trying to embed into G.

Step 1: Preparing G. We apply Theorem 1.9 or Theorem 3.5 to G. From the latter we obtain the pure graph G''. In either case we can define the reduced graph R of G with suitable parameters ϵ and d.

Step 2: Finding structure in R. In an application of the Regularity Lemma we will have some condition on G. From this condition we can then obtain information about R. Typically we may have an edge or degree condition on G. From these conditions we can usually get a similar condition for R. An example of this is Lemma 3.7.

Thus, if we have a bound, involving a slack term, on the minimum degree of, or the number of edges in G, choosing ϵ and d small enough will often give us an equivalent property of R without the slack term. For example, to prove Theorem 2.4 we have to consider a graph G of order n such that $e(G) \geq t_{r-1}(n) + \gamma n^2$ where $\gamma > 0$. Choosing ϵ and d carefully we can deduce that $e(R) \geq t_{r-1}(k)$ where $|R| = k$.

Sometimes, even when we have a condition on G that does not involve a slack term, we may be able to introduce one. For we could be in a situation where proving a result involving a condition with a slack implies the equivalent result but with this slack dropped. Indeed, this is precisely how we prove Theorem 2.8.

Once we have found a suitable property of R we can apply an embedding result to obtain structure in R. For example Turán's Theorem and the Hajnal-Szemerédi Theorem will be applied at this stage in the proofs of Theorem 2.4 and Theorem 2.8 respectively. Notice this is why we put so much stress onto the fact that from a condition on G involving a slack we can obtain a similar property with the slack dropped for R. This will enable us to apply one of our embedding results to R. Note that we may apply the Regularity Lemma to a graph G with a condition different to the form mentioned, and different to that which we wish to obtain in R. However, the general idea of this step is the same: Use properties of G to establish properties of R which then enables us to apply some embedding result.

Step 3: Applying the Key Lemma. In Step 2 we will have embedded graph(s) into R. We could be in a situation where we simply want to show that G contains a copy of a given graph H. Thus, in Step 2 we usually will have embedded H or some graph containing H into R. Provided ϵ is small and $|G|$ is large enough, the Key Lemma implies $H \subseteq G$ as required. Alternatively, we may have embedded a graph K into R which does not contain H but such that a 'blown-up' copy of K contains H. That is $H \subseteq K(s) \subseteq R(s)$ for some $s \in \mathbb{N}$. Again choosing ϵ sufficiently small and $|G|$ large would then force H into G. For example, to prove Theorem 2.4 Step 2 will give us, by Turán's Theorem, that $K_r \subseteq R$ and so $K_r^{|H|} \subseteq R(|H|)$ where $r := \chi(H)$. Thus, for large G, $H \subseteq K_r^{|H|} \subseteq G$.

At this stage of a proof involving the Regularity Lemma, we may wish to repeatedly apply the Key Lemma. We will do this if we wish to embed a graph K 'piece by piece' into G. For example, if we are trying to find an almost perfect H-packing K in G, we cannot simply embed this packing using the Key Lemma in one go. However, we can take the following approach: If $H \subseteq R$ or $H \subseteq R(s)$, provided $|G|$ is large and ϵ is small we can embed H into G. But then removing this copy of H in G we obtain a graph G_1. Thus, given a cluster V_i in our ϵ-regular partition of G most vertices in V_i also belong to the corresponding cluster in G_1. Thus, given two such clusters in G_1, if they correspond to an ϵ-regular pair with density more than d in G, the Slicing Lemma tells us that in G_1 they form an ϵ'-regular pair with density d' where $\epsilon < \epsilon'$ and d' is close to d. Thus, R is a reduced graph of G_1 with parameters ϵ' and d'. Hence, we can apply the Key Lemma to G_1 to embed H into G_1. We can keep on repeating this argument to obtain many disjoint copies of H in G.

This approach suggests it is often possible to find an almost perfect H-packing in G. However, if we have covered most vertices in G with copies of H the remainder G^* of G will be small. So we will be unable to apply the Slicing Lemma to find regular pairs in G^*. Further, G^* may be too small to be able to apply the Key Lemma to it. This implies our approach alone is not sufficient to prove results about perfect packings. So it seems, as one might expect, that it is much harder to find a perfect packing in a large graph G than an almost perfect packing.

Chapter 4

Three proofs using the Regularity Lemma

In the previous chapter we introduced some of the tools that we need to apply the Regularity Lemma. This chapter focuses on three theorems that can be proven using the Regularity Lemma. The aim being to see the techniques that are used in these proofs. Thus, in each section any relevant background information will first be given. Then follows the proof of one of these results, with an analysis of the methods used afterwards.

4.1 The Erdős-Stone Theorem

This section is concerned with the proof of Theorem 2.4. In order to do this we prove another result.

Theorem 4.1 *For all integers $r \geq 2$ and $s \geq 1$, and for every $\epsilon > 0$, there exists an integer $n_0(r, s, \epsilon)$ such that if a graph G has $n \geq n_0$ vertices and*

$$e(G) \geq t_{r-1}(n) + \epsilon n^2,$$

then $K_r^s \subseteq G$.

Recall that $t_{r-1}(n)$ is the number of edges in the Turán graph $T_{r-1}(n)$. It is easy to see that Theorem 4.1 implies Theorem 2.4: Let $\epsilon > 0$ and H be any graph. We define $r := \chi(H)$ and $s := |H|$. Notice Theorem 2.4 is trivial for $\chi(H) = 1$ so we can assume $r \geq 2$. Let n_0 be the output of Theorem 4.1 under input r, s and $\frac{\epsilon}{2}$. Thus, n_0 depends only on ϵ and H. Further, if a graph G on $n \geq n_0$ vertices satisfies $e(G) > (1 - \frac{1}{\chi(H)-1} + \epsilon)\frac{n^2}{2}$ then

$$e(G) > \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} + \frac{\epsilon}{2}n^2 \overset{(2.2)}{\geq} t_{r-1}(n) + \frac{\epsilon}{2}n^2$$

and so Theorem 4.1 tells us that $K_r^s \subseteq G$. But $H \subseteq K_r^s$ so $H \subseteq G$ as required.

We now give a proof of Theorem 4.1 that is based on the one given in [4].

Proof of Theorem 4.1. Suppose we have integers $r \geq 2$, $s \geq 1$ and $\gamma > 0$. Suppose that G is a graph with sufficiently large order n and $e(G) \geq t_{r-1}(n) + \gamma n^2$. We define $d := \gamma$ and $\epsilon > 0$ so that:

(1) $3\epsilon < \gamma$,

(2) $\epsilon \leq \frac{(\gamma-\epsilon)^{(r-1)s}}{2+(r-1)s}$.

Further we let $\delta := \gamma - 3\epsilon > 0$. Now (2.1) implies

$$\lim_{n\to\infty} \frac{t_{r-1}(n)}{n^2/2} = 1 - \frac{1}{r-1}.$$

In particular this implies that for any $\eta > 0$ provided n is large enough then $\frac{t_{r-1}(n)}{n^2/2} + \eta > 1 - \frac{1}{r-1}$. This will be useful later in the proof.

We apply the degree form of the Regularity Lemma to our given graph G with input ϵ and d. Thus, we obtain a partition $\{V_0, V_1, \dots, V_k\}$ of $V(G)$, where $m := |V_i| \leq \lceil \epsilon n \rceil$ (for all $i \in \{1, \dots, k\}$) and a pure graph G''. From this we obtain the reduced graph R with parameters ϵ and d.

Recall that $e(G'') > e(G) - (d+3\epsilon)\frac{n^2}{2}$. Also, each edge in R corresponds to at most m^2 edges in G'' and every edge in G'' must correspond to such an edge in R. So $e(R)m^2 \geq e(G'')$. Hence, provided n is sufficiently large,

$$\begin{aligned} e(R) &> \frac{e(G) - (d+3\epsilon)\frac{n^2}{2}}{m^2} \geq \frac{1}{2}k^2 \left(\frac{t_{r-1}(n) + \gamma n^2 - (d+3\epsilon)\frac{n^2}{2}}{\frac{1}{2}(mk)^2} \right) \\ &\geq \frac{1}{2}k^2 \left(\frac{t_{r-1}(n) + \gamma n^2 - (d+3\epsilon)\frac{n^2}{2}}{\frac{1}{2}n^2} \right) = \frac{1}{2}k^2 \left(\frac{t_{r-1}(n)}{\frac{1}{2}n^2} + 2\gamma - (d+3\epsilon) \right) \\ &= \frac{1}{2}k^2 \left(\frac{t_{r-1}(n)}{\frac{1}{2}n^2} + \delta \right) > \frac{1}{2}k^2 \left(1 - \frac{1}{r-1} \right). \end{aligned}$$

Therefore, by Turán's Theorem, $K_r \subseteq R$. So $K_r^s \subseteq R(s)$. If n is sufficiently large then the choice of ϵ enables us to apply the Key Lemma giving $K_r^s \subseteq G'' \subseteq G$, as required. □

This proof is not the original one. Indeed the first proof of the result was written about 30 years before the proof of the Regularity Lemma. The idea of our proof is as follows: Choosing ϵ and d small enough, and n large we can ensure that the reduced graph R of G still has lots of edges, namely more than $t_{r-1}(k)$. Thus, we can apply Turán's Theorem to R which implies $K_r \subseteq R$. Hence, $K_r^s \subseteq R(s)$. But then provided n is large and ϵ is small, the Key Lemma tells us that $K_r^s \subseteq G$, as desired.

Notice the structure of the proof: We use a property of G to obtain a property of the reduced graph R. From this, applying an embedding result gives some structure in R. Then we apply the Key Lemma to force this structure into G.

We will see the proof of the other two theorems in this chapter have similar, if not, slightly more complicated, constructions.

4.2 Ramsey numbers for graphs of bounded degree

This section goes through a proof in Ramsey Theory that uses our acquired tools. We only need a small amount of background information in the subject before we can proceed to the proof.

Definition 4.2 *(Ramsey numbers) Given $s \in \mathbb{N}$, the Ramsey number $R(s)$ is the smallest $n \in \mathbb{N}$ such that whenever the edges of K_n are coloured red and blue there exists a monochromatic copy of K_s.*

Given a graph H the Ramsey number $R(H)$ is the smallest $n \in \mathbb{N}$ such that whenever the edges of K_n are coloured red and blue there exists a monochromatic copy of H in K_n.

Ramsey [18] in 1930 showed that these numbers exist. One can think of Ramsey numbers in a different way to the above definition. It is equivalent to say that $R(H)$ is the smallest $n \in \mathbb{N}$ such that if G is a graph on n vertices then $H \subseteq G$ or $H \subseteq \overline{G}$. We will use this idea in the proof of the next result.

Theorem 4.3 *(Chvátal, Rödl, Szemerédi and Trotter [3]) For every positive integer Δ there is a constant c such that*

$$R(H) \leq c|H|$$

for all graphs H with $\Delta(H) \leq \Delta$.

Proof. Let $\Delta \geq 1$ be given, and $m := R(\Delta + 1)$. We define $\epsilon > 0$ such that $\epsilon \leq \frac{(1/2-\epsilon)^\Delta}{2+\Delta}$ and $2\epsilon < \frac{1}{m-1} - \frac{1}{m}$. Let

$$c := \frac{M(\Delta + 2)}{(1/2 - \epsilon)^\Delta (1 - \epsilon)},$$

where M is the output of Theorem 1.9 on input ϵ and m. Notice that c depends only on Δ. We claim that if G is a graph on $n \geq c|H|$ vertices then $H \subseteq G$ or $H \subseteq \overline{G}$.

By Theorem 1.9 G has an ϵ-regular partition $\{V_0, V_1, \ldots, V_k\}$ with $k \geq m$. We define an auxiliary graph R with vertices $V_1, \ldots, V_k$ by saying that V_i and V_j are adjacent in R precisely when (V_i, V_j) is an ϵ-regular pair in the partition of G. Notice R is similar to a reduced graph of G except we do not consider the density of ϵ-regular pairs in G. Now R has k vertices and

$$\begin{aligned} e(R) &\geq \binom{k}{2} - \epsilon k^2 = \frac{1}{2}k^2 \left(1 - \frac{1}{k} - 2\epsilon\right) \\ &\geq \frac{1}{2}k^2 \left(1 - \frac{1}{k} - \frac{1}{m-1} + \frac{1}{m}\right) \geq \frac{1}{2}k^2 \left(1 - \frac{1}{m-1}\right). \end{aligned}$$

Thus, by Turán's Theorem R contains a copy K of K_m.

We now define a 2-colouring of the edges of R: we colour an edge red if it corresponds to an ϵ-regular pair (V_i, V_j) of density greater than $1/2$, and blue otherwise. Let R' and R'' be the spanning subgraphs of R formed by the red edges and blue edges respectively. By Lemma 1.3 a pair (V_i, V_j) is ϵ-regular of density d in G precisely when it is ϵ-regular of density $1-d$ in $\overline{G}$. Thus, R' is the reduced graph of G with parameters ϵ and $1/2$. Similarly R'' is the reduced graph of $\overline{G}$ with parameters ϵ and $1/2$.

Now $r := \chi(H) \leq \Delta + 1$, so since $m := R(\Delta+1)$, K contains a red or blue K_r. Thus, $H \subseteq R'(|H|)$ or $H \subseteq R''(|H|)$. Since $n \geq c|H|$, the choice of ϵ is such that we can apply the Key Lemma to obtain $H \subseteq G$ or $H \subseteq \overline{G}$, as required. □

The proof of Theorem 4.3 is based on that given in [4]. Note that the idea of the proof was quite similar to that of Theorem 4.1. Notice, though, we used the Theorem 1.9 not Theorem 3.5.

Theorem 4.3 is quite an important result. It tells us that the Ramsey number of graphs H with bounded maximum degree grows linearly in $|H|$. This is useful since very few Ramsey numbers are known. Thus, it is useful to obtain bounds on these numbers. We know that for $s \geq 4$ we have

$$2^{s/2} < R(s) = R(K_s) < 2^{2s-2}.$$

These general bounds have been improved only slightly. Thus, when considering cliques instead of graphs of bounded maximum degree our general bounds are exponential in $s = |K_s|$.

4.3 Alon-Yuster Theorem on almost perfect packings

We will see that to prove Theorem 2.8 it is sufficient to prove the following:

Theorem 4.4 *For all $\eta > 0$ and graphs H, there exists $n_0(H, \eta)$ such that if*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \eta\right) n$$

and $|G| = n \geq n_0$ then G contains an H-packing which covers all but ηn vertices in G.

Before we prove Theorem 4.4 we will show that it implies Theorem 2.8.

Proof of Theorem 2.8. Let H and $\epsilon > 0$ be given. We may assume $\chi(H) > 1$, as the result is trivial otherwise. Further, it is sufficient to prove the result under the assumption $\epsilon < \frac{1}{2}$. We choose $\epsilon' > 0$ such that $\epsilon' < \frac{\epsilon}{2|H|}$ and thus $\epsilon' < \frac{1}{2\chi(H)}$.

Let G be a graph with $\delta(G) \geq (1 - \frac{1}{\chi(H)})n$ whose order n is sufficiently large. We obtain a new graph from G by adding $\epsilon' n$ vertices to G, connecting all new vertices to each other, and to the vertices G. Call this new graph G^* and let $n^* = (1+\epsilon')n$ be the order G^*. The idea is to obtain a minimum degree condition for G^* involving a slack, in a form suitable to apply Theorem 4.4. We have

$$\begin{aligned}
\delta(G^*) &\geq \left(1 - \frac{1}{\chi(H)} + \epsilon'\right) n = \left(1 - \frac{1}{\chi(H)} + \epsilon'\right) \frac{n^*}{1+\epsilon'} \\
&\geq (1-\epsilon') \left(1 - \frac{1}{\chi(H)} + \epsilon'\right) n^* \\
&= \left(1 - \frac{1}{\chi(H)}\right) n^* + \epsilon' \left(\frac{1}{\chi(H)} - \epsilon'\right) n^* \\
&\geq \left(1 - \frac{1}{\chi(H)}\right) n^* + \epsilon' \left(2\epsilon' - \epsilon'\right) n^* \\
&\geq \left(1 - \frac{1}{\chi(H)} + (\epsilon')^2\right) n^*
\end{aligned}$$

Thus, Theorem 4.4 implies that there is an H-packing that covers all but at most $(\epsilon')^2|G^*| \leq \epsilon' n$ vertices in G^*.

We wish to count how many vertices in G are covered by copies of H that do not contain any vertices in $G^* - G$. At most $\epsilon' n$ vertices in G are not covered by any copy of H in our packing in G^*. Further, at most $\epsilon' n$ copies of H in the H-packing contain vertices in $G^* - G$. So at most $\epsilon' n|H|$ vertices in G are covered by such copies of H. Disregarding all copies of H in the packing that contain any vertex outside of G we obtain an H-packing in G covering all but at most $\epsilon' n + \epsilon' n|H| \leq 2\epsilon' n|H| \leq \epsilon n$ vertices, as required. □

The approach used in this proof is quite common. Indeed later we will see two more identical applications of this method. We are now fully prepared to prove Theorem 4.4 and hence, Theorem 2.8.

Proof of Theorem 4.4. We may assume that $0 < \eta < 1$ and $\chi(H) > 1$ else the result is trivial. Let the graph H and $\eta > 0$ be given. Also, let $|H| = h$, $\chi(H) = r$ and $\epsilon > 0$. We define α and d so that $0 < 3\alpha r \leq 2d \leq \eta$. We choose ϵ to be sufficiently small so as to be able to apply the Key Lemma on input $d - \epsilon$, $\frac{\epsilon}{\alpha}$, and $\Delta = (r-1)h$.

Let G be a graph of sufficiently large order n and $\delta(G) \geq (1 - \frac{1}{\chi(H)} + \eta)n$. We can apply the degree form of the Regularity Lemma with input ϵ and d. Thus, we obtain a partition $\{V_0, V_1, \ldots, V_k\}$ of $V(G)$, where $k \leq M$, $|V_0| \leq \epsilon n$, $m := |V_i| \leq \lceil \epsilon n \rceil$ (for all $i \in \{1, \ldots, k\}$) and a pure graph G''.

Let R be the reduced graph of G with parameters ϵ and d. The choice of d and ϵ gives us, by Lemma 3.7,

$$\delta(R) \geq \left(1 - \frac{1}{\chi(H)} + \eta - 2d\right) |R| \geq \left(1 - \frac{1}{\chi(H)}\right) |R|.$$

Thus, by Theorem 2.7, R has a perfect K_r-packing.

Consider any of these copies of K_r. As $K_r \subseteq R$, this K_r corresponds to some $G^* \subseteq G''$, where G^* is obtained from our K_r by replacing every vertex by the m vertices in the corresponding cluster and replacing every edge of K_r with the ϵ-regular pair between the two corresponding clusters. Thus, each of these ϵ-regular pairs has of density greater than d. The choice of d and ϵ, and since n is sufficiently large, gives us, by the Key Lemma that $K_r^h \subseteq G^*$. Our aim is to delete the vertices in this copy of K_r^h in G^* and then find another copy of K_r^h in G^*. We then continue in this fashion until almost all vertices in G^* are covered. Provided that the fraction of vertices in each cluster in G^* not covered by the chosen disjoint copies of K_r^h is more than α, the Slicing Lemma tells us that each pair of subclusters, obtained by the removal of all vertices covered by those copies of K_r^h which we found so far in G^*, is an $\frac{\epsilon}{\alpha}$-regular pair and has density greater than $d - \epsilon$. Thus, if n is sufficiently large we can repeatedly apply the Key Lemma to find disjoint copies of K_r^h in G^* until all but an α-fraction of vertices in G^* are covered by this K_r^h-packing.

Recall that we chose our K_r (and thus G^*) from the K_r-packing of R arbitrarily. So given any such copy of K_r in R the corresponding subgraph in G has a K_r^h-packing covering all but an α-fraction of its vertices.

All but at most $r-1$ vertices in R are covered by our K_r-packing. So all but $(r-1)m$ vertices in G'' are covered by the disjoint subgraphs G^* of G that correspond to these copies of K_r in R. In other words, all but at most $(r-1)m+\alpha n$ vertices in G'' are covered by our K_r^h-packing. So since, $|V_0| \leq \epsilon n$ all but at most

$$\begin{aligned}(r-1)m + \alpha n + \epsilon n &\leq (r-1)(\epsilon n + 1) + \alpha n + \epsilon n \\ &\leq (r-1)\alpha n + 2\alpha n + (r-1) \\ &\leq 3r\alpha n \leq \eta n\end{aligned}$$

vertices in G are covered by this packing. Each copy of K_r^h in this packing can be perfectly tiled with disjoint copies of H. Hence, we have found the required H-packing in G. $\square$

This proof is simpler than the original, as it uses the Key Lemma (the result was first proved before the Key Lemma was formulated explicitly). We break up the general outline of the proof into steps below.

Step 1: Proving that R has large minimum degree. We consider a graph G with large order and minimum degree as in the hypothesis. We choose ϵ and d small enough so that the reduced graph R has minimum degree at least $(1-\frac{1}{r})|R|$.

Step 2: Applying an embedding result to obtain structure in R. We are now able to apply to apply the Hajnal-Szemerédi Theorem (Theorem 2.7) so as to ascertain that R has a perfect K_r-packing.

Step 3: Understanding the rest of the problem. Each of these copies of K_r corresponds to some $G^* \subseteq G''$. Note though that the graphs G^* in G corresponding to different copies of K_r are not necessarily isomorphic. However, all we need is that each such G^* has structure as it consists of r vertex classes such that any two of them form a regular pair. If we can tile G^* with an almost perfect H-packing then the whole of G'' and, therefore, G will have an almost perfect H-packing. More precisely, if ϵ and d are small then few vertices in G are not contained in G''. Most vertices in G'' are contained in graphs of the form of G^*, each of which is tiled with an almost perfect H-packing. Hence, an almost perfect H-packing of G is obtained.

Thus, all we need to ensure is that each of the graphs G^* contains an almost perfect H-packing.

Step 4: Applying the Key Lemma. We choose ϵ and d small enough so that we can apply the Key Lemma to get a copy of K_r^h in G^*. The idea is to keep on finding disjoint copies of K_r^h in G^* and remove these until most vertices in G^* are known to be covered by one of these copies of K_r^h. We do this as follows: Consider the subgraph G^{**} of G^* that is obtained from G^* by removing all the copies of K_r^h found so far. Whilst not enough vertices in G^* are covered, by the Slicing Lemma the remainders of the clusters in G^* still form regular pairs in G^{**} with density close to that of the original pair in G^*. We then apply the Key Lemma to G^{**} to find another copy of K_r^h. We repeat this argument until enough vertices are covered by a K_r^h-packing in G^*. Each copy of K_r^h can be perfectly tiled with copies of H. Hence, we have our required H-packing.

Notice we repeatedly embedded copies of K_r^h. We could have argued differently to simply embed copies of H. The reason we embedded copies of K_r^h was to make it easier to describe the argument in our proof. We noted that we could keep on applying the Key Lemma to G^* provided we had not already covered at least an α-fraction of each cluster in G^* with disjoint copies of K_r^h. Notice that if $K_r^h \subseteq G^*$ then h vertices from each cluster are covered in G^* by this copy of K_r^h. So once, we have a K_r^h-packing covering all but an α-fraction of one cluster in G^*, we know all such clusters are covered equally. This allowed us to repeatedly use the Key Lemma until all but an α-fraction of vertices in G^* were covered by our K_r^h-packing. We could have embedded copies of H in a way such that all vertex classes have roughly the same number of left-over vertices, and then proceed as above. However, this method is slightly harder to explain.

Chapter 5

The critical chromatic number and Komlós' Theorem on almost perfect packings

5.1 Extremal examples and the critical chromatic number

Throughout this section H and G are graphs, the latter with $|G| = n$. In Chapter 2 we stated results concerning minimum degree conditions on G that ensure a perfect or an almost perfect H-packing in G. These results were Theorems 2.8, 2.9 and 2.10. In the hypothesis of each of these results the minimum degree condition on G depended on $\chi(H)$. The question arises as to whether these results can be strengthened by sharpening the minimum degree conditions on G. In this chapter we will investigate this for Theorem 2.8. We will look at this question for Theorems 2.9 and 2.10 in Chapter 7.

In Proposition 7.2 we will see that for some graphs H the constant C in the minimum degree condition of Theorem 2.10 cannot be omitted. On the other hand, notice we can change the minimum degree condition in Theorem 2.8 such that for any positive constant C the result holds under the degree condition $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n - C$. Indeed, the proof of this is similar to the proof of Theorem 2.8. We add $\chi(H)C$ vertices to G connecting them to all other vertices. We thus obtain a graph G^* on $n^* := n + \chi(H)C$ vertices with

$$\delta(G^*) \geq \left(1 - \frac{1}{\chi(H)}\right) n - C + \chi(H)C = \left(1 - \frac{1}{\chi(H)}\right)(n + \chi(H)C).$$

Let $\epsilon > 0$. If we are considering a graph G as above such that n is sufficiently large, then by Theorem 2.8, G^* has an H-packing covering all but $\frac{\epsilon}{2}n^*$ vertices in G^*. By a similar argument as in the proof of Theorem 2.8, disregarding all copies in the H-packing in G^* with vertices in $G^* - G$ we obtain an H-packing in G covering all but an ϵ-fraction of the vertices in G (provided n is sufficiently large). What we really want therefore, is not to improve the minimum degree condition in Theorem 2.8 by a constant: This was straightforward and just tells us that for a sufficiently large graph G, if its minimum degree is bounded below by something relatively close to $\left(1 - \frac{1}{\chi(H)}\right) n$ then there exists an almost perfect H-packing in G. The question therefore is whether we can replace $\chi(H)$ in the minimum degree condition in Theorem 2.8 with some other quantity dependant on H that sharpens the condition. Rephrasing, is it possible to find a different parameter on H that governs whether a graph G has an almost H-packing?

We cannot embed a graph H into a graph G with lower chromatic number. So consider any H with $\chi(H) = r > 1$. Given any $n \in \mathbb{N}$ such that $(r-1)|n$ let $G := K^s_{r-1}$ where $s = n/(r-1)$. So $H \not\subseteq G$ and $\delta(G) = n - n/(r-1) = (1 - \frac{1}{\chi(H)-1})n$. This shows we cannot replace $\chi(H)$ with anything less than or equal to $\chi(H) - 1$ in the minimum degree condition in Theorem 2.8.

We can work a little harder to find a better bound on what we can replace $\chi(H)$ by. To do this we introduce the following definition.

Definition 5.1 *(Critical chromatic number) Given a graph H with $\chi(H) > 1$ the critical chromatic number of H is defined by*

$$\chi_{cr}(H) = (\chi(H) - 1)\frac{|H|}{|H| - \sigma(H)},$$

where $\sigma(H)$ is defined to be the size of the smallest possible colour class in any r-colouring of H.

Note that only graphs H with $\chi(H) > 1$ have a critical chromatic number. Thus, from now on if we refer to a graph H and $\chi_{cr}(H)$ then we are assuming that $\chi(H) > 1$. Since $\frac{|H|}{|H|-\sigma(H)} > 1$ and $\sigma(H) \leq \frac{|H|}{\chi(H)}$ we have that

$$\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H).$$

If every $\chi(H)$-colouring of H has equal colour class sizes then $\chi_{cr}(H) = \chi(H)$. Conversely if $\chi_{cr}(H) = \chi(H)$ then $\frac{(\chi(H)-1)}{1-\frac{\sigma(H)}{|H|}} = \chi(H)$. So $1 - \frac{1}{\chi(H)} = 1 - \frac{\sigma(H)}{|H|}$ and thus $\sigma(H) = |H|/\chi(H)$. By definition of $\sigma(H)$ this implies in any $\chi(H)$-colouring of H every colour class has size $|H|/\chi(H)$.

Let us now show that we cannot replace $\chi(H)$ with anything less than $\chi_{cr}(H)$ in the minimum degree condition in Theorem 2.8.

Proposition 5.2 *For every graph H with $\chi(H) \geq 2$, and every rational number $0 < a < \chi_{cr}(H)$ there exists an $\eta \in (0, 1)$ and infinitely many graphs G such that*

$$\delta(G) \geq \left(1 - \frac{1}{a}\right)|G|$$

but such that G does not have an H-packing covering all but at most an η-fraction of the vertices in G.

Proof. Let $r := \chi(H)$. Consider any r-colouring of H whose smallest colour class has size $\sigma(H)$. Let $x_1 \leq x_2 \leq \cdots \leq x_r$ denote the sizes of the colour classes. In particular $x_1 = \sigma(H)$. Let $s_1 := (r-1)x_1$ and $s := x_2 + \cdots + x_r = |H| - \sigma(H)$, and $\beta = \frac{1}{a} - \frac{1}{\chi_{cr}(H)}$. Given $k \in \mathbb{N}$ such that $\beta k \in \mathbb{N}$, let G be the complete r-partite graph with vertex classes $V_1, \ldots, V_r$ where $|V_1| = ks_1 - \beta k$, $|V_2| = ks + \beta k$ and $|V_i| = ks$ for every $i \geq 3$. Thus $|G| = k(r-1)|H|$ and

$$\delta(G) = (r-2)ks + ks_1 - \beta k.$$

Now

$$\begin{aligned}\left(1 - \frac{1}{\chi_{cr}(H)}\right)|G| &= k(r-1)|H|\left(1 - \frac{s}{|H|(r-1)}\right) = k(r-1)|H| - ks \\ &= k(r-1)\left(s + \frac{s_1}{r-1}\right) - ks = k(r-1)s + ks_1 - ks \\ &= (r-2)ks + ks_1\end{aligned}$$

and thus

$$\delta(G) = \left(1 - \frac{1}{\chi_{cr}(H)}\right)|G| - \beta k > \left(1 - \frac{1}{a}\right)|G|.$$

However, every copy of H in G contains at most $|H| - x_1 = s$ vertices in $V_2 \cup \cdots \cup V_r$. So any H-packing in G covers at most $s|V_1|/x_1$ vertices in $V_2 \cup \cdots \cup V_r$. But

$$\begin{aligned}s\frac{|V_1|}{x_1} &= \frac{s(ks_1 - \beta k)}{x_1} = \frac{sk(r-1)x_1}{x_1} - \frac{s\beta k}{x_1} = |G| - s_1 k - \frac{s\beta k}{x_1} \\ &= |G| - |V_1| - \beta k - \frac{s\beta k}{x_1} = |G| - |V_1| - \frac{\beta k|H|}{\sigma(H)}.\end{aligned}$$

So at least $\frac{\beta k|H|}{\sigma(H)} = \frac{\beta|G|}{\sigma(H)(r-1)}$ vertices in G are not covered by any H-packing. Thus, the result is proven with $\eta = \eta(H) := \frac{\beta}{\sigma(H)(r-1)}$. $\square$

Komlós [11] proved that one can replace $\chi(H)$ with $\chi_{cr}(H)$ in Theorem 2.8.

Theorem 5.3 *(Komlós [11]) Given any graph H and $\epsilon > 0$ there exists an $n_0(H, \epsilon)$ such that if G is a graph on $n \geq n_0$ vertices with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right)n$$

then G contains an H-packing that covers all but at most ϵn vertices.

In fact, Shokoufandeh and Zhao [19] have proven that the number of leftover vertices in Theorem 5.3 can be improved to a constant dependant only on H. In the next section we will see a proof of a result that is stronger than Theorem 5.3, though not the result by Shokoufandeh and Zhao.

5.2 Proof of Komlós' Theorem on almost perfect packings

In this section we consider the more general question of which minimum degree forces the existence of an H-packing that covers a given proportion of the vertices in G. We will next introduce some notation so we can do this formally.

Given a graph H we define $TT(n, H, M)$ to be the smallest number m such that, if G is a graph on n vertices such that $\delta(G) \geq m$, then there is an H-packing covering at least M vertices in G. Given a real number x such that $0 < x < 1$ we consider the function

$$f_H(x) := \lim_{n\to\infty} \frac{1}{n} TT(n, H, xn).$$

We can extend the definition of f_H by setting

$$f_H(0) := \lim_{x\downarrow 0} f_H(x) \text{ and } f_H(1) := \lim_{x\uparrow 1} f_H(x).$$

Notice that if $\chi(H) = 1$ then if $0 < x < 1$ for sufficiently large n we have $TT(n, H, xn) = 0$. Thus, $f_H(x) = 0$ for $0 < x < 1$ and thus $f_H(0) = 0 = f_H(1)$.

Suppose that H is a graph of chromatic number $\chi(H) = r \geq 2$. Given any $n \in \mathbb{N}$ such that $(r-1)|n$ we let G denote the complete $(r-1)$-partite graph on n vertices such that each vertex class of G has size $n/(r-1)$. We have already seen that $H \not\subseteq G$ and $\delta(G) = (1 - \frac{1}{r-1})n$. Thus, given any $x > 0$ we have that

$$1 - \frac{1}{\chi(H) - 1} \leq \lim_{n\to\infty} \frac{1}{n} TT(n, H, xn). \tag{5.1}$$

Suppose $0 < \epsilon < 1$. Let N be the output of Theorem 2.4 on input H and $\epsilon/2$. Suppose that G is a graph on $n \geq \frac{N}{1-\epsilon}$ vertices and

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right) n.$$

Thus, by Theorem 2.4, $H \subseteq G$. If we remove this copy of H from G we obtain a graph G_1 such that $\delta(G_1) \geq \left(1 - \frac{1}{\chi(H)-1} + \epsilon\right) n - |H|$. We can continue in this fashion to obtain an H-packing in G covering at least $\epsilon n/2$ vertices in G: We remove all disjoint copies of H in G chosen so far to obtain the graph G^*. If less than $\epsilon n/2$ vertices in G have been removed from G to obtain G^* we must have that

$$\delta(G^*) \geq \left(1 - \frac{1}{\chi(H) - 1} + \epsilon/2\right) n > \left(1 - \frac{1}{\chi(H) - 1} + \epsilon/2\right) |G^*|.$$

The choice of n ensures that $|G^*| \geq N$ and so by Theorem 2.4 we have a copy of H in G^*. Thus, repeating this argument we obtain an H-packing in G covering at least an $\epsilon/2$-fraction of the vertices in G. So

$$\lim_{n\to\infty} \frac{1}{n} TT(n, H, \epsilon n/2) \leq \frac{1}{\chi(H) - 1} + \epsilon. \tag{5.2}$$

From (5.1) and (5.2) we thus deduce that

$$f_H(0) = 1 - \frac{1}{\chi(H) - 1}.$$

Also, Theorem 2.8 implies

$$f_H(1) \leq 1 - \frac{1}{\chi(H)}.$$

The next result determines $f_H(x)$ for every $0 < x \leq 1$. In particular, it will imply Theorem 5.3.

Theorem 5.4 *(Komlós [11]) Given a graph H with $\chi(H) > 1$ we define*

$$g(x) := x\left(1 - \frac{1}{\chi_{cr}(H)}\right) + (1-x)\left(1 - \frac{1}{\chi(H)-1}\right) \quad \textit{for } x \in (0,1).$$

Then, for all $x \in (0,1)$,

$$f_H(x) := \lim_{n\to\infty} \frac{1}{n} TT(n, H, xn) = g(x).$$

In particular,

$$f_H(1) = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} TT(n, H, (1-\epsilon)n) = 1 - \frac{1}{\chi_{cr}(H)}.$$

Note that if the graph H associated with the function g is not explicit then we denote g by g_H.

Firstly we show that the first part of Theorem 5.4 implies the second part of the result. We have that

$$f_H(1) = \lim_{x\uparrow 1} f_H(x) = \lim_{\epsilon\to 0} f_H(1-\epsilon) = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} TT(n, H, (1-\epsilon)n)$$

and

$$\lim_{\epsilon\to 0} f_H(1-\epsilon) = \lim_{\epsilon\to 0} g(1-\epsilon) = 1 - \frac{1}{\chi_{cr}(H)}$$

thus proving the second part of the theorem.

We now show that Theorem 5.4 implies Theorem 5.3.

Proof of Theorem 5.3. Let $\epsilon > 0$. Given any graph H such that $\chi(H) > 1$, the first part of Theorem 5.4 gives us that

$$\lim_{n\to\infty} \frac{1}{n} TT(n, H, (1-\epsilon)n) = g(1-\epsilon).$$

Thus, there exists an $n_0(H, \epsilon)$ such that if $n \geq n_0$ then

$$\frac{1}{n} TT(n, H, (1-\epsilon)n) - g(1-\epsilon) < \epsilon.$$

Since $\chi(H) - 1 < \chi_{cr}(H)$ we have

$$x\left(1 - \frac{1}{\chi_{cr}(H)}\right) + (1-x)\left(1 - \frac{1}{\chi(H)-1}\right) < 1 - \frac{1}{\chi_{cr}(H)} \quad \text{for all } x \in (0,1).$$

Hence,

$$TT(n, H, (1-\epsilon)n) < \left(1 - \frac{1}{\chi_{cr}(H)} + \epsilon\right) n.$$

This proves that there exists an $n_0(H, \epsilon)$ such that if G is a graph on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \epsilon\right) n$$

then G contains an H-packing that covers all but at most ϵn vertices. By arguing precisely as in the proof of Theorem 2.8 but replacing $\chi(H)$ with $\chi_{cr}(H)$ throughout the proof, we see that this implies our result. $\square$

We still need to prove the first part of Theorem 5.4 in order to show that the whole result is true. We will do this by proving the following two results.

Proposition 5.5 *Let H be a graph such that $\chi(H) > 1$. Then for all $0 < M \leq n$ we have*

$$TT(n, H, M) \geq M\left(1 - \frac{1}{\chi_{cr}(H)}\right) + (n - M)\left(1 - \frac{1}{\chi(H) - 1}\right).$$

Hence, for $0 < x < 1$ we have

$$f_H(x) \geq x\left(1 - \frac{1}{\chi_{cr}(H)}\right) + (1 - x)\left(1 - \frac{1}{\chi(H) - 1}\right)$$

and

$$f_H(1) \geq 1 - \frac{1}{\chi_{cr}(H)}.$$

Theorem 5.6 *Let H be a graph and $x \in (0, 1)$. Given any $\eta > 0$ there exists an $n_0(H, x, \eta)$ such that if G is a graph on $n \geq n_0$ vertices and*

$$\delta(G) \geq (g(x) + \eta)\, n$$

then G contains an H-packing covering at least xn vertices.

Proposition 5.5 shows us that for $x \in (0, 1)$ we have $f_H(x) \geq g(x)$. Theorem 5.6 implies for any $\eta > 0$, $TT(n, H, xn) \leq (g(x) + \eta)n$ if n is sufficiently large and thus, $f_H(x) \leq g(x)$. Hence, together they prove Theorem 5.4.

Proof of Proposition 5.5. Recall we let $\sigma = \sigma(H)$ denote the smallest possible colour class size in any $\chi(H)$-colouring of H. Let us write $h = |H|$, $r = \chi(H)$, $\chi_{cr} = \chi_{cr}(H)$ and let m be the smallest integer strictly greater than

$$\frac{n - M\sigma/h}{r - 1} = \frac{M}{\chi_{cr}} + \frac{n - M}{r - 1}.$$

(The equation above arises from the definition of χ_{cr}.) Let G be the complete r-partite graph on n vertices with $r-1$ colour classes of size m and a leftover colour class C. Now C is the colour class of smallest size. For if not $|C| > \frac{M}{\chi_{cr}} + \frac{n-M}{r-1}$. Thus,

$$|G| = n > r\frac{M}{\chi_{cr}} + r\frac{n - M}{r - 1} > M + (n - M) = n,$$

a contradiction.

The class C has size $n - (r - 1)m < M\sigma/h$. Given a copy of H in G at least σ vertices of this copy of H lie in C. So at least a σ/h-fraction of all vertices in an H-packing in G lie in C. In particular, G cannot have an H-packing that covers M vertices since we would need at least $M\sigma/h$ vertices in C. Thus,

$$\begin{aligned} TT(n, H, M) \geq \delta(G) + 1 = n - (m - 1) &\geq n - \left(\frac{M}{\chi_{cr}} + \frac{n - M}{r - 1}\right) \\ &= M\left(1 - \frac{1}{\chi_{cr}}\right) + (n - M)\left(1 - \frac{1}{r - 1}\right). \end{aligned}$$

The last two parts of the result follow immediately from this. □

In order to prove Theorem 5.6 we must introduce some more notation. Firstly, we show we only need to prove Theorem 5.6 for a special type of graph.

Definition 5.7 *(Bottle-graphs) A bottle-graph B of chromatic number r is a complete r-partite graph with vertex classes of size $\sigma, \omega, \omega, \dots, \omega$ where $\sigma \leq \omega$. The number σ is called the neck of B and ω is the width of B.*

Notice a bottle-graph B is such that

$$\chi_{cr}(B) = \frac{(r-1)|B|}{|B| - \sigma} = \frac{|B|}{\omega} = r - 1 + \frac{\sigma}{\omega}. \tag{5.3}$$

Given an r-chromatic graph H we say that an r-chromatic bottle-graph B *is a bottle-graph of* H if B has neck $s|B|$ and width $t|B|$ where $s = \sigma(H)/|H|$ and $t = (1-s)/(r-1)$. In this case we have $\chi(H) = \chi(B)$ and $\sigma(B) = \frac{\sigma(H)}{|H|}|B|$, that is $\frac{\sigma(H)}{|B|} = \frac{\sigma(H)}{|H|}$. Since the critical chromatic number of a graph G depends only on $\chi(G)$ and $\frac{\sigma(G)}{|G|}$ we have $\chi_{cr}(B) = \chi_{cr}(H)$.

In particular we consider the complete r-partite graph B with $r-1$ vertex classes of size $|H| - \sigma(H)$ and one class of size $(r-1)\sigma(H)$. Then B is a bottle-graph of H. Further we can think of the vertex classes of B as being obtained from $r-1$ disjoint copies of H as follows: Consider any r-colouring of H such the smallest colour class has size $\sigma(H)$. The smallest vertex class of B consists of $r-1$ copies of the smallest colour class in H. Each of the large vertex classes of B are obtained from the union of the $r-1$ larger colour classes of H. This shows that B has a perfect H-packing. In particular, since $\chi_{cr}(B) = \chi_{cr}(H)$, this implies it is sufficient to prove Theorem 5.6 for bottle-graphs. Indeed, given any graph H and the corresponding bottle-graph B, if we can find an almost perfect B-packing in a graph G then this provides us with an almost perfect H-packing in G.

Thus, in the rest of this section we will consider H to be an r-chromatic bottle-graph with neck σ and width ω. Further, we will see that it is sufficient to prove Theorem 5.6 under the extra assumption that $\sigma < \omega$. Indeed, assuming the result holds under these assumptions, then given any graph K whose vertex classes do not all have equal size, the aforementioned bottle-graph associated with K has width not equal to its neck. Hence, this implies Theorem 5.6 holds under the assumption that H is a graph whose vertex classes do not all have equal size. We now show this in turn implies the result holds for a bottle-graph H with $\sigma = \omega$, and thus, for all graphs that have equally sized vertex classes.

Consider such a bottle graph H with $r := \chi(H) = \chi_{cr}(H) > 1$, and let $x \in (0,1)$, $\eta > 0$ be given. We define $x' > x$ such that $x' < x + \eta/2$ and $x' < 1$. Consider a graph G such that

$$\delta(G) \geq (g_H(x) + \eta)|G|.$$

In particular, by definition of x' we have that

$$\delta(G) \geq (g_H(x') + \eta/2)|G|.$$

We now define the complete r-partite graph H^* so that it has $r-1$ vertex classes of size $k\sigma(H)$ and one vertex class of size $k\sigma(H)+1$, where k is sufficiently large so that $\left(1 - \frac{1}{k|H|+1}\right) x' \geq x$. Notice that $\sigma(H)$ is the size of every vertex class in H, so $|H^*| = k|H|+1$. Further, $\chi_{cr}(H^*) < r = \chi_{cr}(H)$, so $g_H(x') > g_{H^*}(x')$. Hence,

$$\delta(G) \geq (g_{H^*}(x') + \eta/2)|G|.$$

Thus, under our assumptions, Theorem 5.6 holds for input H^*, x' and $\eta/2$. So provided $|G|$ is sufficiently large G has an H^*-packing covering at least $x'|G|$ vertices. Since H^* has an H-packing covering all but 1 vertex, we obtain an H-packing in G covering at least $x'|G| - \frac{x'}{|H^*|}|G| \geq x|G|$ vertices, as required. So indeed, it is sufficient to prove Theorem 5.6 under assumption that H is a bottle-graph with neck not equal to its width.

We will use an auxiliary graph H' which is obtained from H by removing one vertex from each of the colour classes of size ω. Notice that H' is a bottle-graph, as is K_r. The next result is vital in the proof of Theorem 5.6. It tells us if we have an H-packing in a graph G which is not too large, then we can 'improve' on this and tile G with vertex-disjoint copies of H, H' and K_r, covering more vertices than the H-packing.

Lemma 5.8 *Let H be a bottle-graph on h vertices with $\chi(H) = r$, $\chi_{cr}(H) = \chi_{cr}$ and width ω and neck $\sigma \neq \omega$. Given $x, \epsilon \in (0,1)$, let $n \geq n_0(H, x, \epsilon)$ and let G be a graph on n vertices such that $\delta := \delta(G) \geq g(x)n$. Suppose the maximum number of vertices in G covered by an H-packing is $M \leq (1-\epsilon)xn$. Then there exists some $\epsilon'' = \epsilon''(H, x, \epsilon) > 0$ such that G has a tiling with vertex-disjoint copies of H, H' and K_r that covers at least $M + \epsilon'' n$ vertices.*

Proof. We choose some $\epsilon' > 0$ sufficiently small for our calculations below to work. Consider an H-packing in G which covers M vertices. Let $\mathcal{L}$ denote the set of vertices not covered by this H-packing. So $|\mathcal{L}| = n - M =: L$. Let $Z := (1-1/(r-1)+\epsilon')L^2/2$. Then if L is sufficiently large (i.e. if n is sufficiently large) then by Theorem 2.4 we must have that $e(G[\mathcal{L}]) \leq Z$.

If more than $(1-\epsilon')L$ vertices in $\mathcal{L}$ have degrees more than $\frac{2Z}{(1-\epsilon')L}$ into $\mathcal{L}$ then

$$e(G[\mathcal{L}]) > \frac{2Z}{(1-\epsilon')L} \cdot \frac{(1-\epsilon')L}{2} = Z$$

a contradiction. So at least $\epsilon' L$ of the vertices in $\mathcal{L}$ have degree at most $\frac{2Z}{(1-\epsilon')L}$ into $\mathcal{L}$. Pick such a $v \in \mathcal{L}$. By hypothesis, $d_G(v) \geq \delta \geq g(x)n$. Let a be the number of copies of H in our H-packing in which v has more than $h - \omega$ neighbours. Then,

$$ah + \left(\frac{M}{h} - a\right)(h-\omega) \geq d_G(v) - d_{\mathcal{L}}(v) \geq \delta - \frac{2Z}{(1-\epsilon')L}.$$

Rearranging gives

$$\begin{aligned} a &\geq \left[\delta - \frac{2Z}{(1-\epsilon')L} - \frac{h-\omega}{h}M\right]\frac{1}{\omega} \\ &\geq \left[g(x)n - \left(1 - \frac{1}{r-1} + \epsilon'\right)\frac{L}{1-\epsilon'} - \frac{h-\omega}{h}M\right]\frac{1}{\omega} =: C. \end{aligned}$$

Notice that $\frac{h-\omega}{h} = 1 - \frac{\omega}{h} = 1 - \frac{1}{\chi_{cr}}$ as H is a bottle-graph (see equation (5.3)). Further $L + M = n$. Thus,

$$
\begin{aligned}
\omega C &= \left[g(x) - \left(1 - \frac{1}{r-1} + \epsilon'\right)\frac{1}{1-\epsilon'}\right] n - \left[-\left(1 - \frac{1}{r-1} + \epsilon'\right)\frac{1}{1-\epsilon'} + \left(1 - \frac{1}{\chi_{cr}}\right)\right] M \\
&= \left[xn\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right) + \left(1 - \frac{1}{r-1}\right)n - \left(1 - \frac{1}{r-1} + \epsilon'\right)\frac{1}{1-\epsilon'}(M+L)\right] \\
&- \left[-\left(1 - \frac{1}{r-1} + \epsilon'\right)\frac{1}{1-\epsilon'}M + \left(1 - \frac{1}{\chi_{cr}}\right)M\right] \\
&\geq \left[\epsilon xn\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right) + M\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right)\right] + \left(1 - \frac{1}{r-1}\right)n \\
&- L\left(1 - \frac{1}{r-1} + \epsilon'\right)(1+2\epsilon') - M\left(1 - \frac{1}{\chi_{cr}}\right) \\
&= \epsilon xn\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right) - M\left(1 - \frac{1}{r-1}\right) + \left(1 - \frac{1}{r-1}\right)n - L\left(1 - \frac{1}{r-1} + \epsilon'\right)(1+2\epsilon') \\
&= \epsilon xn\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right) - 2\epsilon' L\left(\frac{3}{2} - \frac{1}{r-1} + \epsilon'\right) \\
&\geq \epsilon xn\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right) - 3\epsilon' n \geq \omega\epsilon' n,
\end{aligned}
$$

since $\frac{1}{1-\epsilon'} \leq 1 + 2\epsilon'$ as ϵ' is sufficiently small. The last of these inequalities follows as we can let $\epsilon' < \frac{1}{3h}\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right)\epsilon x$ and thus,

$$
\left(\frac{1}{r-1} - \frac{1}{\chi_{cr}}\right)\epsilon xn - 3\epsilon' n > 3h\epsilon' n - 3\epsilon' n \geq (3\omega - 3)\epsilon' n \geq \omega\epsilon' n
$$

as $\omega > \sigma \geq 1$.

Hence, each of our vertices v is connected to at least $\epsilon' n \geq \epsilon' L$ copies of H by more than $h - \omega$ edges. Thus, for at least $\epsilon' L$ such vertices v we can assign a distinct such copy $H(v)$ of H. Since fewer than ω edges are 'missing' from v to $H(v)$ in G there exists an edge from v to each colour class of $H(v)$ of size ω. So 'connecting' v to one vertex in each such class, we obtain a copy of K_r and a copy of H' from $H(v) \cup \{v\}$. Thus, we obtain a tiling of G with vertex-disjoint copies of H, H' and K_r that covers at least $M + \epsilon'' n$ vertices where $\epsilon'' := \epsilon'(1 - (1-\epsilon)x) > 0$. □

There is a very important subtlety in the proof of Lemma 5.8. The maximum number of vertices covered by an H-packing in G was denoted by M. The statement of the lemma is such that we did not rule out the possibility that $M = 0$. Indeed, in the proof of Lemma 5.8 we did not need to assume $M > 0$. However, we established that there are vertices $v \in \mathcal{L}$ which are connected to more than $\epsilon n'$ copies of H in G. So $M \neq 0$. This essentially stems from the fact that there are vertices $v \in \mathcal{L}$ such that $d_{\mathcal{L}}(v) \leq \frac{2Z}{(1-\epsilon')L}$: If $V(G) = \mathcal{L}$ then $d_G(v) \leq \frac{2Z}{(1-\epsilon')L} = \left(1 - \frac{1}{r-1} + \epsilon'\right)n$. But $g(x) > 1 - \frac{1}{r-1}$ and $d_G(v) \geq g(x)n$, so choosing $\epsilon' > 0$ sufficiently small we have that $\left(1 - \frac{1}{r-1} + \epsilon'\right)n \geq d_G(v) \geq$

$g(x)n$ but $g(x) > 1 - \frac{1}{r-1} + \epsilon'$, a contradiction. So indeed, $M \neq 0$. Thus, if we have a graph G satisfying the conditions in the hypothesis of Lemma 5.8 this implies that G has an H-packing. In particular, when we apply Lemma 5.8 to a reduced graph R in the proof of Theorem 5.6, we can immediately talk about an H-packing in R.

We will now prove a result about H-packings in 'blown-up' copies of H, H' and K_r.

Lemma 5.9 *Let H be a bottle-graph on h vertices with the usual parameters σ and ω. Let $t := (\omega - \sigma)h$. Then $H(t)$, $H'(t)$ and $K_r(t)$ all have perfect H-packings.*

Proof. We can consider $H'(t)$ as follows: $H'(t)$ contains $(\omega - 1 - \sigma)h$ vertex-disjoint copies of H such that the class of size σ of each copy of H goes into the vertex class of size σt of $H'(t)$. All other vertex classes of these copies of H are distributed equally between the other $r-1$ classes of $H'(t)$. So this accounts for all but $\sigma t - \sigma(\omega - 1 - \sigma)h = \sigma h$ vertices in the class of size σt of $H'(t)$. Further, this accounts for all but $(\omega - 1)t - \omega(\omega - 1 - \sigma)h = \sigma h$ vertices in each class of size $(\omega - 1)t$. But then these remaining vertices can be considered as vertices of σr copies of H as follows: we distribute the σr vertex classes of size σ equally among the r vertex classes of $H'(t)$ (so σ of these classes are in each class of $H'(t)$). All larger vertex classes of each copy of H are also distributed equally among the vertex classes of $H'(t)$. Thus, this gives us a perfect H-packing of $H'(t)$.

We can split each vertex class of $K_r(t)$ into $\omega - \sigma$ parts of size σ and $(\omega - \sigma)(r - 1)$ parts of size ω. Thus, from this we obtain a perfect H-packing in K_r. Similarly $H(t)$ has a perfect H-packing. $\square$

We need one more lemma before we can prove Theorem 5.6. Its proof repeatedly applies Theorem 3.8.

Lemma 5.10 *Let $0 < \beta < 1/2$ and H be a bottle-graph with the usual parameters. Let $m \in \mathbb{N}$ and $d \in (0, 1)$. Then there exists an $\epsilon' > 0$ such that for all $\epsilon \leq \epsilon'$ the following holds. Let G be the graph obtained from H by replacing every vertex of H with m vertices and replacing the edges of H with ϵ-regular pairs of density at least d. Then G has an H-packing covering all but a β-fraction of the vertices in G.*

Proof. We choose ϵ so that $0 < \epsilon/\beta < d - \epsilon$, $\epsilon < \beta$ and $\frac{\epsilon}{\beta} < \frac{(d-\epsilon-\frac{\epsilon}{\beta})^{\Delta}}{2+\Delta}$ where $\Delta := \Delta(H) = h - \sigma$. These conditions ensure we can apply the Key Lemma (Theorem 3.8) sufficiently many times. We will refer to a set of m vertices in G corresponding to a vertex in H as a *cluster*.

By definition of G, Theorem 3.8 implies $H \subseteq G$. In particular the proof of Theorem 3.8 tells us that this copy of H is such that each vertex v in this copy of H is embedded into the cluster of size m in G which corresponds to v. So let G_1 denote the subgraph of G obtained by deleting the vertices in this copy of H. Thus, G_1 contains a subcluster of size $m - 1$ for each cluster in G. If the copy of H in G covers all but a β-fraction of the vertices in G we are done.

Else, the remainder of each cluster of G in G_1 contains more than a β-fraction of the vertices of the original cluster. Thus, by Lemma 1.6, the subclusters in G_1 of an ϵ-regular pair in G of density at least d form an ϵ/β-regular pair of density at least $d-\epsilon$. In particular G_1 is obtained from H by replacing every vertex of H with $m-1$ vertices and replacing the edges of H with ϵ/β-regular pairs of density at least $d-\epsilon$. Thus, by the choice of ϵ, Theorem 3.8 implies $H \subseteq G_1$.

We can keep on repeating this argument until all but a β-fraction of vertices in G are covered by an H-packing: If so far we have already found M disjoint copies of H in G which together do not cover a β-fraction of vertices, then the subclusters, formed by the removal of these copies of H, are such that an ϵ-regular pair of density at least d in G corresponds to a pair of subclusters which is ϵ/β-regular and has density at least $d-\epsilon$. So we have a graph $G^* \subseteq G$ which is obtained from H by replacing every vertex of H with $m-M$ vertices and replacing the edges of H with ϵ/β-regular pairs of density at least $d-\epsilon$. Thus, we can apply the Key Lemma to find another disjoint copy of H in G. So indeed we can find our desired H-packing in G. $\square$

We are now in a position to prove Theorem 5.6. The idea of the proof is as follows: Given a reduced graph R of a graph G consider an H-packing in R that does not cover slightly more than an x-fraction of the vertices. By Lemma 5.8 we can find a tiling of R with vertex-disjoint copies of H, H' and K_r that covers substantially more vertices in R than the H-packing. If we 'blow-up' R to obtain $R(t)$, our tiling of R with vertex-disjoint copies of H, H' and K_r corresponds to a tiling of $R(t)$ with vertex-disjoint copies of $H(t)$, $H'(t)$ and $K_r(t)$ (where t is as defined in Lemma 5.9). But by Lemma 5.9 each copy of $H(t)$, $H'(t)$ and $K_r(t)$ contains a perfect H-packing. Thus, we obtain an H-packing of $R(t)$ covering a higher proportion of vertices than our original H-packing in R. If we repeat this argument we will end up with a graph R' which is obtained from R by repeatedly blowing-up R. R' will have the property that it has an H-packing covering slightly more than an x-fraction of the vertices in R'. As R was the reduced graph of G, G contains an almost spanning subgraph G^* which can be obtained from R' as in the process described in Lemma 5.10. So Lemma 5.10 implies G^* has an H-packing covering slightly more than an x-fraction of the vertices in G^*. This packing will then give us an H-packing in G covering an x-fraction of the vertices in G.

Proof of Theorem 5.6. We are given a bottle-graph H on h vertices, $x \in (0,1)$ and $\eta > 0$. We choose n sufficiently large and let G be a graph on n vertices such that $\delta(G) \geq (g(x)+\eta)n$. We apply Theorem 3.5 to obtain the reduced graph R of G with parameters $0 < \epsilon < d < 1$ which are chosen sufficiently small compared to $1/|H|$, x, $1-x$ and η. (Where conditions on these parameters are needed, they will be stated in the proof.) As indicated above, our first aim is to show that a suitable blow-up of R contains some H-packing that covers slightly more than an x-fraction of the vertices.

We choose ϵ and d so that we can apply Lemma 3.7 to obtain $\delta(R) \geq (g(x)+\eta/2)|R|$. Now $x<1$ so we choose an $\epsilon' > 0$ sufficiently small such that $x+2\epsilon' < 1$ and $\delta(R) \geq (g(x)+\eta/2)|R| \geq g(x')|R|$, where $x' := x+2\epsilon'$. In particular we have that $x + \epsilon' \leq (1 - \epsilon')x'$. So if we have an H-packing in R covering more than $(1-\epsilon')x'|R|$ vertices then it certainly covers more than $(x+\epsilon')|R|$ vertices, which is what we want. If not then as n is sufficiently large, by Lemma 5.8, we obtain a tiling of R with disjoint copies of H, H' and K_r, covering substantially more vertices in R than the largest H-packing in R. Let $t = (\omega - \sigma)h$. So the tiling in R consisting of disjoint copies of H, H' and K_r corresponds to a tiling of $R(t)$ with vertex-disjoint copies of $H(t)$, $H'(t)$ and $K_r(t)$. But by Lemma 5.9 each copy of $H(t)$, $H'(t)$ and $K_r(t)$ contains a perfect H-packing. So we obtain an H-packing in $R(t)$ covering a substantially higher proportion of vertices than the original H-packing in R. If our new H-packing covers more than $(1-\epsilon')x'|R(t)|$ vertices we are done. If not, we have $\delta(R(t)) \geq g(x')|R(t)|$ so can apply Lemma 5.8 again, obtaining a tiling of $R(t)$ with disjoint copies of H, H' and K_r, covering substantially more vertices in $R(t)$ than the H-packing. Thus, we can continue in this fashion: if we have not got our required H-packing in our current graph R^* we apply Lemma 5.8 and Lemma 5.9 to find an H-packing in $R^*(t)$ covering a substantially higher proportion of vertices than the previous H-packing. After applying this algorithm a finite number, $Z = Z(H, x, \eta)$ times, we obtain a graph $R' = R(t^Z)$ with an H-packing covering more than $(1-\epsilon')x'|R'| \geq (x+\epsilon')|R'|$ vertices in R'. Notice that $Z \leq 1/\epsilon''$ where $\epsilon'' = \epsilon''(H, x', \epsilon')$ is the output of Lemma 5.8.

Next we claim that R' is a reduced graph of an almost spanning subgraph of G. We know that R corresponds to a pure graph $G'' \subseteq G$ where $|G''| \geq (1-\epsilon)|G|$. We split up each cluster in G'' into t^Z equal subclusters. Thus, less than t^Z vertices in each cluster are discarded. So in total less than Mt^Z vertices in G'' are discarded, where M is the output of Theorem 3.5 on input ϵ (M is an upper bound on the number of clusters in G''). We denote the $|R|t^Z$-partite subgraph thus obtained from G'' by G^*. So each subcluster of G^* is larger than an $\frac{1}{1+t^Z}$-fraction of the cluster it originally belonged to in G''. Thus, with $\alpha := \frac{1}{1+t^Z}$ and since we have chosen ϵ sufficiently small (i.e. $\epsilon < \alpha$), the Slicing Lemma tells us pairs of subclusters of G^* that originate from an ϵ-regular pair of density at least d form an ϵ/α-regular pair with density greater than $d - \epsilon$. Thus, R' is a reduced graph of G^* with parameters ϵ/α and $d - \epsilon$. We define $\beta > 0$ so that $(1-\beta)(x+\epsilon') \geq (x + \epsilon'/2)$. If ϵ is sufficiently small then with input β we can apply Lemma 5.10 with ϵ/α playing the role of ϵ to see that each copy of H in R' corresponds to a subgraph of G^* which has an H-packing covering all but a β-fraction of its vertices. Since R' has an H-packing on at least $(x+\epsilon')|R'|$ vertices, the choice of β when applying Lemma 5.10 ensures $G^* \subseteq G$ has an H-packing covering at least $(1-\beta)(x+\epsilon')|G^*| \geq (x+\epsilon'/2)|G^*|$ vertices. Now, $|G^*| \geq (1-\epsilon)n - Mt^Z$. So choosing ϵ small and n sufficiently large we can ensure $(x+\epsilon'/2)|G^*| \geq xn$. Thus, we have obtained an H-packing in G covering at least xn vertices, as required. $\square$

Chapter 6

Ore-type degree conditions for almost perfect packings

In this chapter we will prove the analogous results of Theorems 2.8 and 5.4 for so-called Ore-type degree conditions. Such a condition on a graph G is of the following form:

If $d(x) + d(y) \geq \ldots$ for all $x \neq y \in V(G)$ such that $xy \notin E(G)$ then...

We will prove the following analogue of Theorem 2.8.

Theorem 6.1 *For every graph H and every $\epsilon > 0$ there exists an $n_0(H, \epsilon)$ such that if G is a graph on $n \geq n_0$ vertices and*

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi(H)}\right) n$$

for all $x \neq y \in V(G)$ such that $xy \notin E(G)$ then G has an H-packing covering all but at most ϵn vertices.

Recall that in the proof of Theorem 2.8 we used Theorem 2.7. In the proof of Theorem 6.1 we will use an Ore-type analogue of Theorem 2.7 due to Kierstead and Kostochka.

Theorem 6.2 *(Kierstead, Kostochka [9]) Let G be a graph on n vertices. If*

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{r}\right) n$$

for all $x \neq y \in V(G)$ such that $xy \notin E(G)$ then G has a perfect K_r-packing.

We also used Lemma 3.7 in the proof of Theorem 2.8. Again we have a similar result for an Ore-type degree condition.

Lemma 6.3 *Let G be a graph such that $d_G(x) + d_G(y) \geq c|G|$ for all $x \neq y \in V(G)$ such that $xy \notin E(G)$, where c is a constant. Suppose we have applied the degree form of the Regularity Lemma to G and have defined from this a reduced graph R with parameters $\epsilon > 0$ and $d \in [0, 1)$. Then $d_R(V_i) + d_R(V_j) > (c - 2d - 4\epsilon)|R|$ for all $V_i \neq V_j \in V(R)$ such that $V_iV_j \notin E(R)$.*

Proof. We apply the degree form of the Regularity Lemma (Theorem 3.5) to obtain a partition $\{V_0, V_1, \dots, V_k\}$ of G with exceptional set V_0 and $|V_i| = m$ for all $i \in [k]$. Thus we obtain a pure graph G'' of G. Recall our convention that G'' is such that it satisfies all the properties of a pure graph with $e(G'')$ maximised. In particular this means that given $i \neq j$, if $V_iV_j \notin E(R)$ then (V_i, V_j) is not an ϵ-regular pair of density more than d in G. Thus, there exists some $x \in V_i$ and $y \in V_j$ such that $xy \notin E(G)$. So $d_G(x) + d_G(y) \geq c|G|$ which implies

$$d_{G''}(x) + d_{G''}(y) > c|G| - 2(d + 2\epsilon)|G| = (c - 2d - 4\epsilon)|G|.$$

Assume that $d_R(V_i) + d_R(V_j) \leq (c - 2d - 4\epsilon)|R|$. So the number of clusters which together with V_i form an ϵ-regular pair of density more than d in G'', added to the number of clusters which together with V_j forms an ϵ-regular pair of density more than d in G'' is at most $(c-2d-4\epsilon)|R|$. However, by definition of G'' each cluster which contains a neighbour of x in G'' must form an ϵ-regular pair of density greater than d together with V_i. The same is true for the clusters containing the neighbours of y. As each cluster has size m it follows that $d_{G''}(x) + d_{G''}(y) \leq (c - 2d - 4\epsilon)|R|m \leq (c - 2d - 4\epsilon)|G|$, a contradiction. So $d(V_i) + d(V_j) > (c - 2d - 4\epsilon)|R|$, as required. □

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. Similarly to the proof of Theorem 2.8, to prove Theorem 6.1 it is sufficient to show the following:

For every graph H and every $\eta > 0$ there exists an $n_0(H, \eta)$ such that if G is a graph on $n \geq n_0$ vertices and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi(H)} + \eta\right) n$$

for all $x \neq y \in V(G)$ such that $xy \notin E(G)$ then G has an H-packing covering all but at most ηn vertices.
(The same argument that is used to derive Theorem 2.8 from Theorem 4.4 shows why it sufficient to prove this.)

Given $\eta > 0$, choose $\epsilon > 0$ and d as in the proof of Theorem 4.4 but with the extra condition that $\eta > 2d + \epsilon$. Arguing identically as in the proof of Theorem 4.4 but replacing the minimum degree condition with our Ore-type condition, and applying Lemma 6.3 and Theorem 6.2 instead of Lemma 3.7 and Theorem 2.7, proves the theorem. □

In order to state the analogue of Theorem 5.4, we first introduce the following notation. We define $TT_0(n, H, M)$ to be the minimum number m such that, if G is a graph on n vertices such that $d(x) + d(y) \geq m$ for all $x \neq y \in V(G)$ where $xy \notin E(G)$, then there is an H-packing covering at least M vertices in G. We also define the function

$$f_H^0(x) := \lim_{n\to\infty} \frac{1}{n} TT_0(n, H, xn)$$

for $0 < x < 1$. We extend this definition by setting $f_H^0(1) := \lim_{x\uparrow 1} f_H(x)$. We will thus prove the following analogue of Theorem 5.4 for the function f_H^0.

Theorem 6.4 *Given a graph H with $\chi(H) > 1$, we have for all $x \in (0,1)$,*

$$f_H^0(x) := \lim_{n\to\infty} \frac{1}{n} TT_0(n, H, xn) = 2g(x).$$

In particular,

$$f_H^0(1) = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} TT_0(n, H, (1-\epsilon)n) = 2\left(1 - \frac{1}{\chi_{cr}(H)}\right).$$

Further, for every graph H and every $\epsilon > 0$ there exists an $n_0(H, \epsilon)$ such that if G is a graph on $n \geq n_0$ vertices and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi_{cr}(H)}\right) n$$

for all $x \neq y \in V(G)$ such that $xy \notin E(G)$ then G has an H-packing covering all but at most ϵn vertices.

Recall that $g(x)$ was defined in Theorem 5.4. Throughout this chapter this is the function we are considering when we are referring to $g(x)$.

Note that Theorem 6.4 implies Theorem 6.1. However, we have included the proof of Theorem 6.1 as using Theorem 6.2 makes this short. We will see though that we can prove Theorem 6.4 without using Theorem 6.2. Notice also that Theorems 6.1 and 6.4 are results strengthening Theorems 2.8 and 5.4 respectively. For, if a graph G satisfies $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n$ it also satisfies the condition in Theorem 6.1. This observation is also true when considering the conditions involving $\chi_{cr}(H)$.

We now turn our attention to the proof of Theorem 6.4. By arguing identically as we did for Theorem 5.4 just replacing f_H and TT with f_H^0 and TT_0 respectively, we see that the first part of Theorem 6.4 implies the second part of the theorem, and the following: Given any graph H and $\epsilon > 0$ there exists an $n_0(H, \epsilon)$ such that if G is a graph on $n \geq n_0$ vertices with

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi_{cr}(H)} + \epsilon\right) n$$

for all $x \neq y \in V(G)$ such that $xy \notin E(G)$ then G has an H-packing covering all but at most ϵn vertices.

But again, we can argue precisely as in the proof of Theorem 2.8, replacing the minimum degree condition with our Ore-type condition in Theorem 6.4, to see that this in turn implies the last part of Theorem 6.4.

To prove the first part of Theorem 6.4 it is sufficient to prove the following two results.

Proposition 6.5 *Let H be a graph such that $\chi(H) > 1$. Then for all $0 < M \leq n$ we have*

$$TT_0(n, H, M) \geq 2M\left(1 - \frac{1}{\chi_{cr}(H)}\right) + 2(n - M)\left(1 - \frac{1}{\chi(H) - 1}\right) - 1.$$

Hence, for $0 < x < 1$ *we have*

$$f_H^0(x) \geq 2x\left(1 - \frac{1}{\chi_{cr}(H)}\right) + 2(1-x)\left(1 - \frac{1}{\chi(H)-1}\right)$$

and

$$f_H^0(1) \geq 2\left(1 - \frac{1}{\chi_{cr}(H)}\right).$$

Proof. The graph G on n vertices in the proof of Theorem 5.5 is such that it does not have an H-packing on at least M vertices. So, $TT_0(n, H, M) \geq 2\delta(G) + 1 \geq 2M\left(1 - \frac{1}{\chi_{cr}(H)}\right) + 2(n-M)\left(1 - \frac{1}{\chi(H)-1}\right) - 1$. The rest of the result follows from this. $\square$

Theorem 6.6 *Let* H *be a graph and* $x \in (0, 1)$. *Given any* $\eta > 0$ *there exists an* $n_0(H, x, \eta)$ *such that if* G *is a graph on* $n \geq n_0$ *vertices and*

$$d(x) + d(y) \geq 2\,(g(x) + \eta)\,n$$

for all $x \neq y \in V(G)$ *such that* $xy \notin E(G)$ *then* G *contains an* H*-packing covering at least* xn *vertices.*

As with the proof of Theorem 5.6 it is sufficient to prove Theorem 6.6 under the assumption that H is a bottle-graph with neck smaller than width. We now introduce an analogue of Lemma 5.8 for an Ore-type condition on the vertex degrees.

Lemma 6.7 *Let* H *be a bottle-graph on* h *vertices with* $\chi(H) = r$, $\chi_{cr}(H) = \chi_{cr}$, *width* ω *and neck* $\sigma \neq \omega$. *Given* $x, \epsilon \in (0, 1)$, *let* $n \geq n_0(H, x, \epsilon)$ *and let* G *be a graph on* n *vertices such that* $d(x)+d(y) \geq 2g(x)n$ *for all distinct* $x, y \in V(G)$ *such that* $xy \notin E(G)$. *Suppose the maximum number of vertices in* G *covered by an* H*-packing is* $M \leq (1-\epsilon)xn$. *Then there exists some* $\epsilon'' = \epsilon''(H, x, \epsilon) > 0$ *such that* G *has a tiling with vertex-disjoint copies of* H, H' *and* K_r *that covers at least* $M + \epsilon'' n$ *vertices.*

Proof. As in the proof of Lemma 5.8 we choose $\epsilon' > 0$ and define Z, $\mathcal{L}$ and L similarly. Following the proof of Lemma 5.8, we obtain at least $\epsilon' L$ vertices in $\mathcal{L}$ with at most $\frac{2Z}{(1-\epsilon')L}$ neighbours in $\mathcal{L}$. We denote this set of vertices by $\mathcal{L}'$ and put $L' := |\mathcal{L}'|$.

Let $Z' := (1 - 1/(r-1) + \epsilon')(L')^2/2$. Since $L' = |\mathcal{L}'| \geq \epsilon' L \geq \epsilon'(1-(1-\epsilon)x)n$ we can ensure that L' is large by choosing n sufficiently large. Thus Theorem 2.4 implies that $e(G[\mathcal{L}']) \leq Z'$. So there exists some $\beta > 0$ such that there are $\beta(L')^2/2$ pairs of distinct vertices that do not have an edge between them in $G[\mathcal{L}']$. In particular there are $\beta L'/2$ disjoint pairs of vertices of this form (i.e. we have a matching in $\overline{G[\mathcal{L}']}$ covering $\beta L'$ vertices). Consider such a pair of vertices (x, y). Since $xy \neq E(G)$, $d_G(x) + d_G(y) \geq 2g(x)n$. So x or y has degree at least $g(x)n$ in G. Thus, there are at least $\beta L'/2 \geq \beta\epsilon'(1-(1-\epsilon)x)n/2$ vertices in $\mathcal{L}$ with degree at least $g(x)n$ in G but degree at most $2Z/((1-\epsilon')L)$ in $\mathcal{L}$.

Following the proof of Lemma 5.8 we can pair these vertices off with vertex-disjoint copies of H in G, in such a way as to obtain a tiling of G with vertex-disjoint copies of H,H' and K_r covering at least $M + \epsilon'' n$ vertices where $\epsilon'' := \beta\epsilon'(1-(1-\epsilon)x)/2 > 0$. $\square$

Proof of Theorem 6.6. We argue as in the proof of Theorem 5.6. The only difference is that once we have defined a reduce graph R of G we apply Lemma 6.3 instead of Lemma 3.7 to obtain that $d_R(V_i) + d_R(V_j) \geq 2(g(x) + \eta/2)|R|$ for all distinct $V_i, V_j \in V(R)$ such that $V_iV_j \notin E(R)$, and we apply Lemma 6.7 instead of Lemma 5.8. $\square$

Chapter 7

Proof of two results concerning perfect packings

7.1 Results and extremal examples for perfect packings in graphs

In this chapter we will prove the Alon-Yuster Theorem on perfect packings which was introduced in Chapter 2 (Theorem 2.9). Before this we will prove a result that improves the minimum degree condition in Theorem 2.9 when we consider perfect H-packings for a certain type of graphs H. We will see that this result can be used to prove Theorem 2.9. We begin, though, by considering some extremal examples.

Proposition 7.1 *Let p_1, p_2 and p_3 be distinct odd numbers such that $p_1 + p_2 = 2p_3$. Consider the complete bipartite graph H with vertex classes A' and B' where $|A'| = p_1$ and $|B'| = p_2$. So $|H| = 2p_3$. Let G be the complete bipartite graph on $n := 2p_3 p$ vertices (where p is odd) with vertex classes A and B of equal size. Then G does not have a perfect H-packing.*

Proof. Suppose not. Then the H-packing must cover all vertices in G since $|H|$ divides $|G|$. Each copy of H in G is such that A' is contained entirely in A or B and B' is contained entirely in the vertex class of G not containing the copy of A'. So if we have x copies of A' embedded into A and y copies of B' embedded into A we must have x copies of B' embedded into B, and y copies of A' embedded into B. Since $|A| = |B|$ we have $xp_1 + yp_2 = xp_2 + yp_1$. That is $(x-y)p_1 = (x-y)p_2$ which implies $x = y$. But then A and B both must have size $x(p_1 + p_2)$. In particular $p_1 + p_2$ divides $|A|$, a contradiction as $|A| = p_3 p$ is odd and $p_1 + p_2$ is even. So G does not have a perfect H-packing. $\square$

Notice in Proposition 7.1 we have that $\delta(G) = \frac{n}{2} = (1 - \frac{1}{\chi(H)})n$, and since p was an arbitrary odd number we can define G so that n is arbitrarily large. Thus, this example shows that for some graphs H we cannot omit the constant in the minimum degree condition in Theorem 2.10, or indeed the ϵn term in the corresponding condition in Theorem 2.9 entirely.

Proposition 7.2 *Given any odd $l \geq 3$ and any natural number $r \geq 2$ there exist infinitely many graphs G such that $|G|$ is divisible by rl,*

$$\delta(G) \geq \left(1 - \frac{1}{r}\right)|G| + (l-3)$$

but G does not contain a perfect K_r^l-packing.

Proof. We firstly remark that by considering random s-regular graphs it is possible to show that given any even $s \in \mathbb{N}$ and any sufficiently large $m \in \mathbb{N}$ there exists an s-regular C_4-free graph on m vertices.

We first consider the case when r is even. Let $n \in \mathbb{N}$ be sufficiently large and let G be the graph obtained from the complete r-partite graph with $r/2$ vertex classes of size $ln - 1$ and $r/2$ vertex classes of size $ln + 1$ by adding a C_4-free $(l-1)$-factor in each of the vertex classes. So $|G| = lrn$ and $\delta(G) = lrn - (ln+1) + (l-1) = (1 - \frac{1}{r})|G| + (l-2)$. Let $V_1, \dots, V_r$ denote the vertex classes of the complete graph that G was obtained from. Given any copy K of K_r^l in G, if one of its vertex classes consists entirely of vertices from V_i (for some $i \in [r]$) then since no vertex in V_i is adjacent to l other vertices in V_i no other such vertices can lie in K. So there are only l vertices from V_i in this copy of K_r^l. So if there are at least $l+1 \geq 4$ vertices from V_i in K then at most $l-1$ of these vertices can lie in the same vertex class of K. But then this implies there is a copy of C_4 in G consisting of vertices from V_i, a contradiction. So any copy K of K_r^l in G meets each V_i in at most l vertices. So precisely l vertices from each V_i lie in such a K. But since not all V_i and V_j have equal size this implies that G does not have a perfect K_r^l-packing.

The case when r is odd is similar except we obtain G from the complete r-partite graph with $(r-3)/2$ vertex classes of size $ln+1$, one of size $ln+2$, and $(r+1)/2$ classes of size $ln-1$ by adding C_4-free $(l-1)$-factors in each of the vertex classes. Again here $n \in \mathbb{N}$ has to be sufficiently large. So $|G| = lrn$ and $\delta(G) = lrn - (ln+2) + l - 1 = (1 - \frac{1}{r})|G| + (l-3)$. We can then argue as in the case when r is even to show that G does not have a perfect K_r^l-packing, as required. □

Proposition 7.2 shows that for certain complete r-partite graphs H there is a positive constant C dependant on H such that there exist arbitrarily large graphs G with $\delta(G) \geq (1 - \frac{1}{r})|G| + C$ that do not have perfect H-packings. Thus this proposition gives us a lower bound on the constant in the minimum degree condition of Theorem 2.10 for such graphs H.

We now turn our attention to improving the minimum degree condition in Theorem 2.9 for some graphs H. In order to do this we must introduce some notation.

Definition 7.3 *(χ-highest common factor) Suppose H is a graph with $\chi(H) = r$. Given an r-colouring c of H, let $x_1 \leq x_2 \leq \cdots \leq x_r$ denote the sizes of the colour classes of c. We write*

$$\mathcal{D}(c) := \{x_{i+1} - x_i | \ i = 1, \dots, r-1\},$$

and let $\mathcal{D}(H)$ denote the union of all the sets $\mathcal{D}(c)$ for all r-colourings c of H. The χ-highest common factor $hcf_\chi(H)$ of H is the highest common factor of all integers in $\mathcal{D}(H)$. Note that if $\mathcal{D}(H) = \{0\}$ then we define $hcf_\chi(H) := \infty$.

Notice that $\mathcal{D}(H) = \{0\}$ if and only if in any r-colouring of H all colour classes are of equal size. But recall that in Chapter 5 we saw this occurred precisely when $\chi(H) = \chi_{cr}(H)$. The next results shows that for non-bipartite graphs H with $hcf_\chi(H) = 1$ we can replace $\chi(H)$ in the minimum degree condition in Theorem 2.9 with $\chi_{cr}(H)$.

Theorem 7.4 *(Kühn and Osthus [15]) Consider any graph H with $hcf_\chi(H) = 1$ and $\chi(H) \geq 3$. Let $\eta > 0$. Then there exists an $n_0(H, \eta)$ such that for every graph G on $n \geq n_0$ vertices with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \eta\right) n$$

G contains a perfect H-packing.

Theorem 7.4 cannot be extended to consider any other type of non-bipartite graph H. Indeed, the next proposition shows that if $hcf_\chi(H) > 1$ then we cannot replace $\chi(H)$ with anything smaller in the minimum degree condition of Theorem 2.9. So in this sense Theorem 2.9 is best possible when considering perfect H-packings for graphs H that do not have χ-highest common factor 1.

Proposition 7.5 *Let H be a graph such that $\chi(H) > 2$ and $hcf_\chi(H) > 1$. Then there are infinitely many graphs G whose order n is divisible by $|H|$ and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n - 1$$

but which do not contain a perfect H-packing.

Proof. Let $r := \chi(H)$ and $h' := hcf_\chi(H)$. Given any $k \in \mathbb{N}$, let G denote the complete r-partite graph with vertex classes $V_1, \dots, V_r$ where $|V_1| = k|H| - 1$, $|V_2| = k|H| + 1$ and $|V_i| = k|H|$ for all $i \geq 3$. So $|G| = kr|H|$ and

$$\delta(G) = (r-1)k|H| - 1 = \left(1 - \frac{1}{\chi(H)}\right)|G| - 1.$$

Consider disjoint copies $H_1, \dots, H_j$ in G, and define $x_j := |V_1 \backslash V(H_1 \cup \dots \cup H_j)|$ and $y_j := |V_3 \backslash V(H_1 \cup \dots \cup H_j)|$. We will now show that $y_j - x_j \equiv 1 \mod h'$. Assume that $j = 1$. In this case we are only considering one copy of H, namely H_1. Since G is r-partite, each vertex class in G contains precisely one colour class of H_1 for some r-colouring of H_1. Let C_1^1 and C_3^1 denote these colour classes for V_1 and V_3 respectively. Thus, by definition of the highest common factor h' of H, $|C_3^1| - |C_1^1| \equiv 0 \mod h'$. Since $|V_3| - |V_1| = 1$ we have that $y_1 - x_1 \equiv 1 \mod h'$. Now assume that $j > 1$ and $y_s - x_s \equiv 1 \mod h'$ for all $1 \leq s < j$. In particular, $y_{j-1} - x_{j-1} \equiv 1 \mod h'$. Again, each vertex class in G contains precisely one colour class of H_j for some r-colouring of H_j. Let C_1^j and

C_3^j denote these colour classes for V_1 and V_3 respectively. As $x_j = x_{j-1} - |C_1^j|$, $y_j = y_{j-1} - |C_3^j|$ and $|C_3^j| - |C_1^j| \equiv 0 \mod h'$ we have that $y_j - x_j \equiv 1 \mod h'$. So indeed by induction we have shown that $y_j - x_j \equiv 1 \mod h'$.

Since $h' > 1$ this implies that at least one of x_j and y_j are nonzero. This shows that no matter how many disjoint copies of H we have in G, they do not cover all of $V_1 \cup V_3$. So G does not have a perfect H-packing. □

Let H be a graph with $\chi(H) \geq 2$ and let $0 < b < \chi_{cr}(H)$. Then there exists an $\epsilon > 0$ such that $0 < a := \frac{b}{1-\epsilon b} < \chi_{cr}(H)$. Proposition 5.2 implies there exists an $\eta \in (0, 1)$ and infinitely many graphs G with

$$\delta(G) \geq \left(1 - \frac{1}{a}\right)|G| = \left(1 - \frac{1}{b} + \epsilon\right)|G|$$

but such that G does not contain an H-packing covering all but at most $\eta|G|$ vertices. So for sufficiently large such graphs G we do not have a perfect H-packing in G. In particular, this shows that for non-bipartite graphs H with $hcf_\chi(H) = 1$ we cannot replace $\chi_{cr}(H)$ with anything smaller in the minimum degree condition in Theorem 7.4. So this result is essentially best possible in this sense as well.

7.2 The Blow-up Lemma

In the next section we will turn our attention to the proof of Theorems 7.4 and 2.9. However, we will need to introduce some more tools to do this. In this section we introduce the so-called Blow-up Lemma which allows us to embed a graph H into another graph G. The Key Lemma allows us to embed certain graphs H of relatively small order into a graph G. However, somewhat surprisingly the Blow-up Lemma allows for embedding a spanning graph H of bounded maximum degree.

Theorem 7.6 *(Blow-up Lemma – Komlós, Sárközy and Szemerédi [12])*
Given a graph R with $V(R) = \{1, \dots, r\}$ and $d, \Delta > 0$, there is a parameter $\epsilon_0(d, \Delta, r) > 0$ such that the following holds. Given $L_1, \dots, L_r \in \mathbb{N}$ and $0 < \epsilon \leq \epsilon_0$, let R^* be the graph obtained by R by replacing each vertex $i \in V(R)$ with a set V_i of L_i new vertices and joining all vertices in V_i to all vertices in V_j precisely when $ij \in E(R)$. Let G be a spanning subgraph of R^* such that for every $ij \in E(R)$ the bipartite graph $(V_i, V_j)_G$ is (ϵ, d)-super-regular. Then G contains a copy of every subgraph H of R^* with $\Delta(H) \leq \Delta$.

The Blow-up Lemma essentially says that dense super-regular pairs behave like complete bipartite graphs with respect to containing subgraphs of bounded degree. That is, if a pair (V_1, V_2) of vertex classes is (ϵ, d)-super-regular for $d > 0$ and sufficiently small $\epsilon > 0$, then this graph will contain the same subgraphs as the complete bipartite graph with vertex classes V_1 and V_2 provided these subgraphs do not have too large maximum degree.

Recall that in a typical application of the Key Lemma (Theorem 3.8) to a graph G, we usually apply the degree form of the Regularity Lemma (Theorem 3.5) with sufficiently small $\epsilon, d > 0$, to obtain the pure graph G'' of G. In particular, G'' consists of clusters that either have no edges between them or form an ϵ-regular pair with density greater than d. From this, we obtain a reduced graph R of G with parameters ϵ and d. We can thus apply the Key Lemma to embed some subgraph H of R (or of blown-up copy of R) into G''. (However, to do this the order of H has to be sufficiently small compared to the size of the clusters.) Removing this copy of H from G'' and applying the Slicing Lemma, we can repeatedly apply this argument to cover almost all of G'' with disjoint copies of H. Thus, the Key Lemma is useful for finding almost perfect H-packings in large dense graphs G.

However, to find perfect H-packings in such graphs G we have to apply the Blow-up Lemma. Typically, our first step is again to apply the degree form of the Regularity Lemma to a graph G to obtain the pure graph G'' with sufficiently small parameters $\epsilon, d > 0$. Consider the reduced graph R corresponding to G''. We will have some structure in R, maybe a perfect K-packing for some graph K. We may then argue as in Lemma 1.8 so that for any such copy K' of K in R, by removing a small fraction of vertices from each cluster in K' we can ensure that adjacent clusters in K' correspond to (ϵ', d')-super-regular pairs where $\epsilon' = 2\epsilon$ and d' is close to d. This gives us a graph G^* contained in G''. Notice that the clusters in G^* may not have equal size. Further, in some applications of the Blow-up Lemma, we may, at this point, remove a bounded number of further vertices from G^*. Provided we only remove a small proportion of vertices from each cluster the super-regularity of pairs of cluster will be preserved. We may have removed these further vertices to 'pair them off' with vertices in $G - G^*$ to obtain disjoint copies of H covering all vertices outside of G^*. We may also have removed whole clusters from G'' and thus G^*. For example, in the proof of Theorem 7.4, we will remove all clusters in G'' whose corresponding vertex in the reduced graph R of G'' is not covered by a B'-packing in R for some graph B'. Thus, the 'new' reduced graph of G'' has a perfect B'-packing.

Let R^* be the complete graph whose vertex classes have the same sizes as the clusters in G^*. If the 'blown-up' copies of each K in R^* themselves have perfect H-packings, then R^* has a perfect H-packing. But since ϵ' is sufficiently small (no bigger than the parameter $\epsilon_0(d, \Delta(H), |K|)$ in Theorem 7.6) the subgraphs of G^* corresponding to the copies of K in R have perfect H-packings. So G^* has a perfect H-packing. Thus, combined with the copies of H in $G - G^*$ we obtain a perfect H-packing in G.

7.3 Proof of Theorem 7.4

7.3.1 Preliminaries and outline of the proof

The aim of this section is to give a proof of Theorem 7.4. However, to do this we require a lot of preparation.

Let $\eta > 0$. Suppose G and H are graphs as in the hypothesis of Theorem 7.4.

We will prove the result with the added assumption that $|H|$ divides $|G|$. Indeed, given any sufficiently large graph G satisfying the minimum degree condition in Theorem 7.4, by removing at most $|H|-1$ vertices from G, we obtain the graph G' whose order is divisible by $|H|$ and which satisfies

$$\delta(G') \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \frac{\eta}{2}\right)|G'|.$$

Thus, if the result holds under our assumption then G' has a perfect H-packing. So G has a perfect H-packing.

Let $r := \chi(H)$. Consider some r-colouring c of H with a colour class of size $\sigma(H)$. Let $\sigma(H) = x_1 \leq x_2 \leq \cdots \leq x_r$ denote the sizes of the colour classes of c. Let $z_1 := (r-1)\sigma(H) = (r-1)x_1$ and $z := |H| - \sigma(H) = x_2 + \cdots + x_r$. We define $\gamma := \frac{z_1}{z}$. Notice that since $x_1 \neq x_r$ we have that $0 < \gamma < 1$. Let B^* denote the complete r-partite graph with one vertex class of size z_1 and $r-1$ vertex classes of size z. We have seen in Section 5.2 that such a graph is a bottle-graph of H with the property that B^* has a perfect H-packing (covering all vertices of B^*). Further,

$$\chi_{cr}(B^*) = \chi_{cr}(H) = (r-1)\frac{|H|}{|H| - \sigma(H)} = r - 1 - \frac{(r-1)\sigma(H)}{|H| - \sigma(H)} = r - 1 + \gamma.$$

Let $s \in \mathbb{N}$ and $\lambda > 0$. Let B' denote the complete r-partite graph with a vertex class of size $s_1 := \gamma(1+\lambda)s$ and the remaining classes having size s. We choose λ sufficiently small so that $\sigma(B') = s_1$. Thus,

$$\chi_{cr}(B') = (r-1)\frac{|B'|}{|B'| - s_1} = r - 1 + \gamma(1+\lambda).$$

Notice that the proportion of vertices in the smaller class of B' compared to one of the larger classes is slightly larger than the corresponding proportion γ associated with B^*. Thus, we can choose s sufficiently large and λ small in such a way that B' has a perfect B^*-packing (covering all vertices of B').

Both B^* and B' will be useful graphs when proving Theorem 7.4. In particular, we will need to consider B' when applying the following lemma. It shows that if a sufficiently large r-partite graph G has vertex classes of similar size to some bottle-graph B of H, with the smallest vertex class in G being slightly larger than the corresponding vertex class in B, then G has a perfect H-packing. In particular, if G (roughly) looks like a blow-up of B' then it has a perfect H-packing.

Lemma 7.7 *Let H be a graph with $r := \chi(H) \geq 3$ and $hcf_\chi(H) = 1$. Let B^*, z, z_1 and γ be as defined earlier in this section (so $\gamma = z_1/z$). There exists a positive constant $\beta_0(H) \ll \gamma$ such that for every $0 < \beta \leq \beta_0$ there exists an integer $u_0(H, \beta)$ for which the following holds. Let λ be a positive constant so that $\lambda \gg \beta$ and $\lambda \leq 1 - \gamma$. Suppose that G is a complete r-partite graph with vertex classes $U_1, \dots, U_r$ such that $|H|$ divides $|G|$. Let $u_i := |U_i|$ for all $i \in [r]$. Suppose that $u_0 \leq u_1 \leq \cdots \leq u_{r-1} \leq (1+\beta)u_1$ and $u_r = (1+\lambda)\gamma u_1$. Then G contains a perfect H-packing.*

We will prove Lemma 7.7 in Section 7.5.

The idea of the proof of Theorem 7.4 is as follows. We let G be a sufficiently large graph satisfying the minimum degree condition given in Theorem 7.4 on input H for some $\eta > 0$. We define the reduced graph R of G with sufficiently small parameters ϵ and d. This will almost inherit the minimum degree of G. In particular, as λ is chosen sufficiently small, and as $\chi_{cr}(H) = r - 1 + \gamma$ and $\chi_{cr}(B') = r - 1 + \gamma(1 + \lambda)$ we will see that

$$\delta(R) \geq \left(1 - \frac{1}{\chi_{cr}(B')}\right) |R|.$$

Thus, by Theorem 5.3, R has an almost perfect B'-packing. We then add all vertices belonging to clusters in R not covered by this B'-packing to the exceptional set. Furthermore, we add a small fraction of the vertices in each cluster in R to the exceptional set. This will ensure that, in any of our copies of B' in R, adjacent clusters will correspond to super-regular pairs in G.

We will then remove disjoint copies of H from G, each covering one vertex in the exceptional set. We will do this in such a way that all exceptional vertices are covered by this H-packing, whilst ensuring that clusters adjacent in one of our copies of B' still form a super-regular pair in G.

For each copy B'_i of B' in our B'-packing in R we wish to find an H-packing covering all vertices belonging to the clusters in B'_i. We will do this by applying the Blow-up Lemma. Let F_i^* denote the complete r-partite graph whose jth vertex class is the union of all clusters in the jth vertex class of B'_i. A necessary condition for us to apply the Blow-up Lemma is that F_i^* has a perfect H-packing. We will ensure such an H-packing by applying Lemma 7.7. We will remove a bounded number of copies of H from G to ensure for each B'_i we have that $|H|$ divides $|F_i^*|$. This process will leave the $r - 1$ larger vertex classes of F_i^* having roughly the same size. Furthermore, the choice of B' is such that the ratio of the size of the smallest class of B' to the size of the other classes is $\gamma(1 + \lambda)$. This will ensure that the ratio of the size of the smallest class of F_i^* to the size of any other vertex class satisfies the condition in Lemma 7.7. Thus, we will apply Lemma 7.7 to find our desired perfect H-packing in F_i^*. So the Blow-up Lemma will ensure that, together with the copies of H covering our exceptional vertices, we obtain a perfect H-packing in G.

In the forthcoming five subsections we give the proof of Theorem 7.4.

7.3.2 Applying the Regularity Lemma and modifying the reduced graph

To ease notation, in the rest of this chapter we take the convention that if we have removed a set of vertices from some graph K, we still denote this new graph by K. Let H and $\eta > 0$ be as in the statement of the theorem and let s, λ, B^*, B' and γ be as defined in Section 7.3.1. We define further constants satisfying

$$0 < \epsilon \ll d \ll \eta_1 \ll \beta \ll \alpha \ll \lambda \ll \eta, \gamma, 1 - \gamma. \tag{7.1}$$

We also define η_1 so that

$$\eta_1 \ll \frac{1}{|B'|}. \tag{7.2}$$

Let G be a graph with sufficiently large order n such that $|H|$ divides n and where

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \eta\right) n.$$

Applying the degree form of the Regularity Lemma (Theorem 3.5) we obtain clusters and an exceptional set V_0, and furthermore, the reduced graph R of G with parameters ϵ and d. Since ϵ and d are sufficiently small Lemma 3.7 implies that

$$\delta(R) \geq \left(1 - \frac{1}{\chi_{cr}(B^*)} + \frac{\eta}{2}\right) |R|.$$

Recall that $\chi_{cr}(B^*) = r - 1 + \gamma$ and $\chi_{cr}(B') = r - 1 + (1 + \lambda)\gamma$. As $\lambda \ll \eta$ this implies therefore that

$$\delta(R) \geq \left(1 - \frac{1}{\chi_{cr}(B')}\right) |R|.$$

In Section 3.2 we remarked that choosing ϵ small and $|G|$ sufficiently large ensures $|R|$ is large. So our choice of n and ϵ ensures that $|R| \geq n_0(\eta_1, B')$ where n_0 is the output of Komlós' Theorem on almost perfect packings (Theorem 5.3). Thus, we can apply Theorem 5.3 to find a B'-packing in R covering all but an η_1-fraction of the vertices. We remove the clusters in R that are not covered by this B'-packing and put them in the exceptional set V_0. Initially this set had size at most ϵn. So now we have that $|V_0| \leq \epsilon n + \eta_1 n \leq 2\eta_1 n$ by (7.1).

So now R has a perfect B'-packing. Let $B'_1, \dots, B'_k$ denote these copies of B' in R. In turn each B'_i has a perfect B^*-packing. Hence, we obtain a perfect B^*-packing $\mathcal{B}^*$ of R. Let $B^*_1, \dots, B^*_{k'}$ denote these copies of B^* in R. Note that we still have that

$$\delta(R) \geq \left(1 - \frac{1}{\chi_{cr}(B^*)} + \frac{\eta}{4}\right) |R|. \tag{7.3}$$

Let m denote the size of the non-exceptional clusters in G. Given any B'_i, we can replace each cluster V_a in B'_i with a subcluster of size $m' := (1 - \epsilon|B'|)m$ so that every edge V_aV_b in B'_i is such that the chosen subclusters of V_a and V_b form a $(2\epsilon, d/2)$-super-regular pair. Indeed, to see this we can argue similarly to the proof of Lemma 1.8: Given any cluster V_a in B'_i we remove all $y \in V_a$ such that $d_{V_b}(y) < (d - \epsilon)|V_b|$ for any neighbour V_b of V_a in B'_i. For each such cluster V_b at most $\epsilon|V_a| = \epsilon m$ such $y \in V_a$ exists (by Lemma 1.4). So we have removed at most $\epsilon|B'|m$ vertices from V_a to obtain a subcluster V'_a say. Note that V_a in B'_i was arbitrary. Since $\epsilon|B'| < 1/2$, we can argue as in the proof of Lemma 1.8, but replacing ϵ with $\epsilon|B'|$ in the argument to see that whenever V_a and V_b are neighbours in B'_i the subclusters V'_a and V'_b form a $(2\epsilon, d - 3\epsilon|B'|)$-super-regular pair. (Note we could have argued more carefully to 'improve' on the value $d - 3\epsilon|B'|$.) So as $\epsilon \ll d$ we obtain an $(2\epsilon, d/2)$-super-regular pair, as required.

All vertices in our clusters not belonging to the chosen subclusters are added to V_0. We thus have removed at most $\epsilon|B'|n$ vertices in total from these clusters, so

$$|V_0| \leq 3\eta_1 n. \tag{7.4}$$

Note that our chosen subclusters are now what we refer to as the clusters of R.

Given any $V_a, V_b \in V(R)$ such that $V_aV_b \in E(R)$, when V_a and V_b had size m they formed an ϵ-regular pair of density more than d. Now we have removed $\epsilon|B'|m$ vertices from each of these clusters. However, since $\epsilon < 1/2$ is sufficiently small the Slicing Lemma (Lemma 1.6) implies that we now have that (V_a, V_b) is a 2ϵ-regular pair with density more than $d/2$.

7.3.3 Partitioning our clusters

We now partition each cluster V_a in R into V_a' and V_a'' where $|\,|V_a'|-|V_a''|\,| \leq \epsilon m'$ and $|\,|N_G(x) \cap V_a'| - |N_G(x) \cap V_a''|\,| \leq \epsilon m'$ for all $x \in V(G)$. To see why such a partition exists we need the following lemma. (For a proof of this lemma consult, for example, [17].)

Lemma 7.8 *(Chernoff type bound) Let $X_1, \ldots, X_n$ be independent 0–1 random variables with $\mathbb{P}(X_i = 1) = p$ for all $i \in [n]$. Denote $X := \sum_{i=1}^n X_i$. Then for all $\zeta > 0$ there exists a positive $\phi(\zeta)$ so that*

$$\mathbb{P}(|X - \mathbb{E}X| > \zeta\mathbb{E}X) \leq 2e^{-\phi\mathbb{E}X}.$$

Consider a random partition V', V'' of V_a which is obtained by including a vertex of V_a into V' with probability $\frac{1}{2}$ (independently of all other vertices in V_a). We let Ω denote the set consisting of the set V_a and all sets $N_{V_a}(x)$ for any $x \in V(G)$. We say a set $X \in \Omega$ is *bad* if $|\,|X \cap V'| - |X \cap V''|\,| > \epsilon m'$. Notice that if we do not have any bad sets in Ω then we have found our desired partition. Clearly if $X \in \Omega$ is such that $|X| \leq \epsilon m'$ then it cannot be a bad set. Otherwise we have that $\frac{\epsilon m'}{2} \leq \mathbb{E}(|X \cap V'|) = \frac{1}{2}|X| \leq \frac{m'}{2}$, and furthermore, Lemma 7.8 implies that

$$\begin{aligned}
\mathbb{P}(X \text{ is bad}) &= \mathbb{P}(|\,|X \cap V'| - \mathbb{E}(|X \cap V'|) + \mathbb{E}(|X \cap V'|) - |X \cap V''|\,| > \epsilon m') \\
&\leq \mathbb{P}(|\,|X \cap V'| - \mathbb{E}(|X \cap V'|)\,| + |\,|X \cap V''| - \mathbb{E}(|X \cap V'|)\,| > \epsilon m') \\
&\leq \mathbb{P}(|\,|X \cap V'| - \mathbb{E}(|X \cap V'|)\,| > \epsilon m'/2) \\
&\leq \mathbb{P}(|\,|X \cap V'| - \mathbb{E}(|X \cap V'|)\,| > \epsilon\mathbb{E}(|X \cap V'|)) \leq 2e^{-\phi(\epsilon)\epsilon m'/2}.
\end{aligned}$$

Recall that the degree form of the Regularity Lemma (Theorem 3.5) implies that $m \geq (1-\epsilon)n/M$ where M is the constant dependent on ϵ in Theorem 3.5. So as $m' = (1-\epsilon|B'|)m$ we have that $m' = Cn$ where C is a constant depending on ϵ and H. Also $|\Omega| \leq n+1$. Thus, we have that the expected number of bad sets in Ω is at most $2|\Omega|e^{-\phi(\epsilon)\epsilon m'/2} \leq 2(n+1)e^{-\phi(\epsilon)\epsilon Cn/2} < 1$ since n is sufficiently large. So there exists some partition V', V'' of V_a for which no set in Ω is bad, as required.

In the next part of the proof we will need to remove vertices from $G - V_0$. When doing this we only remove vertices from one of the partition sets V'_a, V''_a for each cluster V_a in R, namely from V'_a. This is important as it ensures that every edge V_aV_b in any B'_i still corresponds to a $(5\epsilon, d/5)$-super-regular pair in G. Indeed, consider any pair (V_a, V_b) of clusters that are adjacent in some B'_i. So they form a 2ϵ-regular pair of density more than $d/2$ in G. Suppose we remove vertices from V'_a and V'_b. We denote the subclusters of V_a and V_b thus obtained by V^*_a and V^*_b respectively. The Slicing Lemma (Lemma 1.6) implies that V^*_a and V^*_b still form a 5ϵ-regular pair of density more than $d/5$. In particular for any $V_1 \subseteq V^*_a$ and $V_2 \subseteq V^*_b$ satisfying $|V_1| > 5\epsilon|V^*_a|$ and $|V_2| > 5\epsilon|V^*_b|$ we have $e(V_1, V_2) > \frac{d}{5}|V_1||V_2|$. Recall $|\,|N_G(x) \cap V'_a| - |N_G(x) \cap V''_a|\,| \leq \epsilon m'$ and $|\,|N_G(x) \cap V'_b| - |N_G(x) \cap V''_b|\,| \leq \epsilon m'$ for all $x \in V(G)$. Also since (V_a, V_b) is a $(2\epsilon, d/2)$-super-regular pair, $d_{V_b}(x_a) > \frac{d}{2}|V_b| = \frac{d}{2}m'$ and $d_{V_a}(x_b) > \frac{d}{2}m'$ for all $x_a \in V_a$ and $x_b \in V_b$. So together this gives us that $d_{V^*_b}(x^*_a) > \frac{d}{4}m' - \epsilon m' > \frac{d}{5}|V^*_b|$ and similarly $d_{V^*_a}(x^*_b) > \frac{d}{5}|V^*_a|$ for all $x^*_a \in V^*_a$ and $x^*_b \in V^*_b$. So (V^*_a, V^*_b) is a $(5\epsilon, d/5)$-super-regular pair, as required.

7.3.4 Covering all exceptional vertices with disjoint copies of H

If $x \in V_0$ we say a copy $B \in \mathcal{B}^*$ of B^* is *useful for* x if there exists $r - 1$ clusters belonging to different vertex classes of B such that x has at least $\alpha m'$ neighbours in each of these clusters. Let k_x denote the number of useful copies of B^* in $\mathcal{B}^*$. Now x could be adjacent to every vertex in G corresponding to a useful copy of B^*. Further, each of the $|\mathcal{B}^*| - k_x$ copies of B^* that are not useful for x could be such that x is adjacent to all vertices corresponding to such a copy of B^*, except for in two vertex classes of this copy of B^*, x is adjacent to less than $\alpha m'$ vertices in each of the clusters in these vertex classes. Thus,

$$\begin{aligned}&k_x m'|B^*| + (|\mathcal{B}^*| - k_x)(|B^*|m' - (1-\alpha)m'(z_1 + z))\\ &\geq d_G(x) - |V_0| \overset{(7.4)}{\geq} \left(1 - \tfrac{1}{r-1+\gamma} + \eta - 3\eta_1\right) n \overset{(7.1)}{\geq} \left(1 - \tfrac{1}{r-1+\gamma} + \tfrac{\eta}{2}\right) m'|B^*||\mathcal{B}^*|.\end{aligned}$$

Rearranging gives us

$$k_x(1-\alpha)(z_1 + z) \geq \left((1-\alpha)(z_1 + z) - \frac{|B^*|}{r - 1 + \gamma} + \frac{\eta|B^*|}{2}\right)|\mathcal{B}^*|.$$

Now since $(r - 1 + \gamma)(z_1 + z) \geq z_1 + (r-1)z = |B^*|$ we have that

$$1 - \frac{|B^*|}{(r - 1 + \gamma)(1-\alpha)(z_1 + z)} \geq 1 - \frac{1}{1-\alpha}.$$

But $\eta/4 \geq \alpha$ so $\frac{\eta|B^*|}{4(z_1+z)} \geq \alpha$ and hence,

$$1 - \frac{1}{1-\alpha} \geq -\frac{\eta|B^*|}{4(z_1 + z)(1-\alpha)}.$$

Thus

$$k_x \geq \frac{\eta|B^*|}{4(1-\alpha)(z_1+z)}|\mathcal{B}^*| \geq \frac{\eta}{4(1-\alpha)}|\mathcal{B}^*| \geq \frac{\eta|\mathcal{B}^*|}{4}.$$

The choice of η_1 is such that $\eta_1 \ll \beta, \eta, 1/|B^*|$. Thus, $k_x \beta m' \geq \frac{\eta|\mathcal{B}^*|}{4} \beta m' \gg 3\eta_1 n \geq |V_0|$. So we can assign every $x \in V_0$ to some $B_x \in \mathcal{B}^*$ so that B_x is useful for x and for each copy of B^* in $\mathcal{B}^*$ no more than $\beta m'$ such vertices are assigned to it.

Consider such a vertex $x \in V_0$ and the corresponding $B_x \in \mathcal{B}^*$. Consider the induced subgraph K_x of B_x in R which contains the $r-1$ clusters that x has at least $\alpha m'$ neighbours in, together with a cluster V_z in the vertex class of B_x that none of these $r-1$ clusters belong to. So K_x is a copy of K_r. Given any two clusters $V_a, V_b \in V(K_x)$ they form a 2ϵ-regular pair of density more than $d/2$ in G. Recall we have partitions V'_a, V''_a and V'_b, V''_b of V_a and V_b respectively. Our aim is to find a copy of H in G that contains x but only has vertices from the partition sets of the form V'_a for each $V_a \in V(K_x)$.

We have that x is adjacent to at least $\alpha m'$ vertices in $r-1$ of the clusters in K_x. Let V_a and V_b denote such clusters. Thus, by the choice of our partitions of V_a and V_b, x has at least $(\alpha - \epsilon)m'/2 \geq \alpha m'/4$ neighbours in both V'_a and V'_b. Consider the subclusters of all such $V'_a, V'_b \in V(K_x)$ that consist precisely of all those vertices that are adjacent to x in G. By the Slicing Lemma these subclusters form $\sqrt{\epsilon}$-regular pairs of density more than $d/5$. Let G_x denote the subgraph of G induced by all such subclusters belonging to K_x together with V'_z. Let H^- be a copy of H with one vertex removed. For each $x \in V_0$ in turn we apply the Key Lemma to find a copy of H^- in G_x. This together with x forms a copy of H in G. We remove this copy of H from G and repeat the argument for each exceptional vertex. Indeed, since any $B \in \mathcal{B}^*$ is useful for at most $\beta m' \ll \alpha m'/4$ vertices in V_0 we can ensure that if we have removed even $\beta m'$ copies of H from G_x (for some $x \in V_0$) the clusters in G_x are still of sufficient size so that we may apply the Key Lemma to find a copy of H in G containing x. So we obtain disjoint copies of H in G each containing precisely one vertex in V_0. We remove all vertices lying in these copies of H from the clusters they belong to. As remarked earlier each pair of modified clusters that are adjacent in any B'_i in our B'-packing in R will correspond to a $(5\epsilon, d/5)$-super-regular pair.

7.3.5 Making $|V_G(B)|$ divisible by $|H|$ for each $B \in \mathcal{B}^*$

We will need some more notation before we proceed. Given a subgraph $S \subseteq R$ we write $V_G(S)$ to denote the set of all vertices in G that belong to a cluster in S. We have already found disjoint copies of H in G that cover all vertices in V_0. Our aim now is to find, for each copy of B'_i in our B'-packing in R, an H-packing in G that covers all the vertices in $V_G(B'_i)$. Thus, taking the union of these H-packings and the copies of H that contain the vertices in V_0, we will obtain a perfect H-packing in G. If we can ensure that the complete r-partite graph whose jth vertex class is the union of all clusters in the jth vertex class in B'_i has a perfect H-packing, then by the Blow-up Lemma, the subgraph of G corresponding to B'_i will have a perfect H-packing. We will see that by Lemma 7.7 this will happen provided $|H|$ divides $|V_G(B'_i)|$. So our immediate aim is to remove a bounded number of copies of H from G to ensure that for all $i \in [k']$ we indeed have that $|H|$ divides $|V_G(B'_i)|$. This will be achieved by ensuring that $|H|$ divides $|V_G(B)|$ for all $B \in \mathcal{B}^*$.

Consider the graph F whose vertices are precisely the elements of $\mathcal{B}^*$ where $B_1, B_2 \in \mathcal{B}^*$ are adjacent if in R there exists a copy of K_r that has one vertex V_a in B_1 and $r-1$ vertices in B_2 or vice versa. Consider such B_1 and B_2 and assume the corresponding copy of K_r takes the former form. Let V_z be a cluster in B_2 which lies in the vertex class of B_2 avoiding our copy of K_r. So V_z is adjacent to the $r-1$ vertices in our copy of K_r that lie in B_2. Recall that any edge of R corresponds to a 5ϵ-regular pair with density more than $d/5$ in G. Thus, we may apply the Key Lemma to obtain a copy of H in G with one vertex in $V_a \subseteq V_G(B_1)$ and the remaining vertices in $V_G(B_2)$. Further, as usual, since ϵ is sufficiently small, we may apply the Slicing Lemma so that we can use the Key Lemma a further $|H|-2$ times to obtain $|H|-1$ disjoint such copies of H. That is, all these copies of H have precisely one vertex in $V_G(B_1)$ and the rest in $V_G(B_2)$.

Consider the case when F is connected. If we take a spanning tree T of F then consider some B_z in $\mathcal{B}^*$ to be the root of the tree. Let t be the maximum distance from a vertex in T to B_z. Thus we can partition $V(F)$ into sets $S_0, S_1, \dots, S_t$ where an element of $\mathcal{B}^*$ lies in S_i if and only if it has distance i from B_z in T. In particular $S_0 = \{B_z\}$. Consider any $1 \leq i \leq t$. Any $B_1 \in S_i$ is adjacent to some $B_2 \in S_{i-1}$. We may assume that there is a copy of K_r that has one vertex in B_1 and $r-1$ vertices in B_2. So there are $|H|-1$ disjoint copies of H in G that have precisely one vertex in $V_G(B_1)$ and the rest in $V_G(B_2)$. Thus, removing at most $|H|-1$ of these copies of H we obtain that $|V_G(B_1)|$ is divisible by $|H|$. So starting with S_t and continuing with $S_{t-1}, S_{t-2}, \dots$, we can ensure that for any $B_a \in S_1 \cup \dots \cup S_t$, $|V_G(B_a)|$ is divisible by $|H|$. Since $|\bigcup_{B \in \mathcal{B}^*} V_G(B)|$ is divisible by $|H|$, we obtain that $|B_z|$ is divisible by $|H|$. So indeed $|H|$ divides $|V_G(B)|$ for all $B \in \mathcal{B}^*$.

We now consider the more complex case when F is not connected. Let $\mathcal{C}$ denote the set of all connected components of F. Given some component $C \in \mathcal{C}$ we write $V_R(C)$ to denote the set of all clusters in R that belong to some $B \in \mathcal{B}^*$ where $B \in C$. Further we write $V_G(C) \subseteq V(G)$ to denote the union of all the clusters in $V_R(C)$. The idea now is to remove a bounded number of copies of H so that for each component C in $\mathcal{C}$ we have that $|V_G(C)|$ is divisible by $|H|$. Once this has been achieved we then apply the argument used in the case when F is connected for each component of F. Thus we will obtain that $|H|$ divides $|V_G(B)|$ for all $B \in \mathcal{B}^*$. In order to do this we need to introduce a couple of simple results.

Claim 7.9 *Let $C_1, C_2 \in \mathcal{C}$ and let $x \in V_R(C_2)$. Then*

$$|N_R(x) \cap V_R(C_1)| < \left(1 - \frac{1}{r-1+\gamma} + \frac{\eta}{4}\right) |V_R(C_1)|.$$

Proof. Suppose not. Let $B_0 \in \mathcal{B}^*$ denote the copy of B^* in R that contains x.

So $B_0 \in C_2$. By our assumption there exists some $B \in \mathcal{B}^*$ such that

$$|N_R(x) \cap B| \geq \left(1 - \frac{1}{r-1+\gamma} + \frac{\eta}{4}\right)|B| = |B| - \frac{(r-1)z + z_1}{r-1+z_1/z} + \frac{\eta|B|}{4}$$
$$= |B| - z + \frac{\eta|B|}{4} > |B| - z.$$

Hence x has a neighbour in at least $r-1$ vertex classes of B. So R has a copy of K_r with one vertex, namely x, in B_0 and $r-1$ vertices in B. So B and B_0 are adjacent in F. But they lie in different components of F, a contradiction. So the claim is true. □

We now use Claim 7.9 in the proof of the next result.

Claim 7.10 *There exists a component $C' \in \mathcal{C}$, a copy K of K_r in R and a vertex $x_0 \in V(R)\backslash(V(K) \cup V_R(C'))$ such that K meets $V_R(C')$ in precisely one vertex and so that x_0 is adjacent to all other vertices in K.*

Proof. Since $\delta(R) > |R|/2$ there exists two vertices $x_1, x_2 \in V(R)$ that are adjacent in R but correspond to different components of F. That is $x_1 \in V_R(C_1)$ and $x_2 \in V_R(C_2)$ for distinct $C_1, C_2 \in \mathcal{C}$. From (7.3) we obtain that

$$|N_R(x_1) \cap N_R(x_2)| \geq \left(1 - \frac{2}{r-1+\gamma} + \frac{\eta}{4}\right)|R|.$$

We firstly consider the case when at least $(1 - \frac{2}{r-1+\gamma} + \frac{\eta}{4})|V(R)\backslash V_R(C_1)|$ common neighbours of x_1 and x_2 lie outside $V_R(C_1)$. Let x_3 be a common neighbour of x_1 and x_2 outside of $V_R(C_1)$. By (7.3) and Claim 7.9 we have that $|N_R(x_3) \cap (V(R)\backslash V_R(C_1))| \geq (1 - \frac{1}{r-1+\gamma} + \frac{\eta}{4})$. So the common neighbourhood of x_1, x_2 and x_3 outside of $V_R(C_1)$ has size at least

$$\left(1 - \frac{3}{r-1+\gamma} + \frac{\eta}{4}\right)|V(R)\backslash V_R(C_1)|.$$

Choose such a common neighbour x_4. If we continue in this fashion we obtain distinct vertices $x_2, \dots, x_r$ outside $V_R(C_1)$ that, together with x_1 form a copy K of K_r. Further, using Claim 7.9 and (7.3) we obtain that the common neighbourhood of $x_2, \dots, x_r$ outside of $V_R(C_1)$ is at least of size $\eta|V(R)\backslash V_R(C_1)|/4$. We can take x_0 to be such a common neighbour, and C' to be C_1. So the claim is true in this case.

The only other case to consider is when at least $(1 - \frac{2}{r-1+\gamma} + \frac{\eta}{4})|V_R(C_1)|$ common neighbours of x_1 and x_2 lie in $V_R(C_1)$. By Claim 7.9 and (7.3) we have that every vertex in $V_R(C_1)$ has at least $(1 - \frac{1}{r-1+\gamma} + \frac{\eta}{4})|V_R(C_1)|$ neighbours in $V_R(C_1)$. Thus, we can argue similarly as in our first case to choose $x_3, \dots, x_r$ inside $V_R(C_1)$ such that these vertices, together with x_1 and x_2 form a copy K of K_r with precisely one vertex, namely x_2, outside of $V_R(C_1)$. Further, as before we may choose some x_0 in the common neighbourhood of $x_1, x_3, \dots, x_r$ in $V_R(C_1)$. So the claim holds in this case too (with $C' := C_2$). □

We are now in a position to prove the next claim which shows that indeed we can make $|V_G(B)|$ divisible by $|H|$ for all $B \in \mathcal{B}^*$.

Claim 7.11 *By removing at most $|\mathcal{B}^*||H|$ copies of H in G we can make $|V_G(B)|$ divisible by $|H|$ for all $B \in \mathcal{B}^*$.*

Proof. Our first aim is to remove a bounded number of disjoint copies of H in G to ensure that for every component $C \in \mathcal{C}$, $|H|$ divides $|V_G(C)|$. Applying Claim 7.10 we obtain a component $C_1 \in \mathcal{C}$, a copy K of K_r in R and a vertex $x_0 \in V(R)$ disjoint from our clique K and outside of $V_R(C_1)$, such that K has precisely one vertex x_1 in $V_R(C_1)$ and so that x_0 is adjacent to all vertices in $K - x_1$. Since edges in R correspond to 5ϵ-regular pairs of density more than $d/5$ in G we may apply the Key Lemma to obtain a copy of H in G which has precisely one vertex x in $V_G(C_1)$ and whose other vertices lie in the clusters in $V_R(K - x_1) \cup \{x_0\}$. In fact, our usual argument involving repeated applications of the Slicing Lemma and Key Lemma implies that G contains $|H| - 1$ disjoint such copies of H. Now $|V_G(C_1)| \equiv j \mod |H|$ for some $j \in \{0, \dots, |H| - 1\}$. So removing j of these copies of H in G we obtain that $|V_G(C_1)|$ is divisible by $|H|$.

If $|\mathcal{C}| \geq 3$ consider the graphs $F_1 := F - V(C_1)$ and $R_1 := R - V_R(C_1)$. Claim 7.9 and (7.3) give us that

$$\delta(R_1) \geq \left(1 - \frac{1}{r - 1 + \gamma} + \frac{\eta}{4}\right) |R_1|.$$

We can now argue as in Claim 7.10 but considering F_1 instead of F to obtain a component $C_2 \in \mathcal{C} \backslash \{C_1\}$, a copy K' of K_r in R_1 and a vertex $x'_0 \in R_1$ disjoint from our clique K' and outside of $V_{R_1}(C_2)$, such that K' has precisely one vertex x'_1 in $V_{R_1}(C_2)$ and so that x'_0 is adjacent to all vertices in $K' - x'_1$. So as before we can remove at most $|H| - 1$ copies of H in G to ensure that $|V_G(C_2)|$ is divisible by $|H|$. We can continue in this fashion to ensure that all the $C \in \mathcal{C}$ are such that $|H|$ divides $|V_G(C)|$. (Indeed if we have shown that $|\mathcal{C}| - 1$ such C satisfy this condition then the remaining component of F will satisfy this condition automatically since $|H|$ divides $|\bigcup_{C \in \mathcal{C}} V_G(C)|$.)

In this process we have removed at most $(|\mathcal{C}| - 1)(|H| - 1)$ copies of H in G. For each $C \in \mathcal{C}$ we can proceed precisely as in the case where F is connected to make each $|V_G(B)|$ divisible by $|H|$ for each $B \in \mathcal{B}^*$. Notice that for each such C we remove at most $(|C| - 1)(|H| - 1)$ copies of H to achieve this divisibility condition. Hence in total we have removed at most

$$\begin{aligned} &(|\mathcal{C}| - 1)(|H| - 1) + \sum_{C \in \mathcal{C}} (|C| - 1)(|H| - 1) \\ &= (|\mathcal{C}| - 1)(|H| - 1) + (|\mathcal{B}^*| - |\mathcal{C}|)(|H| - 1) \leq |\mathcal{B}^*||H| \end{aligned}$$

copies of H in G. □

7.3.6 Applying the Blow-up Lemma

For each B'_i in our perfect B'-packing of R let G'_i denote the corresponding subgraph of G. So G'_i is the r-partite subgraph of G whose jth vertex class is the union of all clusters lying in the jth vertex class of B'_i. Note we consider the vertices of B'_i to be the clusters obtained after we have removed our copies of H in Sections 7.3.4 and 7.3.5. Thus, $|G'_i| = |V_G(B'_i)|$ is divisible by $|H|$. In Section 7.3.4 we removed at most $|H|\beta m'$ vertices from each cluster. In Section 7.3.5 only a bounded number of vertices were removed from the clusters. So in total no more than $2|H|\beta m'$ vertices have been removed from each cluster.

Let L'_i denote the complete r-partite graph whose vertex classes are the same as the vertex classes of G'_i. Further, let U_r denote the vertex class in L'_i corresponding to the vertex class in B'_i of size s_1, and let $U_1, \dots, U_{r-1}$ denote the other vertex classes in L'_i. We assume $|U_1| \leq \dots \leq |U_{r-1}|$. Each cluster in R initially had size m' and as mentioned, no more than $2|H|\beta m'$ vertices have been removed from each cluster. So let β_1 be such that $(1-2|H|\beta)(1+\beta_1) = 1$. Thus $|U_1| \leq \dots \leq |U_{r-1}| \leq (1+\beta_1)|U_1|$. Notice that $\beta_1 \ll \gamma$ since $\beta \ll \gamma, 1/|H|$. Furthermore let $\lambda_1 > 0$ be such that $|U_r| = (1+\lambda_1)\gamma|U_1|$. Then $\beta \ll \lambda_1 \ll 1-\gamma$ since $s_1/s = (1+\lambda)\gamma$, and since $\beta \ll \lambda \ll 1-\gamma$. Now since n was chosen sufficiently large this ensures m' is sufficiently large. So U_1 and thus all other vertex classes of L'_i are sufficiently large. Thus, Lemma 7.7 implies that L'_i has a perfect H-packing for all i.

Recall that at the end of Section 7.3.4, all edges in each B'_i corresponded to $(5\epsilon, d/5)$-super-regular pairs. Since in Section 7.3.5 we only removed a bounded number of further vertices from each cluster, all these edges now correspond to $(6\epsilon, d/6)$-super-regular pairs. As ϵ is sufficiently small we may apply the Blow-up Lemma to each G'_i. So since L'_i has a perfect H-packing, so does G'_i. Taking the union of all these H-packings, together with all copies of H in G chosen in Sections 7.3.4 and 7.3.5, we obtain a perfect H-packing in G. Thus we have proven Theorem 7.4.

7.4 Proof of the Alon-Yuster Theorem on perfect packings

As we saw in the last section, the proof of Theorem 7.4 is quite involved. However, we will now see that by applying Theorem 7.4 it is not too difficult to prove Theorem 2.9.

Proof of Theorem 2.9. Let H be a graph on h vertices with $\chi(H) = r$ and let $\epsilon > 0$. We may assume that $\chi(H) > 1$ otherwise the result is trivial. Let $k \in \mathbb{N}$. Let H' denote the complete $(r+1)$-partite graph with one vertex class of size 1, one of size $hk-1$ and $r-1 \geq 1$ vertex classes of size hk. So $|H'| = hkr$, $hcf_\chi(H') = 1$ and H' contains K_r^{kh} as a spanning subgraph and thus has a perfect H-packing. Now $\chi_{cr}(H') = (\chi(H')-1)\frac{|H'|}{|H'|-\sigma(H')} = r\frac{|H'|}{|H'|-1}$, so by choosing k sufficiently large, $\chi_{cr}(H')$ can be made to be arbitrarily close to $\chi(H) = r$. In particular, we can choose k sufficiently large so that $1-\frac{1}{\chi(H)}+\frac{\epsilon}{2} \geq 1-\frac{1}{\chi_{cr}(H')}+\frac{\epsilon}{4}$.

Now consider any graph G of sufficiently large order with

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \epsilon\right) n.$$

As before we may assume that $|H|$ divides $|G|$. For such a graph G we can apply the Erdős-Stone Theorem (Theorem 2.4) to obtain a vertex-disjoint copies of H in G (where $a \leq kr$). In particular a can be chosen so that removing these a copies of H from G we obtain the graph G' where hkr divides $|G'|$ and which satisfies

$$\delta(G') \geq \left(1 - \frac{1}{\chi(H)} + \frac{\epsilon}{2}\right) |G'| \geq \left(1 - \frac{1}{\chi_{cr}(H')} + \frac{\epsilon}{4}\right) |G'|.$$

But further as G' is still sufficiently large we can apply Theorem 7.4 to find a perfect H'-packing in G'. In particular, this induces a perfect H-packing in G'. Thus, together with the copies of H in $G - G'$ removed earlier, this forms a perfect H-packing in G, as required. □

In this chapter we have proved essentially 'best possible' minimum degree conditions that ensure a perfect H-packing in a large graph G (where H is non-bipartite). If $hcf_\chi(H) > 1$ it is the chromatic number of H that governs whether G has a perfect H-packing. However, if $hcf_\chi(H) = 1$ then it is the critical chromatic number of H that governs whether G has a perfect H-packing. It should be noted that Kühn and Osthus [16] have also solved the problem for the case when H is bipartite. We write $hcf_c(H)$ to denote the highest common factor of all the components of H. If $\chi(H) \geq 3$ then we say $hcf(H) = 1$ if $hcf_\chi(H) = 1$. If $\chi(H) = 2$ then we say that $hcf(H) = 1$ if $hcf_c(H) = 1$ and $hcf_\chi(H) \leq 2$. Kühn and Osthus showed that if H is bipartite and $hcf(H) = 1$ then we can replace $\chi(H)$ with $\chi_{cr}(H)$ in the minimum degree condition of Theorem 2.9. However, for all other bipartite graphs Theorem 2.9 is best possible up to the slack term ϵn. Furthermore, they showed that these minimum degree conditions for graphs H of $hcf(H) = 1$ can be improved by replacing the linear slack term (for example ηn in Theorem 7.4) with a constant dependent on H.

We conclude this chapter in the next section by proving Lemma 7.7.

7.5 Proof of Lemma 7.7

Proof of Lemma 7.7. The idea is to remove a number of copies of B^* from G to obtain a graph G' whose vertex classes have approximately the same size. We will then show that G' has a perfect H-packing. So firstly we show there exists a B^*-packing $\mathcal{B}^*$ in G so that the following holds. Let G' denote the subgraph of G obtained by removing all copies of B^* in $\mathcal{B}^*$. For all $i \in [r]$ let $U_i' := U_i \cap V(G')$ and $u_i' := |U_i'|$. Then for all $i \in [r]$ we have

$$\frac{\gamma\lambda u_1}{1-\gamma} \leq u_i' \leq (1+\sqrt{\beta})\frac{\gamma\lambda u_1}{1-\gamma}.$$

Indeed, let $\mathcal{B}^*$ consist of $k := \frac{u_1}{z}(1-\frac{\gamma\lambda}{1-\gamma})$ disjoint copies of B^* in G each having z_1 vertices in U_r and z vertices in each of $U_1, \dots, U_{r-1}$. Thus,

$$u'_r = u_r - kz_1 = (1+\lambda)\gamma u_1 - \gamma\left(1 - \frac{\gamma\lambda}{1-\gamma}\right)u_1 = \left(1 + \frac{\gamma}{1-\gamma}\right)\gamma\lambda u_1 = \frac{\gamma\lambda u_1}{1-\gamma}.$$

Further for all $i \in [r-1]$ we have

$$\begin{aligned}\frac{\gamma\lambda u_1}{1-\gamma} = u_1 - kz \le u'_i \le (1+\beta)u_1 - kz = \beta u_1 + \frac{\gamma\lambda u_1}{1-\gamma} = \left(1 + \frac{\beta(1-\gamma)}{\gamma\lambda}\right)\frac{\gamma\lambda u_1}{1-\gamma} \\ \le \left(1+\frac{\beta}{\gamma}\right)\frac{\gamma\lambda u_1}{1-\gamma} \le (1+\sqrt{\beta})\frac{\gamma\lambda u_1}{1-\gamma}\end{aligned}$$

since $\lambda \le 1-\gamma$ and $\beta \le \gamma^2$, as required.

We now show that G' has a perfect H-packing. Let k' be an integer such that for each $i \in [r]$ we can write $u'_i = k'(r-1)!|H| + a_i$ where $a_i \in \mathbb{Z}$ and $0 \le \sum_{i=1}^r a_i < r!|H|$. Clearly such a k' exists. Indeed, we can find it algorithmically: Starting with $k' = 0$ we either have the desired properties or the corresponding value of each a_i is such that $\sum_{i=1}^r a_i \ge r!|H|$. So then considering $k' = 1$ the corresponding sum of the a_i's decreases by $r(r-1)!|H| = r!|H|$. So repeating this process we will eventually find a k' such that $0 \le \sum_{i=1}^r a_i < r!|H|$.

Notice that since $|H|$ divides $|G|$ and thus $|G'|$ we know that $|H|$ divides $\sum_{i=1}^r a_i$. Thus removing at most $r! - 1$ copies of H we can assume that $\sum_{i=1}^r a_i = 0$. Now $u_1 \gg |H|, 1/\beta$ since we chose u_0 sufficiently large. Hence $u_1 \ge \frac{(r!-1)|H|(1-\gamma)}{\gamma\lambda\sqrt{\beta}}$. So we still have that for all $i \in [r]$

$$(1-\sqrt{\beta})\frac{\gamma\lambda u_1}{1-\gamma} \le \frac{\gamma\lambda u_1}{1-\gamma} - (r!-1)|H| \le u'_i \le (1+\sqrt{\beta})\frac{\gamma\lambda u_1}{1-\gamma}.$$

Since there exists a $j \in [r]$ such that $a_j \le 0$ this in turn implies that $k'(r-1)!|H| \ge u'_j \ge (1-\sqrt{\beta})\gamma\lambda u_1/(1-\gamma)$ and so $|a_i| \le 2\sqrt{\beta}\gamma\lambda u_1/(1-\gamma)$ for all $i \in [r]$. Hence

$$|H||a_i| \le \frac{2|H|\sqrt{\beta}\gamma\lambda u_1}{1-\gamma} \le \frac{2|H|\sqrt{\beta}(r-1)!|H|}{1-\sqrt{\beta}}k' \ll k' \tag{7.5}$$

since we may assume $\beta_0 \ge \beta$ was chosen sufficiently small compared to $|H|$ and r.

We now consider the complete r-partite graph G'' whose vertex classes $U''_1, \dots, U''_r$ each have size $k'(r-1)!|H|$. So $|G'| = |G''|$ and clearly G'' has a perfect H-packing. Our aim is to modify such a perfect H-packing of G'' into an H-packing of G'.

Let $c^1, \dots, c^q$ denote all the r-colourings of H where c^j has colour class sizes $x_1^j \le x_2^j \le \dots \le x_r^j$. Choose $k_1, \dots, k_q \in \mathbb{N}$ as equal as possible so that $k_1 + \dots + k_q = k'$. Let S_r denote the set of all permutations of $\{1, \dots, r\}$. For all $j \in [r]$ we let $\mathcal{H}''_j$ be an H-packing in G' which, for every $s \in S_r$, contains exactly k_j copies of H such that for the colouring c^j the $s(i)$th colour class lies in U''_i (for all $i \in [r]$). So $\mathcal{H}''_j$ consists of $|S_r|k_j = r!k_j$ copies of H. Given any

$i, i^* \in [r]$ there are $(r-1)!$ permutations in S_r that permute i^* to i. Thus there are $k_j(r-1)!|H|$ vertices covered by $\mathcal{H}''_j$ in U''_i. So choosing each $\mathcal{H}''_j$ to be disjoint from one another, the union $\mathcal{H}''$ of $\mathcal{H}''_1, \dots, \mathcal{H}''_q$ is a perfect H-packing in G''.

We will now modify $\mathcal{H}''$ by interchanging some vertex classes of some copies of H in $\mathcal{H}''$ to obtain an H-packing of G'. To do this we introduce a multiset X consisting of ordered pairs $\overline{x} = (x_1, x_2)$ with $1 \leq x_1, x_2 \leq r$ such that for all $l \in [r]$ we have $|\{\overline{x} \in X | x_1 = l\}| = \max\{a_l, 0\}$ and $|\{\overline{x} \in X | x_2 = l\}| = \max\{-a_l, 0\}$. So X consists of elements (l_1, l_2) where $a_{l_2} < 0 < a_{l_1}$. More precisely if $a_l > 0$ then X contains exactly a_l tuples whose first entry is l and whose second entry is some l' where $a_{l'} < 0$. If $a_l < 0$ then X contains exactly $-a_l$ tuples whose second entry is l and the first entry is some l' where $a_{l'} > 0$. If $a_l = 0$ then no tuple in X has an entry l. Notice that since we are considering a multiset and since $\sum_{l=1}^{r} a_l = 0$ such an X exists.

For every $j \leq q$ and $i < r$ we define $d_i^j := x_{i+1}^j - x_i^j$. Since $hcf_\chi(H) = 1$ the highest common factor of all such d_i^j is 1. So, as a consequence of the Euclidean Algorithm, there exists $b_i^j \in \mathbb{Z}$ such that

$$1 = \sum_{j=1}^{q} \sum_{i=1}^{r-1} b_i^j d_i^j$$

where $b_i^j = 0$ if $d_i^j = 0$.

Let $\overline{x} = (l_1, l_2) \in X$. For each $j \leq q$ we now modify the H-packing $\mathcal{H}''_j$ in G''. Given any $i < r$ consider b_i^j. If $b_i^j \geq 0$ we choose b_i^j copies of H in $\mathcal{H}''_j$ such that they have their ith colour class in U''_{l_1} and their $(i+1)$th colour class in U''_{l_2}. We modify each of these copies of H so that we interchange the ith and $(i+1)$th colour classes. That is we have the ith colour class of each of these copies of H now in U''_{l_2} and the $(i+1)$th colour class is contained in U''_{l_1}. So the number of vertices in U''_{l_1} covered by this H-packing increases by $b_i^j d_i^j$ whereas the number of vertices in U''_{l_2} covered by this H-packing decreases by $b_i^j d_i^j$. If $b_i^j < 0$ we proceed similarly: We choose $|b_i^j|$ of the copies of H in $\mathcal{H}''_j$ which in the colouring c^j have their ith colour class in U''_{l_2} and their $(i+1)$th colour class in U''_{l_1}. Interchanging these colour classes between U''_{l_1} and U''_{l_2} yields an H-packing where the number of vertices in U''_{l_2} covered by this H-packing increases by $|b_i^j| d_i^j$ whereas the number of vertices in U''_{l_1} covered by this H-packing decreases by $|b_i^j| d_i^j$.

Notice we immediately know that all these copies of H will be distinct for different pairs i, j. In total we have chosen at most $|X| rq \max\{|b_i^j| \,|\, j \leq q, i < r\}$ copies of H. But recall for each $s \in S_r$ we have exactly k_j copies of H (in $\mathcal{H}''_j$) such that for the colouring c^j the $s(i)$th colour class lies in U''_i (for all $i \in [r]$). But $|X| \leq \sum_{i=1}^{r} |a_i|$ and r, q and each b_i^j only depend on H. So by (7.5) we have that $k_j \geq |X| rq \max\{|b_i^j| \,|\, j \leq q, i < r\}$. Thus we can choose our copies of H so that they are distinct for all tuples $(l_1, l_2) \in X$.

Let $\mathcal{H}'$ denote the H-packing obtained in this way. We wish to show $\mathcal{H}'$ is a perfect H-packing of G'. So for all $l \in [r]$ let n_l denote the number of vertices in the lth vertex class covered by $\mathcal{H}'$. Then

$$n_l = k'(r-1)|H| + a_l \sum_{j=1}^{q} \sum_{i=1}^{r-1} b_i^j d_i^j = k'(r-1)!|H| + a_l = u_l'.$$

So indeed $\mathcal{H}'$ is a perfect H-packing of G', and thus G has a perfect H-packing.
$\square$

Chapter 8

Ore-type degree conditions for perfect packings

8.1 An Ore-type analogue of the Alon-Yuster Theorem on perfect packings

In this chapter we will investigate Ore-type degree conditions for perfect packings in graphs. It would be interesting to find analogues to Theorems 2.9 and 7.4 for Ore-type degree conditions. We will prove an analogue of Theorem 2.9 by modifying our proof of Theorem 7.4.

Theorem 8.1 *For every $\eta > 0$ and each graph H there exists an integer $n_0(H, \eta)$ such that given any graph G of order $n \geq n_0$ with*

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi(H)} + \eta\right) n$$

for all distinct $x, y \in V(G)$ where $xy \notin E(G)$, G contains a perfect H-packing.

Proposition 7.5 shows that when considering perfect H-packings for non-bipartite graphs H with $hcf_\chi(H) \neq 1$ we cannot replace the chromatic number of H in the degree condition of Theorem 2.9 with anything smaller. This also shows that Theorem 8.1 is essentially best possible for such graphs H. Somewhat surprisingly we will see that we cannot establish an analogue of Theorem 7.4 for an Ore-type degree condition. Indeed, there are graphs H with $hcf_\chi(H) = 1$ such that Theorem 8.1 is essentially best possible.

As mentioned before, the proof of Theorem 8.1 is a modification of the proof of Theorem 7.4. If we are trying to find a perfect H-packing in a graph G we define an auxiliary $(\chi(H) + 1)$-partite graph H' with $hcf_\chi(H') = 1$. This will allow us to use Lemma 7.7 and thus Theorem 7.6 in the same way that we used them in Theorem 7.4. We will have to argue a little more carefully at certain stages in the proof in order for everything to run smoothly. Throughout the proof we will take the convention that if we remove vertices from a graph K we still denote the graph obtained by K.

Proof of Theorem 8.1. Let $0 < \eta < 1$ and let H be a graph. If $\chi(H) = 1$ the result is trivial. So suppose $\chi(H) = r \geq 2$. We may assume that H is a complete graph with vertex classes of equal size $t \geq 2$ such that $|H| \geq \frac{4}{\eta} \geq \frac{4}{\eta r}$ (the latter condition will be useful once we have defined H'). Indeed, suppose H is any r-partite graph and assume the result holds when considering perfect $K_r^{|H|l}$-packings, where $l \in \mathbb{N}$ is sufficiently large. Then this implies the result holds when considering perfect H-packings.

We define H' to be the complete $(r+1)$-partite graph with $r-1$ vertex classes of size t, one class of size $t-1$ and the other of size 1. Thus, $|H| = |H'|$ and $H \subseteq H'$. Furthermore $hcf_\chi(H') = 1$. Let $z_1 := r\sigma(H') = r$ and $z := |H'| - \sigma(H') = |H'| - 1$ and

$$\gamma := \frac{z_1}{z} = \frac{r}{|H'|-1} < 1.$$

We define $(r+1)$-partite graphs B^* and B' as in Theorem 7.4 though now with respect to H' instead of H. So B^* is the complete $(r+1)$-partite graph with one vertex class of size z_1 and r vertex classes of size z. So $|B^*| = r|H|$ and B^* has a perfect H'-packing. We choose a sufficiently large integer s and $0 < \lambda \ll \eta, \gamma, 1-\gamma$ so that B' has one vertex class of size $s_1 := \gamma(1+\lambda)s$ and r vertex classes of size s, and such that B' has a perfect B^*-packing (covering all vertices). So $\chi_{cr}(H') = \chi_{cr}(B^*) = r\frac{|H'|}{|H'|-1} = r + \gamma$ and $\chi_{cr}(B') = r + \gamma(1+\lambda)$. We define the following constants so that

$$0 < \epsilon \ll d \ll \eta_1 \ll \beta \ll \alpha \ll \lambda \ll \eta/2, \gamma, 1-\gamma \tag{8.1}$$

and

$$\eta_1 \ll \frac{1}{|B'|}. \tag{8.2}$$

Suppose G is a sufficiently large graph on n vertices such that

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{r} + \eta\right)n$$

for all distinct $x, y \in V(G)$ where $xy \notin E(G)$. Similarly as in the proof of Theorem 7.4 it is sufficient to consider the case when $|H|$ divides $|G|$.

Applying the degree form of the Regularity Lemma (Theorem 3.5) with parameters ϵ and d to G we obtain clusters, an exceptional set V_0 and the reduced graph R. Since ϵ and d are sufficiently small Lemma 6.3 implies that

$$d_R(V_a) + d_R(V_b) \geq 2\left(1 - \frac{1}{r} + \frac{\eta}{2}\right)|R| \geq 2\left(1 - \frac{1}{\chi_{cr}(H')} + \frac{\eta}{4}\right)|R|$$

for all distinct $V_a, V_b \in V(R)$ where $V_aV_b \notin E(R)$. Note that the last of these inequalities follows since $|H'| \geq \frac{4}{\eta r}$ which implies $\frac{\eta}{2} \geq \frac{1}{r|H'|} + \frac{\eta}{4}$ and thus $\frac{-1}{r} + \frac{\eta}{2} \geq -\frac{|H'|-1}{r|H'|} + \frac{\eta}{4}$. As $\lambda \ll \eta$ we have that $d_R(V_a) + d_R(V_b) \geq 2\left(1 - \frac{1}{\chi_{cr}(B')}\right)|R|$ for all distinct $V_a, V_b \in V(R)$ where $V_aV_b \notin E(R)$. Moreover, since G is sufficiently large and ϵ is sufficiently small we have that $|R| \geq n_0(\eta_1, B')$ where n_0 is as defined in Theorem 6.4. So by Theorem 6.4 we obtain a B'-packing covering all but at most an η_1-fraction of vertices in R. Denote the copies of B' in this packing by $B'_1, \dots, B'_k$. Similarly to the proof of Theorem 7.4 we add all clusters not covered by this B'-packing to V_0. So $|V_0| \le 2\eta_1 n$ and we still have that

$$d_R(V_a) + d_R(V_b) \ge 2\left(1 - \frac{1}{r} + \frac{\eta}{4}\right)|R| \tag{8.3}$$

for all distinct $V_a, V_b \in V(R)$ where $V_aV_b \notin E(R)$. Since every copy of B' contains a perfect B^*-packing, we obtain a perfect B^*-packing $\mathcal{B}^*$ in R. Let m denote the size of the clusters in G. As in the proof of Theorem 7.4 we replace each cluster V_a in B'_i by a subcluster of size $m' := (1 - \epsilon|B'|)m$ such that for every edge V_aV_b of B'_i, the corresponding subclusters form a $(2\epsilon, d/2)$-super-regular pair in the pure graph G'' of G (as well as forming a 2ϵ-regular pair of density more than $d/2$). Adding all vertices not in such subclusters into V_0 we have that

$$|V_0| \le 3\eta_1 n. \tag{8.4}$$

We now consider these subcluster as the clusters of R. As in the proof of Theorem 7.4 we have a partition V'_a, V''_a for each cluster V_a so that $|\,|V'_a| - |V''_a|\,| \ge \epsilon m'$ and $|\,|N_G(x) \cap V'_a| - |N_G(x) \cap V''_a|\,| \le \epsilon m'$ for all $x \in V(G)$. In what follows we will remove copies of H from G in such a way that from each cluster V_a we only remove vertices belonging to V'_a. As in the proof of Theorem 7.4 the modified clusters thus obtained will be such that if V_aV_b is an edge of B'_i then the corresponding modified clusters form both a 5ϵ-regular pair of density more than $d/5$, and a $(5\epsilon, d/5)$-super-regular pair in G''.

Recall that given a vertex $x \in V_0$ we call a copy $B \in \mathcal{B}^*$ useful for x if there are $r-1$ clusters belonging to different vertex classes of B so that x has at least $\alpha m'$ neighbours in each of these clusters. We let k_x denote the number of copies of B^* in $\mathcal{B}^*$ which are useful for x. Now x could be adjacent to every vertex in G corresponding to a useful copy of B^*. Also, each of the copies of B^* that are not useful could be such that x is adjacent to all vertices corresponding to such a copy of B^*, except for in three vertex classes of this copy of B^*, x is adjacent to less than $\alpha m'$ vertices in each of the clusters in these vertex classes. Further, since our Ore-type degree condition on G ensures that we have $\delta(G) \ge (1 - \frac{2}{r} + \eta)|G|$ we obtain that

$$\begin{aligned} k_x m'|B^*| + (|\mathcal{B}^*| - k_x)(|B^*|m' - (1-\alpha)m'(z_1 + 2z)) &\ge d_G(x) - |V_0| \\ &\overset{(8.1),(8.4)}{\ge} \left(1 - \frac{2}{r} + \frac{\eta}{2}\right) m'|B^*||\mathcal{B}^*|. \end{aligned}$$

Rearranging we obtain

$$k_x(1-\alpha)(z_1 + 2z) \ge \left((1-\alpha)(z_1+2z) - 2\frac{|B^*|}{r} + \frac{\eta|B^*|}{2}\right)|\mathcal{B}^*|.$$

Notice though that since $z_1 = r$

$$1 - \frac{2|B^*|}{r(1-\alpha)(z_1+2z)} = 1 - \frac{2(z + z_1/r)}{(1-\alpha)(z_1+2z)} = 1 - \frac{2z+2}{(1-\alpha)(2z+r)} \ge 1 - \frac{1}{1-\alpha}.$$

But further as $\eta/4 \geq \alpha$ we have that $\frac{\eta|B^*|}{4(z_1+2z)} \geq \alpha$. This implies that

$$1 - \frac{1}{1-\alpha} \geq \frac{-\eta|B^*|}{4(z_1+2z)(1-\alpha)}.$$

Thus

$$k_x \geq \frac{\eta|B^*|}{4(1-\alpha)(z_1+2z)}|\mathcal{B}^*| \geq \frac{\eta}{4(1-\alpha)}|\mathcal{B}^*| \geq \frac{\eta|\mathcal{B}^*|}{4}.$$

Since $\eta_1 \ll \beta, \eta, 1/|B^*|$ this shows that $k_x \beta m' \gg |V_0|$. Thus we may argue precisely as we did in Section 7.3.4 to find disjoint copies of H in G that cover every vertex in the exceptional set V_0, and so that for every cluster V_a in R at most $\beta|H|m'$ vertices of V_a lie in these copies of H and each such copy avoids V_a''. We remove the vertices in all these copies of H from the clusters they belong to.

So if we can find disjoint copies of H covering all vertices in the (modified) clusters of R then we can find a perfect H-packing in G. If we can remove a bounded number of disjoint copies of H from G to ensure that for each B_i' in our B'-packing of R we have that $|H'|$ divides $|V_G(B_i')|$ then we can apply Lemma 7.7 and then the Blow-up Lemma to find an H'-packing covering all vertices in our clusters. The reasoning for this is precisely as in the proof of Theorem 7.4. Since each copy of H' contains H as a spanning subgraph, these copies of H' together with the disjoint copies of H covering all the other vertices of G form a perfect H-packing in G. So it is sufficient to prove that we can indeed remove a bounded number of disjoint copies of H from G to ensure that for each B_i' in our B'-packing of R we have that $|H'|$ divides $|V_G(B_i')|$. As we did in the proof of Theorem 7.4 we achieve this by ensuring that $|H|$ divides $|V_G(B)|$ for each $B \in \mathcal{B}^*$.

We define the graph F as we did in Section 7.3.5. That is the vertices of F are the elements of $\mathcal{B}^*$ and $B_1, B_2 \in \mathcal{B}^*$ are adjacent in F if R contains a copy of K_r with one vertex in $V_R(B_1)$ and $r-1$ vertices in $V_R(B_2)$ or vice versa. If F is connected then arguing precisely as in Section 7.3.5 we can remove less than $|H|||\mathcal{B}^*|$ copies of H from G to ensure that $|H|$ divides $|V_G(B)|$ for each $B \in \mathcal{B}^*$.

So suppose that F is not connected. Let $\mathcal{C}$ denote the set of all connected components of F.

Claim 8.2 *Let $C_1, C_2 \in \mathcal{C}$ and let $x \in V_R(C_2)$. Then*

$$|N_R(x) \cap V_R(C_1)| < \left(1 - \frac{1}{r} + \frac{\eta}{4}\right)|V_R(C_1)|.$$

Proof. Suppose not. Then there exits a $B \in C_1$ such that

$$|N_R(x) \cap B| \geq \left(1 - \frac{1}{r} + \frac{\eta}{4}\right)|B| \geq |B| - z - z_1 + \frac{\eta|B|}{4}.$$

So x has a neighbour in at least $r-1$ vertex classes of B. So R contains a copy of K_r consisting of x and $r-1$ of such neighbours in B. But this implies that B is adjacent in F to the copy B_0 of B^* in $\mathcal{B}^*$ that x belongs to. But B_0 and B^* belong to different components of F, a contradiction. So our assumption was false, as required. $\square$

This last result is similar to Claim 7.9. We now prove the analogue of Claim 7.10.

Claim 8.3 *There exists a component $C' \in \mathcal{C}$, a copy K of K_r in R and a vertex $x_0 \in V(R) \backslash (V(K) \cup V_R(C'))$ such that K meets $V_R(C')$ in precisely one vertex and so that x_0 is adjacent to all other vertices of K.*

Proof. Suppose that the claim is false. Then clearly there exists $x_1, x_2 \in V(R)$ such that $x_1x_2 \notin E(R)$ and such that x_1 and x_2 correspond to two different components of F. That is there exists distinct $C_1, C_2 \in \mathcal{C}$ such that $x_1 \in V_R(C_1)$ and $x_2 \in V_R(C_2)$. By (8.3) we have that

$$|N_R(x_1) \cap N_R(x_2)| \geq \left(1 - \frac{2}{r} + \frac{\eta}{4}\right) |R|.$$

Firstly we consider the case when at least $(1 - \frac{2}{r} + \frac{\eta}{4})|V(R)\backslash V_R(C_1)|$ common neighbours of x_1 and x_2 lie outside of $V_R(C_1)$. Now as $\eta|H| \geq 4$ certainly $(1 - \frac{2}{r} + \frac{\eta}{4})|V(R)\backslash V_R(C_1)| \geq \frac{\eta}{4}|B^*| = \frac{\eta}{4}r|H| \geq r$. Thus the set of these common neighbours cannot form a clique in R as otherwise we would obtain a copy of K_r in R with $r-1$ vertices in $V(R)\backslash V_R(C_1)$ and with vertex x_1 which lies in $V_R(C_1)$. As x_2 is adjacent to $r-1$ of these vertices in this copy of K_r, we would have a contradiction to our assumption. So there exist two non-adjacent vertices x_3, x_3' in the set of common neighbours of x_1 and x_2 that lie outside of $V_R(C_1)$. So by (8.3) we may assume $d_R(x_3) \geq (1 - \frac{1}{r} + \frac{\eta}{4})|R|$. Together with Claim 8.2 this implies that the number of common neighbours of x_1, x_2 and x_3 outside of $V_R(C_1)$ is at least

$$\left(1 - \frac{3}{r} + \frac{\eta}{4}\right) |V(R)\backslash V_R(C_1)|.$$

If $r \geq 3$ then $\left(1 - \frac{3}{r} + \frac{\eta}{4}\right) |V(R)\backslash V_R(C_1)| \geq r$, so by our previous argument we can choose a common neighbour x_4 of x_1, x_2 and x_3 that lies outside of $V_R(C_1)$ and so that $d_R(x_4) \geq (1 - \frac{1}{r} + \frac{\eta}{4})|R|$. We can continue in this fashion to define $x_3, \dots, x_{r+1}$ such these vertices are adjacent to each other and to x_1 and x_2. Thus $x_3, \dots, x_{r+1}$ together with x_1 form a copy of K_r with precisely one vertex x_1 in $V_R(C_1)$ and so that x_2 is adjacent to all other vertices in this copy of K_r. But this contradicts our assumption.

The only other case to consider is when at least $(1 - \frac{2}{r} + \frac{\eta}{4})|V_R(C_1)|$ common neighbours of x_1 and x_2 lie in $V_R(C_1)$. But since $|V_R(C_1)| \geq |B^*|$ we can argue as in the first case to show these common neighbours cannot form a clique. In particular there exists a common neighbour $x_3 \in V_R(C_1)$ of x_1 and x_2 such that $d_R(x) \geq (1 - \frac{1}{r} + \frac{\eta}{4})|R|$. But then Claim 8.2 implies that there are at least $(1 - \frac{3}{r} + \frac{\eta}{4})|V_R(C_1)|$ common neighbours of x_1, x_2 and x_3 that lie inside $V_R(C_1)$. So we can proceed similarly as in the first case to obtain vertices $x_3, \dots, x_{r+1}$ in $V_R(C_1)$ that are adjacent to each other and to x_1 and x_2. Thus $x_3, \dots, x_{r+1}$ together with x_2 form a copy of K_r with precisely one vertex x_2 in $V_R(C_2)$ and so that x_1 is adjacent to all other vertices in this copy of K_r. But this contradicts our assumption. So in both cases we get a contradiction. Thus the claim must be true. $\square$

We can now proceed similarly as in the proof of Claim 7.11 but applying Claim 8.2 instead of Claim 7.9, and arguing as in the proof of Claim 8.3 instead of Claim 7.10, to remove at most $|B^*||H|$ copies of H in G so that $|V_G(B)|$ is divisible by $|H|$ for all $B \in \mathcal{B}^*$. In particular we firstly ensure that some $C_1 \in \mathcal{C}$ is such that $|V_G(C_1)|$ is divisible by $|H|$. We then consider $F_1 := F - V(C_1)$ and $R_1 := R - V_R(C_1)$ as in the proof of Theorem 7.4. However, by Claim 8.2 and (8.3) we have the Ore-type degree condition

$$\begin{aligned} d_{R_1}(V_a) + d_{R_1}(V_b) &\geq 2\left(1 - \frac{1}{r} + \frac{\eta}{4}\right)|R| - |N_R(V_a) \cap V_R(C_1)| - |N_R(V_b) \cap V_R(C_1)| \\ &> 2\left(1 - \frac{1}{r} + \frac{\eta}{4}\right)|R_1| \end{aligned}$$

for all distinct $V_a, V_b \in V(R_1)$ where $V_aV_b \notin E(R_1)$. So we use this condition to argue as we did in Claim 8.3 to ensure that $|V_{R_1}(C_2)|$ is divisible by $|H|$ for some component C_2 of F_1. As in the proof of Theorem 7.4, continuing in this way and then proceeding as in the case when F is connected ensures that $|V_G(B)|$ is divisible by $|H|$ for all $B \in \mathcal{B}^*$. As mentioned before we can now argue as in Section 7.3.6 to obtain a perfect H-packing in G as required. $\square$

8.2 Extremal examples and open questions

In this section we show that we cannot replace the minimum degree condition in Theorem 7.4 by the corresponding Ore-type condition. We also investigate results for other types of degree conditions.

Proposition 8.4 *Let H be a graph with $\chi(H) \geq 2$ such that in any $\chi(H)$-colouring of H every colour class contains a vertex that is adjacent to every vertex outside that class. Then there exist infinitely many graphs G such that $|H|$ divides $|G|$,*

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi(H)}\right)|G| - 2$$

for all distinct $x, y \in V(G)$ with $xy \notin E(G)$ and such that G does not contain a perfect H-packing.

Proof. Let $r := \chi(H)$ and consider any $t \in \mathbb{N}$ such that $|H|$ divides t. Consider the complete r-partite graph with vertex classes $V_1, V_2, ..., V_r$ where $|V_1| = 1, |V_2| = 2t - 1$ and $|V_i| = t$ for all $i \geq 3$. We obtain the graph G from this graph by adding all possible edges in V_2 so that $G[V_2]$ forms a clique, and removing all edges between V_1 and V_2. Thus, $|G| = rt$.

Given any vertex $x \in V_i$, for $i \geq 3$, we have that $d(x) = (r-1)t = (1 - \frac{1}{\chi(H)})|G|$. So given any distinct $x, y \in V(G)\backslash(V_1 \cup V_2)$ such that $xy \notin E(G)$ we have that $d(x) + d(y) = 2(1 - \frac{1}{\chi(H)})|G|$. The vertex $w \in V_1$ is such that $d(w) = (r-2)t$, and given any $z \in V_2$ we have that $d(z) = rt - 2$. So $d(w) + d(z) = 2(r-1)t - 2 = 2(1 - \frac{1}{\chi(H)})|G| - 2$. Altogether, this shows that for any distinct $x, y \in V(G)$ with $xy \notin E(G)$ we have that

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi(H)}\right)|G| - 2.$$

Suppose we have a perfect H-packing in G. Then there exists a copy H^* of H in G that contains the vertex $w \in V_1$. In any $\chi(H)$-colouring of H^* every colour class contains a vertex that sees every vertex in all the other colour classes. Thus, no two such vertices from different colour classes can both lie in V_i for any $i \geq 3$. Thus, two of these vertices, u and v, lie in $V_1 \cup V_2$. If both u and v lie in V_2 then one of them does not belong to the same colour class of H^* as w. So this vertex should be adjacent to w, a contradiction. The only other possibility is that one of u and v is w. But then u and v are adjacent in H^*. However, w is not adjacent to any vertex in V_2. So we have a contradiction, thus proving the result. □

Note that every complete r-partite graph satisfies the hypothesis of Proposition 8.4. So in particular, there are graphs H with $hcf_\chi(H) = 1$ which satisfy this hypothesis. Notice that $\chi_{cr}(H) < \chi(H)$ in this case. Proposition 8.4 tells us that for any $0 < a < \chi(H)$ there exists an $\eta(H, a) > 0$ such that there are infinitely many graphs G with

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{a} + \eta\right)|G|$$

for all distinct $x, y \in V(G)$ with $xy \notin E(G)$ but such that G does not contain a perfect H-packing. So there are graphs H with $hcf_\chi(H) = 1$ such that Theorem 8.1 gives an essentially best possible Ore-type degree condition for perfect H-packings. In particular, for such graphs H we cannot replace $\chi(H)$ with $\chi_{cr}(H)$ in the Ore-type degree condition of Theorem 8.1. Thus, this example shows us that we cannot replace the minimum degree condition by the corresponding Ore-type condition in Theorem 7.4. Further, Proposition 8.4 implies that for any bipartite graph H with no isolated vertices, Theorem 8.1 gives an essentially best possible Ore-type degree condition for perfect H-packings. So the characterisation of the parameter which governs whether a large graph G has a perfect H-packing differs when considering Ore-type degree conditions and minimum degree conditions. Hence, the following question arises: For which graphs H (if any) can we replace $\chi(H)$ in the Ore-type degree condition in Theorem 8.1 with something smaller? In particular, if we can improve on this Ore-type degree condition what is the smallest value we can replace $\chi(H)$ with?

A further question to ask is whether we can strengthen Theorems 6.4 and 8.1 to involve a different type of degree condition. Let G be a graph on n vertices and suppose $c > 0$. Suppose G satisfies

$$d(x_1) + d(x_2) \geq 2cn \tag{8.5}$$

for all distinct $x_1, x_2 \in V(G)$ with $x_1 x_2 \notin E(G)$. Let $t \geq 2$ be a natural number. Notice that by our degree condition (8.5) G also satisfies

$$d(x_1) + d(x_2) + \cdots + d(x_t) \geq tcn$$

for any t distinct vertices $x_1, x_2, \ldots, x_t \in V(G)$ that form an independent set in G. Indeed this follows since $2tcn \leq d(x_t) + d(x_1) + \sum_{i=1}^{t-1}(d(x_i) + d(x_{i+1})) = 2(d(x_1) + d(x_2) + \cdots + d(x_t))$. We call such a degree condition an *O_t-type degree condition.* Given some natural number $t \geq 3$ we can therefore ask whether O_t-type analogues of Theorems 5.3 and 2.9 exist. The problem with an O_t-type degree condition (for $t \geq 3$) on a graph G is that it tells us something about the degrees of vertices in G provided that G contains independent sets of size at least t. Thus it is not hard to see that for graphs H and G in general there does not exist an O_t-type degree condition on G that ensures a perfect H-packing in G. Indeed, suppose H is a connected graph on h vertices. Let G consist precisely of two vertex-disjoint cliques K_1 and K_2 so that h divides $|G|$ but h divides neither $|K_1|$ nor $|K_2|$. Clearly G does not have a perfect H-packing. However, since G does not contain a set of more than two independent vertices, it trivially satisfies any O_t-type degree condition for $t \geq 3$.

Recall that an Ore-type degree condition on a graph G implies a minimum degree condition on G. For example, when we considered Theorem 8.1, given a graph H and $\eta > 0$, we looked at graphs G that satisfy $d(x) + d(y) \geq 2(1 - \frac{1}{\chi(H)} + \eta)|G|$. This implies that $\delta(G) \geq (1 - \frac{2}{\chi(H)} + \eta)|G|$. This minimum degree condition was useful in the proof of Theorem 8.1. So perhaps we can find packing results concerning O_t-type conditions if we add an extra minimum degree condition.

Although O_t-type degree conditions (for $t \geq 3$) are not useful by themselves for establishing results concerning perfect H-packings, it would be of some interest to see if we could establish O_t-type results for almost perfect H-packings. Indeed, it is still an open question as to whether we can find an O_t-type analogue to Komlós' Theorem (Theorem 5.3).

Chapter 9

Conclusion

Extremal graph theory concerns the study of how various parameters of a graph G force certain substructures within G. Two of the classical results in this area are Turán's Theorem and the Erdős-Stone Theorem (proved in Chapter 4). The former gives a bound on the number of edges in a graph G that ensures it has a copy of a complete graph K_r as a subgraph. If G is sufficiently large the latter result generalises Turán's Theorem by giving a bound on the number of edges in G that ensures some graph H is a subgraph of G.

These two results highlight the general theme of extremal graph theory: They are concerned with ensuring a local substructure of a graph G, namely a subgraph, by considering a global parameter of G, in this case $e(G)$. A natural extension of these problems is to consider whether global parameters of a graph G ensure spanning substructures within G.

A simple illustration of this is that of matchings in graphs. A matching in a graph G is a set of disjoint edges. If all vertices of G are covered by this matching it is called perfect. Tutte's Theorem characterises precisely which graphs have perfect matchings.

The concept of a matching can be generalised to that of an H-packing. Given graphs G and H, an H-packing of G is a collection of vertex-disjoint copies of H in G. If an H-packing covers all but at most $|H| - 1$ vertices in G then we say it is perfect. In the most natural case when $|H|$ divides $|G|$ a perfect H-packing is just an extension of the notion of a perfect matching. Indeed, a perfect matching in such a graph G is precisely a perfect K_2-packing.

Unlike in the case of perfect matchings no result is known that characterises all graphs that have perfect H-packings. Since it is perhaps unlikely that such an analogue of Tutte's Theorem exists for every graph H it is useful to establish sufficient conditions for the existence of H-packings in graphs.

It is not difficult to see that there exists infinitely many dense graphs G that do not contain perfect H-packings for certain graphs H. So rather than considering $e(G)$ it is natural to look for bounds on the minimum degree of G that ensure perfect H-packings in G. In this sense the Hajnal-Szemerédi Theorem (Theorem 2.7) is an analogue of Turán's Theorem, giving a bound on the minimum degree of a graph G that ensures a perfect K_r-packing in G.

For dense graphs G of large minimum degree there are a several important results that guarantee perfect H-packings in G. The Alon-Yuster Theorem on perfect packings (proved in Chapter 7) gives us such a minimum degree condition involving the chromatic number of H. This bound on the minimum degree is best possible for certain H. That is for such H the result determines (up to an error term) the minimum degree which guarantees a perfect H-packing in a large graph G. However as Kühn and Osthus [16] have shown, we may improve on this minimum degree condition for other graphs H, see Chapter 7.

An almost perfect H-packing in a graph G is an H-packing covering all but a small fraction of the vertices in G. Komlós' Theorem (proved in Chapter 5) gives us a condition on the minimum degree guaranteeing an almost perfect H-packing in G. The degree condition involves the so-called critical chromatic number of H. This parameter (as shown in Proposition 5.2) cannot be replaced with anything smaller in this degree condition. So in this sense Komlós' Theorem is best possible.

As well as investigating how the minimum degree of a graph G forces H-packings in G it is of interest to examine how other types of degree conditions can ensure such packings in graphs. For example, Ore-type degree conditions consider the sum of the degrees of non-adjacent vertices in graphs. In Chapter 6 we proved a result (Theorem 6.4) similar to Komlós' Theorem but involving an Ore-type degree condition. As in the case of Komlós' Theorem, this Ore-type result is essentially best possible. In Chapter 8 we saw a proof of an Ore-type analogue (Theorem 8.1) of the Alon-Yuster Theorem on perfect packings. We also showed that such an analogue does not exist when considering the result by Kühn and Osthus. That is we cannot just replace the minimum degree in this result by the corresponding Ore-type degree condition. This leads us to the interesting open question of whether we can find, for some graphs H, a better Ore-type degree condition than the one given in Theorem 8.1 that ensures perfect H-packings in graphs.

Most results concerning H-packings use Szemerédi's Regularity Lemma [20] in their proof. What this result essentially says is that any large dense graph can be approximated by a random-like graph. The Regularity Lemma was initially proved by Szemerédi in order to prove a conjecture of Erdős and Turán [7] that sequences of integers of positive upper density must contain long arithmetic progressions. Apart from graph theory and combinatorial number theory, the Regularity Lemma has many applications in other areas of mathematics and theoretical computer science.

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