Packings in dense regular graphs

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Abstract

We prove that for all $\alpha, c > 0$ and for all bipartite graphs H all but at most αn vertices of every cn-regular graph G whose order n is sufficiently large can be covered by vertex-disjoint copies of H. If the vertex-classes of H have different size, then even all but a constant number of vertices of G can be covered. This implies that for all $c > 0$ and all $r \geq 4$ there exists a constant C such that in every cn-regular graph G all but at most C vertices can be covered by vertex-disjoint subdivisions of K_r. We also show that for $r = 4, 5$ one can take $C = 0$.

1 Introduction, results and open problems

Given two graphs H and G, an *H-packing in G* is a collection of vertex-disjoint copies of H in G. It is *perfect* if all of the vertices of G are covered. Improving earlier bounds of Alon and Yuster [3], Komlós, Sárközy and Szemerédi [13] proved that given a graph H of chromatic number χ, there exists a constant c such that every sufficiently large graph G whose order n is divisible by $|H|$ and whose minimum degree is at least $(1 - 1/\chi)n + c$ has a perfect H-packing. This bound is best possible up to the value of c. However, Komlós [9] showed that if we only want an H-packing covering almost all of the vertices of G, then in many cases a significantly smaller minimum degree suffices: given $\alpha > 0$ and a graph H of chromatic number χ, every sufficiently large graph G of minimum degree at least $(1-1/\chi')n$ has an H-packing covering all but at most αn vertices. Here χ' is the *critical chromatic number* of H, which is defined as $(\chi - 1)|H|/(|H| - \sigma)$, where σ denotes the minimum size of the smallest colour class in an optimal colouring of H. It is easy to see that $\chi - 1 < \chi' \leq \chi$ and that χ' is closer to $\chi-1$ if there is an optimal colouring where one of the colours is used comparatively rarely. Again, the bound on the minimum degree is best possible (but Komlós conjectured that the error term αn can be reduced to a constant depending only on H).

Here we show that if we restrict our attention to packings of bipartite graphs H in (almost) regular graphs G then any linear bound on the minimum degree suffices. Given $a \geq b$, we say that a graph G is *$(a \pm b)$-regular* if its minimum degree is at least $a - b$ and its maximum degree is at most $a + b$.

Theorem 1 *Given a bipartite graph H and constants $0 < c, \alpha \leq 1$, there exist positive numbers $\gamma = \gamma(c, \alpha)$ and $n_0 = n_0(H, c, \alpha)$ such that every $(cn \pm \gamma n)$-regular graph G of order $n \geq n_0$ has an H-packing which covers all but at most αn vertices of G.*

The complete bipartite graph $G = K_{a,2a}$ (and $H := C_4$ say) shows that the restriction to almost regular graphs G is necessary. Also the restriction to bipartite graphs H is obviously necessary if $c \leq 1/2$. Our next result shows that if H has unequal vertex classes, then we can obtain an H-packing covering all but a constant number of vertices.

Theorem 2 *Given a bipartite graph H whose vertex classes have different size and a constant $0 < c \leq 1$, there exist $\gamma = \gamma(H, c) > 0$ and $C = C(H, c)$ such that every $(cn \pm \gamma n)$-regular graph G has an H-packing which covers all but at most C vertices of G.*

As the complete bipartite graph $G = K_{k,\ell}$ (with $k - \ell = \gamma n$ say) shows, we cannot hope for such a result if H has equal vertex classes. However, it could help to restrict one's attention to regular graphs G:

Question A *Is it true that for every $c > 0$ and every bipartite graph H there is a constant $C = C(c, H)$ such that every cn-regular graph G has an H-packing which covers all but at most C vertices of G?*

It is easy to see that in both Question A and Theorem 2 (even if we only consider regular graphs there too) the bound on the number of uncovered vertices must depend on H and c (consider the disjoint union G_r of r complete graphs whose order is $k|H| - 1$ for some positive integer k). Thus in contrast to the result of Komlós, Sárközy and Szemerédi mentioned above one cannot hope for a perfect packing when $|H|$ divides n. Moreover, the graphs G_r (with $r \to \infty$) show that in both Theorem 2 and Question A the requirement that the vertex degrees are linear is necessary. On the other hand, it may be true that the bound on the minimum degree in Theorem 1 can be improved. Since our proof of this theorem relies on Szemerédi's Regularity lemma, it seems that this would require a rather different approach from ours though. Note that if H contains a cycle, the bound on the minimum degree of G must of course depend on n. However, in the special case when H is a tree, it can be chosen to be independent of n: Kelmans, Mubayi and Sudakov [8] proved that for every $\alpha > 0$ and every tree T there exists a constant $d_0 = d_0(\alpha, T)$ such that for all $d \geq d_0$ every d-regular graph G has a T-packing which covers all but at most αn vertices of G.

Theorem 2 is related to a problem of Verstraëte on packings of subdivisions in regular graphs. Given graphs H and G, a TH*-packing* in G is a collection of vertex-disjoint subdivisions of H in G (which are not required to be isomorphic).

Question B (Verstraëte [16]) *Denote by $f(d, H)$ the supremum over all $\beta \leq 1$ for which every d-regular graph G has a TH-packing covering at least $\beta|G|$ of its vertices. Is it true that $\limsup_{d\to\infty} f(d, H) = 1$ for every graph H?*

Verstraëte [16] observed that $\liminf_{d\to\infty} f(d, H) \geq 1/2$ and that by Petersen's 2-factor theorem (see e.g. [15, Thm. 6.2.4]) Question B has a positive answer if H is a triangle. By the result in [8] it also holds for trees. Our results provide further indication that Question B might have a positive answer. In particular, Theorem 2 immediately implies the following (let H be the graph obtained by subdividing each edge of K_r exactly once).

Corollary 3 *Given an integer $r \geq 4$ and a constant $0 < c \leq 1$, there exist $\gamma = \gamma(r, c) > 0$ and $C = C(r, c)$ such that every $(cn \pm \gamma n)$-regular graph G has a TK_r-packing which covers all but at most C vertices of G.* $\square$

For $r = 4, 5$ we can even guarantee a perfect packing:

Theorem 4 *For all $0 < c \leq 1$ there exist positive numbers $\gamma = \gamma(c)$ and $n_0 = n_0(c)$ such that every $(cn \pm \gamma n)$-regular graph G of order $n \geq n_0$ has a perfect TK_r-packing for $r = 4$ and $r = 5$.*

It turns out that for $r \geq 6$ the vertex degrees in Corollary 3 have to be linear and that Theorem 4 does not extend to $r \geq 6$ (Proposition 12). However, as in Question A, it may help to consider only regular graphs G:

Question C *Given $r \geq 6$ and $0 < c \leq 1$, does every cn-regular graph of sufficiently large order n have a perfect TK_r-packing?*

As for Theorem 1, it is possible that in both Theorem 4 and Question C the condition on the minimum degree can be relaxed. On the other hand, at the end of Section 5 we give an example which shows that for all $r \geq 3$ we need a minimum degree of at least $\sqrt{n/2}$, even if G is regular.

A result of Jørgensen and Pyber [7] implies that Question B has an affirmative answer if we do not require the subdivisions of H to be disjoint: Let $t(H)$ be so that every graph of average degree at least $t(H)$ contains a subdivision of H. Jørgensen and Pyber proved that the edges of every graph G can be covered by at most $10^4\, t(H)|G|$ subdivisions of H and edges. Hence if the minimum degree of G is large compared with $t(H)$, then almost all of its edges must lie in a subdivision of H (and thus, if G is regular, also almost all of its vertices). In the case when $H = K_r$, Lemma 2.3 of [7] implies that if G is 2-connected and has minimum degree at least $t(K_{r+1})$, then even all edges (and thus also all vertices) of G lie in a subdivided K_r. Using this, it is not hard to show that for all $\varepsilon > 0$ there exists an $r_0 = r_0(\varepsilon)$ such that for all $r \geq r_0$ every graph G of minimum degree at least $(1 + \varepsilon)t(K_{r+1})$ has at most $\varepsilon|G|$ vertices and at most $\varepsilon e(G)$ edges that do not lie in a subdivision of K_r. (Consider the block tree T of G and use that all vertices and edges which lie in blocks corresponding to leaves of T are contained in a subdivided K_r.) In other words, to cover almost all of the vertices of G with (not necessarily disjoint) subdivisions of K_r we only need a minimum degree which is slightly larger than the average degree required for the mere existence of a subdivided K_{r+1}.

This paper is organized as follows. In the next section we introduce some notation and the tools (Regularity lemma and Blow-up lemma) which we will need later on. In Section 3 we prove Theorem 1. In Section 4 we extend the argument to obtain Theorem 2. In the final section we then derive Theorem 4 from Theorem 1. Since our proofs of Theorems 1–4 can be derandomized by standard techniques (see e.g. [2]) and since both the Regularity lemma and the Blow-up lemma have algorithmic proofs (see [1] and [12]), it is easy to verify that the packings guaranteed by Theorems 1–4 can be found in polynomial time.

2 Notation and tools

Throughout this paper we omit floors and ceilings whenever this does not affect the argument. We write $e(G)$ for the number of edges of a graph G, $|G|$ for its order, $\delta(G)$ for its minimum degree, $\Delta(G)$ for its maximum degree and $d(G) := 2e(G)/|G|$ for its average degree. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and the set of its neighbours by $N_G(x)$. Given disjoint $A, B \subseteq V(G)$, an *A–B edge* is an edge of G with one endvertex in A and the other in B, the number of these edges is denoted by $e_G(A, B)$ or $e(A, B)$ if this is unambiguous. We write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all A–B edges in G. More generally, we often write (A, B) for a bipartite graph with vertex classes A and B. A *subdivision* of a graph H is a graph TH obtained from H by replacing the edges of H with internally disjoint paths. The *branch vertices* of TH are all those vertices that correspond to vertices of H. We usually write 'disjoint' instead of 'vertex-disjoint'.

We will use the following well-known bound on the number of edges in a $K_{s,t}$-free graph (see e.g. [4, Ch. VI, Thm. 2.2]).

Theorem 5 *For all $s \leq t$ there exists a constant $c_{s,t}$ such that every bipartite graph $G = (A, B)$ with at least $c_{s,t}|G|^{2-1/s}$ edges contains a copy of $K_{s,t}$ with s vertices in A and t vertices in B.*

Moreover, we will need the following Chernoff bound (see e.g. [2, Cor. A.14]).

Lemma 6 *Let $X_1, \dots, X_n$ be independent 0-1 random variables with $\mathbb{P}(X_i = 1) = p$ for all $i \leq n$, and let $X := \sum_{i=1}^n X_i$. Then for all $\gamma > 0$ there exists a positive $\beta(\gamma)$ so that*

$$\mathbb{P}(|X - \mathbb{E}X| > \gamma \mathbb{E}X) \leq 2\mathrm{e}^{-\beta(\gamma)\mathbb{E}X}.$$

Our proof of Theorem 1 is based on the Regularity lemma and that of Theorem 2 also employs the Blow-up lemma. In the remainder of this section we collect all the information we need about these results (see [14] and [10] for surveys). Let us start with some more notation. The *density* of a bipartite graph $G = (A, B)$ is defined to be

$$d(A, B) := \frac{e(A, B)}{|A||B|}.$$

Given $\varepsilon > 0$, we say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A, B) - d(X, Y)| < \varepsilon$. Given $d \in [0, 1]$, we say that G is *(ε, d)-super-regular* if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) > d$ and, furthermore, if $d_G(a) > d|B|$ for all $a \in A$ and $d_G(b) > d|A|$ for all $b \in B$.

Proposition 7 *Every ε-regular graph $G = (A, B)$ of density at least $d > 2\varepsilon$ can be made into an $(\varepsilon/(1-\varepsilon), d - 2\varepsilon)$-super-regular graph by deleting $\varepsilon|A|$ vertices of A and $\varepsilon|B|$ vertices of B.*

Proof. As G is ε-regular there are less than $\varepsilon|A|$ vertices in A whose degree is at most $(d-\varepsilon)|B|$; and similarly there are less than $\varepsilon|B|$ vertices in B whose degree is at most $(d-\varepsilon)|A|$. It can be easily checked that the graph obtained from G by deleting these vertices of small degree (as well as possibly some other vertices to make up the required number) is $(\varepsilon/(1-\varepsilon), d-2\varepsilon)$-super-regular. $\square$

We will use the following degree form of Szemerédi's Regularity lemma for bipartite graphs which can be easily derived from the classical version. Proofs of the latter are for example included in [5] and [6]. The fact that the regularity partition can be required to refine the given bipartition (A,B) follows immediately from the proof of the Regularity lemma.

Lemma 8 (Regularity lemma) *For every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for every number $d \in [0,1]$ and for every bipartite graph $G = (A,B)$ with $|A| = |B|$ there exist partitions of A into $A_0, A_1, \dots, A_k$ and of B into $B_0, B_1, \dots, B_k$ and there is a spanning subgraph G' of G such that the following holds:*

- $k \leq N$,
- $|A_0 \cup B_0| \leq \varepsilon|G|$,
- $|A_1| = \cdots = |A_k| = |B_1| = \cdots = |B_k| =: L$,
- $d_{G'}(x) > d_G(x) - (d+\varepsilon)|G|$ *for all vertices* $x \in G$,
- *for all* $1 \leq i,j \leq k$ *the graph* $(A_i, B_j)_{G'}$ *is* ε*-regular and has density either* 0 *or* $> d$.

The sets A_i and B_i ($i \geq 0$) are called *clusters*. $A_0 \cup B_0$ is called the *exceptional set*. Given clusters and G' as in Lemma 8, the *reduced graph* R is the bipartite graph whose vertices are $A_1, \dots, A_k$ and $B_1, \dots, B_k$ (so we omit A_0 and B_0 here) and in which A_i is joined to B_j whenever $(A_i, B_j)_{G'}$ is ε-regular and has density $> d$. Thus A_iB_j is an edge of R if and only if G' has an edge between A_i and B_j.

We will also use the following special case of the Blow-up lemma of Komlós, Sárközy and Szemerédi [11].

Lemma 9 (Blow-up lemma) *For all $d > 0$ and all integers Δ there exists a positive number $\varepsilon_0 = \varepsilon_0(d, \Delta)$ such that for all $\varepsilon \leq \varepsilon_0$, all integers a, b and every subgraph H of $K_{a,b}$ with $\Delta(H) \leq \Delta$ each (ε, d)-super-regular bipartite graph $G = (A,B)$ with $|A| = a$ and $|B| = b$ contains H as a subgraph.*

3 H-packings covering all but a small fraction of vertices

The strategy of our proof of Theorem 1 is as follows. Given G, we will first find a spanning bipartite subgraph $G^* = (A,B)$ whose vertex classes have equal size and which is still 'almost regular' (see Proposition 10; we may assume that $|G|$ is even). We then apply the Regularity lemma to G^*. As G^* is 'almost regular', the reduced graph R has a matching M which misses only a small fraction of its vertices (Lemma 11). But as each edge $e \in M$ corresponds to an ε-regular graph G_e of sufficiently large density, we may successively apply Theorem 5 to pull out disjoint copies of H as long as an ε-fraction of the vertices remains uncovered in each of the vertex classes of G_e. Thus the copies of H can be chosen in such a way that we obtain an H-packing which covers a large fraction of the vertices of each G_e (and hence of G).

Proposition 10 *Given $\gamma, c > 0$ with $2\gamma < c \leq 1$, there exists $n_0 = n_0(\gamma, c)$ such that every $(cn \pm \gamma n)$-regular graph G of even order $n \geq n_0$ contains a spanning bipartite subgraph $G^* = (A, B)$ such that $|A| = |B|$ and $|d_{G^*}(x) - d_G(x)/2| \leq \gamma n$ for every vertex $x \in G$.*

Proof. Consider a random bipartition of $V(G)$ into sets A and $B = V(G) \setminus A$ which is obtained by including every vertex $x \in G$ into A with probability $1/2$ independently of all other vertices of G. Call a vertex $x \in G$ *bad* if $||N_G(x) \cap A| - d_G(x)/2| > \gamma n/2$. As $\mathbb{E}(|N_G(x) \cap A|) = d_G(x)/2$ and $\gamma n/2 \geq \gamma d_G(x)/2$, Lemma 6 implies that

$$\mathbb{P}(x \text{ is bad}) \leq 2\mathrm{e}^{-\beta(\gamma) d_G(x)/2} \leq 2\mathrm{e}^{-\beta(\gamma) cn/4}.$$

Hence if n is sufficiently large, then the expected number of bad vertices is at most $2n\mathrm{e}^{-\beta(\gamma)cn/4} < 1/2$. Thus Markov's inequality implies that

$$\mathbb{P}(\text{there is a bad vertex}) \leq 1/2.$$

Moreover, applying Lemma 6 again, we get

$$\mathbb{P}(|\,|A| - n/2| > \gamma n/2) \leq 2\mathrm{e}^{-\beta(\gamma)n/2} < 1/4.$$

Thus with probability at least $1/2 - 1/4 > 0$ there is an outcome A, B with $|\,|A| - n/2\,| \leq \gamma n/2$ for which no vertex of G is bad. We may assume that $|A| \leq |B|$. But then in the bipartite graph G^* obtained from $(A, B)_G$ by moving $(|B| - |A|)/2 \leq \gamma n/2$ vertices from B to A, every vertex $x \in G$ satisfies

$$|d_{G^*}(x) - d_G(x)/2| \leq \frac{\gamma n}{2} + \frac{\gamma n}{2} = \gamma n,$$

as required. □

Lemma 11 *Given positive numbers c, d, ε and γ with $c > \gamma + d + 2\varepsilon$ and a $(cn \pm \gamma n)$-regular bipartite graph $G = (A, B)$ with $|A| = |B|$, let $A_0, A_1, \ldots, A_k$ and $B_0, B_1, \ldots, B_k$ be the clusters of a partition of $V(G)$ as given in Lemma 8. Let R be the corresponding reduced graph. Then R contains a matching of cardinality at least $k(1 - 2(\gamma + d + \varepsilon)/c)$.*

Proof. Given $I \subseteq \{1,\dots,k\}$, let J be the set of all integers $j \geq 1$ such that in R the vertex B_j lies in the neighbourhood of $\{A_i \,|\, i \in I\}$. We will show that $|J| \geq |I|(1-2(\gamma+d+\varepsilon)/c))$. Then by the defect form of Hall's matching theorem (see e.g. [6, Cor. 2.1.3]) the bipartite graph R contains a matching of cardinality at least $k-2k(\gamma+d+\varepsilon)/c$, as required.

Let $I' \subseteq A$ be the union of all A_i with $i \in I$ and define $J' \subseteq B$ to be the union of all B_j with $j \in J$. Let L and G' be as in Lemma 8. Recall that if $x \in A_i \subseteq I'$ and if y is a neighbour of x in G' lying in B_j for some $j \geq 1$ then A_iB_j is an edge of R and thus $j \in J$ and $y \in J'$. As

$$\delta(G') \geq \delta(G) - (d+\varepsilon)n \geq (c-\gamma-d-\varepsilon)n,$$

this shows that

$$e_{G'}(I',J') \geq \sum_{i\in I} |A_i|[(c-\gamma-d-\varepsilon)n - |B_0|] \geq |I|L(c-\gamma-d-2\varepsilon)n.$$

But on the other hand,

$$e_{G'}(I',J') \leq e_{G'}(A,J') \leq \sum_{j\in J} |B_j|(c+\gamma)n = |J|L(c+\gamma)n$$

and so

$$\begin{aligned}|J| &\geq |I| \cdot \frac{1-\frac{\gamma+d+2\varepsilon}{c}}{1+\frac{\gamma}{c}} \geq |I| \cdot \left(1-\frac{\gamma+d+2\varepsilon}{c}\right)\left(1-\frac{\gamma}{c}\right)\\ &\geq |I| \cdot \left(1-\frac{2(\gamma+d+\varepsilon)}{c}\right),\end{aligned}$$

as required. □

Note that in the proof of Lemma 11 we cannot apply Hall's matching theorem directly to the reduced graph R since R is not necessarily almost regular.

Proof of Theorem 1. Clearly, it suffices to consider the case when $H = K_{s,t}$ where $s \leq t$. By deleting one vertex of G if necessary, we may assume that the order n of G is even. We will prove the assertion for $\gamma := \alpha c/50$ and for sufficiently large n_0. Apply Proposition 10 to G to obtain a spanning bipartite $(cn/2 \pm 2\gamma n)$-regular subgraph $G^* = (A,B)$ with $|A| = |B|$. Set $\varepsilon := \gamma$, $d := 2\varepsilon$ and apply Lemma 8 to G^* to obtain clusters $A_0, A_1, \dots, A_k$ and $B_0, B_1, \dots, B_k$. Let L and G' be as in Lemma 8 and let R be the reduced graph. Then by Lemma 11, R has a matching M missing at most $8k(2\gamma+d+\varepsilon)/c = 40\varepsilon k/c$ vertices of R. Recall that for every edge $e = A_iB_j$ of M, the graph $(A_i,B_j)_{G'}$ is ε-regular and has density $> d = 2\varepsilon$. So if n (and thus L) is sufficiently large, then we may successively apply Theorem 5 to pull out disjoint copies of $K_{s,t}$ from $(A_i,B_j)_{G'}$ as long as there are at least εL vertices left in each of its vertex classes. Additionally, we may require that the vertex class of the $K_{s,t}$ of size s lies alternately in A_i and in B_j. In this way we get a $K_{s,t}$-packing in $(A_i,B_j)_{G'}$ that covers all but at most $2\varepsilon L + t$ of its vertices. Proceeding similarly for each edge of M, we have found a $K_{s,t}$-packing of G for which the number of uncovered vertices is at most

$$\begin{aligned}|A_0 \cup B_0| + L|V(R) \setminus V(M)| + (2\varepsilon L + t) \cdot e(M) &\leq \varepsilon n + \frac{40\varepsilon k L}{c} + 3\varepsilon L k \\ &\leq \frac{44\varepsilon n}{c} < \alpha n\end{aligned}$$

(since $t \leq \varepsilon L$ if n is sufficiently large), as required. $\square$

4 $K_{s,t}$-packings covering all but a constant number of vertices

Before proving Theorem 2, let us first sketch the idea. Similarly as in the proof of Theorem 1 we first choose an almost regular bipartite graph $G^* \subseteq G$, apply the Regularity lemma to G^* and choose a large matching M in the reduced graph R. Then we make every bipartite graph corresponding to some edge A_iB_j of M into an (ε', d')-super-regular graph $(A'_i, B'_j)_{G'}$ by adding a small fraction of vertices to the exceptional set $A_0 \cup B_0$ (see Proposition 7). We could now apply the Blow-up lemma to these super-regular graphs to obtain a $K_{s,t}$-packing that misses only a constant number of vertices in each of them. But then we would still be left with the exceptional vertices. So instead we proceed as follows. We first assign each exceptional vertex $x \in A_0 \cup B_0$ to one of the super-regular graphs in which x has many neighbours in such a way that to each super-regular graph we assign only a small number of exceptional vertices. If $(A'_i, B'_j)_{G'}$ is such a super-regular graph, then successively for each exceptional vertex x assigned to it we fix a $K_{s,t}$ which contains x but has all its other vertices in $(A'_i, B'_j)_{G'}$. Furthermore, all these $K_{s,t}$'s are chosen to be disjoint and so that the sizes of the remaining subsets of A'_i and B'_j differ at most by a constant. Having dealt with the exceptional vertices, our aim now is to apply the Blow-up lemma to the remainders of the super-regular graphs in order to find disjoint $K_{s,t}$'s that cover all but constantly many vertices. The only problem is that although the remainders of the super-regular graphs are still large, they may now contain vertices of small degree and thus need no longer be super-regular. But by being careful in the choice of the $K_{s,t}$'s containing the exceptional vertices, this can also be fixed. So let us now turn to the details.

Proof of Theorem 2. Clearly, it suffices to prove the theorem for the case when $H = K_{s,t}$ where $s < t$. Moreover, by deleting one vertex of G if necessary, we may assume that $|G| =: n$ is even. Set $\gamma := c^3/4000t$, $d := \gamma$ and $\varepsilon := \min\{\gamma/8, \varepsilon_0(d/4, t)/4\}$, where ε_0 is as defined in Lemma 9. By making C larger, we may assume that the order n of G is sufficiently large compared with c, s and t. We will show that G contains a $K_{s,t}$-packing which covers all but at most $2tN(\varepsilon)$ vertices, where $N(\varepsilon)$ is as defined in Lemma 8. As indicated before, we start similarly as in the proof of Theorem 1. First we apply Proposition 10 to G to obtain a spanning bipartite $(cn/2 \pm 2\gamma n)$-regular subgraph $G^* = (A, B)$ with $|A| = |B|$. Next we apply Lemma 8 to G^* to obtain clusters $A_0, A_1, \ldots, A_k$

and $B_0, B_1, \ldots, B_k$. Let L and G' be as in Lemma 8 and let R be the reduced graph. Then by Lemma 11, R has a matching M covering all but at most $8k(2\gamma + d + \varepsilon)/c \leq 32\gamma k/c$ vertices of R. Let $L' := (1 - \varepsilon)L$. As for every edge A_iB_j of R the graph $(A_i, B_j)_{G'}$ is ε-regular of density $> d$, we may apply Proposition 7 to find sets $A'_i \subseteq A_i$ and $B'_j \subseteq B_j$ such that $|A'_i| = |B'_j| = L'$ and such that the graph $(A'_i, B'_j)_{G'}$ is $(2\varepsilon, d - 2\varepsilon)$-super-regular. Denote by M' the set of all pairs $A'_iB'_j$ for which A_iB_j is an edge of M. Let A'_0 be set of all those vertices in A that do not lie in some A'_i and define B'_0 similarly. Then

$$\begin{aligned} |B'_0| = |A'_0| &\leq |A_0| + L \cdot \frac{|V(R) \setminus V(M)|}{2} + \varepsilon L \cdot |M'| \\ &\leq \varepsilon n + \frac{16\gamma k L}{c} + \varepsilon k L \leq \frac{18\gamma n}{c}. \end{aligned}$$

Given a vertex $x \in A'_0 \cup B'_0$ and a pair $A'_iB'_j \in M'$, we say that $A'_iB'_j$ is x-*friendly* if x has at least $cL'/4$ neighbours in $A'_i \cup B'_j$ (in the bipartite graph G^*). Denoting the number of x-friendly pairs in M' by N_x, we have

$$N_xL' + |M'| \cdot \frac{cL'}{4} \geq d_{G^*}(x) - |N_{G^*}(x) \cap (A'_0 \cup B'_0)| \geq \frac{cn}{2} - 2\gamma n - \frac{18\gamma n}{c} > \frac{cn}{4},$$

and therefore, as $|M'| \leq n/2L'$,

$$N_x > \frac{cn}{4L'} - \frac{c|M'|}{4} \geq \frac{cn}{8L'}.$$

But setting $\alpha := 300\gamma/c^2$, this shows that

$$\frac{|A'_0 \cup B'_0|}{\alpha L'} \leq \frac{36\gamma n}{c\alpha L'} < N_x$$

for every $x \in A'_0 \cup B'_0$. This implies that we can successively assign every vertex $x \in A'_0 \cup B'_0$ to an x-friendly pair in M' in such a way that to each pair there will be assigned at most $\alpha L'$ vertices.

Consider any pair $A'_iB'_j \in M'$ and let A''_i (respectively B''_j) denote the set of all those vertices in A'_0 (respectively B'_0) that are assigned to $A'_iB'_j$. So every vertex $x \in A''_i$ has at least $cL'/4$ neighbours in B'_j and similarly every vertex in B''_j has at least $cL'/4$ neighbours in A'_i. Let G_{ij} be the bipartite graph obtained from $(A'_i, B'_j)_{G'}$ by adding all vertices in $A''_i \cup B''_j$ and all edges of G^* between these vertices and $A'_i \cup B'_j$.

We now choose bipartitions S_A, T_A of A'_i and S_B, T_B of B'_j such that in the graph G_{ij} each vertex in $A'_i \cup A''_i$ has at least a third of its neighbours in each of S_B and T_B, and such that similarly each vertex of $B'_j \cup B''_j$ has at least a third of its neighbours in each of S_A and T_A. The following simple probabilistic argument shows the existence of such bipartitions. Consider a random bipartition S_A, T_A of A'_i which is obtained by including every vertex $x \in A'_i$ into S_A with probability $1/2$ independently of all other vertices of A'_i. Given a vertex $y \in B'_j \cup B''_j$, set $n_y := |N_{G_{ij}}(y) \cap A'_i|$. Thus, if $y \in B'_j$, then $n_y > (d - 2\varepsilon)L'$ since $(A'_i, B'_j)_{G'}$ is $(2\varepsilon, d - 2\varepsilon)$-super-regular. Also, if $y \in B''_j$, then $n_y \geq cL'/4 > (d - 2\varepsilon)L'$. Call $y \in B'_j \cup B''_j$ *bad* if either

$|N_{G_{ij}}(y) \cap S_A| < n_y/3$ or $|N_{G_{ij}}(y) \cap S_A| > 2n_y/3$. Since the expected number of neighbours of y in S_A is $n_y/2$, Lemma 6 implies that

$$\mathbb{P}(y \text{ is bad}) \leq 2\mathrm{e}^{-\beta(1/6)n_y/2} \leq 2\mathrm{e}^{-\beta(1/6)(d-2\varepsilon)L'/2} \leq 2\mathrm{e}^{-\beta(1/6)\gamma L'/4}.$$

So if n is sufficiently large, then the expected number of bad vertices in $B'_j \cup B''_j$ is at most $2L' \cdot 2\mathrm{e}^{-\beta(1/6)\gamma L'/4} < 1$. Thus there exists an outcome S_A, T_A for which no vertex in $B'_j \cup B''_j$ is bad, i.e. a bipartition of A'_i having the required properties. The existence of S_B, T_B follows similarly.

Recall that every vertex $x \in A''_i$ (respectively $x \in B''_j$) has at least $n_x/3 \geq cL'/12$ neighbours in S_B (respectively S_A) and

$$\frac{cL'}{12} - |A''_i \cup B''_j|t \geq \frac{cL'}{12} - \alpha t L' = \frac{cL'}{12} - \frac{300\gamma t L'}{c^2} \geq \frac{cL'}{12} - \frac{cL'}{13} \geq 2\varepsilon L'.$$

Moreover, the graph $(A'_i, B'_j)_{G'} \supseteq (S_A, S_B)_{G'}$ is $(2\varepsilon, d-2\varepsilon)$-super-regular, so we may apply Theorem 5 for each vertex in $x \in A''_i \cup B''_j$ in turn to find a $K_{s,t}$ in G_{ij} which contains x and has all its other vertices in $S_A \cup S_B$ and such that all these $K_{s,t}$'s are disjoint for different vertices $x \in A''_i \cup B''_j$. Denote the sets of those vertices in A'_i and B'_j that do not lie in such a $K_{s,t}$ by A^*_i and B^*_j. Note that we may assume that the $K_{s,t}$'s were chosen in such a way that $||A^*_i| - |B^*_j|| \leq t - s$. Since firstly in the graph G' every vertex in A^*_i (respectively B^*_j) has more than $(d-2\varepsilon)L'/3 \geq dL'/4$ neighbours in $T_B \subseteq B^*_j$ (respectively in $T_A \subseteq A^*_i$), since secondly the graph $(A'_i, B'_j)_{G'} \supseteq (A^*_i, B^*_j)_{G'}$ is $(2\varepsilon, d-2\varepsilon)$-super-regular and since thirdly both A^*_i and B^*_j have size at least $L' - \alpha t L' \geq L'/2$, the graph $(A^*_i, B^*_j)_{G'}$ is still $(4\varepsilon, d/4)$-super-regular. So recalling the choice of ε, we may apply Lemma 9 to $(A^*_i, B^*_j)_{G'}$ to find disjoint copies of $K_{s,t}$ that cover all but at most $2(t-1)$ vertices of $(A^*_i, B^*_j)_{G'}$. Together with the $K_{s,t}$'s chosen earlier we have found a $K_{s,t}$-packing in G_{ij} that covers all but at most $2(t-1)$ of its vertices. Proceeding similarly for each pair in M', we have found a $K_{s,t}$-packing in G for which the number of uncovered vertices is at most $|M'|2(t-1) < N(\varepsilon)2t$. $\square$

5 Perfect TK_4- and TK_5-packings

Proof of Theorem 4. We only consider the case when $r = 4$ in detail, the proof for $r = 5$ works similarly. Set $s := \lceil 4/c \rceil$ and $\alpha := c/16s$. We will prove the theorem for $\gamma := \min\{c/2, \gamma(c,\alpha)\}$, where $\gamma(c,\alpha)$ is as defined in Theorem 1, and for sufficiently large n_0. First we apply Theorem 1 to obtain a $K_{s,s}$-packing M which covers all but at most αn vertices of G. Let X be the set of all remaining vertices.

We now successively assign each vertex $x \in X$ to a $K_{s,s}$ in M such that x has at least two neighbours in one of the colour classes of the $K_{s,s}$ and such that these $K_{s,s}$'s are distinct for different vertices in X. This can be done since for each $x \in X$

$$d_G(x) \geq cn - \gamma n \geq \frac{cn}{2} \geq 4s\alpha n + \frac{n}{s} > |N_G(x) \cap X| + 2s|X| + \frac{2n}{|K_{s,s}|}$$

and thus after each step for the vertex $x \in X$ to be considered next there must be a $K_{s,s}$ in M to which no vertex in X has been assigned yet and such that x has at least two neigbours in one of its colour classes.

As $s \geq 3$, it is easy to see that we can choose a spanning subdivision of K_4 in each $K_{s,s}$ of M to which we have assigned no vertex of X. Similarly, for every copy (A, B) of $K_{s,s}$ in M to which we have assigned a vertex $x \in X$ we can choose a subdivision of K_4 which contains x and all vertices in (A, B). Indeed, we may assume that x has two neighbours in A. These will be two of the branch vertices (which will be connected by the path of length 2 through x), we choose any other vertex in A as third branch vertex and any vertex in B as the fourth one (Fig. 1). In the case $r = 5$ we proceed similarly except that we now choose an additional branch vertex from B (Fig. 1). $\square$

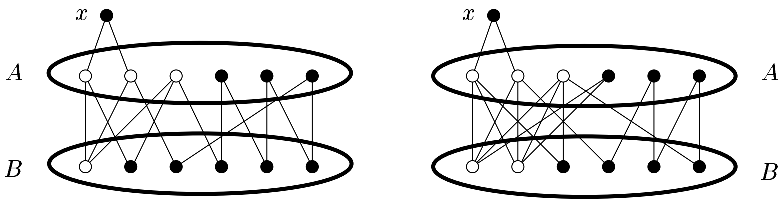

Figure 1: A spanning subdivision of K_4 (left) and K_5 (right). The white vertices are the branch vertices.

The graph $K_{n,n+1}$ immediately shows that we cannot hope for a perfect TK_r-packing if $r = 3$. The following proposition states that this is also the case for all $r \geq 6$ and thus the graph consisting of a disjoint union of $K_{n,n+1}$'s shows that for $r \geq 6$ the vertex degrees in Corollary 3 have to be linear.

Proposition 12 *For all integers n and $r \geq 6$ each TK_r-packing in $K_{n,n+1}$ misses at least one vertex.*

Proof. First consider the case when r is even, say $r = 2\ell$. Let A and B be the vertex classes of $K_{n,n+1}$. Consider any subdivision T of K_r in $K_{n,n+1}$. Let $k \in \mathbb{Z}$ be so that that $\ell + k$ branch vertices of T lie in A and $\ell - k$ branch vertices lie B. Then, no matter how the edges of K_r are subdivided,

$$|V(T) \cap B| - |V(T) \cap A| = \ell - k + \binom{\ell + k}{2} - (\ell + k) - \binom{\ell - k}{2} = (r - 3)k.$$

But this implies that in every TK_r-packing in $K_{n,n+1}$ the difference between the number of covered vertices in A and the number of covered vertices in B is divisible by $r - 3 \geq 3$ and hence cannot be 1. So there exists no perfect TK_r-packing.

In the case when r is odd, $r = 2\ell + 1$ say, we let k be such that $\ell + k + 1$ branch vertices lie in A to obtain $|V(T) \cap B| - |V(T) \cap A| = (2k + 1)(r - 3)/2$ which gives us the same conclusion. $\square$

To conclude this section, we give an example of a regular graph G which shows that for a perfect TK_r-packing (where $r \geq 3$) a minimum degree of at least $\sqrt{n/2}$ is necessary. Let $k \geq 3$ be an odd integer and let H be the graph which is obtained from a $K_{k,k-1}$ by including $(k-1)/2$ independent edges in the larger vertex class. So all vertices of H have degree k except for one vertex of degree $k-1$. Let G be the graph which consists of k disjoint copies of H together with a new vertex x that is joined to the vertices of degree $k-1$ in all copies of H (Fig. 2). So x does not lie in a subdivision of K_r, G is k-regular and $|G| = k(2k-1)+1$.

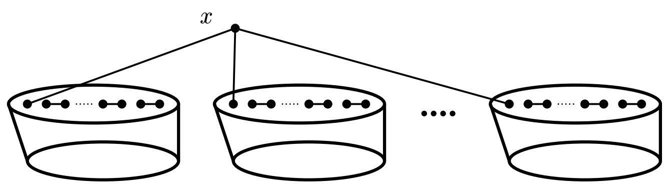

Figure 2: A regular graph of minimum degree at least $\sqrt{n/2}$ and no perfect TK_r-packing.

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