ON PÓSA'S CONJECTURE FOR RANDOM GRAPHS

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Abstract. The famous Pósa conjecture states that every graph of minimum degree at least $2n/3$ contains the square of a Hamilton cycle. This has been proved for large n by Komlós, Sarközy and Szemerédi. Here we prove that if $p \geq n^{-1/2+\varepsilon}$, then asymptotically almost surely, the binomial random graph $G_{n,p}$ contains the square of a Hamilton cycle. This provides an 'approximate threshold' for the property in the sense that the result fails to hold if $p \leq n^{-1/2}$.

1. Introduction

The kth power of a cycle C is obtained by including an edge between all pairs of vertices whose distance on C is at most k. The Pósa-Seymour conjecture states that every graph G on n vertices with minimum degree at least $kn/(k+1)$ contains the kth power of a Hamilton cycle. (Here the case $k = 2$ was conjectured by Pósa and the general case was later conjectured by Seymour.) This beautiful conjecture was proved for large n by Komlós, Sarközy and Szemerédi [12]. The case $k = 1$ of course corresponds to Dirac's theorem [7] on Hamilton cycles. For $k = 2$, there have been significant improvements in the bound on n that is required (see e.g. [5]). More generally, many other recent advances have been made on embedding spanning subgraphs in dense graphs (see e.g. [18] for a survey). For instance, recall that G has an *F-factor* if G contains $\lfloor |G|/|F| \rfloor$ vertex-disjoint copies of F. The famous Hajnal-Szemerédi theorem [8] states that every graph with minimum degree at least $kn/(k+1)$ contains a K_{k+1}-factor. More generally, Kühn and Osthus [19] determined the minimum degree that G needs to have to ensure the existence of an F-factor in G (up to an additive constant depending on F).

It is natural to ask for probabilistic analogues of these results, i.e. given a graph H on n vertices, how large does p have to be to ensure that $G_{n,p}$ a.a.s. contains a copy of H? Here $G_{n,p}$ denotes the binomial random graph on n vertices with edge probability p and we say that a property A holds a.a.s. (asymptotically almost surely), if the probability that A holds tends to 1 as n tends to infinity. (Note that formally one actually needs to ask the above question for a sequence of graphs H_i whose order tends to infinity.)

This turns out to be a surprisingly difficult problem, and the answer is known for very few (families of) graphs H. A notable exception is the seminal result of Johansson, Kahn and Vu [11], who determined the 'approximate' threshold for

Date: July 20, 2012.

D. Kühn was supported by the ERC, grant no. 258345.

the existence of an F-factor. So this is a probabilistic version of the result in [19] mentioned above. Also, Riordan [20] obtained a very general result, which gives a bound that can be applied to every graph H. As a corollary, he obtained the threshold for the existence of a spanning hypercube in $G_{n,p}$ and several kinds of spanning lattices, e.g. the square grid. His result can be applied to powers of Hamilton cycles to give the following result (see Section 8 for the straightforward details):

Theorem 1.1. *Let $k \geq 2$ be fixed. Suppose that $pn^{1/k} \to \infty$ and $pn^{1/3} \to \infty$. Then a.a.s. $G_{n,p}$ contains the kth power of a Hamilton cycle.*

A simple first moment argument shows that this result gives the correct threshold for $k \geq 3$. Indeed, note that the number of edges in the kth power of a cycle of length $n > 2k$ is kn. So if $n > 2k$ and $p \leq n^{-1/k}$, it follows that the expected number of appearances of the kth power of a Hamilton cycle in $G_{n,p}$ is at most $n!p^{kn} \leq (np^k/2)^n = o(1)$.

However, for squares (i.e. when $k = 2$) Theorem 1.1 does not give the correct answer. Indeed, the above first moment argument suggests that the threshold should be close to $n^{-1/2}$. Our main result is an 'approximate' threshold, i.e. our bound on p is tight up to a factor of n^{ε}, where $\varepsilon > 0$ is arbitrary. Our argument works for higher powers in the same way as it does for squares, so we formulate our proof for arbitrary $k \geq 2$.

Theorem 1.2. *Let $\varepsilon > 0$ and $k \geq 2$ be fixed. Suppose that $p = p(n) \geq n^{-1/k+\varepsilon}$. Then a.a.s. $G_{n,p}$ contains the kth power C^k of a Hamilton cycle.*

Note that Theorems 1.1 and 1.2 as well as the result on F-factors in [11] (see Theorem 5.1) imply that the threshold for a K_{k+1}-factor is much smaller than that for the kth power of a Hamilton cycle. So this is different from the 'deterministic' setting described earlier, where the minimum degree conditions are the same.

We now discuss some further related results on embedding spanning subgraphs in $G_{n,p}$. The case of Hamilton cycles (i.e. when $k = 1$) has been studied successfully and in great detail. In particular, a classical result of Komlós and Szemerédi [13] and Korshunov [14] implies that the threshold function for the existence of a Hamilton cycle is $\log n/n$. In fact, much more is true: a celebrated result of Bollobás [2] and Ajtai, Komlós and Szemerédi [1] states that the hitting time for the emergence of a Hamilton cycle on n vertices coincides a.a.s. with the hitting time of the property that the minimum degree is at least 2. (An algorithmic version of this result was later proved by Bollobás, Fenner and Frieze [4].) On the other hand, the expected number of Hamilton cycles already tends to infinity when $np \to \infty$. So the existence of vertices of degree less than two in $G_{n,p}$ can be viewed as a 'local obstruction' to the existence of a Hamilton cycle in $G_{n,p}$. For $k \geq 3$, Theorem 1.1 shows that there are no 'local obstructions'. It seems natural to conjecture that the case of squares is similar, i.e. that the threshold for the square of a Hamilton cycle in $G_{n,p}$ is at $p = n^{-1/2}$.

Another class of subgraphs which has received much attention is that of spanning trees. The best general result is due to Krivelevich [17], who showed (amongst other results) that if T is any tree on n vertices of bounded maximum degree and $p \geq n^{-1+\varepsilon}$, then a.a.s. $G_{n,p}$ contains a copy of T. It seems likely that the term n^{ε} in this result can be replaced by a much smaller function. This is supported by several results on certain classes of trees. For instance, the threshold for a Hamilton path is $p = \log n/n$ by the above results on Hamilton cycles. As another example, Hefetz, Krivelevich, and Szabó [9] showed that $p = \log n/n$ is the (sharp) threshold for a tree T having a linear number of leaves.

In the probabilistic setting, it is also natural to ask for 'universality' results. Again, this is a question where much progress has been made recently. Given a graph G and a family of graphs $\mathcal{H}$, we say that a graph G is *$\mathcal{H}$-universal* if G contains every member of $\mathcal{H}$ as a subgraph. An important case is when $\mathcal{H} = \mathcal{H}(n, \Delta)$ consists of all graphs on n vertices with maximum degree at most Δ. Here the best bound is due to Dellamonica, Kohayakawa, Rödl and Ruciński [6], who showed that if $p \gg n^{-1/2\Delta} \log^{1/\Delta} n$, then a.a.s. $G_{n,p}$ is $\mathcal{H}(n, \Delta)$-universal. Note that the kth power of a Hamilton cycle on $n > 2k$ vertices has maximum degree $2k$. So the bounds one obtains for this case are significantly weaker than the ones given by Theorems 1.1 and 1.2.

The proof in [20] is based on the second moment method. Instead, our proof is based on the 'absorbing method', which was introduced as a general method by Rödl, Ruciński and Szemerédi [21] (the underlying idea was also used earlier, e.g. by Krivelevich [16]). The method has proved to be an extremely versatile tool for embedding various types of spanning subgraphs in dense graphs. Though additional difficulties arise in the context of (sparse) random graphs, we believe that the method has significant further potential in this setting.

This paper is organized as follows. After introducing some notation, we define an 'absorber', which will be the crucial concept for extending the kth power of an almost spanning cycle into the kth power of a Hamilton cycle. We then describe the proof of Theorem 1.2 in Section 4, under the assumption that Lemmas 4.1, 4.2 and 4.3 hold. Section 4 also contains an informal overview of the proof. These lemmas are proved in the subsequent sections. In the short final section, we show how Theorem 1.1 follows from the more general result in [20].

2. Notation

We write $|G|$ and sometimes also v_G for the number of vertices of a graph G. We write $e(G)$ and sometimes also e_G for the number of edges of G. We say that two graphs H and G are disjoint if they are vertex-disjoint. Given graphs G and H, an H-factor in G is a collection of $\lfloor |G|/|H| \rfloor$ pairwise disjoint copies of H in G.

We denote the path on n vertices by P_n. The *distance* between two vertices x and y in a graph G is the length (i.e. the number of edges) of the shortest path between x and y. The *kth power* of a graph G is the graph G^k whose vertex set is $V(G)$ and in which two vertices $x, y \in V(G)$ are joined by an edge if and only if the distance between x and y in G is at most k. So P_n^k denotes the kth power of P_n. Suppose that $n \geq 2k$ and that $P_n = x_1 \dots x_n$. We will view x_1 as the first vertex of P_n and x_n as its final vertex. The *initial endsequence* of P_n^k is the sequence $x_1, \dots, x_k$ and the *final endsequence* of P_n^k is the sequence $x_{n-k+1}, \dots, x_n$.

Suppose that $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ are two sequences of vertices such that all these $2k$ vertices are distinct from each other. An (A, B)*-linkage* R is defined as follows: let R' be the kth power of a path such that the initial endsequence of R' is A and the final endsequence of R' is B. Then we obtain R by removing all edges within A and within B. We will use the notion of linkages to join up kth powers of paths into longer ones. More precisely, suppose that Q and Q' are kth powers of paths which are pairwise disjoint, that A is the final endsequence of Q, that B is the initial endsequence of Q' and that R is an (A, B)-linkage which meets $V(Q) \cup V(Q')$ only in $A \cup B$. Then $Q \cup R \cup Q'$ is again the kth power of a path.

We will omit floors and ceilings whenever this does not affect the argument. We write $\log n$ for the natural logarithm and $\log^a n := (\log n)^a$.

3. Absorbers

The aim of this section is to define an *absorber*, which is the main tool in our proof of Theorem 1.2. Roughly speaking, an absorber A will be the union of the kth power P^k of a path P and the kth power $(P')^k$ of a path P' such that the following two properties are satisfied:

- The two endsequences of P^k are the same as the two endsequences of $(P')^k$.
- $V(P')$ is obtained from $V(P)$ by adding one extra vertex v (which we call the absorbtion vertex).

Thus if we can find the kth power C^* of some cycle which contains P^k as a subgraph but does not contain v, then we can 'absorb' v into C^* by replacing P^k with $(P')^k$. When defining the absorber, we have to make sure that our random graph $G_{n,p}$ a.a.s. contains many disjoint copies of this absorber. A result of Johansson, Kahn and Vu (Theorem 5.1 below) implies that the latter will be the case if the 1-densities of all subgraphs of the absorber are not too large. (This will turn out to be true if the parameters j and ℓ below satisfy $k \ll j \ll \ell$.)

More precisely, for all $k \geq 2$, $j \geq 3$ and $\ell \geq 2k$, we will now define the (j, ℓ, k)-*absorber* $A_{j,\ell,k}$. Consider first a path P on s vertices, where $s := j(2\ell + 4) + \ell$, and a vertex v that does not belong to P. We call P the *spine* of the absorber and v its *absorbtion vertex*. We will view one endvertex of P as its first vertex and the other endvertex of P as its last vertex. This induces an order of the vertices on P. Split P into $j+1$ consecutive disjoint segments $S_1, \dots, S_{j+1}$ such that S_i has $2\ell + 4$ vertices for each $i = 1, \dots, j$ and S_{j+1} consists of the final ℓ vertices of P. For $i = 1, \dots, j$, in S_i we label the $(\ell + 1)$st, the $(\ell + 2)$nd, the $(2\ell + 3)$rd and the $(2\ell + 4)$th vertices by $a_{i,1}$, $a_{i,2}$, $b_{i,1}$ and $b_{i,2}$, respectively. We call these special vertices *junctions*.

We add the edges $a_{1,1}v$ and $vb_{1,2}$. For every $i = 1, \dots, j-2$, we add the edges $a_{i,2}b_{i+1,2}$ and $b_{i,1}a_{i+1,1}$. Finally, we add the edges $a_{j,2}b_{j,2}$, $a_{j-1,2}a_{j,1}$ and $b_{j-1,1}b_{j,1}$. We will be referring to the resulting graph (consisting of the spine P, the absorbtion vertex v and the edges incident to the junctions and to v which we added) as the *skeleton* of the absorber.

It is not hard to see that the graph P' obtained from the skeleton by deleting the edges $a_{i,1}a_{i,2}$ and $b_{i,1}b_{i,2}$ for all $i = 1, \ldots, j$ is a path with $V(P') = V(P) \cup \{v\}$ and with the same endvertices as the spine P (see Figure 1 for the case when $j = 4$). We call P' the *augmented path* of the absorber and the edges in $E(P') \setminus$

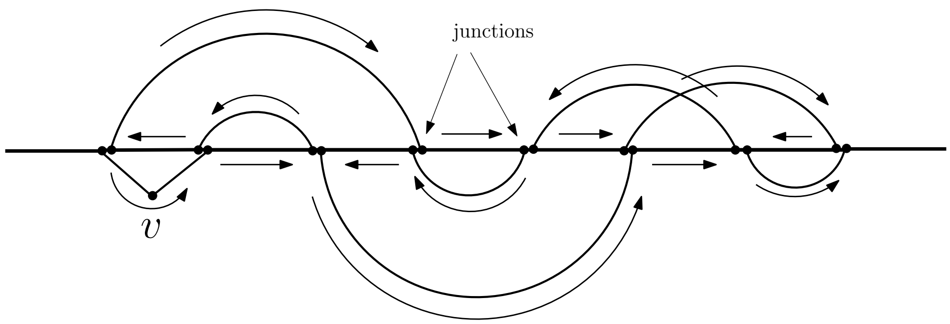

FIGURE 1. The skeleton of a $(4, \ell)$-absorber. The path P' is indicated by the arrows.

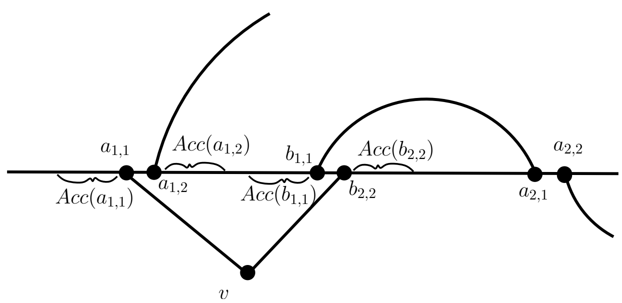

FIGURE 2. Junctions and access vertices of a $(4, \ell)$-absorber.

$(E(P) \cup \{a_{1,1}v, vb_{1,2}\})$ the *junction edges.* We define the (j, ℓ, k)*-absorber* $A_{j,\ell,k}$ to be $P^k \cup (P')^k$. The *first endsequence of* $A_{j,\ell,k}$ is the first endsequence of P^k (and thus of $(P')^k$) and the *final endsequence of* $A_{j,\ell,k}$ is the final endsequence of P^k (and thus of $(P')^k$).

Given a junction a, let $\mathrm{Acc}(a)$ be the set consisting of a as well as all the $k-1$ vertices that have distance at most $k-1$ from a in *both* P and P' (see also Figure 2, where these sets are marked for four of the junctions). Call the vertices in $\mathrm{Acc}(a)$ *access vertices* associated with a. Note that the following properties hold:

(A1) Let ab be any junction edge, where a is the predecessor of b on P'. Then the subpath Q_a of P' induced by a and the ℓ vertices preceding a on P' is also a subpath of P and $\mathrm{Acc}(a)$ is the set of all those vertices having distance at most $k-1$ from a on Q_a. Similarly, the subpath Q_b of P'

induced by b and the ℓ vertices succeeding b on P' is also a subpath of P and $\mathrm{Acc}(b)$ is the set of all those vertices having distance at most $k-1$ from b on Q_b.

(A2) $a_{1,1}vb_{1,2}$ is a subpath of P'. The subpath $Q_{a_{1,1}}$ of P' induced by $a_{1,1}$ and the ℓ vertices preceding $a_{1,1}$ on P' is also a subpath of P and $\mathrm{Acc}(a_{1,1})$ is the set of all those vertices having distance at most $k-1$ from $a_{1,1}$ on $Q_{a_{1,1}}$. Similarly, the subpath $Q_{b_{1,2}}$ of P' induced by $b_{1,2}$ and the ℓ vertices succeeding $b_{1,2}$ on P' is also a subpath of P and $\mathrm{Acc}(b_{1,2})$ is the set of all those vertices having distance at most $k-1$ from $b_{1,2}$ on $Q_{b_{1,2}}$.

(A3) The graph consisting of all junction edges, of the path $a_{1,1}vb_{1,2}$ and of all the edges $a_{i,1}a_{i,2}$, $b_{i,1}b_{i,2}$ (for all $i=1,\dots,j$) is a cycle.

(A1) and (A2) together with the fact that $\ell \geq 2k$ imply that every edge $e \in E(A_{j,\ell,k}) \setminus E(P^k)$ satisfies precisely one of the following conditions:

- There is precisely one junction edge ab such that e joins some vertex in $\mathrm{Acc}(a)$ to some vertex in $\mathrm{Acc}(b)$.
- e joins some vertex in $\mathrm{Acc}(a_{1,1}) \cup \{v\}$ to some vertex in $\mathrm{Acc}(b_{1,2}) \cup \{v\}$.

In the first case we say that e is *associated with* ab (so ab itself is associated with ab) and in the second case we say that e is *associated with* v. Note that for every junction edge ab there are precisely $\binom{k+1}{2}$ edges associated with ab. Indeed, let $a_k := a$ and for each $i=1,\dots,k-1$ let a_i be the vertex of distance i from a on Q_a, where Q_a is as defined in (A1). (So $\mathrm{Acc}(a) = \{a_1,\dots,a_k\}$.) Then a_i has precisely i neighbours in $\mathrm{Acc}(b)$. Similarly, precisely $\binom{k+1}{2}+k$ edges are associated with v. Since there are $2j-1$ junction edges, altogether this shows that

$$e(A_{j,\ell,k}) = e(P^k) + 2j\binom{k+1}{2} + k. \tag{3.1}$$

4. Proof of Theorem 1.2

Since the property of containing the kth power of a Hamilton cycle is monotone it suffices to show that a.a.s. G_{n,p^*} contains the kth power of a Hamilton cycle, where

$$p^* = p^*(n) := n^{-1/k+\varepsilon_*}.$$

Here ε_* is fixed and we assume that

$$\varepsilon_* \leq \frac{1}{10^4 k}. \tag{4.1}$$

So in particular $p^* = o(1)$. We shall consider a multiple round exposure of G_{n,p^*}. More precisely, we will expose G_{n,p^*} in four rounds considering four independent random graphs $G_{n,p_1^*},\dots,G_{n,p_4^*}$, where $p_1^* = \dots = p_4^*$. Thus $p_i^* = (1+o(1))p^*/4 \geq n^{-1/k+\varepsilon_*/2}$ for all $i=1,\dots,4$.

Roughly speaking, the strategy of our proof is as follows. We will first use G_{n,p_1^*} to find a collection $\mathcal{A}$ of pairwise disjoint absorbers which cover about $n/3$ vertices. Let A denote the set consisting of all absorbtion vertices of all these absorbers. We use G_{n,p_2^*} to connect the kth powers of the spines of the absorbers in $\mathcal{A}$ into the kth power $Q_{\mathcal{A}}$ of a path. To do this we will only use vertices which are not covered by the absorbers in $\mathcal{A}$. Moreover, $V(Q_{\mathcal{A}}) \cup A$ will contain at most $2n/3$ vertices. Let $S' := [n] \setminus (V(Q_{\mathcal{A}}) \cup A)$ denote the set of uncovered vertices. We will use G_{n,p_3^*} to cover S' by a collection $\mathcal{P}$ consisting of not too many kth powers of pairwise disjoint paths. Finally, we will use G_{n,p_4^*} connect all the paths in $\mathcal{P}$ as well as $Q_{\mathcal{A}}$ into the kth power C^* of a cycle. To do this we will only use vertices in A. Let $A'' \subseteq A$ be the vertices not used in this step. Since A'' consists of absorbtion vertices, we can 'absorb' all the vertices of A'' into C^* to obtain the kth power of a Hamilton cycle. More precisely, for each $v \in A''$ let A_v denote the unique absorber in $\mathcal{A}$ that contains v. Then the subgraph obtained from C^* by replacing the kth power of the spine of A_v with the kth power of the augmenting path of A_v (for each $v \in A''$) is the kth power of a Hamilton cycle in G_{n,p^*}.

After outlining our strategy, let us now return to the actual proof. We will use the next lemma (which is proved in Section 6) in order to find the collection $\mathcal{A}$ of absorbers in G_{n,p_1^*}.

Lemma 4.1. *For each $\varepsilon > 0$ and each integer $k \geq 2$, there exist integers $j \geq 3$ and $\ell_0 \geq 2k$ such that whenever $\ell \geq \ell_0$ and $p = p(n) \geq n^{-1/k+\varepsilon}$, then a.a.s. $G_{n,p}$ contains an $A_{j,\ell,k}$-factor.*

Let $j = j(k, \varepsilon_*/2)$ and $\ell_0 = \ell_0(k, \varepsilon_*/2)$ be as in Lemma 4.1. Set

$$\ell := \max\{\ell_0, \lceil 1/\varepsilon_*^2 \rceil\}. \tag{4.2}$$

Then Lemma 4.1 implies that a.a.s. G_{n,p_1^*} contains an $A_{j,\ell,k}$-factor. So we may assume that such a factor exists. Let $s := j(2\ell+4)+\ell$ and note that $|A_{j,\ell,k}| = s+1$. Let $\mathcal{A}$ be a collection of $n/(3(s+1))$ copies of $A_{j,\ell,k}$ in this $A_{j,\ell,k}$-factor and let A denote the set of absorbtion vertices in all these copies. (So the assertion of Lemma 4.1 is far stronger than we need it to be – see the discussion after Theorem 5.1.) Note that the absorbers in $\mathcal{A}$ cover $n/3$ vertices of G_{n,p^*}. Let S be a set of $n/3$ vertices not covered by these absorbers. As indicated above, our next aim is to use G_{n,p_2^*} in order connect the absorbers in $\mathcal{A}$, using some of the vertices in S. To do this, we will use the following lemma (which we prove in Section 7).

Lemma 4.2. *Suppose that $k \geq 2$, that $0 < \varepsilon < 1/k$, that $p = p(n) \geq n^{-1/k+\varepsilon}$ with $p(n) = o(1)$ and that $f \leq \varepsilon n/(60k)$. For each $i = 1, \dots, f$ let A_i and B_i be sequences, each consisting of k vertices in $[n]$, such that these $2f$ sequences are pairwise disjoint. Then a.a.s. $G_{n,p}$ contains pairwise disjoint (A_i, B_i)-linkages R_i with $|R_i| \leq \lceil 30k/\varepsilon \rceil$ (for all $i = 1, \dots, f$).*

Choose an order of the absorbers in $\mathcal{A}$. For each $i = 1, \dots, |\mathcal{A}|-1$ let A_i denote the final endsequence of the ith absorber in $\mathcal{A}$ and let B_i be the initial endsequence of the $(i+1)$st absorber in $\mathcal{A}$. Let S^* denote the union of S together with all the vertices contained in one of these endsequences A_i or B_i. Note that

$$|\mathcal{A}| = \frac{n}{3(s+1)} = \frac{|S|}{s+1} \leq \frac{|S|}{\ell} \overset{(4.2)}{\leq} \varepsilon_*^2 |S| \overset{(4.1)}{\leq} \frac{\varepsilon_* |S|}{180k} \leq \frac{\varepsilon_* |S^*|}{180k}$$

and $p_2^* \geq n^{-1/k+\varepsilon_*/2} \geq |S^*|^{-1/k+\varepsilon_*/3}$. So we may apply Lemma 4.2 (with $\varepsilon_*/3$ playing the role of ε) to see that a.a.s. the random subgraph of G_{n,p_2^*} induced by S^* contains pairwise disjoint (A_i, B_i)-linkages R_i with $|R_i| \le \lceil 90k/\varepsilon_* \rceil$ for all $i = 1, \dots, |\mathcal{A}|-1$. So we may assume that such linkages exist. Let $Q_\mathcal{A}$ be the union of $R_1, \dots, R_{|\mathcal{A}|-1}$ and of the kth powers of the spines of all absorbers in $\mathcal{A}$. Then $Q_\mathcal{A}$ is the kth power of a path whose initial endsequence is the initial endsequence of the first absorber in $\mathcal{A}$ and whose final endsequence is the final endsequence of the last absorber in $\mathcal{A}$. Moreover, $Q_\mathcal{A}$ avoids the set A of absorbtion vertices.

Let $S' := [n] \setminus (V(Q_\mathcal{A}) \cup A)$ be the set of uncovered vertices. Thus $|S'| \ge n/3$. Our next aim is to cover S' with not too many kth powers of paths. To simplify this step, first let $t := |S'| \mod s^2$. Now remove $s^2 - t$ vertices from A and call the resulting set A'. Add these $s^2 - t$ vertices to S' and call the resulting set S''. So $|S''|$ is divisible by s^2.

The next lemma (which will be proved in Section 5) implies that a.a.s. the random subgraph of G_{n,p_3^*} induced by S'' contains a $P_{s^2}^k$-factor $\mathcal{P}$. So we may assume that such a factor exists.

Lemma 4.3. *Suppose that $\varepsilon > 0$, that $k, r \ge 2$ and that $p = p(n) \ge n^{-1/k+\varepsilon}$. Then a.a.s. $G_{n,p}$ has a P_r^k-factor.*

Since $|S''|$ is divisible by s^2, all the vertices in S'' are covered by $\mathcal{P}$. We will now use G_{n,p_4^*} to connect all the copies of $P_{s^2}^k$ in $\mathcal{P}$ as well as $Q_\mathcal{A}$ into the kth power of a cycle, using some of the vertices in A'. To do this, we choose an order of the copies of $P_{s^2}^k$ in $\mathcal{P}$. For each $i = 1, \dots, |\mathcal{P}| - 1$ let A'_i denote the final endsequence of the ith copy of $P_{s^2}^k$ in $\mathcal{P}$ and let B'_i be the initial endsequence of the $(i+1)$st copy. Let $A'_{|\mathcal{P}|}$ denote the final endsequence of the last copy of $P_{s^2}^k$ in $\mathcal{P}$ and let $B'_{|\mathcal{P}|}$ denote the initial endsequence of $Q_\mathcal{A}$. Finally, let $A'_{|\mathcal{P}|+1}$ denote the final endsequence of $Q_\mathcal{A}$ and let $B'_{|\mathcal{P}|+1}$ denote the initial endsequence of the first copy of $P_{s^2}^k$ in $\mathcal{P}$. Let A^* denote the union of A' together with all the vertices contained in one of the endsequences A'_i or B'_i with $i = 1, \dots, |\mathcal{P}| + 1$. Recall that $|A| = |\mathcal{A}| = n/(3(s+1))$ and so $|A^*| \ge |A'| \ge |A| - s^2 \ge n/(4(s+1))$. Moreover, $s \ge \ell$. Thus

$$|\mathcal{P}| + 1 = \frac{|S''|}{s^2} + 1 \le \frac{n}{s^2} \overset{(4.1),(4.2)}{\le} \frac{\varepsilon_* n}{720k(s+1)} \le \frac{\varepsilon_* |A^*|}{180k}.$$

Moreover, $p_4^* \ge n^{-1/k+\varepsilon_*/2} \ge |A^*|^{-1/k+\varepsilon_*/3}$. So we may apply Lemma 4.2 (with $\varepsilon_*/3$ playing the role of ε) to see that a.a.s. the random subgraph of G_{n,p_4^*} induced by A^* contains pairwise disjoint (A'_i, B'_i)-linkages R'_i with $|R'_i| \le \lceil 90k/\varepsilon_* \rceil$ for all $i = 1, \dots, |\mathcal{P}| + 1$. So we may assume that such linkages exist. Thus the union of C^* of all these linkages R'_i, of all the copies of $P_{s^2}^k$ in $\mathcal{P}$ and of $Q_\mathcal{A}$ forms the kth power of a cycle which covers all vertices apart from some vertices in A'.

Let $A'' \subseteq A' \subseteq A$ denote the set of all uncovered vertices. For each $v \in A''$, let $A_v \in \mathcal{A}$ denote the unique absorber containing v. Let P_v denote the spine of A_v and let P'_v denote its augmented path. Note that C^* contains the kth power P_v^k of P_v as a subgraph. But the kth power $(P'_v)^k$ of P'_v has the same endsequences as P_v^k. Thus the graph obtained from C^* by replacing P_v^k with $(P'_v)^k$ for each $v \in A''$

is the kth power of a Hamilton cycle in G_{n,p^*}. (Note that our construction implies that a.a.s. $G_{n,p*}$ contains C^* as well as $(P'_v)^k$ for every $v \in A$.)

5. Finding a factor of kth powers of paths: Proof of Lemma 4.3

The 1-*density* of a graph H on at least two vertices is defined to be

$$d_1(H) := \frac{e_H}{v_H - 1},$$

where e_H and v_H denote the number of edges and the number of vertices of H. Let

$$d_1^{\max}(H) := \max_{H' \subseteq H,\ v_{H'} \geq 2} d_1(H').$$

Lemma 4.3 will be an easy consequence of the following deep result of Johansson, Kahn and Vu [11], which was already mentioned in the introduction.

Theorem 5.1 (Theorem 2.2 [11])**.** *Fix $\varepsilon > 0$ and a graph H. Suppose that $p(n) \geq n^{-1/d_1^{\max}(H)+\varepsilon}$. Then a.a.s. $G_{n,p}$ contains an H-factor.*

Thus in order to prove Lemma 4.3, it suffices prove the following proposition.

Proposition 5.2. *Let $k, r \geq 2$ be integers. Then $d_1^{\max}(P_r^k) \leq k$.*

Proof. Consider any $H \subseteq P_r^k$ on $v_H \geq 2$ vertices. Thus there is an ordering $x_1, \dots, x_{v_H}$ of the vertices of H such that for all $i = 2, \dots, v_H$ every x_i has at most k neighbours amongst $x_1, \dots, x_{i-1}$. Since $d_1(H[\{x_1, x_2\}]) \leq 2 \leq k$, it follows that $d_1(H) \leq k$. □

It seems likely that our use of Theorem 5.1 is not essential and our arguments can be extended to avoid its use. Indeed, first note that we only use Theorem 5.1 to prove Lemmas 4.1 and 4.3. As mentioned earlier, instead of Lemma 4.1, we only need an assertion which guarantees a linear number of disjoint absorbers. Such an assertion can be deduced from Lemma 6.1 and a 'non-partite' version of Lemma 7.1. Moreover, instead of the factor covering all vertices of S'' guaranteed by Lemma 4.3, one can use this version repeatedly to cover almost all the vertices of S''. The strategy would then be to use Lemmas 4.2 and 7.1 to cover the remaining vertices of S'' by powers of paths which are also allowed to use some vertices in A. But relying on Theorem 5.1 makes these steps unnecessary.

6. Finding a factor of absorbers: Proof of Lemma 4.1

The aim of this section is to show that there are integers $j \geq 3$ and $\ell \geq 2k$ such that the 1-density of any subgraph of $A_{j,\ell,k}$ is not much larger than k (see Lemma 6.1 below). Together with Theorem 5.1 this immediately implies Lemma 4.1.

Lemma 6.1. *For every $k \geq 2$ and every $\delta > 0$, there exist integers $j \geq 3$ and $\ell_0 \geq 2k$ such that whenever $\ell \geq \ell_0$ every subgraph H of $A_{j,\ell,k}$ satisfies $d_1(H) \leq k + \delta$.*

Proof. Choose $j \geq k/\delta + 3$ and $\ell_0 \geq 2jk^4/\delta$. Pick $\ell \geq \ell_0$ and let P and P' be the spine and the augmented path of $A_{j,\ell,k}$. So $A_{j,\ell,k} = P^k \cup (P')^k$. Consider any subgraph H of $A_{j,\ell,k}$ on $v_H \geq 2$ vertices. Let $H^* := H \cap P^k$. We will distinguish the following two cases. Roughly speaking, in the first case H^* 'spans' a large interval of P^k, in which case we can easily deduce that $d_1(H)$ is at most $k + \delta$.

Case 1. *There is a component C of H^* satisfying one of the following properties:*

- $V(C) \cap (\mathrm{Acc}(a_{i,1}) \cup \mathrm{Acc}(a_{i,2})) \neq \emptyset$ *and* $V(C) \cap (\mathrm{Acc}(b_{i,1}) \cup \mathrm{Acc}(b_{i,2})) \neq \emptyset$ *for some* $i \leq j$.
- $V(C) \cap (\mathrm{Acc}(b_{i,1}) \cup \mathrm{Acc}(b_{i,2})) \neq \emptyset$ *and* $V(C) \cap (\mathrm{Acc}(a_{i+1,1}) \cup \mathrm{Acc}(a_{i+1,2})) \neq \emptyset$ *for some* $i < j$.

We assume that the first property holds. The argument for the second property is similar. Note that the distance between $a_{i,2}$ and $b_{i,1}$ on P is $\ell+1$ and so the distance between $\mathrm{Acc}(a_{i,1}) \cup \mathrm{Acc}(a_{i,2})$ and $\mathrm{Acc}(b_{i,1}) \cup \mathrm{Acc}(b_{i,2})$ on P is $\ell + 1 - 2(k-1)$. Thus $|C| \geq (\ell - 2k)/k = \ell/k - 2$. Moreover, Proposition 5.2 implies that $d_1(H^*) \leq k$. Thus

$$\begin{aligned} d_1(H) &= \frac{e_H}{v_H - 1} = \frac{e_{H^*}}{v_H - 1} + \frac{e_{H \setminus E(H^*)}}{v_H - 1} \leq \frac{e_{H^*}}{v_H - 1} + \frac{e(A_{j,\ell,k}) - e(P^k)}{v_H - 1} \\ &\overset{(3.1)}{\leq} k + \frac{2j\binom{k+1}{2} + k}{\ell/k - 3} \leq k + \frac{2jk^4}{\ell} \leq k + \delta, \end{aligned}$$

as required.

Case 2. *There is no component of H^* as in Case 1.*

Let H' be the spanning subgraph of H whose edge set is $E(H) \setminus E(H^*)$. So every edge of H' lies in $E((P')^k) \setminus E(P^k)$. Our first aim is to choose a suitable orientation of the edges of H. If $xy \in E(H^*)$ we orient xy towards y if and only if y succeeds x on P. Recall from (A3) in Section 3 that the subgraph D of $A_{j,\ell,k}$ consisting of all junction edges, of the path $a_{1,1}vb_{1,2}$ and of all the edges $a_{i,1}a_{i,2}$, $b_{i,1}b_{i,2}$ (for all $i = 1, \dots, j$) is a cycle. In order to orient the edges in $E(H') = E(H) \setminus E(H^*)$, we will use an orientation of this cycle D, which we will now choose. Orient $a_{1,1}v$ towards v and $vb_{1,2}$ towards $b_{1,2}$. Since D contains the path $a_{1,1}vb_{1,2}$ we can orient all edges of D to obtain a directed cycle. We now use this orientation of D in order to orient the edges in $E(H')$ as follows. Recall from Section 3 that every edge in $E(A_{j,\ell,k}) \setminus E(P^k) \supseteq E(H')$ is either associated with a unique junction edge or with the absorbtion vertex v of $A_{j,\ell,k}$. For every edge $xy \in E(H')$ which is associated with some junction edge ab, orient xy towards y if and only if $x \in \mathrm{Acc}(a)$ and $y \in \mathrm{Acc}(b)$, where ab is oriented towards b (in the orientation of D). Finally, for every edge $xy \in E(H')$ which is associated with v, orient xy towards y if and only if $x \in \mathrm{Acc}(a_{1,1}) \cup \{v\}$ and $y \in \mathrm{Acc}(b_{1,2}) \cup \{v\}$.

Note that for every $i = 2, \dots, j$, one of the junctions $a_{i,1}, a_{i,2}$ sends out a junction edge while the other junction receives a junction edge (in the orientation of D). Let $a(+,i)$ denote the former junction and let $a(-,i)$ denote the latter one. Similarly, for every $i = 2, \dots, j$ one of the junctions $b_{i,1}, b_{i,2}$ sends out a junction edge while the other junction receives a junction edge. Let $b(+,i)$ denote the former junction and let $b(-,i)$ denote the latter one. Let $a(+,1) := a_{1,1}$, $a(-,1) := a_{1,2}$, $b(+,1) := b_{1,1}$ and $b(-,1) := b_{1,2}$. Then the following property holds for all $i = 1,\dots,j$:

No vertex in $\mathrm{Acc}(a(-,i))$ *sends out an edge in* H' *while no vertex in* $\mathrm{Acc}(a(+,i))$ *receives an edge in* H'*. Similarly, no vertex in* $\mathrm{Acc}(b(-,i))$ *sends out an edge in* H' *while no vertex in* $\mathrm{Acc}(b(+,i))$ *receives an edge in* H'*.* $(*)$

For each $i = 1,\dots,j$, let $C(i,a)$ denote the union of all components of H^* which intersect $\mathrm{Acc}(a_{i,1}) \cup \mathrm{Acc}(a_{i,2})$ and let $C(i,b)$ denote the union of all components of H^* which intersect $\mathrm{Acc}(b_{i,1}) \cup \mathrm{Acc}(b_{i,2})$. (Some of the $C(i,a)$ and $C(i,b)$ might be empty.) Let C^* denote the union of all components of H^* which do not intersect any of $\mathrm{Acc}(a_{i,i'})$ or $\mathrm{Acc}(b_{i,i'})$ for $i' = 1,2$ and $i = 1,\dots,j$. Our assumption of Case 2 implies that the vertex sets of graphs $C(1,a),\dots,C(j,a),C(1,b),\dots,C(j,b),C^*$ form a partition of $V(H^*) = V(H) \setminus \{v\}$.

Consider the vertices of C^* in their order on P. In the graph H^* each of these vertices sends out at most k edges (in our chosen orientation). However, the last vertex of C^* does not send out any edges in H^*. Thus if $C^* \neq \emptyset$ then

$$e(C^*) \leq k|C^*| - k. \tag{6.1}$$

Note also that none of the vertices in C^* are incident to any edges of H', so (6.1) bounds the number of all edges of H incident to vertices of C^*.

Let $r(i,a) := \min\{|C(i,a)|, k\}$. Consider the vertices of $C(i,a)$ in their order on P. In the graph H^* each of these vertices sends out at most k edges (in our chosen orientation). However, the last vertex of $C(i,a)$ does not send out any edges in H^*. More generally, for each $r = 0,\dots,r(i,a)-1$ the vertex of $C(i,a)$ at position $|C(i,a)| - r$ sends out at most r edges in H^*. Thus if $C(i,a) \neq \emptyset$ then

$$e(C(i,a)) \leq k|C(i,a)| - (k + (k-1) + \dots + (k - r(i,a) + 1)). \tag{6.2}$$

Let us now count the number of edges in H' sent out by vertices of $C(i,a)$. $(*)$ implies that no vertex in $C(i,a) - \mathrm{Acc}(a(+,i))$ sends out an edge in H'. But $a(+,i)$ sends out at most k edges in H'. More generally, for each $r = 0,\dots,r(i,a)-1$ the unique vertex x in $\mathrm{Acc}(a(+,i))$ which has distance r from $a(+,i)$ on P is incident to $k-r$ edges in $E(A_{j,\ell,k}) \setminus E(P^k) \supseteq E(H')$. So x sends out at most $k-r$ edges in H' . (Note that some of these $r(i,a)$ vertices x of P might not lie in $C(i,a)$.) Thus we have the following property:

Altogether the vertices in $C(i,a)$ *send out at most* $k + (k-1) + \dots + (k - r(i,a) + 1)$ *edges lying in the graph* H'*.* $(**)$

Clearly, the analogues of (6.2) and $(**)$ also hold for the $C(i,b)$. Moreover, the absorbtion vertex v of $A_{j,\ell,k}$ sends out at most k edges in H. Let $I^* := 1$ if $C^* \neq \emptyset$ and $I^* := 0$ otherwise. Altogether the above shows that

$$e_H \leq kv_H - kI^*. \tag{6.3}$$

We now distinguish three subcases.

Case 2a. *$C^* \neq \emptyset$.*

In this case we have

$$d_1(H) \stackrel{(6.3)}{\leq} \frac{kv_H - k}{v_H - 1} = k,$$

as required.

Case 2b. *H contains at least one edge associated with v as well as at least one edge associated with every junction edge ab.*

In this case we have that $v_H \geq 2j$ since there are $2j-1$ junction edges. Thus

$$d_1(H) \stackrel{(6.3)}{\leq} \frac{kv_H}{v_H - 1} = k + \frac{k}{v_H - 1} \leq k + \frac{k}{2j-1} \leq k + \delta,$$

as required.

Case 2c. *$C^* = \emptyset$. Moreover, H avoids all edges associated with v or there exists a junction edge ab such that H avoids all edges associated with ab.*

We will first show that in this case at least one of the following four properties hold:

(a) There is an i with $2 \leq i \leq j$ such that $C(i,a) \neq \emptyset$ and H avoids all edges associated with the junction edge sent out by $a(+,i)$.
(b) There is an i with $1 \leq i \leq j$ such that $C(i,b) \neq \emptyset$ and H avoids all edges associated with the junction edge sent out by $b(+,i)$.
(c) $C(1,a) \neq \emptyset$ and H avoids all edges associated with v.
(d) $C(1,b) = \emptyset$ and $v \in V(H)$.

To prove that one of (a)–(d) holds, let D' be the cycle obtained from D by replacing the path $a_{1,1}vb_{1,2}$ with a single edge e_v from $a_{1,1}$ to $b_{1,2}$ and contracting each edge $a_{i,1}a_{i,2}$ into a new vertex (i,a) as well as contracting each edge $b_{i,1}b_{i,2}$ into a new vertex (i,b) (for all $i = 1,\dots,j$). Thus every edge of D' apart from e_v corresponds to a junction edge. Moreover, our orientation of D induces one of D'. So we will view D' as a directed cycle. Colour the vertex (i,a) of D' red if $C(i,a) \neq \emptyset$ and colour (i,b) red if $C(i,b) \neq \emptyset$. Colour the edge e_v of D' red if H contains some edge associated with v. Colour each (junction) edge $e \neq e_v$ of D' red if H contains some edge associated with e. Since $v_H \geq 2$ and since we are assuming that $C^* = \emptyset$, it follows that at least one vertex of D' is red. Moreover, our assumption that Case 2c holds implies that not all edges of D' are red.

Let us first consider the case when $v \notin V(H)$. Then both endvertices of a red edge of D' are red. Thus D' contains a red vertex w such that the edge from w to its successor on D' is not red. If $w = (i,a)$ for some $i > 1$ then (a) holds. If $w = (i,b)$ for some $i \geq 1$ then (b) holds. If $w = (1,a)$ then (c) holds. So suppose next that $v \in V(H)$. In this case we can only guarantee that both endvertices of a red edge $e \neq e_v$ of D' are red. We may also assume that $(1,b)$ is red (otherwise (d) holds). If not all edges in $D' - e_v$ are red then D' contains a red vertex $w \neq (1,a)$ such that the edge from w to its successor on D' is not red. Similarly as before this implies that (a) or (b) holds. So suppose that all edges in $D' - e_v$ are red.

This implies that $(1, a)$ is red and e_v is not red. Thus (c) holds. This completes the proof that one of (a)–(d) holds.

Suppose first that (a) holds. Then the vertices in $C(i, a)$ send out no edges lying in the graph H'. Together with (6.2) this implies that instead of (6.3) we have that

$$e_H \leq kv_H - (k + (k-1) + \cdots + (k - r(i, a) + 1)) \leq kv_H - k$$

and so $d_1(H) \leq k$ as required. The arguments for (b) and (c) are similar. So let us now assume that (d) holds. Then v does not send out any edges in the graph H. So instead of (6.3) we have $e_H \leq kv_H - k$ and so $d_1(H) \leq k$ as required. □

7. Linking up kth powers of paths: Proof of Lemma 4.2

A result of Kreuter [15] determines the threshold for the existence of a linear number of disjoint copies of a given graph Q in a random graph $G_{n,p}$. (This threshold is roughly the same as the one in Theorem 5.1.) We will prove an analogue of this result for partite multigraphs (see Lemma 7.1). We will then apply Lemma 7.1 to find disjoint copies of a partite multigraph Q, where each copy of Q_i of Q will correspond to an (A_i, B_i)-linkage R_i (see Lemma 7.5). This allows us to link up a positive fraction of the pairs (A_i, B_i) we are required to link up. Roughly speaking, in the proof of Lemma 4.2 we will apply Lemma 7.5 repeatedly to eventually obtain disjoint linkages for all the pairs (A_i, B_i) that we are required to link up.

We write $[k] := \{1, \dots, k\}$ and $[-k] := \{-1, \dots, -k\}$. Suppose that $p = p(n)$. We define the random graph $\mathcal{G} = \mathcal{G}(n_0, n, t, k, p)$ as follows: Consider the complete $(t+1)$-partite multigraph K with vertex classes $V_1, \dots, V_t$ of size n and one vertex class V_0 of size n_0, and where each edge from V_0 to V_i has multiplicity exactly $2k$ (for all $i = 1, \dots, t$) and all other edges have multiplicity one. Moreover, for all pairs of vertices $x \in V_0$ and $y \in V_1 \cup \cdots \cup V_t$ the $2k$ edges between x and y in K have labels in $[-k] \cup [k]$ and these labels are distinct for different edges between x and y. We obtain $\mathcal{G}$ by including each edge of K into $\mathcal{G}$ with probability p, independently of all other edges.

Let G be any $(t+1)$-partite multigraph with vertex classes $Y_0, \dots, Y_t$ such that $|Y_0| \leq n_0$ and $|Y_i| \leq n$ for all $i = 1, \dots, t$, and where each edge between Y_0 and Y_i has multiplicity at most $2k$ (for all $i = 1, \dots, t$) and all other edges have multiplicity one. Moreover, for all pairs of vertices $a \in Y_0$ and $b \in Y_1 \cup \cdots \cup Y_t$ the edges between a and b in G have labels in $[-k] \cup [k]$ and these labels are distinct for different edges between a and b.

We say that a (not necessarily induced) copy of G in K is a *good copy of* G if for all $i = 0, \dots, t$ each vertex in Y_i is mapped to a vertex in V_i and if each edge of G with label j between some pair $a \in Y_0$ and $b \in Y_1 \cup \cdots \cup Y_t$ of vertices is mapped to the edge of K with label j between the images of a and b in K. Let X_G denote the number of good copies of G in $\mathcal{G}$. Let D_G denote the maximum size of a set of disjoint good copies of G in $\mathcal{G}$. Set

$$\Phi_G := \min\{\mathbb{E}(X_{G'}) \colon G' \subseteq G, e_{G'} > 0\}$$

and

$$\Phi_G^v := \min\{\mathbb{E}(X_{G'})\colon G' \subseteq G, v_{G'} > 0\}.$$

Note that we allow G' to consist of a single vertex in the second definition. Also note that $\Phi_G^v \leq \Phi_G$.

Throughout this section, when using the $O(.)$, $\Theta(.)$ and $\Omega(.)$ notation, we mean that the size n of the vertex classes $V_1, \dots, V_t$ tends to infinity. In most cases n_0 will be a function of n, but we sometimes also allow $n_0 = 1$. The number of vertices in the graph G will always be bounded.

Lemma 7.1. *Suppose that $\Phi_G^v \to \infty$ as $n \to \infty$. Then there is a constant $c > 0$ (depending only on G) such that with probability $1 - O(1/\Phi_G^v)$ we have $D_G \geq c\Phi_G^v$.*

The proof of Lemma 7.1 is essentially the same as that of Theorem 3.29 in [10], which in turn is based on an argument of Kreuter [15]. The difference is that in Theorem 3.29 X_G counts disjoint copies of G in $G_{n,p}$ (rather than good copies of G in $\mathcal{G}$). For completeness, we will give a sketch which only highlights the (very minor) adjustments one has to make. The proof of Lemma 7.1 needs the following proposition, which is proved using a standard application of Chebyshev's inequality (see Lemma 3.5 and Remark 3.7 in [10] for a similar and more detailed argument). Note that Proposition 7.2 does not assume any bounds on n_0. In particular, we will later also apply it in the case when $n_0 = 1$.

Proposition 7.2.

(i) $Var(X_G) = O\left(\mathbb{E}(X_G)^2/\Phi_G\right)$.

(ii) *Suppose that $\Phi_G \to \infty$ as $n \to \infty$ and that $\varepsilon > 0$ is fixed. Then with probability $1 - O(1/\Phi_G)$ we have $X_G = (1 \pm \varepsilon)\mathbb{E}(X_G)$.*

In the next two proofs we will use the following notation: Suppose that H is a $(t+1)$-partite multigraph on a bounded number of vertices with vertex classes $Y_0, \dots Y_t$ such that $|Y_0| \leq n_0$. Then we define

$$\Psi_H := p^{e_H} n_0^{|Y_0|} n^{v_H - |Y_0|}. \tag{7.1}$$

Note that $\mathbb{E}(X_H) = \Theta(\Psi_H)$.

Proof of Proposition 7.2. Given a good copy G' of G in K, let $I_{G'}$ denote the indicator function that G' is contained in $\mathcal{G}$. Below, the summations are always over good copies of the relevant graphs in K. With the above notation, we have

$$\begin{aligned}
\mathbb{E}(X_G^2) = \sum_{G',G''} \mathbb{E}(I_{G'}I_{G''}) &\leq \mathbb{E}(X_G)^2 + \sum_{E(G')\cap E(G'')\neq\emptyset} \mathbb{E}(I_{G'}I_{G''}) \\
&= \mathbb{E}(X_G)^2 + O\left(\sum_{H\subseteq G, e_H>0} \frac{\Psi_G^2}{\Psi_H}\right) \\
&= \mathbb{E}(X_G)^2 + O\left(\sum_{H\subseteq G, e_H>0} \frac{\mathbb{E}(X_G)^2}{\mathbb{E}(X_H)}\right) \\
&= \mathbb{E}(X_G)^2 + O\left(\mathbb{E}(X_G)^2/\Phi_G\right).
\end{aligned}$$

So $Var(X_G) = O(\mathbb{E}(X_G)^2/\Phi_G)$. This proves (i). A straightforward application of Chebyshev's inequality now completes the proof of (ii). $\square$

Proof of Lemma 7.1. The proof begins by considering an auxiliary graph Γ, where the vertices of Γ correspond to good copies of G in $\mathcal{G}$ (rather than to copies in $G_{n,p}$ as in the proof of Theorem 3.29 in [10]), with an edge between two vertices of Γ if the corresponding copies of G share at least one vertex. So Γ has X_G vertices and $\sum_F X_F$ edges, where the sum is taken over all unions $F = G_1 \cup G_2$ of two copies of G sharing at least one vertex, and where F is viewed as a $(t+1)$-partite multigraph whose ith vertex class Y_i^F is the union of the ith vertex classes of G_1 and G_2 (but we include any vertex in $G_1 \cap G_2$ only once). Since $X_F = 0$ if $|Y_i^F| > n_0$, we only sum over all those F for which $|Y_0^F| \leq n_0$.

Note that any independent set of vertices in Γ corresponds to a collection of pairwise disjoint good copies of G in $\mathcal{G}$. So one can use Turán's theorem to show that

$$D_G \geq \frac{X_G^2}{X_G + 2\sum_F X_F}. \tag{7.2}$$

Proposition 7.2(ii) implies that with probability $1 - O(1/\Phi_G)$, we have $\mathbb{E}(X_G)/2 \leq X_G \leq 2\mathbb{E}(X_G)$. Together with (7.2) this implies that it suffices to show that with probability $1 - O(1/\Phi_G^v)$ we have

$$X_F = O\left(\frac{(\mathbb{E}X_G)^2}{\Phi_G^v}\right) = O\left(\frac{\Psi_G^2}{\Phi_G^v}\right), \tag{7.3}$$

where Ψ_G is as defined in (7.1). To prove (7.3), the first step is to observe that if $F = G_1 \cup G_2$ is as above and $H := G_1 \cap G_2$, then

$$\mathbb{E}(X_F) = \Theta(\Psi_F) = \Theta\left(\frac{\Psi_G^2}{\Psi_H}\right) = O\left(\frac{\Psi_G^2}{\Phi_G^v}\right). \tag{7.4}$$

Next, note that Proposition 7.2(i) implies that

$$Var(X_F) = O(\Psi_F^2/\Phi_F). \tag{7.5}$$

To bound this expression, we need the following log-supermodularity property, where H_1 and H_2 are arbitrary $(t+1)$-partite multigraphs. This property follows easily from the definition of Ψ_H (indeed, the overlap between H_1 and H_2 contributes twice to both the left and right hand side).

$$\Psi_{H_1 \cup H_2} \Psi_{H_1 \cap H_2} = \Psi_{H_1} \Psi_{H_2}.$$

Now one can proceed exactly as in the proof of Theorem 3.29: Using repeated applications of the log-supermodularity, one can show that the right hand side of (7.5) is $O(\Psi_G^4/(\Phi_G^v)^3)$. With this bound, Chebyshev's inequality now implies that

$$\mathbb{P}\left(X_F \geq \mathbb{E}(X_F) + \frac{\Psi_G^2}{\Phi_G^v}\right) \leq Var(X_F) \cdot \frac{(\Phi_G^v)^2}{\Psi_G^4} = O\left(1/\Phi_G^v\right).$$

Together with (7.4) this implies that (7.3) holds with the required probability. $\square$

We now apply the above results to find powers of paths. Let $k \geq 2$ and $s \geq 4k$. Recall that P_s^k denotes the kth power of a path $P_s = x_1 \dots x_s$ on s vertices. Let Q be the multigraph obtained from P_s^k by contracting $x_1, \dots, x_k, x_{s-(k-1)}, \dots, x_s$ into a single vertex x_0 and deleting any resulting loops at x_0 (but not removing any of the multiple edges). So Q is a multigraph on $t+1$ vertices, where $t := s-2k$ and where x_0 has degree $k(k+1)$ and all other vertices have degree $2k$. We view Q as a $(t+1)$-partite multigraph with vertex class $Y_0 := \{x_0\}$ and each other vertex class $Y_1, \dots, Y_t$ also consisting of a single vertex. Note that every edge of Q corresponds to a unique edge of P_s^k. We now assign each edge of Q at x_0 a label as follows: For all $i \in [k]$ and every $x \in Y_1 \cup \dots \cup Y_t$ we label an edge of Q between x_0 and x which corresponds to an edge of P_s^k between x_i and x with i. Similarly, for all $i \in [-k]$ and every $x \in Y_1 \cup \dots \cup Y_t$ we label an edge of Q between x_0 and x which corresponds to an edge of P_s^k between x_{s+1+i} and x with i. So for each $i \in [k]$ there are $k-i+1$ edges with labels $i, \dots, k$ between x_0 and x_{k+i}. Similarly, for each $i \in [-k]$ there are $k+1+i$ edges with labels $-k, \dots, i$ between x_0 and $x_{s-(k-1)+i}$.

Lemma 7.3. *Let $s > 8k^2$ and define Q as above. Suppose that $1 \leq n_0 \leq n$ and $p = p(n) \geq n^{-1/k+8k/s}$. Then*

(i) $\Phi_Q = \Omega(n^{8k^2/s})$;
(ii) $\Phi_Q^v = \Omega(n_0)$.

Proof. Note that both assertions follow if we can show that any submultigraph Q' of Q, which contains at least one edge, satisfies $\mathbb{E}(X_{Q'}) = \Omega(n^{8k^2/s} n_0)$. Let $v := v_{Q'}$ and $e := e_{Q'}$.

First suppose that $v \geq s/(2k)$. In this case, it suffices to note that at most one vertex of Q' has degree at most $k(k+1)$ and all others vertices of Q' have degree at most $2k$. Thus $e \leq kv + k^2$ with room to spare. So recalling that $n \geq n_0 \geq 1$, we have

$$\mathbb{E}(X_{Q'}) = \Omega\left(p^e n_0 n^{v-1}\right) = \Omega\left(n^{(v+k)(-1+8k^2/s)+v-1}\right).$$

But

$$(v+k)(-1+8k^2/s) + v - 1 \geq 8k^2 v/s - k - 1 \geq 2,$$

and so the required result follows in this case, with room to spare.

So we may assume that $v \leq s/(2k)$. Consider the ordering $x_0, x_{k+1}, x_{k+2}, \dots, x_{s-k}$ of the vertices of Q. The assumption on v implies that there are k consecutive vertices $x_a, \dots, x_{a+k-1}$ with $k < a \leq s-2k+1$ which are not contained in Q'. Write $x_{s+1} := x_0$. Now for each edge $x_i x_{i'}$ of Q' with $i < i'$ we either have $0 \leq i < i' < a$ or $a+k \leq i < i' \leq s+1$. In the first case, we orient $x_i x_{i'}$ towards x_i and in the second case we orient $x_i x_{i'}$ towards $x_{i'}$. Now it is easy to see that for every vertex x_i of Q', the outdegree of x_i in this orientation of Q' is at most k. Moreover, the above process yields an orientation of all edges of Q' and there is at least one vertex in Q' which has outdegree 0. (If Q' contains $x_0 = x_{s+1}$, then this will be one such vertex. If Q' is disconnected, there will be several such vertices.) Thus

$e \leq k(v-1)$. So using that $n \geq n_0$ and $v \geq 2$, we have

$$\mathbb{E}(X_{Q'}) = \Omega\left(p^e n_0 n^{v-1}\right) = \Omega\left(n_0 (p^k n)^{v-1}\right) = \Omega\left(n_0 p^k n\right) = \Omega\left(n_0 n^{8k^2/s}\right),$$

as required. □

We can now combine Lemma 7.1 and Lemma 7.3(ii) to obtain the following result.

Corollary 7.4. *Let $s > 8k^2$ and define Q as in Lemma 7.3. Suppose that $n_0 \leq n$, that $n_0 \to \infty$ and that $p = p(n) \geq n^{-1/k+8k/s}$. Then there is a constant $c > 0$ (depending only on Q) such that with probability $1 - O(1/n_0)$, we have $D_Q \geq cn_0$.*

Our aim is now to apply Corollary 7.4 to link up given sets of vertices in $G_{n,p}$ by powers of paths. Suppose that $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ are two (ordered) sequences of vertices which are disjoint from each other. Recall that a graph R is an (A, B)-linkage if R is obtained from the kth power of a path whose initial endsequence is A and whose final endsequence is B by deleting all edges within A and within B. We call $\mathcal{A} := \{(A_1, B_1), \dots, (A_f, B_f)\}$ a *set of pairwise disjoint k-sequence pairs* if each A_i and each B_i is a sequence of k vertices and all these $2f$ sequences are pairwise disjoint. A *partial $\mathcal{A}$-linkage of size f' and parameter s* consists of $\mathcal{R} = \{R_1, \dots, R_{f'}\}$ where

- for each $i = 1, \dots, f'$ there is a $j = j(i) \in [f]$ such that R_i is an (A_j, B_j)-linkage;
- the R_i are pairwise disjoint;
- if $j' \neq j(i)$, then R_i avoids $A_{j'} \cup B_{j'}$;
- $|R_i| = s$ for all $i = 1, \dots, f'$.

If $j' \neq j(i)$ for all $i = 1, \dots, f'$, we say that $(A_{j'}, B_{j'})$ *is unlinked by $\mathcal{R}$*.

Lemma 7.5. *For every $0 < \varepsilon < 1/k$ and every $k \geq 2$ there is a constant $c > 0$ such that the following holds: Suppose that $p = p(n) \geq n^{-1/k+\varepsilon}$ and that $\log^2 n \leq f \leq n/(4k)$. Let $\mathcal{A} = \{(A_1, B_1), \dots, (A_f, B_f)\}$ be a set of f pairwise disjoint k-sequence pairs. Then with probability $1 - O(1/\log^2 n)$, we have that $G_{n,p}$ contains a partial $\mathcal{A}$-linkage $\mathcal{R} = \{R_1, \dots, R_{f'}\}$ of size $f' := cf$ and parameter $\lceil 10k/\varepsilon \rceil$.*

Proof. Set $s := \lceil 10k/\varepsilon \rceil$. So each R_i will consist of s vertices (including those in the endsequences of R_i). Note that the number of vertices contained in some A_i or B_i is $2kf \leq n/2$. We will view $G_{n,p}$ as a subgraph of K_n. Let N' consist of $n/2$ vertices of K_n which are not contained in any of the A_i or B_i. Let $t := s - 2k$. Partition N' into t classes $V_1, \dots, V_t$ of equal size $n' := n/(2t)$. For all $j \in [k]$, let V'_j consist of the jth vertex in each of the A_i. So $|V'_j| = f$. For all $j \in [-k]$, let V'_j consist of the $(k+1+j)$th vertex in each of the B_i. Again $|V'_j| = f$. Let K' be the complete s-partite subgraph of K_n induced by the vertex classes $V_1, \dots, V_t$, and all the V'_j for $j \in [-k] \cup [k]$. Let K be the $(t+1)$-partite multigraph obtained from K' by contracting all the vertices in $A_i \cup B_i$ into a single vertex y_i, where any resulting loops at y_i are removed (but we do not remove any multiple edges). So the vertex classes of K are $V_0 := \{y_1, \dots, y_f\}$ and $V_1, \dots, V_t$. Note that each edge e of K corresponds to a unique edge e' of K'. We now label the edges of K as follows: For all $i \in [f]$ and all $j \in [-k] \cup [k]$ we label an edge e of K between y_i and some vertex $x \in V_1 \cup \dots \cup V_t$ with j if the corresponding edge e' of K' joins some vertex in V'_j to x.

Now define a random graph $\mathcal{G}$ as follows: $\mathcal{G}$ is a spanning subgraph of K, where we include an edge e of K into $\mathcal{G}$ if and only if the corresponding edge e' of K' is included in $G_{n,p}$. This means that each edge of K is included in $\mathcal{G}$ with probability p, independently of all other edges. So this corresponds exactly to the setting described at the beginning of the section, with n' playing the role of n and f playing the role of n_0.

Let Q be as defined before Lemma 7.3. Then a good copy of Q in K containing y_i corresponds to an (A_i, B_i)-linkage in K_n. (Thus a good copy of Q in $\mathcal{G}$ containing y_i corresponds to an (A_i, B_i)-linkage in $G_{n,p}$.) Similarly, a set of ℓ disjoint good copies of Q in K (or in $\mathcal{G}$) corresponds to a partial $\mathcal{A}$-linkage of size ℓ and parameter s in K_n (or in $G_{n,p}$).

Also note that $p(n) \geq n^{-1/k+\varepsilon} \geq (n')^{-1/k+4\varepsilon/5} \geq (n')^{-1/k+8k/s}$. So we can apply Corollary 7.4 with n' and f playing the roles of n and n_0 to see that, with with probability $1 - O(1/\log^2 n)$, $G_{n,p}$ contains a partial $\mathcal{A}$-linkage of parameter s and size cf, where c depends only on Q (and thus only on k and ε). □

A simple consequence of the previous arguments is that we can link up a given sequence A of k vertices to a given sequence B of k vertices via the kth power of a sufficiently long path.

Lemma 7.6. *Let $0 < \varepsilon < 1/k$ and $k \geq 2$. Suppose that $p \geq n^{-1/k+\varepsilon}$ and that $A = (a_1 \dots a_k)$ and $B = (b_1 \dots b_k)$ are pairwise disjoint sequences of vertices. Then with probability $1 - O(1/\log^3 n)$, $G_{n,p}$ contains an (A,B)-linkage R with $|R| = \lceil 10k/\varepsilon \rceil$.*

Proof. Let $\mathcal{A} := \{(A,B)\}$ and $s := \lceil 10k/\varepsilon \rceil$. We now define K', K, $\mathcal{G}$ and Q exactly as in the proof of Lemma 7.5. In particular, for all $j \in [k]$, let V'_j consist of the jth vertex in A. For all $j \in [-k]$, let V'_j consist of the $(k+1+j)$th vertex in B. So V_0 consists of a single vertex y and $n_0 = 1$. Again, a good copy of Q in K containing y corresponds to an (A,B)-linkage in K_n (with a similar correspondence between $\mathcal{G}$ and $G_{n,p}$). Moreover, again we have $p(n) \geq (n')^{-1/k+8k/s}$, where $n' := n/(2t)$ and $t := s - 2k$.

Now Lemma 7.3(i) together with Proposition 7.2(ii) imply that with probability $1 - O(n^{-8k^2/s})$, we have $X_Q > 0$. So the error bound is at most $O(1/\log^3 n)$ (with room to spare), as required. □

We can now combine Lemmas 7.5 and 7.6 in order to prove Lemma 4.2.

Proof of Lemma 4.2. Since $p = p(n) = o(1)$, we can view $G_{n,p}$ as a union of $2\log^2 n$ independent random graphs G_{n,p_i}, with $p_i = p'$, where $p' \geq (1 + o(1))p/(2\log^2 n) \geq n^{-1/k+\varepsilon/2}$. Let $s := \lceil 30k/\varepsilon \rceil$ and $\mathcal{A} := \{(A_1, B_1), \dots, (A_f, B_f)\}$. Our strategy is to first apply Lemma 7.5 repeatedly to obtain partial linkages until the number of unlinked pairs in $\mathcal{A}$ is less than $\log^2 n$ (using a different G_{n,p_i} each time). We will then apply Lemma 7.6 repeatedly in order to link the remaining pairs in $\mathcal{A}$ one by one (again, using a different G_{n,p_i} each time).

Let $c = c(k,\varepsilon)$ be as in Lemma 7.5 and let $\mathcal{A}_0 := \mathcal{A}$. Suppose that we have obtained a set $\mathcal{A}_i$ consisting of $(1-c)^i f$ unlinked pairs from $\mathcal{A}$ and that we have found a partial $\mathcal{A}$-linkage $\mathcal{R}_i$ with parameter s which links precisely all the pairs in $\mathcal{A}\setminus\mathcal{A}_i$. Let N_i be obtained from $[n]$ by deleting all the vertices in linkages from $\mathcal{R}_i$. Thus $|N_i| = n - (|\mathcal{A}| - |\mathcal{A}_i|)s \geq n - fs \geq n/2$ and so $p_i = p' \geq n^{-1/k+\varepsilon/2} \geq |N_i|^{-1/k+\varepsilon/3}$. Hence if $|\mathcal{A}_i| = (1-c)^i f > \log^2 n$, we can apply Lemma 7.5 with $\varepsilon/3$ playing the role of ε and with the random subgraph of G_{n,p_i} induced by the set N_i playing the role of $G_{n,p}$. With probability $1 - O(1/\log^2 n)$ this yields a partial linkage $\mathcal{R}'_i$ of size $c|\mathcal{A}_i|$ and parameter s. Let $\mathcal{R}_{i+1} := \mathcal{R}_i \cup \mathcal{R}'_i$ and let $\mathcal{A}_{i+1}$ denote the set of pairs which are still unlinked. So $|\mathcal{A}_{i+1}| = (1-c)^{i+1} f$.

Let $i^* \geq 0$ be the smallest integer for which $(1-c)^{i^*} f \leq \log^2 n$. Thus $i^* \leq \log_{1/(1-c)} n$. Our argument shows that with probability at least $1 - O(i^*/\log^2 n)$ we can find a partial linkage $\mathcal{R}_{i^*}$ of parameter s such that the set $\mathcal{A}_{i^*}$ of unlinked pairs has size $|\mathcal{A}_{i^*}| = (1-c)^{i^*} f$.

We will now link up the remaining pairs one by one. For this, write $\mathcal{A}_{i^*} = \{(A_1^*, B_1^*), \dots, (A_{f^*}^*, B_{f^*}^*)\}$. Thus $f^* \leq \log^2 n$. Let N_{i^*} be obtained from $[n]$ by deleting all the vertices in linkages from $\mathcal{R}_{i^*}$. Suppose that $1 \leq j \leq f^*$ and that we have obtained an (A_i^*, B_i^*)-linkage R_i^* for all $i = 1, \dots, j-1$ such that all the R_i^* are pairwise disjoint, $|R_i^*| = s$, $V(R_i^*) \subseteq N_{i^*}$ and such that R_i^* avoids $(A_{i'}^*, B_{i'}^*)$ for all $i' \neq i$. Let

$$N_j^* := \left(N_{i^*} \setminus \left(V(R_1^*) \cup \dots \cup V(R_{j-1}^*) \cup \bigcup_{i=1}^{f^*} (A_i^* \cup B_i^*) \right) \right) \cup A_j^* \cup B_j^*.$$

Thus $|N_j^*| \geq n - fs \geq n/2$ and so $p' \geq n^{-1/k+\varepsilon/2} \geq |N_j^*|^{-1/k+\varepsilon/3}$. Hence we can apply Lemma 7.6 with $\varepsilon/3$ playing the role of ε and with the random subgraph of $G_{n,p_{i^*+j}}$ induced by the set N_j^* playing the role of $G_{n,p}$. With probability $1 - O(1/\log^3 n)$ this yields a (A_j^*, B_j^*)-linkage R_j^* with $|R_j^*| = s$ and $V(R_j^*) \subseteq N_j^*$.

Since $f^* \leq \log^2 n$, this means that altogether, a.a.s. we can find pairwise disjoint (A_i^*, B_i^*)-linkages for all $i = 1, \dots, f^*$ which only use vertices in N_{i^*} and so are disjoint from the linkages in $\mathcal{R}_{i^*}$. $\square$

8. Deriving Theorem 1.1

Given a graph H on at least three vertices, we define

$$d_2(H) := \frac{e_H}{v_H - 2} \quad \text{and} \quad d_2^{\max}(H) := \max_{H' \subseteq H,\ v_{H'} \geq 3} d_2(H').$$

The purpose of this section is to derive Theorem 1.1 from the following result of Riordan [20]. Actually the result in [20] is more general than the version below, as its formulation in [20] does not require the maximum degree of the H_n to be bounded. Moreover, it is stated for $G_{n,m}$ with $m = p\binom{n}{2}$ instead of $G_{n,p}$. But Theorem 2.2(ii) of [3] allows us to apply it to $G_{n,p}$.

Theorem 8.1. *Let $(H_n)_{n=1}^{\infty}$ be a fixed sequence of graphs such that $n = v_{H_n}$, $e_{H_n} \geq n$ and such that the maximum degree of the H_n is bounded. Let $p = p(n)$ be such that*

$$np^{d_2^{\max}(H)} \to \infty, \quad pn^2 \to \infty \quad \text{and} \quad (1-p)\sqrt{n} \to \infty. \tag{8.1}$$

Then a.a.s. $G_{n,p}$ contains a copy of H_n.

Thus in order to derive Theorem 1.1 from this, it suffices to prove the following proposition. Note that the third condition in (8.1) does not hold if p is very close to 1. But since the property of containing the kth power of a Hamilton cycle is monotonically non-decreasing under the addition of edges, this case follows immediately from the fact that in our case there is some p satisfying all three conditions in (8.1).

Proposition 8.2. *Suppose that $n \geq 4k$ and $k \geq 3$. Then $d_2^{\max}(C_n^k) \leq k + \frac{(k+1)k^2}{n}$. Moreover, $d_2^{\max}(C_n^2) = 3$ if $n \geq 18$.*

Proof. Let us first consider the case when $k \geq 3$. Consider any $H \subseteq C_n^k$ on $v_H \geq 3$ vertices. Suppose first that $H \subseteq P_n^k$. Thus there is an ordering $x_1, \dots, x_{v_H}$ of the vertices of H such that for all $i = 2, \dots, v_H$ every x_i has at most k neighbours amongst $x_1, \dots, x_{i-1}$. Since $d_2(H[\{x_1, x_2, x_3\}]) \leq 3 \leq k$, it follows that $d_2(H) \leq k$. Now suppose that $H \not\subseteq P_n^k$. Then $v_H \geq n/k$ and by deleting at most $\binom{k+1}{2}$ edges from H one can obtain a subgraph H' with $H' \subseteq P_n^k$. Thus

$$d_2(H) \leq d_2(H') + \frac{\binom{k+1}{2}}{v_H - 2} \leq d_2(H') + \frac{\binom{k+1}{2}}{n/k - 2} \leq k + \frac{(k+1)k^2}{n}$$

since $n \geq 4k$. A similar argument shows that $d_2^{\max}(C_n^2) = 3$ if $n \geq 18$. □

9. Acknowledgements

We are extremely grateful to Nikolaos Fountoulakis for helpful discussions throughout the project and comments on the manuscript. We are also indebted to the referees for pointing out an error in an earlier version.

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