Multicoloured Hamilton cycles and perfect matchings in pseudo-random graphs

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Abstract

Given $0 < p < 1$, we prove that a pseudo-random graph G with edge density p and sufficiently large order has the following property: Consider any red/blue-colouring of the edges of G and let r denote the proportion of edges which have colour red. Then there is a Hamilton cycle C so that the proportion of red edges of C is close to r. The analogue also holds for perfect matchings instead of Hamilton cycles. We also prove a bipartite version which is used elsewhere to give a minimum-degree condition for the existence of a Hamilton cycle in a 3-uniform hypergraph.

1 Introduction

1.1 Overview

It is well known that random graphs, pseudo-random graphs and ε-superregular graphs have some strong Hamiltonicity properties in common. For instance, a recent result of Frieze and Krivelevich [10] states that, for every constant $0 < p < 1$, with high probability almost all edges of the random graph $G_{n,p}$ can be packed into edge-disjoint Hamilton cycles. (They derive this from a similar result about ε-superregular graphs.)

Hamiltonicity has also been investigated from the viewpoint of (anti-)Ramsey theory. For example, Albert, Frieze and Reed [1] proved that there is a linear function $k = k(n)$ such that for every edge-colouring of the complete graph K_n on n vertices which uses each colour at most k times there is a Hamilton cycle where each edge has a different colour. This improves bounds by previous authors. A related problem for random graphs was also considered by Cooper and Frieze [6].

Here, we prove a related result about colourings of bipartite ε-superregular graphs (which will imply analogous statements for pseudo-random and random graphs). Roughly speaking, we prove that given a k-colouring of a sufficiently large ε-superregular graph G (where ε is sufficiently small) there is a Hamilton cycle C in G which is strongly multicoloured (or well balanced) in the following sense: for all colours i, the proportion of edges in C of colour i is close to the proportion of edges in G which have colour i. We derive this from a related result about random perfect matchings (Theorem 1.1) which is also a crucial tool in [12], see Section 1.3.

This paper is organized as follows. In Sections 2 and 3.1 we collect some tools which we will need in our proofs. In Section 3.2 we then use these tools to deduce some simple properties of random perfect matchings in ε-superregular graphs. The core result of this paper is Lemma 3.8 in Section 3.3, which proves Theorem 1.1 for special graphs H. In the final section, the remaining results in this paper are easily deduced from Lemma 3.8 and the results in Section 3.2.

1.2 Statement of results

Given a bipartite graph $G = (A, B)$ with vertex classes A and B, we denote the edge set of G by $E(A, B)$ and let $e(G) = e(A, B) = |E(A, B)|$. The *density* of a bipartite graph $G = (A, B)$ is defined to be

$$d(A, B) := \frac{e(A, B)}{|A||B|}.$$

Given $0 < \varepsilon < 1$ and $d \in [0, 1]$, we say that G is (d, ε)*-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $(1 - \varepsilon)d < d(X, Y) < (1 + \varepsilon)d$. We say that G is (d, ε)*-superregular* if it is (d, ε)-regular and, furthermore, if $(1-\varepsilon)d|B| < d_G(a) < (1+\varepsilon)d|B|$ for all vertices $a \in A$ and $(1 - \varepsilon)d|A| < d_G(b) < (1 + \varepsilon)d|A|$ for all $b \in B$. This is more or less equivalent to the traditional notions of ε-regularity and ε-superregularity—see Section 2.

Theorem 1.1 *For all positive constants $d, \nu_0, \eta \leq 1$ there is a positive $\varepsilon = \varepsilon(d, \nu_0, \eta)$ and an integer $N_0 = N_0(d, \nu_0, \eta)$ such that the following holds for all $n \geq N_0$ and all $\nu \geq \nu_0$. Let $G = (A, B)$ be a (d, ε)-superregular bipartite graph whose vertex classes both have size n and let H be a subgraph of G with $e(H) = \nu e(G)$. Choose a perfect matching M uniformly at random in G. Then with probability at least $1 - \mathrm{e}^{-\varepsilon n}$ we have*

$$(1 - \eta)\nu n \leq |M \cap E(H)| \leq (1 + \eta)\nu n.$$

At first sight it may seem surprising that the only parameter of H that is relevant here is the number of its edges. However, this is quite natural in view of the fact that the assertion would be trivial if instead of a perfect matching one would choose n edges independently and uniformly at random.

The case when H is a sufficiently large induced subgraph of G was proved earlier by Rödl and Ruciński [13] as a tool in their alternative proof of the Blow-up Lemma of Komlós, Sárközy and Szemerédi.

From Theorem 1.1 we will also deduce a (weaker) analogue for Hamilton cycles:

Theorem 1.2 *For all integers k and all positive constants $d, \nu, \eta \leq 1$ there is a positive $\varepsilon = \varepsilon(d, \nu, \eta)$ and an integer $N_1 = N_1(k, d, \nu, \eta)$ such that the following holds for all $n \geq N_1$. Let $G = (A, B)$ be a (d, ε)-superregular bipartite graph whose vertex classes both have size n. For each $1 \leq i \leq k$ let H_i be a subgraph of G with $e(H_i) = \nu_i e(G)$, where $\nu_i \geq \nu$. Then G contains a Hamilton cycle C such that for all $1 \leq i \leq k$*

$$(1 - \eta)2\nu_i n \leq |C \cap E(H_i)| \leq (1 + \eta)2\nu_i n.$$

Theorems 1.1 and 1.2 can in turn be used to deduce analogues for non-bipartite graphs (see the final section for details). For this, we need to modify the notion of (d, ε)-superregularity as follows. Given $0 < \varepsilon < 1$ and $d \in [0, 1]$, we say that a graph G with n vertices is *(d, ε)-regular* if for all disjoint sets $X, Y \subseteq V(G)$ with $|X|, |Y| \geq \varepsilon n$ we have $(1 - \varepsilon)d < d(X, Y) < (1 + \varepsilon)d$. We say that G is *(d, ε)-superregular* if it is (d, ε)-regular and, furthermore, if $(1 - \varepsilon)dn < d_G(x) < (1 + \varepsilon)dn$ for all vertices x of G.

Theorem 1.3 *For all integers k and all positive constants $d, \nu, \eta \leq 1$ there is a positive $\varepsilon = \varepsilon(d, \nu, \eta)$ and an integer $N_2 = N_2(k, d, \nu, \eta)$ such that the following holds for all $n \geq N_2$. Let G be a (d, ε)-regular graph with n vertices. For each $1 \leq i \leq k$, let H_i be a subgraph of G with $e(H) = \nu_i e(G)$, where $\nu_i \geq \nu$ for all $i \geq k$. Then*

(i) *G contains a Hamilton cycle C such that for all i*
$(1 - \eta)\nu_i n \leq |C \cap E(H_i)| \leq (1 + \eta)\nu_i n$;

(ii) *if n is even then G contains a perfect matching M such that for all i*
$(1 - \eta)\nu_i n/2 \leq |M \cap E(H_i)| \leq (1 + \eta)\nu_i n/2$.

Note that the assertion is not even trivial (but much easier to prove) in the special case where G is the complete graph K_n. Moreover, Let $G_{n,p}$ be a random graph on n vertices obtained by connecting each pair of vertices with probability p (independently of all the other pairs). For given $0 < p < 1$ and n sufficiently large, $G_{n,p}$ is (p, ε)-superregular with high probability (in fact the probability that this is not the case is easily seen to decrease exponentially in n). Thus the assertion of Theorem 1.3 holds with high probability in this case. Also, if G is dn-regular and the second eigenvalue of the adjacency matrix is at most λdn for sufficiently small λ, then G is (d, ε)-superregular (see e.g. Chung [7], Theorem 5.1) so the result applies in this case, too (such graphs are often called pseudo-random graphs).

1.3 Application: Loose Hamilton cycles in 3-uniform hypergraphs

A fundamental theorem of Dirac states that every graph on n vertices with minimum degree at least $n/2$ contains a Hamilton cycle. In [12], we prove an analogue of this for 3-uniform hypergraphs, which we describe below. All the results proved in this paper except Theorems 1.2 and 1.3 and Lemma 3.8 are used as a tool in [12].

One way to extend the notion of the minimum degree of a graph to that of a 3-uniform hypergraph $\mathcal{H}$ is as follows. Given two distinct vertices x and y of $\mathcal{H}$, the *neighbourhood $N(x, y)$ of (x, y)* in $\mathcal{H}$ is the set of all those vertices z which form a hyperedge together with x and y. The *minimum degree $\delta(\mathcal{H})$* is defined to be the minimum $|N(x, y)|$ over all pairs of vertices of $\mathcal{H}$.

We say that a 3-uniform hypergraph $\mathcal{C}$ is a *cycle of order n* if there exists a cyclic ordering $v_1, \dots, v_n$ of its vertices such that every consecutive pair $v_i v_{i+1}$ lies in a hyperedge of $\mathcal{C}$ and such that every hyperedge of $\mathcal{C}$ consists of 3 consecutive vertices. A cycle is *tight* if every three consecutive vertices form a hyperedge. A cycle of order n is *loose* if it has the minimum possible number of hyperedges among all cycles on n vertices. A *Hamilton cycle* of a 3-uniform hypergraph $\mathcal{H}$ is a subhypergraph of $\mathcal{H}$ which is a cycle containing all its vertices. The following result is proved in [12].

Theorem 1.4 *For each $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon)$ such that every 3-uniform hypergraph $\mathcal{H}$ with $n \geq n_0$ vertices and minimum degree at least $n/4 + \varepsilon n$ contains a loose Hamilton cycle.*

The bound on the minimum degree is essentially best possible in the sense that there are hyergraphs with minimum degree $\lceil n/4 \rceil - 1$ which do not even contain some (not necessarily loose) Hamilton cycle. Recently, Rödl, Ruciński and Szemerédi [14] proved that if the minimum degree is at least $n/2 + \varepsilon n$ and n is sufficiently large, then one can even guarantee a tight Hamilton cycle. This is also best possible up to the error term (they announced in [14] that the error term εn can in fact be omitted).

2 Notation and a probabilistic estimate

Given a graph G, we write $N_G(x)$ for the neighbourhood of a vertex x in G and let $d_G(x) := |N_G(x)|$. Given $\varepsilon > 0$, we say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A,B) - d(X,Y)| < \varepsilon$. This (more traditional) notion of regularity is more or less equivalent to the one defined in the introduction. Indeed, clearly every (d,ε)-regular graph is also $2\varepsilon d$-regular (and thus 2ε-regular). Conversely, if $d = d(A,B) \geq \sqrt{\varepsilon}$ then every ε-regular bipartite graph (A,B) is $(d, \sqrt{\varepsilon})$-regular.

Given a positive number ε and sets $A, Q \subseteq T$, we say that A *is split ε-fairly by Q* if

$$\left| \frac{|A \cap Q|}{|Q|} - \frac{|A|}{|T|} \right| \leq \varepsilon.$$

Thus, if ε is small and A is split ε-fairly by Q, then the proportion of all those elements of T which lie in A is almost equal to the proportion of all those elements of Q which lie in A. We will use the following version of the well-known fact that if Q is random then it tends to split large sets ε-fairly. It is an easy consequence of standard large deviation bounds for the hypergeometric distribution, see e.g. [12] for a proof.

Proposition 2.1 *For each $0 < \varepsilon < 1$ there exists an integer $q_0 = q_0(\varepsilon)$ such that the following holds. Given $t \geq q \geq q_0$ and a set T of size t, let Q be a subset of T which is obtained by successively selecting q elements uniformly at random without repetitions. Let $\mathcal{A}$ be a family of at most q^{10} subsets of T such that $|A| \geq \varepsilon t$ for each $A \in \mathcal{A}$. Then with probability at least $1/2$ every set in $\mathcal{A}$ is split ε-fairly by Q.*

3 Perfect matchings in superregular graphs

In this section, we collect and prove several results about (random) perfect matchings in bipartite superregular graphs G which we will all need to prove

Theorems 1.1 and 1.2. Moreover, Lemmas 3.6 and 3.7 will also be used in [12]. The main result of this section is Lemma 3.8. Given a reasonably regular small subgraph H of G, it gives precise bounds on the likely number of all those edges of H that are contained in a random perfect matching M of G. This is proved in the third subsection. In the first subsection, we collect some tools which we will need in the other two subsections. In the second subsection, we give likely upper bounds on the number of all those edges of an arbitrary sparse subgraph H of G that are contained in a random perfect matching and on the number of cycles in the union of two random perfect matchings in G.

3.1 Known results on counting perfect matchings

We use the following version of Stirling's inequality (the bound is a weak form of a result of Robbins, see e.g. [4]):

Proposition 3.1 *For all integers $n \geq 1$ we have*

$$\left(\frac{n}{\mathrm{e}}\right)^n \leq n! \leq 3\sqrt{n}\left(\frac{n}{\mathrm{e}}\right)^n. \tag{1}$$

We will frequently use the following immediate consequence of the lower bound in Stirling's inequality:

$$\binom{n}{k} \leq \left(\frac{\mathrm{e}n}{k}\right)^k. \tag{2}$$

We will also use that

$$1 - x \geq \mathrm{e}^{-x-x^2} \text{ for all } 0 < x < 0.45 \tag{3}$$

(see e.g. [4, Section 1.1]).

We also need the following result of Brégman [5] which settles a conjecture of Minc on the permanent of a 0-1 matrix. (A short proof of it was given by Schrijver [15], see also [3]). We state this result in terms of an upper bound on the number of perfect matchings of a bipartite graph.

Theorem 3.2 *The number of perfect matchings in a bipartite graph $G = (A, B)$ is at most*

$$\prod_{a \in A} (d_G(a)!)^{1/d_G(a)}.$$

An application of Stirling's inequality (Proposition 3.1) to Theorem 3.2 immediately yields the following.

Corollary 3.3 *For all $\varepsilon > 0$ there is an integer $d = d_0(\varepsilon)$ so that the following holds: Let $G = (A, B)$ be a bipartite graph with $|A| = |B| = n$ and let $m(G)$ denote the number of perfect matchings in G. Then*

$$m(G) \leq (1 + \varepsilon)^n \prod_{a \in A} \frac{\max\{d_G(a), d_0\}}{\mathrm{e}}.$$

A very useful lower bound on the number of perfect matchings in a k-regular bipartite graph is provided by the following result by Egorichev [8] and Falikman [9], which was formerly known as the van der Waerden conjecture.

Theorem 3.4 *Let G be a k-regular bipartite graph whose vertex classes have size n. Then the number of perfect matchings in G is at least $(k/n)^n n!$.*

To bound the number of perfect matchings in superregular graphs, we will use the following theorem of Alon, Rödl and Ruciński [2]. (Actually, we will only apply the lower bound, which is based on Theorem 3.4. The upper bound in Theorem 3.5 is an easy consequence of Corollary 3.3.) Note that their result is stated slightly differently in [2] as the definition of (d, ε)-superregularity in [2] is slightly different.

Theorem 3.5 *For every $0 < \varepsilon < 1/4$ there exists an integer $n_1 = n_1(\varepsilon)$ such that whenever $d > 0$ and G is a (d, ε)-superregular bipartite graph whose vertex classes both have size $n \geq n_1$, then the number $m(G)$ of perfect matchings in G satisfies*

$$(d(1 - 4\varepsilon))^n n! \leq m(G) \leq (d(1 + 4\varepsilon))^n n!.$$

3.2 Simple properties of random perfect matchings

Based on the results in Section 3.1, we can easily deduce the next lemma, which implies that if we are given a (super-)regular graph G and a 'bad' subgraph F of G which is comparatively sparse, then a random perfect matching of G will probably only contain a few bad edges. The 'moreover' part will only be used in [12], the assertion about (d, ε)-regular graphs will be used in [12] and the proof of Theorem 1.1.

Lemma 3.6 *For all positive constants ε and d with $d \leq 1$ and $\varepsilon \leq 1/6$ there exists an integer $n_0 = n_0(\varepsilon, d)$ such that the following holds. Let G be a (d, ε)-superregular graph whose vertex classes A and B satisfy $|A| = |B| =: n \geq n_0$. Let M be a perfect matching chosen uniformly at random from the set of all perfect matchings of G. Let F be a subgraph of G such that all but at most $\Delta' n$ vertices in F have degree most $\Delta' dn$, where $1/2 \geq \Delta' \geq 18\varepsilon$. Then the probability that M contains at least $9\Delta' n$ edges of F is at most $\mathrm{e}^{-2\varepsilon n}$. Moreover, the statement also holds if we assume that G is dn-regular, where $dn \in \mathbb{N}$.*

Proof. First suppose that F has maximum degree at most $\Delta' dn$. Let $F' \supseteq F$ be a subgraph of G such that $d_{F'}(a) = \Delta' dn$ for each vertex $a \in A$. (Such an F' exists since $d_G(a) \geq (1 - \varepsilon)dn \geq \Delta' dn$ as G is (d, ε)-superregular.) Given a set $A' \subseteq A$, we denote by $F'_{A'}$ the bipartite graph with vertex classes A and B in which every vertex $a \in A'$ is joined to all the vertices $b \in N_{F'}(a)$ while every vertex $a \in A \setminus A'$ is joined to all the vertices $b \in N_G(a) \setminus N_{F'}(a)$. For an integer $q \geq \mathrm{e}^2 \Delta' n$, let $m(q)$ denote the number of perfect matchings in G which contain precisely q edges from F'. Every such matching M' can be obtained by first fixing a q-element set $A' \subseteq A$ and then choosing a perfect matching in the graph $F'_{A'}$. (So the elements of A' correspond to the q endvertices of the edges in $M' \cap E(F')$.) If we apply Corollary 3.3 to $F'_{A'}$ we now obtain

$$\begin{aligned} m(q) &\le \binom{n}{q}(1+\varepsilon)^n \left(\frac{\Delta' dn}{\mathrm{e}}\right)^q \left(\frac{(1+\varepsilon)dn}{\mathrm{e}}\right)^{n-q} \\ &\overset{(2)}{\le} \left(\frac{\mathrm{e}n}{q}\right)^q \left(\frac{dn}{\mathrm{e}}\right)^n (\Delta')^q (1+\varepsilon)^{2n}. \end{aligned}$$

Let $m(G)$ denote the number of perfect matchings in G. Then the lower bound in Theorem 3.5 implies that

$$m(G) \ge (d(1-4\varepsilon))^n n! \overset{(1)}{\ge} \left(\frac{(1-4\varepsilon)dn}{\mathrm{e}}\right)^n.$$

Thus the probability $m(q)/m(G)$ that M contains exactly q edges from F' is at most

$$\left(\frac{\mathrm{e}n\Delta'}{q}\right)^q (1+5\varepsilon)^{3n} \le \mathrm{e}^{-q}(1+5\varepsilon)^{3n} \le \mathrm{e}^{(15\varepsilon-\Delta')n} \le \mathrm{e}^{-3\varepsilon n}.$$

(To see the first inequality, use that $q \ge \mathrm{e}^2\Delta' n$.) By summing this bound over all $q \ge \mathrm{e}^2\Delta' n$, we find that the probability that M contains at least $\mathrm{e}^2\Delta' n$ edges of F' is at most $n\mathrm{e}^{-3\varepsilon n} \le \mathrm{e}^{-2\varepsilon n}$. Since $F \subseteq F'$, this implies that with probability at most $\mathrm{e}^{-2\varepsilon n}$ the matching M contains at least $\mathrm{e}^2\Delta' n$ edges of F. If F is now allowed to have up to $\Delta' n$ vertices whose degree is larger than $\Delta' dn$, this can increase the number of edges of F in M by at most $\Delta' n$, which implies the result.

The same proof also works in the case were G is dn-regular. We now use the lower bound $m(G) \ge d^n n! \ge (dn/\mathrm{e})^n$ which follows from Theorem 3.4 and inequality (1). □

In the following lemma we will use Theorems 3.4 and 3.5 to show that a randomly chosen 2-factor in a (super-)regular graph G will typically only contain few cycles. We will need this fact in the proof of Theorem 1.2 (and in [12] again, as mentioned earlier). A similar observation was also used in Frieze and Krivelevich [10]. The 'moreover' part will only be used in [12].

Lemma 3.7 *For all positive constants $\varepsilon < 1/64$ and $d \le 1$ there exists an integer $n_0 = n_0(\varepsilon, d)$ such that the following holds. Let G be a (d,ε)-superregular graph whose vertex classes A and B satisfy $|A| = |B| =: n \ge n_0$. Let M_1 be any perfect matching in G. Let M_2 be a perfect matching chosen uniformly at random from the set of all perfect matchings in $G - M_1$. Let $R = M_1 \cup M_2$ be the resulting 2-factor. Then the probability that R contains more than $n/(\log n)^{1/5}$ cycles is at most e^{-n}. Moreover, the statement also holds if we assume that G is dn-regular, where $dn \in \mathbb{N}$, and that G and M_1 are disjoint.*

Proof. Let $G' := G - M_1$. Let $m(G')$ denote the number of perfect matchings in G'. Since the deletion of a perfect matching from G still leaves a $(d, 2\varepsilon)$-superregular graph, Theorem 3.5 implies that

$$m(G') \ge ((1-8\varepsilon)d)^n n! \overset{(1),(3)}{\ge} \mathrm{e}^{-9\varepsilon n}\left(\frac{dn}{\mathrm{e}}\right)^n.$$

Let $k := n/(\log n)^{1/2}$ and $\ell' := (\log n)^{1/4}$. Given an integer $\ell \le \ell'$, let $f_{k,\ell}$ denote the number of 2-factors of G which contain M_1 and have at least k cycles of length 2ℓ. We will now find an upper bound on $f_{k,\ell}$. For this, note that the number of possibilities for choosing a set $C_{k,\ell}$ of k disjoint cycles of length 2ℓ in G where every second edge is contained in M_1 is at most

$$\frac{1}{k!} n^{\ell k} \overset{(1)}{\le} \left(\frac{\mathrm{e}}{k} n^{\ell}\right)^k =: c_{k,\ell}.$$

(Indeed, each such cycle of length 2ℓ is determined by an ordered choice of ℓ edges in M_1.) By Corollary 3.3, given some $C_{k,\ell}$ as above, the number of matchings on the remaining vertices of $G - M_1$ is at most

$$(1+\varepsilon)^{2n} \left(\frac{dn}{\mathrm{e}}\right)^{n-k\ell} \le \mathrm{e}^{2\varepsilon n} \left(\frac{dn}{\mathrm{e}}\right)^{n-k\ell} =: d_{k,\ell}.$$

Hence we have that $f_{k,\ell} \le c_{k,\ell} d_{k,\ell}$. Altogether, this implies that the probability $f_{k,\ell}/m(G')$ that a random 2-factor R (chosen as in the statement of the lemma) contains at least k cycles of length 2ℓ can be bounded as follows.

$$\frac{f_{k,\ell}}{m(G')} \le \mathrm{e}^{11\varepsilon n} \left(\frac{\mathrm{e}}{k} n^{\ell}\right)^k \left(\frac{\mathrm{e}}{dn}\right)^{k\ell} = \mathrm{e}^{11\varepsilon n} \left(\frac{\mathrm{e}^{\ell+1}}{kd^{\ell}}\right)^k \le \mathrm{e}^{11\varepsilon n} k^{-k/2} \le \mathrm{e}^{-2n}.$$

To derive the third inequality, we used the fact that $(\mathrm{e}/d)^{\ell'}$ (and thus $(\mathrm{e}/d)^{\ell}$) is small compared to k. For the final one, we used that $k \log k$ is large compared to n.

Hence the probability that there is an $\ell \le \ell'$ such that the random 2-factor R contains at least k cycles of length 2ℓ is at most $\ell' \mathrm{e}^{-2n} \le \mathrm{e}^{-n}$. Note that the number of cycles of length at least $2\ell'$ in R is at most $2n/(2\ell')$. Thus with probability at least $1 - \mathrm{e}^{-n}$ the number of cycles in R is at most $k\ell' + n/\ell' = 2n/(\log n)^{1/4}$, which implies the first part of the lemma.

The proof of the 'moreover' part of Lemma 3.7 is almost the same, except that we use the lower bound $m(G) \ge (dn/\mathrm{e})^n$ on the number of perfect matchings in G which follows from Theorem 3.4 by an application of (1). $\square$

3.3 Counting perfect matchings which contain a given number of edges of an almost regular subgraph

Lemma 3.8 *For each positive constant $\beta \ne 1$ there is a constant $f(\beta)$ with $0 < f(\beta) \le 1$ such that the following holds. Suppose that $\alpha, \varepsilon, \xi, c'$ and d are positive constants with $\varepsilon \ll \alpha, c', d \le 1$ and $\alpha, c' \ll \xi \ll f(\beta) \le 1$. There exists an integer $n_0 = n_0(\alpha, \varepsilon, \xi, c', d)$ for which the following is true. Let G be a bipartite (d, ε)-superregular graph whose vertex classes V and W satisfy $|V| = |W| =: n \ge n_0$. Let H be a subgraph of G with vertex classes $C \subseteq V$ and $D \subseteq W$ where $c'n \le |C| = cn \le 2c'n$ and*

$$\alpha dn \le d_H(v) \le (1+\xi)\alpha dn \text{ for all vertices } v \in C.$$

Let M be a perfect matching chosen uniformly at random from the set of all perfect matchings in G. Then

(i) $\mathbb{P}\left(|M \cap E(H)| \leq \beta\alpha cn\right) \leq \mathrm{e}^{-f(\beta)\alpha cn}$ *if* $\beta < 1$,

(ii) $\mathbb{P}\left(|M \cap E(H)| \geq \beta\alpha cn\right) \leq \mathrm{e}^{-f(\beta)\alpha cn}$ *if* $\beta > 1$.

The intuition behind this result is the following (see also the remark after Theorem 1.1): If the inclusion of the edges of G into the random perfect matching M would be mutually independent and equally likely, then the probability that a given edge e is contained in M would be close to $|M|/e(G)$. Thus the expected value of $|M \cap E(H)|$ would be close to $ne(H)/e(G)$ which in turn is close to $n(\alpha dn)(cn)/(dn^2) = \alpha cn$. The above result would thus immediately follow by an application of some large deviation bound on the tail of the binomial distribution.

The basic strategy of the proof is similar to that of [13], where the authors assume that H is a sufficiently large induced subgraph of G. The main difficulty of our proof is due to the fact that H is assumed to be rather small compared to G.

Proof. Let $m(G)$ denote the total number of perfect matchings in G. If we apply Stirling's formula (1) to the lower bound in Theorem 3.5, we obtain

$$m(G) \geq \left(\frac{(1-4\varepsilon)dn}{\mathrm{e}}\right)^n \overset{(3)}{\geq} \left(\frac{dn}{\mathrm{e}}\right)^n \mathrm{e}^{-5\varepsilon n}. \tag{4}$$

Given $a \leq cn$, let $m(a)$ be the number of perfect matchings in G which meet $E(H)$ in precisely a edges. Our aim is to show that $m(a)$ is much smaller than $m(G)$ if a is significantly smaller or larger than αcn. Let $\sum_J$ denote the summation over all matchings J in H of cardinality a. Given such a matching J, let $m(J)$ denote the number of perfect matchings M' in $G(J) := G - V(J) - E(H)$. Thus M' together with J forms a perfect matching of G which intersects H in exactly a edges and so $m(a) = \sum_J m(J)$. We claim that for all matchings J as above, we have

$$m(J) \leq \left(\frac{dn}{\mathrm{e}}\right)^{n-a} \mathrm{e}^{-\alpha cn - a} \mathrm{e}^{\xi a + 5\varepsilon n}. \tag{5}$$

The first term is the roughly the bound we would get if we would just use the fact that $G(J)$ has maximum degree $(1+\varepsilon)dn$. The second term is a small but crucial improvement on this estimate. The third term is an insignificant error term.

We now prove (5). By Corollary 3.3, we have

$$m(J) \leq (1+\varepsilon)^{n-a} \prod_{v \in V \setminus V(J)} \frac{\max\{d_{G(J)}(v), d_0(\varepsilon)\}}{\mathrm{e}}, \tag{6}$$

where $d_0(\varepsilon)$ is the integer defined in Corollary 3.3. Thus we have reduced the problem of bounding $m(J)$ to that of finding accurate upper bounds on the degrees of the vertices in $G(J)$. Recall that the vertex classes of H are C and D and that $\Delta(G) \leq (1+\varepsilon)dn$ since G is (d,ε)-superregular. For a vertex $v \in C \setminus V(J)$ we have

$$d_{G(J)}(v) \leq dn(1+\varepsilon-\alpha) =: q_H.$$

We say that a vertex $v \in V \setminus V(H)$ is *average for J* if in the graph G it has at least $(1-\varepsilon)d(a-\varepsilon n)$ neighbours in $W \cap V(J)$. Let V^{av} be the set of such vertices. For $v \in V^{av}$, we have

$$d_{G(J)}(v) \leq dn(1+\varepsilon-(1-\varepsilon)(a/n-\varepsilon)) =: q_J.$$

Since G is (d,ε)-superregular, we have that $|V^{av}| \geq n - cn - \varepsilon n$ if $a \geq \varepsilon n$. If $a \leq \varepsilon n$, then trivially every vertex in $v \in V \setminus V(H)$ is average for J, so the above bound on $|V^{av}|$ holds in this case, too. Moreover, note that both $q_H \geq d_0(\varepsilon)$ and $q_J \geq d_0(\varepsilon)$ since n is sufficiently large compared to ε. Thus, inserting all these bounds into (6) gives

$$m(J) \leq (1+\varepsilon)^n \mathrm{e}^{a-n}(q_H)^{|C\setminus V(J)|}(q_J)^{|V^{av}|}((1+2\varepsilon)dn)^{n-a-|C\setminus V(J)|-|V^{av}|}.$$

Now note that $q_J \leq (1+2\varepsilon)dn$ to deduce that the right hand side is maximized if $|V^{av}|$ is minimized. Thus

$$\begin{aligned} m(J) &\leq \mathrm{e}^{\varepsilon n}\mathrm{e}^{a-n}(q_H)^{cn-a}(q_J)^{(1-c-\varepsilon)n}((1+2\varepsilon)dn)^{\varepsilon n} \\ &\leq \left(\frac{dn}{\mathrm{e}}\right)^{n-a}\exp Q, \end{aligned} \tag{7}$$

where $Q := \varepsilon n + Q_H + Q_J + 2\varepsilon(\varepsilon n)$ and

$$\begin{aligned} Q_H :=& (\varepsilon-\alpha)(cn-a), \\ Q_J :=& [\varepsilon-(1-\varepsilon)(a/n-\varepsilon)][(1-c-\varepsilon)n]. \end{aligned}$$

Note that, we made use of the fact that $1+x \leq \mathrm{e}^x$ three times in order to obtain (7). Now observe that

$$\begin{aligned} Q_H &\leq -\alpha cn + \alpha a + \varepsilon n, \\ Q_J &\leq \varepsilon n - a(1-c-\varepsilon)(1-\varepsilon) + \varepsilon n \leq -a(1-2c) + 2\varepsilon n. \end{aligned}$$

Altogether, we thus have

$$Q \leq \varepsilon n - \alpha cn + \alpha a + \varepsilon n - a + 2ac + 2\varepsilon n + \varepsilon n \leq -\alpha cn - a + \xi a + 5\varepsilon n,$$

which proves (5).

Let p_a denote the probability that a perfect matching which is chosen uniformly at random in the set of all perfect matchings in G contains exactly a edges of H. Thus $p_a = m(a)/m(G) = \sum_J m(J)/m(G)$. Let $|\sum_J|$ denote the number of summands, i.e. the number of matchings in H of cardinality a. Each matching of cardinality a in H can be obtained by first choosing a subset of a vertices in C and then choosing one neighbour in H for each vertex in this subset. Thus, writing $(x/0)^0 := 1$ for all $x > 0$, it follows that

$$|\sum_J| \leq \binom{cn}{a}((1+\xi)\alpha dn)^a \overset{(2)}{\leq} \left(\frac{\mathrm{e}^{1+\xi}\alpha cdn^2}{a}\right)^a. \tag{8}$$

Since the bound (5) on $m(J)$ is independent of J, we can now combine (4) and (5) to obtain

$$\begin{aligned} p_a &= \sum_J \frac{m(J)}{m(G)} \leq |\sum_J| \left(\frac{\mathrm{e}}{dn}\right)^a \mathrm{e}^{5\varepsilon n}\mathrm{e}^{-\alpha cn - a}\mathrm{e}^{\xi a + 5\varepsilon n} \\ &\stackrel{(8)}{\leq} \left(\frac{\mathrm{e}\alpha cn}{a}\right)^a \mathrm{e}^{-\alpha cn}\mathrm{e}^{2\xi a+10\varepsilon n}. \end{aligned}$$

Now define β' by $a = \beta'\alpha cn$ and let $g(\beta') := \log\{(\mathrm{e}/\beta')^{\beta'}/\mathrm{e}\}$. Then

$$p_a \leq \left(\left(\frac{\mathrm{e}}{\beta'}\right)^{\beta'} \mathrm{e}^{-1}\right)^{\alpha cn} \mathrm{e}^{2\xi a+10\varepsilon n} \leq \exp\left\{\alpha cn(g(\beta') + 2\xi\beta' + \xi)\right\}.$$

Now set $\mu := \alpha cn$ to obtain

$$p_a \leq \exp\{\mu(g(\beta') + \xi(1 + 2\beta'))\}.$$

(Note that if $\xi = 0$ and $\beta' < 1$, this would be exactly the standard Chernoff bound on the probability that $X \leq \beta'\mu$, where X has a binomial distribution with mean μ, see e.g. Theorem A.12 in [3].) It is easy to check that $g(\beta') < 0$ if $\beta' \neq 1$.

The assertion (i) (i.e. the case $\beta < 1$) of the lemma now follows with $f(\beta) := -g(\beta)/4$ by summing over all values of a between 1 and $\beta\mu$. Indeed, as $g(\beta')$ is negative and increasing for $\beta' < 1$, we have

$$\mathbb{P}\left(|M \cap E(H)| \leq \beta\alpha cn\right) \leq \beta\mu \exp\{\mu g(\beta) + 3\xi\} \leq \beta\mu \exp\{\mu g(\beta)/2\},$$

as required. To prove the assertion (ii) of the lemma, we first consider the case $1 < \beta \leq \beta' \leq \mathrm{e}^2$. As $g(\beta')$ is negative and decreasing for $\beta' > 1$, it follows that

$$p_a \leq \exp\{\mu(g(\beta) + 17\xi)\} \leq \exp\{\mu g(\beta)/2\}.$$

Next consider the case that $\beta' \geq \mathrm{e}^2$. It is easy to check that $g(\beta') \leq -\beta'$. Thus

$$p_a \leq \exp\{\mu(-\beta' + \xi(1 + 2\beta'))\} \leq \exp\{-\mu\beta'/2\}.$$

Similarly to the case (i), the assertion of the lemma in case (ii) now follows by summing the bounds on p_a over all values of a between $\beta\mu$ and cn. $\square$

4 Proof of Theorems 1.1–1.3

We will prove Theorem 1.1 by decomposing H into small 'almost regular' subgraphs H_{ij} and a small remainder F. We will apply Lemma 3.8 to each of the H_{ij} separately and then use Lemma 3.6 to show that a random perfect matching contains only a negligible number of edges of F.

Proof of Theorem 1.1. By adding all the vertices in $V(G) \setminus V(H)$ to H, we may assume that H is a spanning subgraph of G. Set $\beta := 1 + \eta/4$, define

$f(\beta)$ as in the statement of Lemma 3.8 and choose parameters $\alpha, \varepsilon, \xi, c'$ so that $0 < \varepsilon \ll \alpha, c', d \leq 1$ and $c' \ll \alpha \ll \xi \ll \nu, \eta, f(\beta)$. Thus the restrictions in the statement of Lemma 3.8 are satisfied. Choose N_0 to be sufficiently large compared to both $1/\varepsilon$ and the integer $n_0(\alpha, \varepsilon, \xi, c', d)$ defined in Lemma 3.8. Finally, fix a constant c such that $cn \in \mathbb{N}$ and $c' \leq c \leq 2c'$.

First, we prove the upper bound in Theorem 1.1. Let ℓ be the smallest integer so that $\mathrm{e}^{\xi\ell/2}\alpha > 1 + \varepsilon$. Thus

$$\ell \leq \frac{2}{\xi}\log(2/\alpha) \leq 1/\sqrt{c}. \tag{9}$$

Let A_0 be the set of vertices in A with $d_H(a) < \alpha dn$. For all $i \geq 1$, let $\alpha_i := \mathrm{e}^{\xi(i-1)/2}\alpha$. Thus

$$\alpha_{i+1} \leq (1+\xi)\alpha_i \tag{10}$$

since $\mathrm{e}^{\xi/2} \leq 1 + \xi$ (see e.g. [4, Section 1.1]). Moreover,

$$1 + \varepsilon < \alpha_{\ell+1} \leq 2. \tag{11}$$

For all i with $1 \leq i \leq \ell$, let A_i be the set of vertices in $a \in A$ with $\alpha_i dn \leq d_H(a) < \alpha_{i+1} dn$. Since G is (d,ε)-superregular and thus $d_H(a) \leq d_G(a) \leq (1+\varepsilon)dn$ for each $a \in A$, it follows that the A_i with $0 \leq i \leq \ell$ give a partition of A.

We now define a partition of the edge set of H into graphs H_{ij}. Given $1 \leq i \leq \ell$, define q_i by $|A_i| = q_i cn$ and let $q(i) := \lfloor q_i \rfloor$. We partition the vertices in A_i into $q(i)+1$ parts A_{ij} with $0 \leq j \leq q(i)$ as follows: the partition is arbitrary except that we require that $|A_{ij}| = cn$ for all $j \geq 1$. Thus $|A_{i0}| < cn$ and so

$$\sum_{i=1}^{\ell} |A_{i0}| \leq \ell cn \leq \sqrt{c}n \leq \alpha n. \tag{12}$$

Let H_{ij} be the subgraph of H induced by A_{ij} and B. Then for all $a \in A_{ij}$, we have

$$\alpha_i dn \leq d_{H_{ij}}(a) < \alpha_{i+1} dn \overset{(10)}{\leq} (1+\xi)\alpha_i dn. \tag{13}$$

Let H_{00} be the subgraph of H which is induced by A_0 and B. Given $1 \leq i \leq \ell$, let H_{i0} be the subgraph of H which is induced by A_{i0} and B. Let F denote the union of all the H_{i0} with $0 \leq i \leq \ell$. Then

$$e(F) \leq \alpha dn|A_0| + \sum_{i=1}^{\ell} |A_{i0}|\alpha_{i+1} dn \overset{(11),(12)}{\leq} \alpha dn^2 + 2\alpha dn^2 \leq 4\alpha e(G) \leq \eta e(H)/4. \tag{14}$$

Let M be a perfect matching chosen uniformly at random from the set of all perfect matchings in G. Let $X_{ij} := |M \cap E(H_{ij})|$ and $\mu_i := \alpha_i cn$. (Note that μ_i can be thought of as roughly the expected value of X_{ij}). Then for all i,j with $i,j \geq 1$ we can apply Lemma 3.8(ii) to H_{ij} to see that with probability at least $1 - \mathrm{e}^{-f(\beta)\mu_i}$ we have $X_{ij} \leq \beta\mu_i$ (apply the lemma with α_i taking on the role of the parameter α there). Moreover, we can apply Lemma 3.6 to F as follows: Let $\Delta' := \alpha$. Then (12) implies that at most $\Delta' n$ vertices of F have degree more than $\Delta' dn$. Thus Lemma 3.6 implies that with probability at least $1-\mathrm{e}^{-2\varepsilon n}$ we have

$$|M \cap E(F)| \leq 9\alpha n \leq \eta\nu n/2.$$

But F and the sets $E(H_{ij})$ with $i,j \geq 1$ form a partition of $E(H)$ and so with probability at least $1-\mathrm{e}^{-2\varepsilon n} - \sum_{i=1}^{\ell} q(i)\mathrm{e}^{-f(\beta)\mu_i} \geq 1-\mathrm{e}^{-\varepsilon n}$ we have

$$|M \cap E(H)| \leq \eta\nu n/2 + \beta\sum_{i=1}^{\ell} q(i)\mu_i \leq \eta\nu n/2 + \beta\sum_{i=1}^{\ell} |A_i|\alpha_i.$$

Now use the fact that $\sum_{i=1}^{\ell} |A_i|\alpha_i dn \leq e(H) \leq (1+\varepsilon)\nu dn^2$ to see that $|M \cap E(H)| \leq \eta\nu n/2 + \beta(1+\varepsilon)\nu n \leq (1+\eta)\nu n$, as required.

The proof of the lower bound is almost exactly the same: in this case, we let $\beta = 1-\eta/4$. The graphs H_{ij} are defined as before. We now apply Lemma 3.8(i) to H_{ij} to see that with probability at least $1-\sum_{i=1}^{\ell} q(i)\mathrm{e}^{-f(\beta)\mu_i} \geq 1-\mathrm{e}^{-\varepsilon n}$ we have $X_{ij} \geq \beta\mu_i$ for all i,j with $i \geq 1$. Thus with probability at least $1-\mathrm{e}^{-\varepsilon n}$, we have

$$\begin{aligned}|M \cap E(H)| &\geq \beta\sum_{i=1}^{\ell} q(i)\mu_i \geq \beta\sum_{i=1}^{\ell}(|A_i|-cn)\alpha_i \overset{(11)}{\geq} \beta\sum_{i=1}^{\ell}|A_i|\alpha_i - 2\beta\ell cn \\ &\overset{(9)}{\geq} \beta\sum_{i=1}^{\ell}|A_i|\alpha_i - 4\sqrt{c}n \geq \beta\sum_{i=1}^{\ell}|A_i|\alpha_i - \eta\nu dn/2. \end{aligned} \tag{15}$$

But

$$\begin{aligned}\sum_{i=1}^{\ell}|A_i|\alpha_i dn &\overset{(10)}{\geq} (1-2\xi)\sum_{i=1}^{\ell}|A_i|dn\alpha_{i+1} \geq (1-2\xi)(e(H)-e(F)) \\ &\overset{(14)}{\geq} (1-2\xi)(1-\eta/4)e(H) \geq (1-\eta/2)\nu dn^2,\end{aligned}$$

which implies the result together with (15). □

We can now easily deduce Theorem 1.2 from Theorem 1.1 and Lemma 3.7.

Proof of Theorem 1.2. Put $\varepsilon := \min\{1/64, d/5, \varepsilon(d,\nu,\eta/2)/2\}$ where $\varepsilon(d,\nu,\eta/2)$ is as defined in Theorem 1.1. Let N_1 be sufficiently large compared to $1/\eta$, $1/\nu$ and k as well as larger than $n_0(\varepsilon,d)$ and $N_0(d,\nu,\eta/2)$ defined in Lemma 3.7 and Theorem 1.1 respectively.

Choose a perfect matching M_1 uniformly at random in G and then choose a perfect matching M_2 uniformly at random in $G-M_1$. Lemma 3.7 implies that with probability at least $1-\mathrm{e}^{-n}$ the resulting 2-factor $R = M_1 \cup M_2$ contains at most $(n/\log n)^{1/5}$ cycles. Moreover, Theorem 1.1 implies that we may assume that

$$(1-\eta/2)2\nu_i n \leq |R \cap E(H_i)| \leq (1+\eta/2)2\nu_i n \tag{16}$$

for all $i \leq k$. Thus it suffices to prove that there is a Hamilton cycle C in G which has sufficiently many edges in common with R. This is achieved using a standard argument based on expansion properties of G.

Let C' be any cycle in R with the property that there are adjacent vertices x and y on C' such that x has a neighbour z outside C'. (Using that G is (d,ε)-superregular, it is easy to see that such a cycle always exists unless R is already a Hamilton cycle. Indeed, since $\delta(G) \geq (1-\varepsilon)dn$, each cycle in R of length at most dn will have a neighbour outside and thus can be taken to be C'. On the other hand, $|N_G(X)| \geq (1-\varepsilon)n$ for any set X of size at least $dn/2 \geq \varepsilon n$ which lies in one of the vertex classes of G. This implies that if all the cycles in R have length at least dn and R is not a Hamilton cycle then we can take for C' any cycle of R.)

Let C'' denote the cycle in R which contains z. Let P denote the path obtained from $C' \cup C''$ by adding the edge xz and deleting xy as well as one of the edges on C'' adjacent to z. Note that the length of P is odd. If one of the endpoints of P has a neighbour outside P, we can further enlarge P in a similar way. So suppose we can no longer enlarge P in this way and view P as a directed path whose first vertex is denoted by x and whose final vertex is denoted by y. Thus all the neighbours of x and y lie on P. Moreover, since P is odd, x and y lie in different vertex classes of G.

We claim that there is a cycle C^* which has the same vertex set as P. Let X_1 be the set consisting of the first $\lfloor d_G(x)/2 \rfloor$ neighbours of x on P and let X_2 consist of all other neighbours. Define Y_1 and Y_2 similarly. It is easily seen that either (i) all vertices in Y_1 come before all those in X_2 or (ii) all vertices in X_1 come before those in Y_2. Suppose first that (i) holds. Note that $|X_i|, |Y_j| \geq \delta(G)/4 \geq (1-\varepsilon)dn/4 \geq \varepsilon n$ and so the (d,ε)-superregularity of G implies that there is an edge $e \in E(G)$ between a predecessor p of some vertex $y_1 \in Y_1$ and a successor s of some vertex $x_2 \in X_2$. We thus obtain a cycle C^* whose vertex set is $V(P)$ by removing the edges py_1 and x_2s from P and adding the three edges e, xx_2 and yy_1. The case (ii) is identical except that we now consider the predecessors of the vertices in X_1 and the successors of the vertices in Y_2.

Altogether, we have now constructed a 2-factor where the number of cycles has decreased. Continuing in this way, we eventually arrive at a Hamilton cycle C. It is easy to check that the symmetric difference of C and R contains only at most $5(n/\log n)^{1/5} \leq \eta\nu n/2$ edges. Together with (16) this shows that C is as required in the theorem. $\square$

It remains to deduce Theorem 1.3 from Theorems 1.1 and 1.2.

Proof of Theorem 1.3. First suppose that n is even. Set $n' := n/2$. Consider a random partition of the vertex set of G into two sets A and B of equal size. Let G' be the bipartite subgraph of G between A and B. Lemma 2.1 implies that we may assume that the graph G' is $(d, 2\varepsilon)$-superregular (in the bipartite sense) if n is sufficiently large compared to ε. Also, Lemma 2.1 implies that we may assume that the density of the bipartite subgraph of H_i between A and B is still close to $\nu_i d$ for all $i \leq k$. Thus we can apply Theorems 1.1 and 1.2 in this case.

Now suppose that n is odd and set $n' := \lfloor n/2 \rfloor$. Delete any vertex x from the vertex set of G. Again, Lemma 2.1 implies that we may assume that the bipartite graph $G' = (A, B)$ constructed as above on the remaining $2n'$ vertices is $(d, 3\varepsilon)$-superregular if n is sufficiently large compared to ε. Moreover, we may assume that for all $i \le k$ the density of the bipartite subgraph of H_i between A and B is still very close to that of H_i i.e. close to $\nu_i d$. Thus we may apply Theorem 1.2 to obtain a Hamilton cycle C' which satisfies

$$(1-\eta/2)2\nu_i n' \le |C' \cap E(H_i - x)| \le (1+\eta/2)2\nu_i n'.$$

Let P be a Hamilton path obtained from C' by adding an edge between x and some vertex $y \in C'$ and deleting one of the two edges on C' incident to y. As in the proof of Theorem 1.2, one can easily show that one can transform P into a Hamilton cycle C by deleting two and adding three edges. Then C is as required in Theorem 1.3(i). $\square$

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