OPTIMAL PACKINGS OF BOUNDED DEGREE TREES

FELIX JOOS, JAEHOON KIM, DANIELA KÜHN, AND DERYK OSTHUS

ABSTRACT. We prove that if $T_1, \ldots, T_n$ is a sequence of bounded degree trees so that T_i has i vertices, then K_n has a decomposition into $T_1, \ldots, T_n$. This shows that the tree packing conjecture of Gyárfás and Lehel from 1976 holds for all bounded degree trees (in fact, we can allow the first $o(n)$ trees to have arbitrary degrees). Similarly, we show that Ringel's conjecture from 1963 holds for all bounded degree trees. We deduce these results from a more general theorem, which yields decompositions of dense quasi-random graphs into suitable families of bounded degree graphs. Our proofs involve Szemerédi's regularity lemma, results on Hamilton decompositions of robust expanders, random walks, iterative absorption as well as a recent blow-up lemma for approximate decompositions.

1. INTRODUCTION

For a collection of graphs $\mathcal{H} = \{H_1, \ldots, H_s\}$ and a graph G, we say $\mathcal{H}$ *packs* into G if there are pairwise edge-disjoint copies of $H_1, \ldots, H_s$ in G. If every edge of G lies in one of these copies, then we have a *decomposition* of G into $\mathcal{H}$. Packing and decomposition problems are central to combinatorics and related areas. Famous early instances of such problems involve Steiner triple systems as well as Hamilton decompositions: Kirkman's theorem on the existence of Steiner triple systems translates into decompositions of cliques into triangles (subject to divisibility conditions) and Walecki's construction provides an analogue of this for decompositions into Hamilton cycles. A celebrated theorem of Wilson [31, 32, 33] generalizes Kirkman's theorem to decompositions of cliques into arbitrary subgraphs F – this forms one of the cornerstones of design theory. These results now are part of a major area, with connections to coding and information theory, as well as extremal combinatorics and algorithms.

Classical results on packings and decompositions have often been limited to symmetric structures, as these allow for the exploitation of these symmetries or the use of algebraic techniques. Probabilistic approaches are having an increasing impact on the area, and enable the construction of packings in more complex or general settings. In particular, in this paper we build on such approaches to obtain optimal results on packing suitable families of bounded degree graphs into quasi-random graphs. This provides the first instance of optimal packing and decomposition results involving general families of large (but sparse) structures which are not necessarily symmetric.

1.1. Packing trees into complete graphs.
The famous tree packing conjecture of Gyárfás and Lehel has driven a large amount of research in the area.

Conjecture 1.1 (Gyárfás and Lehel [15]). *Given $n \in \mathbb{N}$ and trees $T_1 \ldots, T_n$ with $|T_i| = i$, the complete graph K_n has a decomposition into copies of $T_1, \ldots, T_n$.*

The conjecture has been verified for several very special classes of trees (see e.g. [7, 15, 16, 29, 35]). Bollobás [5] showed that one can pack $T_1, \ldots, T_{n/\sqrt{2}}$ into K_n (and that a better bound would follow from the Erdős-Sós conjecture). Balogh and Palmer [4] showed that one can pack $T_{n-n^{1/4}/10}, \ldots, T_{n-1}$ into K_n. Zak [35] showed (amongst others) that one can pack the final five trees into K_n. Fishburn [11] verified that the set of degree sequences of the trees $T_1, \ldots, T_n$ can be "packed" into the degree sequence of K_n. Böttcher, Hladký, Piguet and Taraz [6] were able

to prove that we can obtain a near-optimal packing if one restricts to bounded degree trees. More generally, they showed that if $T_1, \ldots, T_s$ have bounded degree, satisfy $|T_i| \leq (1-\varepsilon)n$, and $\sum_{i=1}^{s} e(T_i) \leq (1-\varepsilon)\binom{n}{2}$, then $T_1, \ldots, T_s$ pack into K_n. Messuti, Rödl and Schacht [27] generalized this to separable families of graphs (which also includes minor-closed families of graphs). This was further improved by Ferber, Lee and Mousset [10], who obtained near-optimal packings of spanning copies of separable bounded degree graphs into K_n. Finally, Kim, Kühn, Osthus and Tyomkyn [19] proved that such a result in fact holds for all bounded degree graphs: if $H_1, \ldots, H_s$ have bounded degree and $\sum_{i=1}^{s} e(H_i) \leq (1-\varepsilon)\binom{n}{2}$, then $H_1, \ldots, H_s$ pack into K_n. Note that if we place no further restrictions on the H_i, then we cannot ask for an actual decomposition. However, in the setting of the tree packing conjecture, we do obtain such a decomposition in the current paper.

Theorem 1.2. *For all $\Delta \in \mathbb{N}$, there are $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $n \geq N$ the following holds. Suppose that for each $i \in [n]$, we have a tree T_i with $|T_i| = i$ and suppose $\Delta(T_i) \leq \Delta$ for all $i > \varepsilon n$. Then K_n decomposes into $T_1, \ldots, T_n$.*

Note that this implies Conjecture 1.1 for all bounded degree trees. (In fact, we do not require the first εn trees to have bounded degree.) We will deduce Theorem 1.2 from Theorem 1.7 in Section 10.

For bounded degree trees, we actually obtain the following more general result which has less restrictive assumptions on $|T_i|$. For a family $\mathcal{H} = \{H_1, \ldots, H_s\}$ of graphs we write $e(\mathcal{H}) := \sum_{i=1}^{s} e(H_i)$.

Theorem 1.3. *For all $\Delta \in \mathbb{N}$ and $\delta > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ the following holds. Suppose that $\mathcal{T}$ is a collection of trees such that*

    (i) *$|T| \leq n$ and $\Delta(T) \leq \Delta$ for all $T \in \mathcal{T}$,*
    (ii) *there are at least $(1/2 + \delta)n$ trees $T \in \mathcal{T}$ such that $\delta n \leq |T| \leq (1 - \delta)n$, and*
    (iii) *$e(\mathcal{T}) = \binom{n}{2}$.*

Then K_n decomposes into $\mathcal{T}$.

Another beautiful open problem in the area of tree decompositions is Ringel's conjecture (which in turn would follow from the notorious graceful labelling conjecture).

Conjecture 1.4 (Ringel [28])**.** *Given $n \in \mathbb{N}$ and a tree T on $n + 1$ vertices, the complete graph K_{2n+1} has a decomposition into $2n + 1$ copies of T.*

The conjecture has been verified for several very special classes of trees. A dynamic survey is maintained by Gallian [12]. Note the results in [6, 10, 19, 27] imply an approximate version of Conjecture 1.4 for bounded degree trees. Recent progress on Conjecture 1.4 (and its bipartite analogue) for random trees was obtained in [8, 26]. Snevily [30] showed that Ringel's conjecture holds if we replace K_{2n+1} by K_m, where $m = \Omega(n^3)$, which improved a bound of Yuster [34]. Recently Ringel's conjecture was generalized to allow for more than one tree by Böttcher, Hladký, Piguet and Taraz.

Conjecture 1.5 ([6])**.** *Given $n \in \mathbb{N}$, suppose $\mathcal{T}$ is a collection of trees such that $|T| \leq n + 1$ for all $T \in \mathcal{T}$. If $e(\mathcal{T}) \leq e(K_{2n+1})$, then $\mathcal{T}$ packs into K_{2n+1}.*

Note that Theorem 1.3 immediately implies that Ringel's conjecture holds for all bounded degree trees. In fact, by concatenating small trees into large ones if necessary, one can easily use Theorem 1.3 to show that the more general Conjecture 1.5 holds for all bounded degree trees, too.

Corollary 1.6. *For all $\Delta \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ the following holds. Suppose $\mathcal{T}$ is a collection of trees such that $|T| \leq n + 1$ and $\Delta(T) \leq \Delta$ for all $T \in \mathcal{T}$. If $e(\mathcal{T}) \leq e(K_{2n+1})$, then $\mathcal{T}$ packs into K_{2n+1}.*

1.2. **Quasi-random graphs.** We actually prove our results in the setting of dense quasi-random graphs, which we now define. Let G be a graph and let $u, v \in V(G)$. We denote by $d_G(u)$ the degree of u and by $d_G(u, v)$ the size of the common neighbourhood of u and v. We say a graph

G on n vertices is (ε, p)-*quasi-random* if $d_G(u) = (1 \pm \varepsilon)pn$ and $d_G(u, v) = (1 \pm \varepsilon)p^2 n$ for all distinct vertices $u, v \in V(G)$.

The next theorem is the main result of this paper (and implies all other results stated in the introduction). It states that we can decompose a dense quasi-random graph into a family of bounded degree graphs as long as this family contains sufficiently many trees which are not too small and not too large.

Theorem 1.7. *For all $\Delta \in \mathbb{N}$ and $\delta > 0$, there are $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $n \geq N$ and all $p \in [0, 1]$ the following holds. Suppose G is an (ε, p)-quasi-random graph on n vertices, and $\mathcal{H}, \mathcal{T}$ are sets of graphs satisfying*

(i) $|J| \leq n$ and $\Delta(J) \leq \Delta$ for all $J \in \mathcal{H} \cup \mathcal{T}$,
(ii) for all $T \in \mathcal{T}$, the graph T is a tree and $\delta n \leq |T| \leq (1 - \delta)n$,
(iii) $|\mathcal{T}| \geq (1/2 + \delta)n$, and
(iv) $e(\mathcal{H}) + e(\mathcal{T}) = e(G)$.

Then G decomposes into $\mathcal{H} \cup \mathcal{T}$.

Observe that $p \geq \delta$ by (ii)–(iv) and that Theorem 1.7 immediately implies Theorem 1.3.

A construction which shows that the upper bound in (ii) cannot be omitted completely (even if $\mathcal{H} = \emptyset$) is described in [6]. The bound in (iii) also arises naturally: suppose $\mathcal{T}$ consists of $n/2 - 1$ paths, the collection of graphs $\mathcal{H}$ consists only of Eulerian graphs, and G is a regular graph of odd degree on n vertices. Then we can satisfy $e(\mathcal{H} \cup \mathcal{T}) = e(G)$, but G will not have a decomposition into $\mathcal{H} \cup \mathcal{T}$.

In Section 10.2 we use Theorem 1.7 to deduce Corollary 1.8. Corollary 1.8 implies that for trees of bounded maximum degree (an analogue of) Ringel's conjecture holds even in the dense quasi-random setting. Note that Corollary 1.8 immediately implies Corollary 1.6.

Corollary 1.8. *For all $\alpha, \Delta, p_0 > 0$, there are $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $n \geq N$ and $p \geq p_0$ the following holds. Suppose G is an (ε, p)-quasi-random graph on n vertices. Suppose that $\mathcal{T}$ is a collection of trees such that each $T \in \mathcal{T}$ satisfies $|T| \leq (1 - \alpha)pn$, $\Delta(T) \leq \Delta$ and $e(\mathcal{T}) \leq e(G)$. Then $\mathcal{T}$ packs into G.*

A related result of [19] guarantees a near-optimal packing of a collection of bounded degree graphs (which are allowed to be spanning) into quasi-random graphs. The results in [19] are actually proved in the even more general setting of ε-regular graphs (so they significantly strengthen the classical Blow-up lemma of Komlós, Sárközy and Szemerédi [22]). In combination with Szemerédi's regularity lemma these results will be a crucial tool in our proof (see Theorem 6.8 and Theorem 10.2 for the precise statement of the results from [19] that we use).

Note that the results of this paper hold for dense quasi-random graphs. It would be very interesting to find out if it is possible to extend them to the sparse (quasi-)random setting (see e.g. [3] for partial results).

2. Outline of the argument

2.1. **The tree packing conjecture.** Suppose our aim is to pack trees $T_1, \ldots, T_n$ with $|T_i| = i$ and $\Delta(T_i) \leq \Delta$ into $G = K_n$, i.e. we have the setting of Theorem 1.2 with uniformly bounded degrees. Our proof develops an "iterative absorbing approach" going back to [21].

Roughly speaking, an absorbing approach to find an optimal packing means that we first find and remove an absorbing graph G_{abs} from the host graph G. In our case, we would then aim to find a near-optimal packing ϕ of most of the trees T_i into $G - E(G_{\mathrm{abs}})$, which leaves some sparse remainder G_{rem} of G uncovered. The properties of G_{abs} then should ensure that we can find an optimal packing of the remaining trees into $G_{\mathrm{abs}} \cup G_{\mathrm{rem}}$, altogether leading to an (optimal) packing of $T_1, \ldots, T_n$ into G.

A recent result of [19] allows us to find the near-optimal packing required in the second step above. However, it is far from clear how to construct such an absorbing graph G_{abs}. The iterative absorbing approach replaces the single absorbing step with a sequence of steps, each designed (amongst others) to reduce the number of uncovered "leftover edges". The argument

is still concluded with an absorbing step, but the small size of the leftover and the added control over its location now makes this step feasible.

To be more precise, we consider a sequence $V(G) = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_\Lambda$ of sets with $|A_{i+1}| \ll |A_i|$ and such that $|A_\Lambda| \approx n^{1/3}$ (see Section 10.1). After the ith step of the iteration, we will ensure that any edges of G not covered so far lie inside A_i and that this leftover is still sufficiently dense, when viewed as having vertex set A_i.

We can achieve this as follows. We split $\{T_1, \ldots, T_n\}$ into sets $\mathcal{T}_0, \ldots, \mathcal{T}_{\Lambda-1}$ such that each $\mathcal{T}_i$ contains approximately $|A_i|$ trees of order at most $|A_i|$. Assume we have packed $\mathcal{T}_0 \cup \ldots \cup \mathcal{T}_{i-1}$ in such a way that

  (a) all edges in $G_{i-1} := G[A_{i-1}] - E(G[A_i])$ are covered and
  (b) most edges in $G[A_i]$ are not covered.

Our aim then is to find a packing of most of the trees in $\mathcal{T}_i$ into $G_i := G[A_i] - E(G[A_{i+1}])$ using the results from [19]. However, a direct application of the results in [19] to G_i would produce a leftover which is too dense. Thus instead we apply Szemerédi's regularity lemma to G_i in order to obtain a very efficient decomposition of G_i into a bounded number of blown-up cycles (see Section 5). We can then use a random walk algorithm as well as the results in [19] to obtain a very efficient packing of most of $\mathcal{T}_i$ into these blown-up cycles (see Section 6 and Step 2 in Section 8) and thus into G_i.

Roughly speaking we next use the "unpacked" trees in $\mathcal{T}_i$ to cover the remaining edges of G_i greedily. To achieve this, we will make use of a small number of edges from G_{i+1}, but need to be very careful that we do not affect the structure of G_{i+1} too much. This task of covering the remaining edges is divided into several steps. For example, one step consists of covering all remaining edges induced by $A_i \setminus A_{i+1}$. Another step consists of adjusting the parity of the degrees of the remaining graph. (We achieve this by embedding a leaf onto a vertex of "incorrect" parity.) The tools for carrying out these steps are provided in Section 7. Eventually, we have now packed $\mathcal{T}_i$ in such a way that (a) and (b) hold with i replaced by $i+1$.

After Λ iterations we arrive at a final leftover graph G_Λ^* on A_Λ, which is almost complete. So we have a very restricted leftover, but have so far no method for carrying out the actual absorption step. In order to prepare for this, we actually run the iterative process on a slightly modified set of trees: we choose an arbitrary collection $\mathcal{T}^*$ of m trees from $T_2, \ldots, T_{n/2}$ (where m is slightly smaller than $\binom{|A_\Lambda|}{2}$) and remove a leaf ℓ_{T^*} from each $T^* \in \mathcal{T}^*$. Let z_{T^*} be the vertex incident to ℓ_{T^*}. When carrying out the above iteration we do not embed $T^* \in \mathcal{T}^*$ but instead we embed $T^* - \ell_{T^*}$ in such a way that z_{T^*} is embedded into A_Λ (and no other vertex of T^* is embedded into A_Λ).

This means that after Λ iterations, we have embedded all (modified) trees $T_1, \ldots, T_n$ except for the leaves ℓ_{T^*} (and their incident edges) of the trees in $\mathcal{T}^*$. Also, the uncovered leftover graph G_Λ^* has exactly $m = |\mathcal{T}^*|$ edges. Thus in order to complete the absorption step it remains to assign these edges to the images of $T^* - \ell_{T^*}$ in a suitable way. Hence each edge of G_Λ^* will be the image of $z_{T^*}\ell_{T^*}$ for some $T^* \in \mathcal{T}^*$. (This is the reason why we embedded z_{T^*} into A_Λ.) This assignment is carried out by considering a suitable "out-regular" orientation of G_Λ^*. We prove the existence of such an orientation in Section 9. Note that in this process, the role of the final absorbing graph G_{abs} is played by the images of the modified trees in $\mathcal{T}^*$.

2.2. The general setting.
We now discuss the additional ideas needed to prove Theorem 1.7. It turns out that considering a quasi-random graph G instead of K_n does not affect the argument significantly. Moreover, the family of graphs $\mathcal{H}$ of bounded degree in Theorem 1.7 can easily be packed (with a quasi-random remainder) using the results of [19]. What does make a difference is that the family $\mathcal{T}$ of trees in Theorem 1.7 consists entirely of trees of linear size whereas the iterative absorption argument outlined in the previous section only makes sense if in the ith iteration the trees to be embedded have order at most $|A_i| = |G_i|$.

To overcome this problem, we run the iterative absorption with suitable subtrees of the original trees. More precisely, in the 0th iteration, we start by packing almost all of the trees, but from a small proportion of the trees $F \in \mathcal{T}$, we cut off a small subforest F^* and denote by $\mathcal{T}^1$ the set consisting of all the F^*. We then embed each $F - V(F^*)$ into $G[A_0 \setminus A_1]$. In the first iteration,

we proceed similarly for $\mathcal{T}^1$, i.e. most forests in $\mathcal{T}^1$ are embedded in their entirety, but from the others we cut off small subforests which together form $\mathcal{T}^2$, most of which we pack in the second iteration and so on.

A significant difficulty with this approach is of course that if we embed a tree in several iterations, then the subforests embedded in the various rounds must fit together. To achieve this we carefully embed the "intersection points" (or "roots") of these subforests in a separate step.

2.3. **Organisation of the paper.** In Section 3, we introduce some notation, collect some probabilistic tools, introduce Szemerédi's regularity lemma, and prove some basic results on packings and decompositions. In Section 4, we prove a result on near-optimal decompositions of suitable graphs into long cycles. In Section 5, we combine this with Szemerédi's regularity lemma to obtain near-optimal decompositions of suitable graphs into blown-up cycles. In Section 6, we combine the results of [19] with a random walk algorithm to find efficient packings of trees into blown-up cycles. In Section 7, we provide tools for covering a sparse set of leftover edges with trees. In Section 8, we then combine the results from Section 3–7 to prove the iteration lemma, which forms the core of the proof of Theorem 1.7. In Section 9, we prove an orientation lemma which will be important in the final absorbing step. In Section 10, we repeatedly apply the iteration lemma and then finally apply the orientation lemma to prove a version of Theorem 1.7 where we require $\mathcal{H} = \emptyset$ (see Theorem 10.1). We then derive Theorem 1.7 itself (using a result from [19]). We also derive Corollary 1.8 and Theorem 1.2 from Theorem 1.7.

3. Preliminaries

3.1. **Basic definitions.** For $a, b, c \in \mathbb{R}$ we write $a = b \pm c$ if $b - c \leq a \leq b + c$. In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $1/n \ll a \ll b \leq 1$ (where $n \in \mathbb{N}$ is typically the order of a graph), then this means that there are non-decreasing functions $f^* : (0, 1] \to (0, 1]$ and $g^* : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b \leq 1$ and all $n \in \mathbb{N}$ with $a \leq f^*(b)$ and $1/n \leq g^*(a)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way.

For $N \in \mathbb{N}$, we define $[N] := \{1, \ldots, N\}$. We define $\binom{X}{k} := \{A \subseteq X : |A| = k\}$.

We simply refer to "graphs" when we consider simple, undirected and finite graphs, and refer to "multigraphs" as graphs with potentially parallel edges, but without loops.

Let G be a multigraph and let $u, v \in V(G)$ and $U, V \subseteq V(G)$ such that $U \cap V = \emptyset$. We write $G[U, V]$ to denote the bipartite (multi-)subgraph of G induced by the edges joining U and V and let $e_G(U, V) := e(G[U, V])$. We extend $e_G(U, V)$ to sets U, V which are not necessarily disjoint by defining

$$e_G(U, V) := e_G(U \setminus V, V) + e_G(V \setminus U, U \cap V) + 2e(G[U \cap V]).$$

In addition, let $\text{den}_G(U, V) := e_G(U, V)/(|U||V|)$.

We define $d_{G,U}(v) := |\{e \in E(G) : e = uv, u \in U\}|$. If G is simple, we let $d_{G,U}(u, v) := |\{w \in U : uw, vw \in E(G)\}|$. We denote by $N_G(v)$ the set of all neighbours of v and by $N_G(u, v)$ the set of all common neighbours of u, v. We define $N_G[v] := N_G(v) \cup \{v\}$.

Given $E \subseteq E(G)$, we write $V(E)$ for the set of all endvertices of edges in E. It will sometimes be convenient to identify a set of edges $E \subseteq E(G)$ with the subgraph H of G such that $V(H) = V(E)$ and $E(H) = E$. In particular, we write $\Delta(E)$ for $\Delta(H)$.

We say a set $I \subseteq V(G)$ in a graph G is *k-independent* if for any two distinct $u, v \in I$, the distance between u and v in G is at least k. Thus a 2-independent set is just an independent set. We say an edge set M is a *k-independent matching* if any two edges in M have distance at least k. Note that a 2-independent matching is an induced matching.

Given a packing of $\mathcal{H} = \{H_1, \ldots, H_s\}$ into G, we can naturally associate a function $\phi : \bigcup_{H \in \mathcal{H}} V(H) \cup E(H) \to V(G) \cup E(G)$ with such a packing and we also say ϕ packs $\mathcal{H}$ into G. For simplicity, we write $\phi(\mathcal{H}) := \bigcup_{H \in \mathcal{H}} \phi(H)$. Given $H \in \mathcal{H}$, we sometimes abuse notation a

little and view $\phi(H)$ both as a subgraph of G and as a subset of $V(G)$. We say $\mathcal{H}$ *decomposes* G or G *has a decomposition into* $\mathcal{H}$ if $\mathcal{H}$ packs into G and $e(G) = e(\mathcal{H})$ where $e(\mathcal{H}) := \sum_{H \in \mathcal{H}} e(H)$.

We say that a function f on domain $\mathrm{dom}(f)$ is *consistent* with a function g if $f(x) = g(x)$ for any $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. We say that a function f is *consistent* with a collection of functions $\{g_1, \ldots, g_m\}$ if f is consistent with g_i for every $i \in [m]$.

We say that a bipartite graph G with vertex partition (A, B) is ε-*regular* if for all sets $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, we have

$$|\mathrm{den}_G(A', B') - \mathrm{den}_G(A, B)| < \varepsilon.$$

Moreover, G is (ε, d)-*regular* if for all sets $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, we have

$$|\mathrm{den}_G(A', B') - d| < \varepsilon.$$

If G is (ε, d)-regular and $d_G(a) = (d \pm \varepsilon)|B|$ for all $a \in A$ and $d_G(b) = (d \pm \varepsilon)|A|$ for all $b \in B$, then we say that G is (ε, d)-*super-regular*.

We say a multigraph G is (β, α)-*dense* if any two (not necessarily disjoint) sets $U, V \subseteq V(G)$ with $|U|, |V| \geq \beta|G|$ satisfy

$$\mathrm{den}_G(U, V) \geq \alpha.$$

A bipartite graph with vertex partition (A, B) is (ε, p)-*quasirandom* if for any two distinct vertices $u, v \in A$, we have $d(u) = (1 \pm \varepsilon)p|B|$ and $d(u, v) = (1 \pm \varepsilon)p^2|B|$ and for any two distinct vertices $u, v \in B$, we have $d(u) = (1 \pm \varepsilon)p|A|$ and $d(u, v) = (1 \pm \varepsilon)p^2|A|$.

We say a partition $(V_1, \ldots, V_k)$ of a set V is an *equitable partition* if $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. We refer to a graph G as having equitable partition $(V_1, \ldots, V_k)$ if $(V_1, \ldots, V_k)$ is an equitable partition of $V(G)$. Note that in this case we still allow G to have edges joining two vertices in V_i for $i \in [k]$.

For $k, \ell \in \mathbb{N}$ with $\ell \geq 3$, we define an ℓ-*cycle k-blow-up* as the graph $C(\ell, k)$ with vertex set consisting of the disjoint union of $V_1, \ldots, V_\ell$ such that $|V_i| = k$ for all $i \in [\ell]$. Moreover, the edge set consists of all edges uv with $u \in V_i, v \in V_{i+1}$ for $i \in [\ell]$, where we consider the index i modulo ℓ (and we will always do this in such a case). We refer to the sets V_i as the *clusters* of $C(\ell, k)$, and refer to ℓ as the *length* of the cycle blow-up. Sometimes we omit the "ℓ" if we speak about an ℓ-cycle k-blow-up for some ℓ.

A spanning subgraph G_1 of $C(\ell, k)$ is *internally k'-regular* for some $k' \in \mathbb{N}$ if $G_1[V_i, V_{i+1}]$ is k'-regular for all $i \in [\ell]$. A spanning subgraph G_2 of $C(\ell, k)$ is an (ε, d)-*(super)-regular ℓ-cycle k-blow-up* if $G_2[V_i, V_{i+1}]$ is (ε, d)-(super)-regular for all $i \in [\ell]$.

3.2. Probabilistic tools.

We frequently use random processes to show that (sub-)graphs with certain properties exist. To this end, we will use several concentration inequalities which we introduce now. A sequence $X_0, \ldots, X_N$ of random variables is a *martingale* if X_0 is a fixed real number and $\mathbb{E}[X_n \mid X_0, \ldots, X_{n-1}] = X_{n-1}$ for all $n \in [N]$. We say that the martingale $X_0, \ldots, X_N$ is c-*Lipschitz* if $|X_n - X_{n-1}| \leq c$ holds for all $n \in [N]$. Our applications of Azuma's inequality will involve *exposure martingales* (also known as Doob martingales). These are martingales of the form $X_i := \mathbb{E}[X \mid Y_1, \ldots, Y_i]$, where X and $Y_1, \ldots, Y_i$ are some previously defined random variables.

Theorem 3.1 (Azuma's inequality [2, 17])**.** *Suppose that $\lambda, c > 0$ and that $X_0, \ldots, X_N$ is a c-Lipschitz martingale. Then*

$$\mathbb{P}[|X_N - X_0| \geq \lambda] \leq 2e^{\frac{-\lambda^2}{2Nc^2}}.$$

For $m, n, N \in \mathbb{N}$ with $m, n < N$, the *hypergeometric distribution* with parameters N, n and m is the distribution of the random variable X defined as follows. Let S be a uniformly chosen random subset of $[N]$ of size n and let $X := |S \cap [m]|$. The first part of the following lemma is essentially same as Corollary A.1.7 in [1] and the second part is from [18].

Lemma 3.2 (see [1, 18])**.** *Suppose $X_1, \ldots, X_n$ are independent random variables such that $\mathbb{P}[X_i = 0] = p_i$ and $\mathbb{P}[X_i = 1] = 1 - p_i$ for all $i \in [n]$. Let $X := X_1 + \cdots + X_n$. Then for all $t > 0$, $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2e^{-t^2/(2n)}$. Suppose Y has a hypergeometric distribution with parameters N, n, m, then $\mathbb{P}[|Y - \mathbb{E}[Y]| \geq t] \leq 2e^{-t^2/(3n)}$.*

We say that $(X_t)_{t \geq 0}$ is a *symmetric random walk* on $[\ell]$ if $X_0 \in [\ell]$ and for all $t \geq 1$, we have $\mathbb{P}[X_t = X_{t-1} + 1] = \mathbb{P}[X_t = X_{t-1} - 1] = 1/2$, independently of $X_0, \ldots, X_{t-1}$, where X_t is considered modulo ℓ.

Lemma 3.3 (see [20]). *Let $\ell \geq 3$ be an odd integer. If $(X_t)_{t \geq 0}$ is a symmetric random walk on $[\ell]$, then there is a $0 < \gamma < 1$ and an integer $t_0 > 0$ such that for every $i \in [\ell]$ and every $t \geq t_0$,*

$$\mathbb{P}[X_t = i] = \frac{1 \pm \gamma^t}{\ell}.$$

3.3. **Regularity and quasi-randomness tools.** In this section we collect several basic statements involving graph regularity and quasi-randomness as well as some of their consequences. The following three propositions follow easily from the definition of (ε, d)-regularity.

Proposition 3.4. *Let $0 < \varepsilon \leq \delta \leq d \leq 1$. Suppose G is an (ε, d)-regular bipartite graph with vertex partition (A, B) and $A' \subseteq A, B' \subseteq B$ with $|A'|/|A|, |B'|/|B| \geq \delta$. Then $G[A', B']$ is $(\varepsilon/\delta, d)$-regular.*

Proposition 3.5. *Suppose $n \in \mathbb{N}$ with $1/n \ll \varepsilon \ll d \leq 1$. If G is an (ε, d)-regular bipartite graph with vertex partition (A, B) and $n = |A| \leq |B|$, then there is a matching of size at least $(1 - \varepsilon)n$ in G.*

Proposition 3.6. *Suppose G is an (ε, d)-regular bipartite graph with vertex partition (A, B) and $B' \subseteq B$ with $|B'| \geq \varepsilon|B|$. Then all but at most $2\varepsilon|A|$ vertices in A have degree $(d \pm \varepsilon)|B'|$ in B'.*

The following result follows easily from Proposition 3.4 and 3.6.

Proposition 3.7. *Suppose $\ell, n \in \mathbb{N}$ and $1/n \ll \varepsilon \ll d \leq 1$. Then every (ε, d)-regular ℓ-cycle n-blow-up contains a $(6\varepsilon, d)$-super-regular ℓ-cycle $(1 - 4\varepsilon)n$-blow-up.*

We will also use the next result from [9]. (In [9] it is proved in the case when $|A| = |B|$ with $16\varepsilon^{1/5}$ instead of $\varepsilon^{1/6}$. The version stated below can be easily derived from this.)

Theorem 3.8. *Suppose $n \in \mathbb{N}$ with $1/n \ll \varepsilon \ll \alpha, p \leq 1$. Suppose G is a bipartite graph with vertex partition (A, B) such that $|A| = n$, $\alpha n \leq |B| \leq \alpha^{-1}n$ and at least $(1 - 5\varepsilon)n^2/2$ pairs $u, v \in A$ satisfy $d(u), d(v) \geq (p - \varepsilon)|B|$ and $d(u, v) \leq (p + \varepsilon)^2|B|$. Then G is $\varepsilon^{1/6}$-regular. In particular, if G is (ε, p)-quasi-random, it is $(\varepsilon^{1/6}, p)$-super-regular.*

The next three propositions involve basic properties of quasi-random and of (β, α)-dense graphs which follow easily from the definitions.

Proposition 3.9. *Suppose $\varepsilon \leq 1/10$. Let G be a (ε, p)-quasi-random graph on n vertices. Let $U \subseteq V(G)$ be a set of vertices with $|U| \geq (1 - \varepsilon)n$ and let E_1, E_2 be collections of edges on $V(G)$ such that $\Delta(E_1) \leq \varepsilon n$ and $\Delta(E_2) \leq \varepsilon n$. Then $(G \cup E_1)[U] - E_2$ is $(10p^{-2}\varepsilon, p)$-quasi-random.*

Proposition 3.10. *Suppose $n \in \mathbb{N}$ with $1/n \ll \beta \ll p \leq 1$. If G is a (β, p)-quasi-random graph on n vertices, then it is $(\beta^{1/6}, p - \beta^{1/6})$-dense.*

Proof. Let $V := V(G) = \{v_1, \ldots, v_n\}$ and $V' = \{v_1', \ldots, v_n'\}$. Consider the bipartite graph H with bipartition $V \cup V'$ such that $v_i v_j' \in E(H)$ if and only if $v_i v_j \in E(G)$. Observe that H is (β, p)-quasi-random. Hence, by Theorem 3.8, H is $(\beta^{1/6}, p)$-super-regular. Let $U_1, U_2 \subseteq V$ with $|U_1|, |U_2| \geq \beta^{1/6}n$. Let $U_2' = \{v_i' : v_i \in U_2\}$. We conclude that $\text{den}_G(U_1, U_2) = \text{den}_H(U_1, U_2') \geq p - \beta^{1/6}$. Thus G is $(\beta^{1/6}, p - \beta^{1/6})$-dense. $\square$

Proposition 3.11. *Suppose $0 < \alpha, \beta < 1$ and $C \geq 1$. If G is a (β, α)-dense graph on n vertices and G' is a spanning subgraph of G such that $d_G(v) - d_{G'}(v) \leq C\beta n$ for all $v \in V(G)$, then G' is $(C\beta^{1/2}, \alpha - \beta^{1/2})$-dense.*

The next result is a slight extension of Szemerédi's regularity lemma. We will apply it in Section 5.

Lemma 3.12. *Suppose $M, M', n, s, t \in \mathbb{N}$, $1/n \ll 1/M \ll 1/M' \ll \varepsilon \ll 1/t \ll \beta \ll \alpha, 1/s, d \leq 1$. Suppose G is a graph on n vertices with equitable partition $(U_1, \ldots, U_s)$. Then there is a $k \in \mathbb{N}$ such that G has a vertex partition $(V_0, V_1, \ldots, V_k)$ satisfying the following, where we write $d_{ij} := \mathrm{den}_G(V_i, V_j)$:*

> *(I) $M' \leq k \leq M$,*
> *(II) $|V_0| \leq \varepsilon n$,*
> *(III) $|V_1| = \ldots = |V_k|$,*
> *(IV) for every $i \in [k]$, the graph $G[V_i, V_j]$ is (ε, d_{ij})-regular except for at most εk indices $j \in [k]$.*
> *(V) for every $i \in [k]$, there exists a unique $j \in [s]$ such that $V_i \subseteq U_j$ and $L := |\{i : V_i \subseteq U_j\}| = |\{i : V_i \subseteq U_{j'}\}|$ is odd for all $j, j' \in [s]$.*

Let R_t be the multigraph on $[k]$ such that R_t contains an edge ij with multiplicity $\lfloor t d_{ij} \rfloor$ if and only if $G[V_i, V_j]$ is (ε, d_{ij})-regular and $d_{ij} \geq 1/t^{1/2}$. Then the following hold:

> *(VI) For each $j \in [s]$, let $U_j^R := \{i \in [k] : V_i \subseteq U_j\}$. If $d_{G,U_j}(v) = (d \pm \beta)|U_j|$ for every $v \in V(G)$, then for every $i \in [k]$ we have*
> $$d_{R_t, U_j^R}(i) = (d \pm 3\beta) t |U_j^R|.$$

> *(VII) If G is (β, α)-dense, then R_t is $(2\beta, t\alpha/2)$-dense.*

Proof. As (I)–(V) are standard, we omit the proof. Let $d'_{ij} := d_{ij}$ if $G[V_i, V_j]$ is (ε, d_{ij})-regular and $d_{ij} \geq 1/t^{1/2}$ and let $d'_{ij} = 0$ otherwise. We first show (VI). So assume that $d_{G,U_j}(v) = (d \pm \beta)|U_j|$ for all $v \in V(G)$. Let e_{ij} be the number of edges between V_i and U_j in G and let $\overline{n} := |V_1| = \cdots = |V_k|$. Note that $\overline{n} = (1 \pm \varepsilon)n/k$. Then

$$
\begin{aligned}
e_{ij} &= \sum_{v \in V_i} d_{G,U_j}(v) \pm e_G(V_i) = (d \pm 3\beta/2)|U_j||V_i| \\
&= (d \pm 3\beta/2)(|U_j^R|\overline{n} \pm \varepsilon n)\overline{n} = (d \pm 2\beta)|U_j^R|\overline{n}^2.
\end{aligned}
\tag{3.1}
$$

By (IV), there are at most $\varepsilon k \overline{n}^2$ edges between V_i and all those $V_{j'}$ for which $G[V_i, V_{j'}]$ is not $(\varepsilon, d_{ij'})$-regular. Moreover, there are at most $k\overline{n}^2/t^{1/2}$ edges between V_i and all those $V_{j'}$ for which $d_{ij'} \leq 1/t^{1/2}$. Furthermore, $e_G(V_i, U_j \cap V_0) \leq \varepsilon \overline{n} n$. Thus

$$
\begin{aligned}
e_{ij} &= \sum_{j' \in U_j^R} d'_{ij'} \overline{n}^2 \pm (\varepsilon k \overline{n}^2 + k\overline{n}^2/t^{1/2} + \varepsilon \overline{n} n) \\
&= \frac{\overline{n}^2}{t} \sum_{j' \in U_j^R} \lfloor t d'_{ij'} \rfloor \pm \frac{2k\overline{n}^2}{t^{1/2}} = d_{R_t, U_j^R}(i)\frac{\overline{n}^2}{t} \pm \frac{2k\overline{n}^2}{t^{1/2}}.
\end{aligned}
\tag{3.2}
$$

By combining (3.1) and (3.2), we obtain

$$d_{R_t, U_j^R}(i) = (d \pm 3\beta) t |U_j^R|,$$

which implies (VI).

To prove (VII), let $W, W' \subseteq V(R_t)$ with $|W|, |W'| \geq 2\beta|R_t| = 2\beta k$. Thus $|\bigcup_{i \in W} V_i|, |\bigcup_{i \in W'} V_i| \geq \beta n$. Since G is (β, α)-dense, this implies that

$$
e_G\left(\bigcup_{i \in W} V_i, \bigcup_{i \in W'} V_i\right) \geq \alpha \left|\bigcup_{i \in W} V_i\right|\left|\bigcup_{i \in W'} V_i\right| \geq \alpha |W||W'|\overline{n}^2.
\tag{3.3}
$$

On the other hand,

$$
\begin{aligned}
e_G \left(\bigcup_{i \in W} V_i, \bigcup_{i \in W'} V_i \right) \overset{(IV)}{\leq} & \sum_{i \in W, i' \in W'} d'_{ii'} \overline{n}^2 + \varepsilon k |W| \overline{n}^2 + \frac{k|W| \overline{n}^2}{t^{1/2}} \\
\leq & \sum_{i \in W, i' \in W'} (\lfloor t d'_{ii'} \rfloor + 1) \frac{\overline{n}^2}{t} + \frac{2k|W| \overline{n}^2}{t^{1/2}} \\
\leq & \ e_{R_t}(W, W') \frac{\overline{n}^2}{t} + \frac{|W||W'| \overline{n}^2}{t^{1/3}}.
\end{aligned}
\tag{3.4}
$$

By combining (3.3) and (3.4), we obtain

$$
e_{R_t}(W, W') \geq \alpha t |W||W'| - t^{2/3} |W||W'| \geq t\alpha |W||W'|/2.
$$

Thus R_t is $(2\beta, t\alpha/2)$-dense. $\qquad \square$

In Section 3.5 and Section 10 we will use the following version of the Blow-up lemma of Komlós, Sárközy and Szemerédi (see Remark 8 in [22]).

Theorem 3.13. *Suppose $r, n, \Delta \in \mathbb{N}$ and $1/n \ll \varepsilon \ll d, d_0, 1/\Delta, 1/r \leq 1$. Let H be an r-partite graph with $\Delta(H) \leq \Delta$ with vertex partition $(X_1, \ldots, X_r)$ and let G be an r-partite graph with vertex partition $(V_1, \ldots, V_r)$ such that $|X_i| = |V_i| \in \{n, n+1\}$ for all $i \in [r]$. Suppose that $G[V_i, V_j]$ is (ε, d)-super-regular for each $i \neq j \in [r]$. Let X' be a subset of $V(H)$ with $|X'| \leq \varepsilon n$ and for each $i \in [r]$ and $x \in X' \cap X_i$, let $A_x \subseteq V_i$ with $|A_x| \geq d_0 n$. Then there exists an embedding $\phi : H \to G$ such that $\phi(x) \in A_x$ for all $x \in X'$.*

3.4. **Graph packing and decomposition tools.** In this subsection we will collect some results about packings and decompositions of graphs that we will use later on. We start with three simple results about splitting an (ε, d)-regular or a (β, α)-dense graph into suitable edge-disjoint subgraphs.

Proposition 3.14. *Let $k, n \in \mathbb{N}$ and $1/n \ll \varepsilon \ll d, d', 1/k \leq 1$. Let G be a bipartite graph with vertex partition (A, B) such that $|A| = |B| = n$. If G is (ε, d)-regular and $d'k \leq d$, then there exist edge-disjoint $(2\varepsilon, d')$-regular spanning subgraphs $G_1, \ldots, G_k$ of G such that $d_{G'}(u) \leq (1 - kd'/d)n + n^{2/3}$ for all $u \in A \cup B$, where $G' := G - \bigcup_{i=1}^{k} E(G_i)$.*

Proof. We colour each edge of G with colour $i \in [k]$ with probability d'/d and with colour 0 with probability $1 - kd'/d$ at random independently from all other edges. Let G_i be the graph consisting of the edges of colour i. Straightforward applications of Lemma 3.2 show that $G_1, \ldots, G_k$ are $(2\varepsilon, d')$-regular and $d_{G'}(u) \leq (1 - kd'/d)n + n^{2/3}$ for all $u \in A \cup B$ with positive probability, in particular, such graphs exist. $\qquad \square$

Proposition 3.15. *Suppose $k, n \in \mathbb{N}$ with $1/n \ll 1/k, \alpha, \beta, d \leq 1$, and $\beta \ll \alpha, d$. Let G be a $(\beta, k\alpha)$-dense multigraph on n vertices whose edge multiplicity is at most k. If $d_G(v) = k(d \pm \beta)n$ for all $v \in V(G)$, then G can be decomposed into k spanning edge-disjoint (simple) graphs $G_1, \ldots, G_k$ such that for each $i \in [k]$, the graph G_i is $(\beta, \alpha - \beta)$-dense and $d_{G_i}(v) = (d \pm 2\beta)n$ for all $v \in V(G)$.*

Proof. For every set E of ℓ parallel edges of G, we choose an ℓ-subset of $[k]$ uniformly at random and label every edge in E with a distinct member of this ℓ-subset. For each $i \in [k]$, let G_i be the spanning subgraph of G consisting of all edges with label i. Lemma 3.2 shows that the G_i have the desired properties with probability at least $1/2$, in particular, such a decomposition exists. $\qquad \square$

In Section 5, we will need to find an approximate decomposition of an almost regular graph into long but not quite spanning cycles. As an intermediate step, the following lemma gives a decomposition into "almost" spanning almost regular subgraphs.

Lemma 3.16. *Suppose $D, k, n \in \mathbb{N}$ with $1/n \ll d, 1/k, \alpha, \beta, 1/D \leq 1$, and $\beta \ll \alpha, d, 1/D$, and $D \geq 6$. Let G be a $(\beta, k\alpha)$-dense multigraph on n vertices whose edge multiplicity is at most k.*

Suppose that $(U_1, \ldots, U_D)$ is an equitable partition of $V(G)$ such that $d_{G, U_i}(v) = (d \pm \beta)k|U_i|$ for all $v \in V(G)$ and $i \in [D]$. Then we can decompose G into $\{G_S : S \in \binom{[D]}{D-3}\}$ such that for all $S \in \binom{[D]}{D-3}$

(I) G_S is a $(2\beta, k\alpha D^{-3})$-dense multigraph with $V(G_S) = \bigcup_{i \in S} U_i$, and

(II) $d_{G_S}(v) = (d \pm 2\beta)k\binom{D-1}{3}^{-1}n$ for all $v \in \bigcup_{i \in S} U_i$.

In particular, $e(G_S) = (1 \pm \beta^{2/3})\binom{D}{3}^{-1}e(G)$.

Proof. We show that a suitable random decomposition produces a decomposition as desired with positive probability. We randomly label every edge of G with a set from $\binom{[D]}{D-3}$ according to the following rules. For each $i \in [D]$ and each edge in $G[U_i]$, choose a set uniformly at random from $\{S : i \in S \in \binom{[D]}{D-3}\}$. For $i, j \in [D]$ with $i \neq j$ and each edge in $G[U_i, U_j]$, choose a set uniformly at random from $\{S : i, j \in S \in \binom{[D]}{D-3}\}$.

For each $S \in \binom{[D]}{D-3}$, let G_S be the random (multi-)subgraph of G with vertex set $\bigcup_{i \in S} U_i$ whose edge set is the set of all edges with label S. Therefore, for all $S \in \binom{[D]}{D-3}$ and $u \in U_i$ with $i \in S$, we obtain

$$\mathbb{E}[d_{G_S}(u)] = \binom{D-1}{D-4}^{-1} d_{G, U_i}(u) + \binom{D-2}{D-5}^{-1} d_{G, \bigcup_{j \in S \setminus \{i\}} U_j}(u)$$

$$= \frac{6(d \pm \beta)kn}{D(D-1)(D-2)(D-3)} + \frac{6(D-4)(d \pm \beta)kn}{D(D-2)(D-3)(D-4)}$$

$$= \binom{D-1}{3}^{-1}(d \pm \beta)kn.$$

Let $S \in \binom{[D]}{D-3}$ and $W, W' \subseteq \bigcup_{i \in S} U_i$ with $|W|, |W'| \geq 2\beta|\bigcup_{i \in S} U_i| \geq \beta n$. Observe that every edge in $G[\bigcup_{i \in S} U_i]$ is contained in G_S with probability at least $2D^{-3}$. Thus, as G is $(\beta, k\alpha)$-dense,

$$\mathbb{E}[e_{G_S}(W, W')] \geq 2D^{-3}k\alpha|W||W'|.$$

Straightforward applications of Lemma 3.2 lead to the desired result. □

We call a path of length 2 a *seagull* and a graph consisting of the vertex-disjoint union of k seagulls a *flock of seagulls of size k* or just a *flock of seagulls* or a *flock*. Moreover, let the vertices of degree one of a seagull be its *wings*. The following simple proposition guarantees a decomposition of a suitable bipartite graph into edge-disjoint flocks of seagulls.

Proposition 3.17. *Let G be a bipartite graph with vertex partition (A, B). Suppose every vertex in B has even degree. Then G can be decomposed into at most $3\Delta(G)$ edge-disjoint flocks of seagulls with wings in A.*

Proof. Since every vertex $u \in B$ has even degree, we can decompose the edges incident to u into $d_G(u)/2$ edge-disjoint seagulls with wings in A. Let $\{S_1, \ldots, S_t\}$ be a corresponding decomposition of G into seagulls. Let $\mathcal{H}$ be the graph with vertex set $\{S_1, \ldots, S_t\}$ and edge set $\{S_i S_j : V(S_i) \cap V(S_j) \neq \emptyset\}$. Thus we have $\Delta(\mathcal{H}) \leq 3\Delta(G) - 1$. Hence we can properly colour $\mathcal{H}$ by using $3\Delta(G)$ colours, and each colour class gives rise to a flock of seagulls. Thus there exists a decomposition of G into $3\Delta(G)$ edge-disjoint flocks of seagulls. □

The following observation will be used to find suitable partitions (e.g. of flocks of seagulls).

Proposition 3.18. *Let X be a set of size n and let $w : X \to [0, M]$. Let $w(X') := \sum_{x \in X'} w(x)$ for all $X' \subseteq X$. For each $m \leq n$, there is a partition of X into $X_1, \ldots, X_m$ such that $|X_i| \leq \lceil 2n/m \rceil$ and $w(X_i) \leq 2w(X)/m + M$ for all $i \in [m]$.*

Proof. Let $X_1', \ldots, X_m'$ be disjoint subsets of X such that $|X_i'| \leq \lceil 2n/m \rceil$ and $w(X_i') \leq 2w(X)/m + M$ for all $i \in [m]$ and subject to these conditions such that $|\bigcup_{i=1}^m X_i'|$ is maximal. Observe that such sets exist as the collection of m empty sets satisfy the first two conditions.

If $\bigcup_{i=1}^m X_i' = X$, there is nothing to show. Otherwise, let $x \in X \setminus \bigcup_{i=1}^m X_i'$. Since $\sum_{i=1}^m |X_i'| \leq |X| - 1$, there are more than $m/2$ indices $i \in [m]$ such that $|X_i'| < \lceil 2n/m \rceil$. Since $\sum_{i=1}^m w(X_i') \leq w(X)$, there are at least $m/2$ indices $i \in [m]$ such that $w(X_i') \leq 2w(X)/m$. Thus there exists $i \in [m]$ such that $|X_i' \cup \{x\}| \leq \lceil 2n/m \rceil$ and $w(X_i' \cup \{x\}) \leq 2w(X)/m + M$. However, then $X_1', \ldots, X_{i-1}', X_i' \cup \{x\}, X_{i+1}', \ldots, X_m'$ contradicts the choice of $X_1', \ldots, X_m'$. $\square$

3.5. **Extending a partial embedding.** In this section we provide three results that allow us to extend partial embeddings to proper embeddings. The first one is only for forests and requires only very mild assumptions. The second one is based on the Blow-up lemma and requires stronger assumptions but works for general graphs. The third one provides a tool for embedding a collection of edges where one endpoint has already been embedded.

Lemma 3.19. *Suppose $n, k \in \mathbb{N}$. Let G be a graph on n vertices such that $d_G(u, v) \geq k$ for every pair of vertices u, v. Let F be a forest on k vertices. Let $I \subseteq V(F)$ be a 3-independent set in F. Then for any injection $\phi' : I \to V(G)$, there exists a function ϕ consistent with ϕ' which embeds F into G.*

Proof. Let $v_1, \ldots, v_k$ be an ordering of $V(F)$ such that for every i, the vertex v_i has at most one neighbour in $\{v_1, \ldots, v_{i-1}\}$, say $v_{a(i)}$, if this neighbour exists. Note that for all $i \in [k]$, we have $d_{F,I}(v_i) \leq 1$, since I is 3-independent. We define $\phi(v) := \phi'(v)$ for every $v \in I$. We sequentially determine $\phi(v_i)$ for the remaining vertices according to the order $v_1, \ldots, v_k$. So assume that $v_i \notin I$ and we have already determined $\phi(v)$ for each $v \in \{v_1, \ldots, v_{i-1}\} \cup I$.

If $d_{F,I}(v_i) = 0$, then we let $\phi(v_i)$ be some vertex in $V(G) \setminus \phi(\{v_1, \ldots, v_{i-1}\} \cup I)$ if $v_{a(i)}$ does not exist and otherwise let $\phi(v_i)$ be some vertex in $N_G(\phi(v_{a(i)})) \setminus \phi(\{v_1, \ldots, v_{i-1}\} \cup I)$, which exists as $\delta(G) \geq k$.

If $N_F(v_i) \cap I = \{u\}$ for some vertex u, then we let $\phi(v_i)$ be some vertex in $N := N_G(\phi(u)) \setminus \phi(\{v_1, \ldots, v_{i-1}\} \cup I)$ if $v_{a(i)}$ does not exist and otherwise let $\phi(v_i)$ be some vertex in $N' := N_G(\phi(v_{a(i)}), \phi(u)) \setminus \phi(\{v_1, \ldots, v_{i-1}\} \cup I)$. Such a choice is always possible, because $\min\{|N|, |N'|\} \geq k - (k - 1) \geq 1$. $\square$

Lemma 3.20. *Suppose $n, \Delta \in \mathbb{N}$ and $1/n \ll \varepsilon \ll p, 1/\Delta \leq 1$. Let G be an (ε, p)-quasi-random graph on n vertices. Let H be a graph on n vertices with $\Delta(H) \leq \Delta$. Let $I \subseteq V(H)$ be a 3-independent set in H with $|I| \leq \varepsilon n$. Then for any injection $\phi' : I \to V(G)$, there exists a function ϕ consistent with ϕ' which embeds H into G.*

Proof. Let $U := \phi'(I)$. As I is a 3-independent set, $d_{H,I}(x) \leq 1$ for any $x \in V(H)$ and $d_{H,I}(x) = 1$ for at most $\Delta|I| \leq \Delta\varepsilon n$ vertices x. Apply the Hajnal-Szemerédi theorem to obtain an equitable partition $(X_1, \ldots, X_{\Delta+1})$ of $H - I$ such that X_i is an independent set of H for all $i \in [\Delta + 1]$. Consider a partition $(V_1, \ldots, V_{\Delta+1})$ of $V(G) \setminus U$ such that $|X_i| = |V_i|$ for $i \in [\Delta + 1]$ chosen uniformly at random. Straightforward applications of Lemma 3.2 and Theorem 3.8 show that the following hold with probability at least $1/2$:

(GV1) $G[V_i, V_j]$ is $(\varepsilon^{1/7}, p)$-super-regular for all distinct $i, j \in [\Delta + 1]$, and
(GV2) $d_{G,V_i}(u) \geq pn/(2(\Delta + 1))$ for all $u \in U$ and $i \in [\Delta + 1]$.

In particular, there exists a partition $(V_1, \ldots, V_{\Delta+1})$ of $V(G) \setminus U$ such that $|X_i| = |V_i|$ and both (GV1) and (GV2) hold.

For each $x \in X_i$ for which $x \in N_H(y)$ for some $y \in I$, we let $A_x := N_G(\phi'(y)) \cap V_i$. Thus $|A_x| \geq pn/(2(\Delta + 1))$. Apply the Blow-up lemma (Theorem 3.13) to find an embedding ϕ'' of $H - I$ into $G - U$ such that $\phi''(x) \in A_x \subseteq N_G(\phi'(y))$ whenever $x \in N_H(y)$ for some $y \in I$. Then we obtain the desired embedding ϕ by defining $\phi := \phi' \cup \phi''$. $\square$

The next result will be used to embed edges of trees where one endpoint has already been embedded. The role of the graph H is to help us avoid collisions between edges belonging to the same tree.

Proposition 3.21. *Suppose $\Delta, m, s \in \mathbb{N}$. Let G be a graph, let $A \subseteq V(G)$, and let $u_1, \ldots, u_m$ be a sequence of (not necessarily distinct) vertices of G. Let $W_1, \ldots, W_m$ be sets of vertices and let H be a graph on $[m]$ satisfying the following:*

(i) $d_{G,A}(u_i) - |W_i| \geq 3\Delta + m/s + s$ for all $i \in [m]$,
(ii) $|\{i \in [m] : v = u_i\}| \leq \Delta$ for any $v \in V(G)$, and
(iii) $\Delta(H) \leq \Delta$.

Then we can choose distinct edges $u_1v_1, \ldots, u_mv_m$ in G such that $|\{i \in [m] : v = v_i\}| \leq s$ for all $v \in V(G)$ and $v_i \in A \setminus W_i$ for all $i \in [m]$ and $v_i \notin \{u_j, v_j\}$ whenever $ij \in E(H)$.

Proof. We sequentially choose v_i. Assume that we have chosen $v_1, \ldots v_{i-1}$ such that

(a)$_{i-1}$ $v_j \in (N_G(u_j) \cap A) \setminus W_j$ and $u_jv_j \neq u_{j'}v_{j'}$ for any $j \neq j' \leq i-1$,
(b)$_{i-1}$ $|\{j \leq i-1 : v = v_j\}| \leq s$ for any $v \in V(G)$, and
(c)$_{i-1}$ $v_j \notin \{u_{j'}, v_{j'}\}$ whenever $jj' \in E(H)$ and $j \neq j' \leq i-1$.

Consider the set $(N_G(u_i) \cap A) \setminus W_i$, which contains at least $3\Delta + m/s + s$ vertices by (i). Among those are

- at most $\Delta - 1$ vertices w such that $w = v_j, u_i = u_j$ for some $j \leq i-1$, by (ii),
- at most s vertices w such that $w = u_j, u_i = v_j$ for some $j \leq i-1$, by (b)$_{i-1}$,
- at most m/s vertices w such that $|\{j \leq i-1 : w = v_j\}| = s$, and
- at most 2Δ vertices w such that $w \in \{u_j, v_j\}$ for $j \in N_H(i)$, by (iii).

Therefore, by (i), we can choose a vertex v_i satisfying (a)$_i$, (b)$_i$ and (c)$_i$. □

4. Approximate cycle decomposition

The aim of this section is to show that every (β, α)-dense almost regular graph on an odd number of vertices has an approximate decomposition into Hamilton cycles and a few very long odd cycles (see Lemma 4.5). In Section 5, we will apply this result to the reduced graph obtained from Szemerédi's regularity lemma.

The main idea for the proof is based on that of Theorem 4.1 in [14]. We will use results from [23],[24] and [25] which imply that robustly expanding graphs enjoy very strong Hamiltonicity properties. To state these, we first need to introduce the concept of robust expansion. For $0 < \nu \leq \tau < 1$, a graph G on n vertices and $S \subseteq V(G)$, we define the *ν-robust neighbourhood* $RN_{\nu,G}(S)$ of S as the set of all those vertices of G which have at least νn neighbours in S. We call G a *robust (ν, τ)-expander* if

$$|RN_{\nu,G}(S)| \geq |S| + \nu n \text{ for all } S \subseteq V(G) \text{ with } \tau n \leq |S| \leq (1 - \tau)n.$$

We will use the following two results about Hamilton cycles in robust expanders.

Theorem 4.1 ([25]). *Suppose $n \in \mathbb{N}$ and $1/n \ll \nu \leq \tau \ll \alpha \leq 1$. Then any robust (ν, τ)-expander G on n vertices with $\delta(G) \geq \alpha n$ contains a Hamilton cycle.*

The following result guarantees a Hamilton decomposition in an even-regular robust expander. It is derived in [24] from a digraph version in [23].

Theorem 4.2 ([23, 24]). *Suppose $n \in \mathbb{N}$ and $1/n \ll \nu \leq \tau \ll \alpha \leq 1$. If G is an αn-regular robust (ν, τ)-expander on n vertices such that αn is even, then G can be decomposed into Hamilton cycles.*

The next observation shows that in a (β, α)-dense graph every not too small set of vertices induces a robust expander.

Proposition 4.3. *Suppose $n \in \mathbb{N}$ and $1/n \ll \beta \ll \alpha, \gamma \leq 1$. Let G be a (β, α)-dense graph on n vertices. If $U \subseteq V(G)$ with $|U| \geq \gamma n$, then $G[U]$ is a robust $(\alpha\beta, 2\beta/\gamma)$-expander.*

Proof. Consider a set $S \subseteq U$ with $2\beta n \leq (2\beta/\gamma)|U| \leq |S| \leq (1 - 2\beta/\gamma)|U| \leq |U| - 2\beta n$. Let $T := \{v \in U : d_{G,S}(v) < \alpha\beta n\}$. This implies that

$$\text{den}_G(S, T) \leq \frac{\alpha\beta n|T|}{|S||T|} \leq \frac{\alpha\beta n}{2\beta n} \leq \frac{\alpha}{2}.$$

As G is (β, α)-dense, we conclude that $|T| < \beta n$. Thus

$$|RN_{\alpha\beta,G[U]}(S)| = |U \setminus T| \geq |U| - \beta n \geq |S| + \alpha\beta|U|.$$

Therefore, G is a robust $(\alpha\beta, 2\beta/\gamma)$-expander. □

Next, we show that every (β, α)-dense graph can be "approximated" by an Eulerian graph.

Proposition 4.4. *Suppose $n \in \mathbb{N}$ and $1/n \ll \beta \ll \alpha \leq 1$. Let G be a (β, α)-dense graph on n vertices with $\delta(G) \geq \alpha n$. Then G contains an Eulerian subgraph G' such that $\Delta(G - E(G')) \leq 3$. In particular, $e(G - E(G')) \leq 3n/2$.*

Proof. Let M be a maximal matching in the subgraph of G induced by the vertices of odd degree. Let X be the set of all vertices of odd degree which are not covered by M. Thus X is an independent set (of even size) and hence $|X| < \beta n$ as G is (β, α)-dense. We write $X = \{x_1, y_1, \ldots, x_k, y_k\}$. Since $\delta(G) \geq \alpha n$ and G is (β, α)-dense, for every $i \in [k]$, there are at least $\alpha^3 n^2/3$ edges joining $N(x_i)$ and $N(y_i)$ which do not lie in M. Thus there are at least $\alpha^3 n/6$ internally vertex-disjoint paths of length 3 between x_i and y_i which are edge-disjoint from M. Since $\alpha^3 n/6 \geq \beta n \geq 2k$, we can choose vertex-disjoint paths $P_1, \ldots, P_k$ in $G - M$ such that P_i is a path of length 3 between x_i and y_i. Then $G' := G - (M \cup \bigcup_{i=1}^{k} E(P_i))$ is an Eulerian subgraph of G with $\Delta(G - E(G')) \leq 3$. □

Lemma 4.5. *Suppose $n, r \in \mathbb{N}$ and $1/n \ll \beta \ll \alpha, 1/r < 1$ and n is odd. Let G be a (β, α)-dense Eulerian graph on n vertices with $\delta(G) \geq \alpha n$ and $\Delta(G) - \delta(G) \leq \beta n$. Then G can be decomposed into Hamilton cycles and at most $2r\beta n$ odd cycles of length at least $(1 - \frac{1}{r-1})n$.*

To prove Lemma 4.5, we will sequentially remove the edges of suitable odd cycles until the remaining graph is regular. We can then obtain a Hamilton decomposition of the remaining graph via Theorem 4.2. The argument builds on ideas from [14].

Proof. Given any $V' \subseteq V(G)$ such that $|V'|$ is even, we say $(V_1, \ldots, V_k)$ is a *fair k-partition* of V' with respect to G if the following hold:

    (a) $|V_i|$ is even for every $i \in [k]$,
    (b) $||V_i| - |V_j|| \leq 2$, and
    (c) $d_{G,V_i}(v) \geq d_{G,V'}(v)/k - n^{2/3}$ for all $v \in V(G)$ and $i \in [k]$.

First, note that any set $V' \subseteq V(G)$ such that $|V'|$ is even has a fair k-partition. (To see this, choose a partition satisfying (a) and (b) uniformly at random and apply Lemma 3.2 to show that (c) holds with probability at least $1/2$.) Now we sequentially construct nested collections $\mathcal{C}_0 := \emptyset \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_s$ of edge-disjoint odd cycles and spanning subgraphs $G_0 := G \supseteq G_1 \supseteq \cdots \supseteq G_s$ of G such that G_s is regular and for all $0 \leq i < s$ the following hold:

    (i)$_i$ $E(G_i) - E(G_{i+1}) = \bigcup_{C \in \mathcal{C}_{i+1} \setminus \mathcal{C}_i} E(C)$ and $|\mathcal{C}_{i+1} \setminus \mathcal{C}_i| \leq 2r$,
    (ii)$_i$ $d_{G_i}(v) - d_{G_{i+1}}(v) \leq 4r - 2$ for every $v \in V(G)$,
    (iii)$_i$ $(\Delta(G_i) - \delta(G_i)) - (\Delta(G_{i+1}) - \delta(G_{i+1})) = 2$, and
    (iv)$_i$ $|C| \geq (1 - \frac{1}{r-1})n$ for every $C \in \mathcal{C}_{i+1}$.

Suppose that for some $j \geq 0$ we have already defined collections $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_j$ of edge-disjoint odd cycles and a sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_j$ of graphs satisfying (i)$_i$–(iv)$_i$ for all $0 \leq i < j$. If G_j is regular, then we may set $s := j$, so assume that G_j is not regular. Since $\Delta(G) - \delta(G) \leq \beta n$, (iii)$_i$ for $i < j$ implies that $j \leq \beta n/2$. This together with (ii)$_i$ for $i < j$ implies that

$$d_G(v) - d_{G_j}(v) \leq 2r\beta n. \tag{4.1}$$

By Proposition 3.11, the graph G_j is $(2r\beta^{1/2}, \alpha/2)$-dense. Thus Proposition 4.3 implies that

> *for any set $V' \subseteq V(G)$ with $|V'| \geq \alpha n/5$ and any graph $G' \subseteq G_j[V']$ with $\Delta(G_j[V'] - E(G')) \leq 4r$, the graph $G'[V']$ is a robust $(\alpha r \beta^{1/2}/2, \beta^{1/3})$-expander.* (4.2)

Let M_j be the set of all vertices of maximum degree in G_j and let $Z_j := V(G) \setminus M_j$. Note that $d_{G_j}(u) - d_{G_j}(v) \geq 2$ for every pair of vertices $u \in M_j, v \in Z_j$, since G (and thus G_j) is Eulerian. Pick $x \in M_j$ and let

$$Z' := \begin{cases} Z_j & \text{if } |M_j| \text{ is odd,} \\ Z_j \cup \{x\} & \text{if } |M_j| \text{ is even.} \end{cases}$$

Note that $|Z'|$ is even, because n is odd. Let $(V_1, \ldots, V_r)$ be a fair r-partition of Z' with respect to G. Next we show how to define G_{j+1} and $\mathcal{C}_{j+1}$. We will choose the cycles in $\mathcal{C}_{j+1} \setminus \mathcal{C}_j$ one by one. Assume we have already defined edge-disjoint odd cycles $C_1, \ldots, C_{t-1}$ in G_j for some $t \in [r]$. Consider the graph

$$G_{j,t} := G_j[V(G) \setminus V_t] - \bigcup_{i=1}^{t-1} E(C_i).$$

By (4.1), part (c) in the definition of an r-fair partition, and $\delta(G) \geq \alpha n$, we conclude that $d_{G_j[V(G)\setminus V_t]}(v) \geq \alpha n/3$. Hence $\delta(G_{j,t}) \geq \alpha n/3 - 2r$. This together with (4.2) implies that we can use Theorem 4.1 to obtain a Hamilton cycle C_t in $G_{j,t}$. We repeat this procedure until we have defined $C_1, \ldots, C_r$. Note that $|C_t| \geq (1 - 1/r)n - 2 \geq (1 - 1/(r-1))n$ for each $t \in [r]$.

If $|M_j|$ is odd, then we let $\mathcal{C}_{j+1} := \mathcal{C}_j \cup \{C_1, \ldots, C_r\}$, and $G_{j+1} := G_j - \bigcup_{i=1}^{r} E(C_i)$. Then we have

$$d_{G_j}(v) - d_{G_{j+1}}(v) = \begin{cases} 2r - 2 & \text{if } v \notin M_j, \\ 2r & \text{if } v \in M_j, \end{cases}$$

so each of (i)$_j$–(iv)$_j$ holds.

If $|M_j|$ is even, we will define additional cycles $C_{r+1}, \ldots, C_{2r}$ as follows. Let $(U_1, \ldots, U_r)$ be a fair r-partition of $V(G) \setminus \{x\}$. Assume for some $r + 1 \leq t \leq 2r$ we have already defined additional edge-disjoint odd cycles $C_{r+1}, \ldots, C_{t-1}$ in $G_j - \bigcup_{i=1}^{r} E(C_i)$. Consider the graph

$$G'_{j,t} := G_j[V(G) \setminus U_{t-r}] - \bigcup_{i=1}^{t-1} E(C_i).$$

By (c), we obtain $\delta(G'_{j,t}) \geq \alpha n/3 - 4r$. As before we can use Theorem 4.1 to obtain a Hamilton cycle C_t in $G'_{j,t}$. We repeat this process until we defined $C_1, \ldots, C_{2r}$. Let $\mathcal{C}_{j+1} := \mathcal{C}_j \cup \{C_1, \ldots, C_{2r}\}$, and $G_{j+1} := G_j - \bigcup_{i=1}^{2r} E(C_i)$. Thus

$$d_{G_j}(v) - d_{G_{j+1}}(v) = \begin{cases} 4r - 4 & \text{if } v \notin M_j, \\ 4r - 2 & \text{if } v \in M_j, \end{cases}$$

so each of (i)$_j$–(iv)$_j$ holds.

Hence, by repeating this procedure, we can obtain a regular graph G_s and a collection $\mathcal{C}_s$ of edge-disjoint odd cycles of length at least $(1 - \frac{1}{r-1})n$. Also $|\mathcal{C}_s| \leq 2rs \leq 2r\beta n$, since $s \leq \beta n/2 + 1$. Observe that G_s is Eulerian as G is Eulerian. By (4.2) and Theorem 4.2, G_s can be decomposed into a collection $\mathcal{C}'$ of Hamilton cycles. Now $\mathcal{C} := \mathcal{C}_s \cup \mathcal{C}'$ is the desired decomposition of G. $\square$

5. Decomposing a graph into cycle blow-ups

The main result in the current section (Lemma 5.1) guarantees a near-optimal packing of blow-ups of long cycles into a (β, α)-dense graph which is close to being regular. Results in Section 6 will allow us to obtain a near-optimal packing of trees into blow-ups of long cycles. So together these results guarantee a near-optimal packing of trees into a (β, α)-dense graph which is close to being regular. The fact in (R1) that none of the cycle blow-ups is (close to) spanning will be important when we apply Lemma 5.1 in the iteration lemma in Section 8.

Lemma 5.1. *Suppose $D, M, M', n, t \in \mathbb{N}$ with $1/n \ll 1/M \ll 1/M' \ll \varepsilon \ll 1/t \ll \beta \ll \alpha \leq d, 1/D \leq 1$ and let D be even with $D \geq 6$. Let $\mathcal{I} := \binom{[D]}{D-3}$. Suppose that G is a (β, α)-dense graph on n vertices with equitable partition $(U_1, \ldots, U_D)$ such that $d_{G,U_i}(v) = (d \pm \beta)D^{-1}n$ for all $v \in V(G)$ and all $i \in [D]$. Then there exist a set $V_0 \subseteq V(G)$ with $|V_0| \leq \varepsilon n$, an integer Γ with $M' \leq \Gamma \leq M$, and for each $S \in \mathcal{I}$ a set $\mathcal{C}_S$ such that the following properties hold (writing $\mathcal{C} := \bigcup_{S \in \mathcal{I}} \mathcal{C}_S$):*

(R1) for every $S \in \mathcal{I}$, the set $\mathcal{C}_S$ is a set of $(2\varepsilon, 1/t)$-super-regular cycle $(1-\varepsilon)\frac{n}{\Gamma}$-blow-ups such that $C \subseteq G$ for each $C \in \mathcal{C}_S$, the length of each $C \in \mathcal{C}_S$ is odd and at least $(1 - 7/(2D))\Gamma$ and such that $V(C) \subseteq (\bigcup_{i \in S} U_i) \setminus V_0$ for each $C \in \mathcal{C}_S$,

(R2) the cycle blow-ups in $\mathcal{C}$ are pairwise edge-disjoint,

(R3) $e(\mathcal{C}_S) = (1 \pm \beta^{1/2})\binom{D}{3}^{-1}e(G)$,

(R4) $d_G(v) - \sum_{C \in \mathcal{C}} d_C(v) \leq 3n/t^{1/2}$ for every $v \in V(G) \setminus V_0$. In particular, $e(\mathcal{C}) \geq e(G) - 2n^2/t^{1/2}$.

To prove Lemma 5.1, we apply Szemerédi's regularity lemma to obtain a reduced multigraph R^m. We then apply Lemma 4.5 to obtain an approximate decomposition of R^m into long odd cycles. This then translates into the existence of the desired super-regular cycle blow-ups in G. The reliance on Szemerédi's regularity lemma has the drawback that we have a small exceptional set V_0 which is not part of the cycle blow-ups. On the other hand the packing is very efficient: by (R4) the leftover density can be chosen to be much smaller than the input parameters α, β of G.

Proof. In this proof, given a graph H, we set $d_H(v) := 0$ whenever $v \notin V(H)$. We apply the regularity lemma (Lemma 3.12) to the graph G with ε^2 playing the role of ε to obtain a partition $(V_0', V_1', \ldots, V_\Gamma')$ of $V(G)$ such that, writing $d_{ij} := \mathrm{den}_G(V_i', V_j')$,

(i) $M' \leq \Gamma \leq M$,
(ii) $|V_0'| \leq \varepsilon^2 n$,
(iii) $|V_1'| = \ldots = |V_\Gamma'| =: n'$,
(iv) for each $i \in [\Gamma]$, the graph $G[V_i', V_j']$ is (ε^2, d_{ij})-regular except for at most $\varepsilon^2 \Gamma$ indices $j \in [\Gamma]$, and
(v) for each $i \in [\Gamma]$, there exists a unique $j \in [D]$ such that $V_i' \subseteq U_j$ and, writing $U_j^R := \{i : V_i' \subseteq U_j\}$, we have $|U_1^R| = |U_j^R|$ for all $j \in [D]$ and $|U_1^R|$ is odd.

Moreover, let R be the graph with vertex set $[\Gamma]$ where two vertices i, j are joined by an edge if $d_{ij} \geq 1/t^{1/2}$ and $G[V_i', V_j']$ is (ε^2, d_{ij})-regular. Construct a multigraph R^m from R by replacing each edge ij in R by $\lfloor d_{ij} t \rfloor$ edges. Observe that $(U_1^R, \ldots, U_D^R)$ is an equitable partition of $V(R^m)$. Then

(vi) $d_{R^m, U_j^R}(i) = (d \pm 3\beta) t |U_j^R|$ for all $i \in [\Gamma]$ and $j \in [D]$ and
(vii) R^m is $(2\beta, t\alpha/2)$-dense.

For every $ij \in E(R)$, we use Proposition 3.14 to find $\lfloor d_{ij} t \rfloor$ disjoint edge sets $E_{ij}^1, \ldots, E_{ij}^{\lfloor d_{ij} t \rfloor}$ of $G[V_i', V_j']$ such that E_{ij}^k induces a $(2\varepsilon^2, 1/t)$-regular graph for every $k \in [\lfloor d_{ij} t \rfloor]$.

Let $E_{ij}' := E(G[V_i', V_j']) \setminus \bigcup_{k=1}^{\lfloor d_{ij} t \rfloor} E_{ij}^k$ for any $ij \in E(R)$. Note that, also by Proposition 3.14, for every edge $ij \in E(R)$

$$d_{E_{ij}'}(v) \leq \frac{3n'}{2t^{1/2}}. \tag{5.1}$$

Let $\phi : E(R^m) \to \{E_{ij}^k : ij \in E(R), k \in \lfloor d_{ij} t \rfloor\}$ be a bijection such that for every $e \in E(R^m)$ which joins i and j, we have $\phi(e) = E_{ij}^k$ for some $k \in [\lfloor d_{ij} t \rfloor]$.

The proof strategy is now as follows. We first use Lemma 3.16 to decompose R^m into a collection $\{R_S^m\}_{S \in \mathcal{I}}$ of simple graphs such that $V(R_S^m) = \bigcup_{i \in S} U_i^R$ for each $S \in \mathcal{I}$. For each $S \in \mathcal{I}$, we then apply Lemma 4.5 to find a set $\mathcal{D}_S$ of edge-disjoint long odd cycles which together cover almost all edges of R_S^m. Via the map ϕ each cycle $C \in \mathcal{D}_S$ corresponds to a $(2\varepsilon^2, 1/t)$-regular cycle blow-up $\phi(C)$. The set $\mathcal{C}_S$ is then obtained from $\{\phi(C) : C \in \mathcal{D}_S\}$ by modifying each $\phi(C)$ slightly to obtain a cycle blow-up which is super-regular and in which all the clusters have equal size. In this last step we have to be careful since we need to ensure that (R4) holds.

By Lemma 3.16 and (vi) as well as (vii), we can decompose the multigraph R^m into a collection of graphs $\{R_S^m\}_{S \in \mathcal{I}}$, such that for each $S \in \mathcal{I}$ the following holds:

(a) R_S^m is $(6\beta, D^{-3} t\alpha/2)$-dense with $V(R_S^m) = \bigcup_{i \in S} U_i^R$,
(b) $d_{R_S^m}(j) = (d \pm 6\beta) \binom{D-1}{3}^{-1} t\Gamma = (d \pm 6\beta) \binom{D-1}{3}^{-1} tD|R_S^m|/(D-3)$ for every $j \in V(R_S^m)$, and
(c) $e(R_S^m) = (1 \pm 3\beta^{2/3}) \binom{D}{3}^{-1} e(R^m)$.

We next use Proposition 3.15 to decompose R_S^m into t simple spanning graphs $R_{S,1}, \ldots, R_{S,t}$ such that for all $S \in \mathcal{I}$ and $i \in [t]$ the following hold:

(a′) $R_{S,i}$ is $(6\beta, D^{-3}\alpha/4)$-dense, and

(b') $d_{R_{S,i}}(j) = (d \pm \beta^{2/3})\binom{D-1}{3}^{-1} D|R_{S,i}|/(D-3)$ for every $j \in V(R_{S,i})$.

Using Proposition 4.4 for each $S \in \mathcal{I}$, we obtain Eulerian graphs $R'_{S,1}, \dots, R'_{S,t}$ such that $R'_{S,i} \subseteq R_{S,i}$ for all $i \in [t]$ and

$$\Delta(R_{S,i} - E(R'_{S,i})) \leq 3. \tag{5.2}$$

Note that since D is even and $|U_1^R| = \Gamma/D$ is odd (by (v)), it follows that $|R'_{S,i}| = (D-3)\Gamma/D$ is odd for all $S \in \mathcal{I}$ and $i \in [t]$. Observe that $\delta(R'_{S,i}) \geq D^{-3}\alpha|R_{S,i}|/5$ for every $i \in [t]$ by (b'), and that $R'_{S,i}$ is still $(\beta^{3/4}, D^{-3}\alpha/5)$-dense. Thus for every $S \in \mathcal{I}$, we can apply Lemma 4.5 with $2D+1$, $\beta^{3/5}$ and $D^{-3}\alpha/5$ playing the roles of r, β and α to decompose $R'_{S,1}, \dots, R'_{S,t}$ into a set $\mathcal{D}_S$ of odd cycles of length at least $(1 - 1/(2D))(D-3)|U_1^R| \geq (1 - 7/(2D))\Gamma$. Let $\mathcal{D} := \bigcup_{S \in \mathcal{I}} \mathcal{D}_S$. Thus all the cycles in $\mathcal{D}$ are pairwise edge-disjoint.

Recall that for every $e \in R^m$ joining i and j, we have $\phi(e) = E_{ij}^k$ for some $k \in [\lfloor d_{ij}t \rfloor]$. For every $C \in \mathcal{D}$, let $\phi(C) := \bigcup_{e \in E(C)} \phi(e)$ be the cycle blow-up in G induced by the edges in $\bigcup_{e \in E(C)} \phi(e)$. Then $\phi(C)$ is a $(2\varepsilon^2, 1/t)$-regular cycle n'-blow-up. Moreover, for all $S \in \mathcal{I}$ and $v \in V(G)$, we have

$$\sum_{C \in \mathcal{D}_S} d_{\phi(C)}(v) \overset{(5.2)}{=} \sum_{e \in R_S^m} d_{\phi(e)}(v) \pm 3tn'. \tag{5.3}$$

In the remainder of the proof we show how to modify the cycle blow-ups $\phi(C)$ (for $C \in \mathcal{D}$) so that they satisfy (R4). First observe that

$$|\mathcal{D}| \leq 2t\Gamma. \tag{5.4}$$

Indeed, every $C \in \mathcal{D}$ has length at least $(1 - 7/(2D))\Gamma \geq 5\Gamma/12$ and thus $\phi(C)$ contains at least $n^2/(3t\Gamma)$ edges while $e(G) \leq n^2/2$.

By Proposition 3.7, for each $C \in \mathcal{D}$, the graph $\phi(C)$ contains a $(12\varepsilon^2, 1/t)$-super-regular cycle $(1 - 8\varepsilon^2)n'$-blow-up $\phi^s(C)$. For every $i \in [\Gamma]$, let

$$W_i := \{u \in V_i' : |\{C \in \mathcal{D} : u \in V(\phi(C)) \setminus V(\phi^s(C))\}| \geq \varepsilon^{1/2}\Gamma\}.$$

Thus the sets W_i contain all those vertices in V_i' that get deleted too often when making the cycle blow-ups super-regular. Then

$$|W_i|\varepsilon^{1/2}\Gamma \leq 8\varepsilon^2 n'|\mathcal{D}| \overset{(5.4)}{\leq} 16\varepsilon^2 t\Gamma n'$$

and hence $|W_i| \leq \varepsilon n/(2\Gamma)$. Let $V_0 := W_1 \cup \dots \cup W_\Gamma \cup V_0'$ and let $V_i := V_i' \setminus V_0$. Thus $|V_0| \leq \varepsilon n$, and by construction for each $v \in V(G) \setminus V_0$, we obtain

$$|\{C \in \mathcal{D} : v \in \phi(C)\}| - |\{C \in \mathcal{D} : v \in V(\phi^s(C)) \setminus V_0\}| \leq \varepsilon^{1/2}\Gamma. \tag{5.5}$$

Also for all $C \in \mathcal{D}$ and $v \in V(\phi^s(C)) \setminus V_0$, we have

$$d_{\phi(C)}(v) = d_{\phi^s(C) - V_0}(v) \pm \frac{2\varepsilon n}{\Gamma}. \tag{5.6}$$

Write $\mathcal{D} = \{C_1, \dots, C_m\}$. For each $r \in [m]$, let $C'_r := \phi^s(C_r) - V_0$. Note that for each $j \in V(C_r)$, we obtain

$$|V(C'_r) \cap V_j| \geq |V(\phi^s(C_r)) \cap V_j'| - |W_j| \geq (1 - 8\varepsilon^2)n' - \frac{\varepsilon n}{2\Gamma} > (1 - \varepsilon)\frac{n}{\Gamma}. \tag{5.7}$$

We claim that we can find cycle $(1 - \varepsilon)\frac{n}{\Gamma}$-blow-ups $C''_1, \dots, C''_m$ such that for all $r \in [m]$ the following hold:

$(\alpha 1)_r$ C''_r is an induced subgraph of C'_r,
$(\alpha 2)_r$ C''_r is a $(2\varepsilon, 1/t)$-super-regular cycle blow-up, and
$(\alpha 3)_r$ $|\{r' \in [r] : u \in V(C'_{r'}) \setminus V(C''_{r'})\}| \leq 10\varepsilon t\Gamma$ for all $u \in V(G) \setminus V_0$.

Suppose that for some $r \in [m]$ we have already constructed $C_1'', \ldots, C_{r-1}''$ such that $(\alpha 1)_{r'}-(\alpha 3)_{r'}$ hold for all $r' < r$. Next we construct C_r''. For each $i \in V(C_r)$, consider the set $B_i^r \subseteq V_i$ defined by

$$B_i^r := \{u \in V_i : |\{r' < r : u \in V(C_{r'}') \setminus V(C_{r'}'')\}| \geq 8\varepsilon t \Gamma\}.$$

Since

$$|B_i^r| \cdot 8\varepsilon t \Gamma \leq \sum_{r'=1}^{r-1} |(V(C_{r'}') \setminus V(C_{r'}'')) \cap V_i| \leq \sum_{r'=1}^{r-1} \varepsilon n/\Gamma \overset{(5.4)}{\leq} 2t\varepsilon n,$$

we have $|B_i^r| \leq n/(4\Gamma)$. Thus we can choose any set B_i' of size $|V(C_r') \cap V_i| - (1-\varepsilon)n/\Gamma$ in $(V(C_r') \cap V_i) \setminus B_i^r$ and define

$$C_r'' := C_r' - \bigcup_{i \in V(C_r')} B_i'.$$

Then C_r'' satisfies $(\alpha 1)_r-(\alpha 3)_r$. (In order to check $(\alpha 2)_r$, use that $|B_i'| \leq \varepsilon n/\Gamma$ and $|W_i| \leq \varepsilon n/(2\Gamma)$.) By repeating this procedure, we obtain a collection $\{C_1'', \ldots, C_m''\}$ satisfying $(\alpha 1)_r-(\alpha 3)_r$ for all $r \in [m]$.

Let $\mathcal{C} := \{C_1'', \ldots, C_m''\}$ and let $\mathcal{C}_S := \{C_r'' : C_r \in \mathcal{D}_S\}$. We now verify that $\mathcal{C}$ and $\mathcal{C}_S$ with $S \in \mathcal{I}$ satisfy (R1)–(R4). Note that $C_1'', \ldots, C_m''$ are edge-disjoint subgraphs of G and that for each $r \in [m]$ and every $v \in V(G)$, we trivially have $d_{C_r''}(v) \leq d_{C_r'}(v) \leq d_{\phi(C_r)}(v) \leq 3n/\Gamma$. Thus, for fixed $S \in \mathcal{I}$ and $v \in V(G) \setminus V_0$, we obtain

$$\sum_{C_r'' \in \mathcal{C}_S} d_{C_r''}(v) \overset{(\alpha 1)_r,(\alpha 3)_r}{=} \sum_{C_r'' \in \mathcal{C}_S} \left(d_{C_r'}(v) \pm \frac{\varepsilon n}{\Gamma} \right) \pm 10\varepsilon t\Gamma \cdot \frac{3n}{\Gamma}$$

$$\overset{(5.4),(5.5),(5.6)}{=} \sum_{C_r \in \mathcal{D}_S} \left(d_{\phi(C_r)}(v) \pm \frac{2\varepsilon n}{\Gamma} \right) \pm \left(\varepsilon^{1/2}\Gamma \cdot \frac{3n}{\Gamma} + 2\varepsilon tn + 30\varepsilon tn \right)$$

$$\overset{(5.3),(5.4)}{=} \sum_{e \in R_S^m} d_{\phi(e)}(v) \pm \left(3tn' + \frac{7\varepsilon^{1/2}n}{2} \right)$$

$$= \sum_{e \in R_S^m} d_{\phi(e)}(v) \pm 4\varepsilon^{1/2}n. \tag{5.8}$$

Hence for each $v \in V(G) \setminus V_0$, we obtain

$$\sum_{C_r'' \in \mathcal{C}} d_{C_r''}(v) = \sum_{S \in \mathcal{I}} \sum_{C_r'' \in \mathcal{C}_S} d_{C_r''}(v) \overset{(5.8)}{=} \sum_{S \in \mathcal{I}} \left(\sum_{e \in R_S^m} d_{\phi(e)}(v) \pm 4\varepsilon^{1/2}n \right)$$

$$= \sum_{e \in R^m} d_{\phi(e)}(v) \pm 4|\mathcal{I}|\varepsilon^{1/2}n.$$

Given $v \in V(G) \setminus V_0$, let $i(v) \in \Gamma$ be such that $v \in V_{i(v)}'$. Then

$$d_G(v) - \sum_{C_r'' \in \mathcal{C}} d_{C_r''}(v) \quad \leq \quad d_G(v) - \sum_{e \in R^m} d_{\phi(e)}(v) + 4|\mathcal{I}|\varepsilon^{1/2}n$$

$$\leq \quad d_{G, V_0' \cup V_{i(v)}'}(v) + \sum_{j \in [\Gamma] \setminus N_R[i(v)]} d_{G, V_j'}(v)$$

$$+ \sum_{j \in N_R(i(v))} d_{E_{ij}'}(v) + 4|\mathcal{I}|\varepsilon^{1/2}n$$

$$\overset{(ii),(iv),(5.1)}{\leq} \quad 2\varepsilon^2 n + \Gamma n'/t^{1/2} + \varepsilon^2 \Gamma \cdot n' + \Gamma \cdot 3n'/(2t^{1/2}) + 4|\mathcal{I}|\varepsilon^{1/2}n$$

$$\leq \quad 3n/t^{1/2}. \tag{5.9}$$

This shows (R4).

Since $|V_0| \leq \varepsilon n$ and by (5.8), for every $S \in \mathcal{I}$ we conclude that $e(\mathcal{C}_S) = (1 \pm \beta)e(\phi(R_S^m))$. Furthermore, by (c), we obtain that $e(\phi(R_S^m)) = (1 \pm 4\beta^{2/3}) \binom{D}{3}^{-1} e(\phi(R^m))$. Since $e(G) =$

$e(\phi(R^m)) \pm 3n^2/t^{1/2}$ by (5.9), altogether this implies (R3). Note that (R1), (R2) follow directly from our construction of the cycle blow-ups. □

6. Packing trees into cycle blow-ups

The main result of this section is Lemma 6.9, which states that we can obtain a near-optimal packing of a given collection of bounded degree forests into an ε-regular cycle blow-up. The key ingredient for this is Theorem 6.8 which is (a special case of) the main result in [19]. Theorem 6.8 achieves a near-optimal packing of suitable regular bounded degree graphs. In order to be able to apply this, we need to pack bounded degree trees into regular bounded degree subgraphs of cycle blow-ups. This is achieved by Lemma 6.6 and 6.7.

At the beginning of this section, we also make some simple observations about trees that are used in later sections. Let us start with some notation. We write (T, r) for a rooted tree T with root r. For a vertex v in (T, r), let $T(v)$ be the subtree of T containing all vertices below v (including v), that is, all vertices u such that the path between u and r in T contains v. Moreover, let $T(v, t)$ be the subtree of $T(v)$ induced by all vertices in distance at most t from v.

6.1. **Simple results about trees.** In this subsection we collect simple results concerning k-independent sets in trees and partitions of trees into certain subtrees.

Proposition 6.1. *Suppose $n, t, k, \Delta \in \mathbb{N} \setminus \{1\}$. Suppose that T is a tree on n vertices with at most t leaves, $\Delta(T) \le \Delta$, and $X \subseteq V(T)$ is a k-independent set in T. Then*

    (I) *T contains at least $n - 2t$ vertices of degree 2, and*

    (II) *T contains a k-independent set $Y \supseteq X$ of size at least $(n - 2t)/\Delta^k$ such that $Y \setminus X$ consists of vertices of degree 2.*

Proof. To prove (I), note that if T has at most t leaves, it has at most $t - 2$ vertices of degree at least 3. To prove (II), we greedily add vertices of degree 2 in T to X to obtain a maximal k-independent set Y with this property. Clearly, $|Y| \ge \max\{(n - 2t - |X|\Delta^k)/\Delta^k, 0\} + |X| \ge (n - 2t)/\Delta^k$. □

The next proposition follows easily from a greedy argument.

Proposition 6.2. *Suppose $n, k, \Delta \in \mathbb{N} \setminus \{1\}$ and let G be a graph on n vertices with $\Delta(G) \le \Delta$ and $\delta(G) \ge 1$. Suppose X is a k-independent set of G and $X \subseteq Z \subseteq V(G)$. Then the following hold.*

    (I) *There exists a k-independent set Y such that $X \subseteq Y \subseteq Z$ and $|Y| \ge |Z|/\Delta^k$.*

    (II) *G contains a k-independent matching M with $|M| \ge e(G)/(2\Delta^k) \ge n/(4\Delta^k)$.*

The next result guarantees a subtree $T(y)$ of roughly prescribed order inside a rooted tree (T, x) such that the distance between $T(y)$ and x is not too small.

Proposition 6.3. *Suppose $n, \Delta, k \in \mathbb{N}$ and $1/n \ll \alpha < 1$. Let (T, x) be a rooted tree on n vertices with $\Delta(T) \le \Delta$. If $\alpha \le \Delta^{-k}/2$, then there exists a vertex $y \in V(T)$ such that the subtree $T(y)$ of T satisfies*

    (I) *$\alpha n \le |T(y)| \le \alpha \Delta n$, and*

    (II) *the distance between x and y is at least k.*

Note that the definition of $T(y)$ implies that $T - V(T(y))$ is always a tree.

Proposition 6.4. *Suppose $\Delta, n \in \mathbb{N}$ with $1/n \ll \beta \ll \alpha \le 1/\Delta$ and $c \in [2]$. Let $\mathcal{F}$ be a collection of forests such that*

    (i) *each $F \in \mathcal{F}$ consists of two components K_F^1, K_F^2,*

    (ii) *K_F^i has a root x_F^i for every $i \in [2]$ and every $F \in \mathcal{F}$,*

    (iii) *$|K_F^1|, |K_F^2| \ge \alpha n$, and*

    (iv) *$|F| \le n$ and $\Delta(F) \le \Delta$.*

Then for every $F \in \mathcal{F}$, there exists a subforest T_F of F such that

    (I) *T_F consists of c components $T_F^1, \ldots, T_F^c$,*

(II) for each $i \in [c]$, we have $T_F^i = K_F^i(y_F^i)$ for some $y_F^i \in V(K_F^i)$, in particular, $F - V(T_F)$ is a forest consisting of two components,

(III) T_F has distance at least 5 from $\{x_F^1, x_F^2\}$,

(IV) $\Delta^{-1}\beta n/2 \le |T_F^i| \le \Delta\beta n$ for all $F \in \mathcal{F}$ and $i \in [c]$, and

(V) $\sum_{F \in \mathcal{F}} e(T_F) = \beta n |\mathcal{F}| \pm n$.

Proof. For each $F \in \mathcal{F}$ and $j \in [c]$, we apply Proposition 6.3 to K_F^j to obtain vertices $y_F'^j, \hat{y}_F^j \in V(K_F^j)$ and subtrees $T_F'^j = K_F^j(y_F'^j)$ and $\hat{T}_F^j = K_F^j(\hat{y}_F^j)$ such that

(1) $c^{-1}\Delta^{-1}\beta n \le |T_F'^j| \le c^{-1}\beta n$ and $c^{-1}\beta n \le |\hat{T}_F^j| \le c^{-1}\Delta\beta n$, and

(2) both $T_F'^j$ and $\hat{T}_F^j$ have distance at least 5 from x_F^j.

Let $T_F' := \bigcup_{j=1}^{[c]} T_F'^j$ and $\hat{T}_F := \bigcup_{j=1}^{[c]} \hat{T}_F^j$. We write $\mathcal{F} = \{F_1, \ldots, F_{|\mathcal{F}|}\}$ and consider $S_i := \sum_{j=1}^{i} e(\hat{T}_{F_j}) + \sum_{j=i+1}^{|\mathcal{F}|} e(T_{F_j}')$. Then $0 \le S_{i+1} - S_i = e(\hat{T}_{F_{i+1}}) - e(T_{F_{i+1}}') \le n$ and $S_0 \le \beta n |\mathcal{F}| \le S_{|\mathcal{F}|}$ by (1). Thus there exists $0 \le t \le |\mathcal{F}|$ such that $S_t = \beta n |\mathcal{F}| \pm n$. Let

$$T_{F_i} := \begin{cases} \hat{T}_{F_i} & \text{if } i \le t, \\ T_{F_i}' & \text{if } i \ge t+1. \end{cases}$$

Then $\sum_{F \in \mathcal{F}} e(T_F) = S_t = \beta n |\mathcal{F}| \pm n$, so we obtain (V). Conditions (I)–(IV) follow from (1) and (2). $\square$

6.2. **Trees in a cycle blow-up.** In this subsection, we find a near-optimal packing of a set $\mathcal{F}$ of bounded degree forests into an ε-regular cycle blow-up. The first step is to find a suitable embedding of a single tree (see Lemma 6.6). To analyze the corresponding embedding process, we consider a suitable partition into subtrees, which is given by Proposition 6.5.

Proposition 6.5. *Suppose $n, \Delta \in \mathbb{N} \setminus \{1\}$ and $n \ge t \ge 1$. Then for any rooted tree (T, r) on n vertices with $\Delta(T) \le \Delta$, there exists a collection $\mathcal{S}$ of pairwise vertex-disjoint rooted subtrees such that*

(I) $S \subseteq T(s)$ for every $(S, s) \in \mathcal{S}$,

(II) $t \le |S| \le 2\Delta t$ for every $(S, s) \in \mathcal{S}$, and

(III) $\bigcup_{(S,s) \in \mathcal{S}} V(S) = V(T)$.

Proof. We use induction on n for fixed t. If $t \le n \le 2\Delta t$, then $\mathcal{S} := \{(T, r)\}$ is as desired. So let $m > 2\Delta t$ and assume the proposition holds for all $n \le m - 1$. Consider a rooted tree (T, r) on m vertices. Choose y at maximal distance from r in T subject to $|T(y)| \ge t$. This implies that $t \le |T(y)| \le \Delta t$. Consider $T' := T - V(T(y))$. Since $t \le m - \Delta t \le |T'| \le m - 1$, by induction, we obtain a collection $\mathcal{S}'$ of pairwise vertex-disjoint rooted subtrees satisfying (I)–(III) with T' playing the role of T. Then it is clear that $\mathcal{S} := \mathcal{S}' \cup \{(T(y), y)\}$ satisfies (I)–(III). Thus the proposition holds for all $n \ge t$. $\square$

Recall from Section 3 that we write $C(m, \ell)$ for an ℓ-cycle m-blow-up. The next result describes how to embed a bounded-degree tree into a cycle blow-up in a uniform way, i.e. we embed approximately the same number of edges into every "blown-up edge" of $C(m, \ell)$. We will achieve this by considering a symmetric random walk on the cycle of length ℓ. Note that for Lemma 6.6 it is crucial that ℓ is odd.

Lemma 6.6. *Suppose $n, \Delta, \ell \in \mathbb{N}$ with $1/n \ll 1/\Delta, 1/\ell < 1$, where ℓ is odd. Let $m := \frac{n}{\ell} + \frac{n}{\log^2 n}$. Let T be a tree on n vertices with $\Delta(T) \le \Delta$ and let $G := C(\ell, m)$. Then there is an embedding ϕ of T into G such that $\phi(T)$ contains at most m edges between any two clusters of G.*

Proof. Choose an additional constant δ such that $1/n \ll \delta \ll 1/\ell$. Pick $u \in V(T)$ and apply Proposition 6.5 to (T, u) with $n^{1-\delta}$ playing the role of t to obtain a collection $\mathcal{S} = \{(S_1, s_1), \ldots, (S_k, s_k)\}$ of vertex-disjoint rooted subtrees such that

- $S_i \subseteq T(s_i)$ for every $i \in [k]$,
- $n^{\delta}/(2\Delta) \le k \le n^{\delta}$,
- $|S_i| \le 2\Delta n^{1-\delta}$, and

- $\bigcup_{i=1}^{k} V(S_i) = V(T)$.

We may assume that the (S_i, s_i) are labelled in such a way that the distance of $s_1, \ldots, s_k$ from the root u of T is non-decreasing. We will assign the vertices of T one by one to some cluster of G in such a way that the vertices of S_i are assigned before the vertices of S_{i+1}, and such that within S_i we assign the vertices in breadth-first order, starting with s_i. To choose our assignments, we consider the following random process. For every edge e of T we pick a label $X_e \in \{-1, 1\}$ independently and uniformly at random. We assign s_1 to a cluster of G uniformly at random. Assume that we have already assigned some vertices of T to clusters of G and next wish to assign x to a cluster. Let V_i be the cluster assigned to the ancestor y of x. Assign x to V_{i-1} if $X_{xy} = -1$ and to V_{i+1} otherwise. Note that this assignment of vertices of T to clusters induces an assignment of the edges of T to the pairs (V_i, V_{i+1}) of clusters. We will show that with positive probability for all $i \in [\ell]$ both the number of vertices of T assigned to V_i and the number of edges of T assigned to the pair (V_i, V_{i+1}) is at most m. This then implies that the assignment corresponds to the required embedding of T into G.

We first show that with probability at least $1 - n^{-1}$ the number of vertices assigned to V_r is at most m for fixed $r \in [\ell]$. (Then a union bound over all $r \in [\ell]$ completes the proof for the statement about the vertices.) We denote by $\phi : V(T) \to \{V_1, \ldots, V_\ell\}$ the assignment of the vertices produced by the random process described above. For each $i \in [k]$, let $S_i' := S_i - V(T(s_i, (\log_\Delta n)/2))$. Thus S_i' is a subforest of S_i and $|S_i| - |S_i'| \leq \Delta^{(\log_\Delta n)/2} = n^{1/2}$ and

$$\sum_{i=1}^{k} |S_i'| = n \pm k n^{1/2} = n \pm n^{1/2+\delta}. \tag{6.1}$$

Let X be the total number of vertices assigned to V_r and for each $i \in [k]$ define X_i to be the number of vertices of S_i' assigned to V_r. Note that

$$X \leq \sum_{i=1}^{k} |S_i \setminus S_i'| + \sum_{i=1}^{k} X_i \leq n^{1/2+\delta} + \sum_{i=1}^{k} X_i. \tag{6.2}$$

Consider the exposure martingale $Y_i := \mathbb{E}[\sum_{j=1}^{k} X_j \mid X_1, \ldots, X_i]$. Thus $Y_0 = \mathbb{E}[\sum_{j=1}^{k} X_j] = \sum_{j=1}^{k} |S_j'|/\ell$. Given any assignment $\phi(s_1), \ldots, \phi(s_i)$ of $s_1, \ldots, s_i$, by Lemma 3.3, there exists $0 < \gamma = \gamma(\ell) < 1$ such that for each $v \in \bigcup_{i'=i}^{k} S_{i'}'$,

$$\mathbb{P}[\phi(v) = V_r \mid \phi(s_1), \ldots, \phi(s_i)] = \frac{1 \pm \gamma^{(\log_\Delta n)/2}}{\ell}.$$

Note that since $\delta \ll 1/\ell$ and $\gamma = \gamma(\ell)$, we have $\gamma^{(\log_\Delta n)/2} = n^{\frac{\log \gamma}{2 \log \Delta}} \leq n^{-\delta}$. Hence for any $i, j \in [k]$ with $i < j$,

$$\mathbb{E}[X_j \mid X_1, \ldots, X_i] = \frac{(1 \pm n^{-\delta})|S_j'|}{\ell}.$$

Let $i \in [k]$ and $x_1, \ldots, x_i, x_i' \in \mathbb{N} \cup \{0\}$ be such that $x_j \leq |S_j'|$ for each $j \in [i]$ and $x_i' \leq |S_i'|$. Let $\mathcal{E}'$ be the event that $X_1 = x_1, \ldots, X_i = x_i$ occurs, and let $\mathcal{E}''$ be the event that $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i = x_i'$. Then

$$\left| \mathbb{E}\left[\sum_{j=1}^{k} X_j \mid \mathcal{E}' \right] - \mathbb{E}\left[\sum_{j=1}^{k} X_j \mid \mathcal{E}'' \right] \right| \leq |x_i - x_i'| + \sum_{j=i+1}^{k} \left| \mathbb{E}[X_j \mid \mathcal{E}'] - \mathbb{E}[X_j \mid \mathcal{E}''] \right|$$

$$\leq |S_i'| + \sum_{j=i+1}^{k} \frac{1}{\ell} \cdot \left| (1 \pm n^{-\delta})|S_j'| - (1 \pm n^{-\delta})|S_j'| \right|$$

$$\leq 2\Delta n^{1-\delta} + n^{-\delta} \sum_{j=i+1}^{k} |S_j'| \overset{(6.1)}{\leq} 4\Delta n^{1-\delta}.$$

Therefore, Y_i is a $4\Delta n^{1-\delta}$-Lipschitz martingale. Thus by Lemma 3.1 and (6.1), we conclude that

$$\mathbb{P}\left[\sum_{i=1}^{k} X_i \neq \frac{n}{\ell} \pm \frac{n}{2\log^2 n}\right] \leq e^{-\frac{(n/(3\log^2 n))^2}{2k(4\Delta n^{1-\delta})^2}} \leq \frac{1}{n}.$$

Together with (6.2), this implies that with probability at least $1 - n^{-1}$, the assignment ϕ uses at most m vertices in V_r.

In order to prove the corresponding statement involving the edges of T, observe that by Lemma 3.3 for any edge $xy \in \bigcup_{i'=i}^{k} E(S'_{i'})$ such that x is an ancestor of y, we have

$$\mathbb{P}[\phi(e) \in E(V_r, V_{r+1}) \mid \phi(s_1), \ldots, \phi(s_i)]$$
$$= \mathbb{P}[\phi(x) = V_r, X_{xy} = 1 \mid \phi(s_1), \ldots, \phi(s_i)] + \mathbb{P}[\phi(x) = V_{r+1}, X_{xy} = -1 \mid \phi(s_1), \ldots, \phi(s_i)]$$
$$= \frac{1 \pm \gamma^{(\log_\Delta n)/2}}{\ell}.$$

Similar arguments as before lead to the desired statement. $\square$

The previous lemma shows that a tree can be embedded "uniformly" into a blow-up of an odd cycle. The next lemma allows us to combine such "uniform" embeddings of several trees and pack them together into a graph which is an internally regular blow-up of a cycle. The lemma is a special case of Lemma 7.1 in [19]. Recall that a spanning subgraph H of $C(\ell, m)$ is internally k-regular if $H[V_i, V_{i+1}]$ is k-regular for all $i \in [\ell]$ (where $V_1, \ldots, V_\ell$ are the clusters of $C(\ell, m)$).

Lemma 6.7 ([19])**.** *Suppose $n, \Delta, \ell, k, s \in \mathbb{N}$ with $1/n \ll 1/s \ll 1/\Delta$ and $1/n \ll 1/k$, and ℓ divides n. Suppose that $0 < \xi < 1$ is such that $s^{2/3} \leq \xi k$. Let $G = C(\ell, n/\ell)$ and let $V_1, \ldots, V_\ell$ be the clusters of G. Suppose that for each $j \in [s]$, the graph L_j is a subgraph of $C(\ell, n/\ell)$ with clusters $X_1^j, \ldots, X_\ell^j$ such that $\Delta(L_j) \leq \Delta$ and for each $i \in [\ell]$, we have*

$$\sum_{j=1}^{s} e(L_j[X_i^j, X_{i+1}^j]) = (1 - 3\xi \pm \xi)\frac{kn}{\ell} \tag{6.3}$$

where $X_{\ell+1}^j := X_1^j$. Then there exist an internally k-regular subgraph H of G and a function ϕ which packs $\{L_1, \ldots, L_s\}$ into H such that $\phi(X_i^j) \subseteq V_i$. Moreover, writing $J_i := H[V_i, V_{i+1}] - \phi(L_1 \cup \cdots \cup L_s)$ for each $i \in [\ell]$, we have $\Delta(J_i) \leq 4\xi k + 2s^{2/3}$.

Theorem 6.8 is a special case of Theorem 6.1 in [19]. Theorem 6.8 guarantees a near-optimal packing of the internally regular graphs given by Lemma 6.7 into a super-regular cycle blow-up. An important feature is that Theorem 6.8 (iii) also allows for a small proportion of vertices to be embedded into a given "target set".

Theorem 6.8 ([19])**.** *Suppose $n, k, \ell, s \in \mathbb{N}$ with $1/n \ll \varepsilon \ll \alpha, d, d_0, 1/k \leq 1$ and $1/n \ll 1/\ell$, and ℓ divides n. Suppose $s \leq \frac{d}{k\ell}(1 - \alpha/2)n$ and the following hold.*

(i) *G is a (ε, d)-super-regular ℓ-cycle $\frac{n}{\ell}$-blow-up with clusters $V_1, \ldots, V_\ell$.*

(ii) *$\mathcal{H} = \{H_1, \ldots, H_s\}$, where each H_i is an internally k-regular subgraph of $C(\ell, n/\ell)$ with clusters $X_1, \ldots, X_\ell$.*

(iii) *For all $j \in [s]$ and $i \in [\ell]$, there is a set $Y_i^j \subseteq X_i$ with $|Y_i^j| \leq \varepsilon n/\ell$ and for each $y \in Y_i^j$, there is a set $A_y^j \subseteq V_i$ with $|A_y^j| \geq d_0 n/\ell$.*

Then there is a function ϕ packing $\mathcal{H}$ into G such that for all $j \in [s]$ and $i \in [\ell]$ the vertices of H_j in X_i are mapped to V_i and such that $\phi(y) \in A_y^j$ for all $y \in Y_1^j \cup \cdots \cup Y_\ell^j$.

We deduce the next lemma from Lemma 6.6, Lemma 6.7 and Theorem 6.8. It guarantees a near-optimal packing of a set of rooted forests into a cycle blow-up which is compatible with given embeddings of the roots into a "root set" R.

Lemma 6.9. *Suppose $n, \ell, \Delta \in \mathbb{N}$ with $1/n \ll 1/\ell \ll \varepsilon \ll \alpha, \eta, 1/\Delta, d, d_0 \leq 1$, and ℓ is odd and divides n. Suppose that G is a graph, $\mathcal{F}$ is a collection of forests, and R is a set with $V(G) \cap R = \emptyset$ satisfying the following:*

(a1) G is an (ε, d)-super-regular ℓ-cycle $\frac{n}{\ell}$-blow-up with clusters $V_1, \ldots, V_\ell$,

(a2) G' is a graph with $R \subseteq V(G') \subseteq V(G) \cup R$ such that $|N_{G'}(u) \cap V(G)| \geq d_0 n$ for each $u \in R$,

(a3) for every $F \in \mathcal{F}$, we have $\eta n \leq |F| \leq (1-\eta)n$, F has $c(F)$ components where $c(F) \leq 2\Delta$ and $\Delta(F) \leq \Delta$,

(a4) $e(\mathcal{F}) = (1 - 2\alpha \pm \alpha)e(G)$,

(a5) for every $F \in \mathcal{F}$, there is a set $r(F) = \{r_F^1, \ldots, r_F^{c(F)}\}$ of vertices belonging to distinct components of F,

(a6) there is a set $U' \subseteq V(G)$ with $|U'| \leq n^{1/2}$ and for every $F \in \mathcal{F}$, there is a set $X_F' \subseteq V(F)$ with $|X_F'| \leq n^{1/2}$, and

(a7) there is a function $\phi' : \{r_F^c : c \in [c(F)], F \in \mathcal{F}\} \to R$ such that $|\phi'^{-1}(v)| \leq \ell^{1/2}$ for every $v \in R$ and $\phi'(r_F^c) \neq \phi'(r_F^{c'})$ whenever $c \neq c'$.

Then there exists a function ϕ which is consistent with ϕ' and packs $\mathcal{F}$ into $G \cup G'$ satisfying the following:

(A1) $\Delta(G - E(\phi(\mathcal{F}))) \leq 10\alpha dn/\ell$,

(A2) $\phi(V(F) \setminus r(F)) \cap R = \emptyset$ for every $F \in \mathcal{F}$, and

(A3) $\phi(X_F') \cap U' = \emptyset$ for every $F \in \mathcal{F}$.

To prove Lemma 6.9 we apply Lemma 6.6 separately to all the components of $F - r(F)$ for all $F \in \mathcal{F}$ to obtain a "uniform" embedding into a blown-up cycle. We also split $\mathcal{F}$ into $\mathcal{F}_1, \ldots, \mathcal{F}_t$ and use Lemma 6.7 for all $i \in [t]$ to pack together these uniform embeddings of $F \in \mathcal{F}_i$ into internally regular graphs H_i. The graphs H_i are then packed into G via Theorem 6.8.

Proof. For each $F \in \mathcal{F}$, let $F^* := F - r(F)$. Then the forest F^* has most $2\Delta^2$ components. Choose an integer k and a constant ξ such that $\varepsilon \ll 1/k \ll \xi \ll \alpha, \eta$. Let

$$t := \frac{\sum_{F \in \mathcal{F}} e(F^*)}{(1-3\xi)kn}. \tag{6.4}$$

We partition $\mathcal{F}$ into $\mathcal{F}_1, \ldots, \mathcal{F}_t$ such that $\sum_{F \in \mathcal{F}_i} e(F^*) = (1 - 3\xi \pm \xi/2)kn$ for all $i \in [t]$. Note that $\eta n|\mathcal{F}_i| \leq e(\mathcal{F}_i) \leq (1-\xi)kn$ by (a3) and so

$$|\mathcal{F}_i| \leq k\eta^{-1}. \tag{6.5}$$

Note that by (a4), we have

$$t = \frac{(1 - 2\alpha \pm 4\alpha/3)e(G)}{(1-3\xi)kn} \leq \frac{(1 - 2\alpha/3)(1+\varepsilon)dn^2/\ell}{(1-3\xi)kn} \leq \frac{d}{k\ell}(1 - \alpha/2)n. \tag{6.6}$$

We claim that for each r_F^c there is an index $j(F, c) \in [\ell]$ such that

(i) $|N_{G'}(\phi'(r_F^c)) \cap V_{j(F,c)}| \geq d_0|V_{j(F,c)}|/2$ and

(ii) $j(F, c) \neq j(F', c')$ if one of the following holds:

(j1) $F, F' \in \mathcal{F}_i$ for some $i \in [\ell]$ and $(F, c) \neq (F', c')$,

(j2) $\phi'(r_F^c) = \phi'(r_{F'}^{c'})$ and $(F, c) \neq (F', c')$.

Indeed, we can choose the $j(F, c)$ greedily. When choosing $j(F, c)$ for $F \in \mathcal{F}_i$, the number of choices excluded by (j1) is at most $2\Delta|\mathcal{F}_i|$ and the number of choices excluded by (j2) is at most $\ell^{1/2}$ by (a7). Hence the total number of excluded choices is at most $2\Delta|\mathcal{F}_i| + \ell^{1/2} \leq 2\Delta k\eta^{-1} + \ell^{1/2} \leq d_0\ell/3$ by (6.5). On the other hand, by (a2), for each r_F^c there are at least $d_0\ell/2$ indices $j \in [\ell]$ such that $|N_{G'}(\phi'(r_F^c)) \cap V_j| \geq d_0|V_j|/2$. This proves the claim.

For each $F \in \mathcal{F}_i$ and each component J of F^*, we apply Lemma 6.6 to J in order to find a partition $(X_1^J, \ldots, X_\ell^J)$ of $V(J)$ such that for each $j \in [\ell]$,

- $|X_j^J| \leq \frac{|J|}{\ell} + \frac{n}{\log n}$.
- $e(J[X_j^J, X_{j'}^J]) = \begin{cases} \frac{e(J)}{\ell} \pm \frac{n}{\log n} & \text{if } |j - j'| = 1 \pmod{\ell}, \\ 0 & \text{otherwise.} \end{cases}$

From this, by rotating the indices and taking disjoint unions of the components, we obtain a partition $(X_1^F, \ldots, X_\ell^F)$ of $V(F^*)$ such that

(α1) $|X_j^F| \leq n/\ell$,

(α2) $e(F^*[X_j^F, X_{j'}^F]) = \begin{cases} (1 \pm \varepsilon)\frac{e(F^*)}{\ell} & \text{if } |j - j'| = 1 \ (\mathrm{mod} \ \ell), \\ 0 & \text{otherwise,} \end{cases}$

(α3) $N_F(r_F^c) \subseteq X_{j(F,c)}^F$.

Note that for each $j \in [\ell]$,

$$\sum_{F \in \mathcal{F}_i} e(F^*[X_j^F, X_{j+1}^F]) \overset{(\alpha 2)}{=} \sum_{F \in \mathcal{F}_i} (1 \pm \varepsilon)\frac{e(F^*)}{\ell} = (1 - 3\xi \pm \xi)\frac{kn}{\ell}.$$

Moreover, $|\mathcal{F}_i|^{2/3} \leq \xi k$ by (6.5). Thus we can apply Lemma 6.7 with $|\mathcal{F}_i|$ playing the role of s, in order to obtain a function τ_i packing $\{F^* : F \in \mathcal{F}_i\}$ into H_i, where H_i is an internally k-regular subgraph H_i of $C(\ell, n/\ell)$, such that τ_i maps X_j^F into the jth cluster X_j of $C(\ell, n/\ell)$ for every $F \in \mathcal{F}_i$ and every $j \in [\ell]$. Also for all $x \in V(H_i)$ we have

$$d_{H_i}(x) - \sum_{F \in \mathcal{F}_i} d_{\tau_i(F^*)}(x) \leq 8\xi k + 4|\mathcal{F}_i|^{2/3} \leq 12\xi k. \tag{6.7}$$

Note that $\tau_i(N_F(r_F^c)) \cap \tau_i(N_{F'}(r_{F'}^{c'})) = \emptyset$ for all $F \neq F' \in \mathcal{F}_i$ and $c \in [c(F)], c' \in [c(F')]$ by (j1). For all $F \in \mathcal{F}_i$, $c \in [c(F)]$ and $x \in N_F(r_F^c)$, let

$$A_F^c(x) := N_{G'}(\phi'(r_F^c)) \cap V_{j(F,c)}.$$

Thus $|A_F^c(x)| \geq d_0 n/(2\ell)$ by (i). For all $i \in [t]$, let

$$Y_i := \bigcup_{F \in \mathcal{F}_i} \left(\tau_i(X_F' \setminus r(F)) \cup \bigcup_{c \in [c(F)]} \tau_i(N_F(r_F^c)) \right).$$

Then $Y_i \subseteq V(H_i)$ and $|Y_i| \leq |\mathcal{F}_i|(n^{1/2} + 2\Delta^2) \leq \varepsilon n/\ell$ by (a6) and (6.5).

For all $i \in [t]$, $j \in [\ell]$ and $y \in Y_i \cap X_j$, we let

$$A_y^i = \begin{cases} A_F^c(\tau_i^{-1}(y)) \setminus U' & \text{if } y \in \tau_i(N_F(r_F^c)) \text{ for } F \in \mathcal{F}_i, c \in [c(F)], \\ V_j \setminus U' & \text{otherwise.} \end{cases}$$

Note that (α3) implies $\tau_i(N_F(r_F^c)) \subseteq X_{j(F,c)}$. So the sets A_y^i are well-defined since the pair (F, c) is unique by (j1). Moreover, $|A_y^i| \geq d_0 n/(3\ell)$ for each $y \in Y_i$. Together with (a1) and (6.6), this means that we can apply Theorem 6.8 to G and $\mathcal{H} := \{H_1, \ldots, H_t\}$ with $d_0/3$ playing the role of d_0 in order to find a function ϕ'' which packs $\mathcal{H}$ into G such that for all $i \in [t]$ and $j \in [\ell]$ the vertices of H_i in X_j are mapped to V_j and such that $\phi''(y) \in A_y^i$ for all $y \in Y_i$. For $x \in \bigcup_{F \in \mathcal{F}} V(F)$, we let

$$\phi(x) = \begin{cases} \phi''(\tau_i(x)) & \text{if } x \in V(F) \setminus r(F) \text{ for some } F \in \mathcal{F}_i, i \in [t], \\ \phi'(x) & \text{if } x \in r(F) \text{ for some } F \in \mathcal{F}_i, i \in [t]. \end{cases}$$

Then for all $F \in \mathcal{F}_i$ and $x \in N_F(r_F^c)$,

$$\phi(x) = \phi''(\tau_i(x)) \in A_{\tau_i(x)}^i \subseteq A_F^c(x) \subseteq N_{G'}(\phi(r_F^c)).$$

Moreover, if $F, F' \in \mathcal{F}$, $c \in [c(F)], c' \in [c(F')]$, $(F, c) \neq (F', c')$ and $\phi(r_F^c) = \phi(r_{F'}^{c'})$, then ($j$2) implies

$$\phi(N_F(r_F^c)) \cap \phi(N_{F'}(r_{F'}^{c'})) = \emptyset.$$

Thus ϕ packs $\mathcal{F}$ into $G \cup G'$ such that ϕ is consistent with ϕ' and (A2) holds. Again, (A3) also holds since $A_{\tau_i(x)}^i \cap U' = \emptyset$ for all $x \in X_F'$. To show (A1), let $G'' := G - E(\phi(\mathcal{F}))$. Then for each vertex $v \in V(G)$

$$d_{G''}(v) = d_G(v) - \sum_{F \in \mathcal{F}} d_{\phi(F)}(v) \overset{(6.7)}{\leq} \frac{2(d + \varepsilon)n}{\ell} - \sum_{i=1}^{t} (2k - 12\xi k)$$

$$\overset{(6.4)}{\leq} \frac{2(d + \varepsilon)n}{\ell} - \frac{(2 - 12\xi)e(\mathcal{F})}{(1 - 2\xi)n} \overset{(a4)}{\leq} \frac{10\alpha dn}{\ell}.$$

Thus (A1) also holds.   □

7. Covering parts of graphs with trees

In this section we present four results which will be used in Section 8. In each of these lemmas we use a collection of trees or forests to cover certain given sets of edges of a graph G. In the first lemma, we are given a small "exceptional" set $V_0 \subseteq V(G)$ and a collection $\mathcal{F}$ of small forests, each consisting of two components. The lemma guarantees a packing of $\mathcal{F}$ which covers almost all edges of G incident to V_0, but not too many edges incident to any vertex in $V(G) \setminus V_0$.

Throughout the section, we write $S(c)$ for a star with centre c.

Lemma 7.1. *Suppose* $n, \Delta \in \mathbb{N}$ *and* $1/n \ll \varepsilon \ll \eta \ll \delta_2 \ll 1/\Delta \leq 1/2$. *Let* G *be a graph on* n *vertices and let* $V_0 \subseteq V(G)$ *with* $|V_0| \leq \varepsilon n$. *Let* $\mathcal{F}$ *be a collection of forests with* $|\mathcal{F}| = (1/2 \pm \varepsilon)n$ *and* $\Delta(F) \leq \Delta$ *for every* $F \in \mathcal{F}$. *Let* $\{W_F\}_{F \in \mathcal{F}}$ *be a collection of vertex sets with* $W_F \subseteq V(G)$ *for every* $F \in \mathcal{F}$. *Suppose*

(c1.1) $d_G(u,v) \geq \delta_2^2 n/2$ *for all* $u, v \in V(G) \setminus V_0$,
(c1.2) *for every* $F \in \mathcal{F}$, *the forest* F *consists of two components* T_F^1, T_F^2 *such that* $\eta n/\Delta \leq |T_F^i| < \eta^{3/4} n$ *for all* $i \in [2]$,
(c1.3) *for every* $F \in \mathcal{F}$, *there is a* 5-*independent set* $R_F := \{y_F^1, y_F^2, z_F^1, z_F^2\}$ *such that* $y_F^i, z_F^i \in V(T_F^i)$ *and there is an injective function* $\tau_F' : R_F \to V(G) \setminus (V_0 \cup W_F)$ *such that* $|\{F : u \in \tau_F'(R_F)\}| \leq \varepsilon n$ *for each* $u \in V(G)$,
(c1.4) $|W_F| \leq \Delta^2$, *and*
(c1.5) $|\{F \in \mathcal{F} : u \in W_F\}| \leq \varepsilon n$ *for every* $u \in V(G)$.

Then there is a function τ *packing* $\mathcal{F}$ *into* G *which is consistent with* $\{\tau_F'\}_{F \in \mathcal{F}}$ *such that*

(C1.1) $\tau(F) \cap W_F = \emptyset$ *for all* $F \in \mathcal{F}$,
(C1.2) $d_{\tau(\mathcal{F})}(v) \leq \eta^{1/3} n$ *for all* $v \in V(G) \setminus V_0$, *and*
(C1.3) $d_{\tau(\mathcal{F})}(v) \geq d_G(v) - \varepsilon^{1/2} n$ *for all* $v \in V_0$.

When we apply Lemma 7.1 in Section 8, F will be a part of a larger forest. The remaining parts of this larger forest are attached to F at the "root set" R_F.

Proof. We write $\mathcal{F} = \{F_1, \ldots, F_t\}$, $\tau_i' := \tau_{F_i}'$, and $p_R(u, i) := |\{i' \in [i] : u \in \tau_{i'}'(R_{F_{i'}})\}|$ for all $i \in [t]$ and $u \in V(G)$. Thus (c1.3) implies that for all $i \in [t]$ and $u \in V(G)$,

$$p_R(u, i) \leq \varepsilon n. \tag{7.1}$$

We use an algorithmic approach and iteratively find an embedding of $F_i \in \mathcal{F}$ into G.

To be precise, we will construct embeddings $\tau_1, \ldots, \tau_t$ and define sets $A_i, X_i \subseteq V(G) \setminus (V_0 \cup W_{F_i})$ for all $i \in [t]$ such that for all $i \in [t]$ the following hold:

$(A1)_i$ $|A_i| = \eta^{2/3} n$.
$(A2)_i$ For all $u \in V(G) \setminus V_0$, let

$$p_A(u, i) := |\{i' \in [i] : u \in A_{i'} \setminus (X_{i'} \cup \tau_{i'}'(R_{F_{i'}}))\}| \text{ and } p_X(u, i) := |\{i' \in [i] : u \in X_{i'}\}|.$$

Then $p_A(u, i) \leq \eta^{1/2} n$ and $p_X(u, i) \leq \varepsilon n$.
$(A3)_i$ Let $G_i := G - \bigcup_{j=1}^{i-1} E(\tau_j(F_j))$. Then $d_{G_i, A_i}(u, v) \geq 2\eta^{3/4} n$ for every $u, v \in V(G) \setminus V_0$.
$(A4)_i$ τ_i embeds F_i into $G_i[(V_0 \setminus W_{F_i}) \cup A_i \cup X_i]$.
$(A5)_i$ Let $W_i' := \{v \in V_0 \setminus W_{F_i} : d_{G_i, V(G_i) \setminus (V_0 \cup W_{F_i})}(v) \geq \Delta \varepsilon n + 4\}$. Then $W_i' \subseteq \tau_i(F_i)$ and for every $w \in W_i'$, the vertex $\tau_i^{-1}(w)$ is a non-leaf of F_i.
$(A6)_i$ $X_i = N_{\tau_i(F_i)}(W_i')$.

We will then show that $\tau := \bigcup_{j=1}^t \tau_i$ satisfies (C1.1)–(C1.3). In particular, $(A5)_1$–$(A5)_t$ will ensure that (C1.3) holds.

Assume that for some $i \in [t]$ and all $j < i$ we have constructed τ_j, A_j, X_j satisfying $(A1)_j$–$(A6)_j$. Note that for $u, v \in V(G) \setminus V_0$, we have

$$d_{G_i}(u,v) \overset{(A4)_1-(A4)_{i-1}}{\geq} d_G(u,v) - \sum_{w \in \{u,v\}} \Delta(p_A(w, i-1) + p_X(w, i-1) + p_R(w, i-1))$$

$$\overset{(c1.1),(7.1),(A2)_{i-1}}{\geq} \delta_2^2 n/2 - 2\Delta\eta^{1/2}n - 4\Delta\varepsilon n \geq \delta_2^2 n/3. \tag{7.2}$$

Next we show how to choose an embedding τ_i and sets $A_i, X_i \subseteq V(G) \setminus (V_0 \cup W_{F_i})$ satisfying $(A1)_i$–$(A6)_i$. Let $B := \{w \in V(G) \setminus V_0 : p_A(w, i-1) \geq \eta^{1/2}n/2\}$. Then $|B| \leq 2(\eta^{1/2}n)^{-1} \sum_{j<i} |A_j| \leq 2\eta^{1/6}n$, because $(A1)_j$ holds for all $j < i$ and since $i \leq t \leq n$. Consider a set A' of $\eta^{2/3}n - 4$ vertices in $V(G) \setminus (V_0 \cup B \cup W_{F_i} \cup \tau_i'(R_{F_i}))$ chosen uniformly at random. Note that for any two distinct vertices $u, v \in V(G_i) \setminus V_0$, we have

$$d_{G_i - (V_0 \cup B \cup W_{F_i} \cup \tau_i'(R_{F_i}))}(u,v) \geq d_{G_i}(u,v) - 2\eta^{1/6}n - |V_0 \cup W_{F_i} \cup \tau_i'(R_{F_i})|$$

$$\overset{(7.2),(c1.4)}{\geq} \delta_2^2 n/3 - 2\eta^{1/6}n - \varepsilon n - \Delta^2 - 4$$

$$\geq \delta_2^2 n/4.$$

A straightforward application of Lemma 3.2 shows that for any two distinct vertices $u, v \in V(G) \setminus V_0$,

$$d_{G_i, A'}(u,v) \geq \eta^{2/3}\delta_2^2 n/8 > 2\eta^{3/4}n \tag{7.3}$$

with probability at least $1 - n^{-3}$. Thus there is a choice of A' that satisfies (7.3) for all $u, v \in V(G) \setminus V_0$. Let $A_i := A' \cup \tau_i'(R_{F_i})$ for this choice. Then clearly $(A1)_i$ and $(A3)_i$ hold.

We write $W_i' = \{w_1, \ldots, w_r\}$. Note that $r \leq |V_0| \leq \varepsilon n$. Consider the set $V_{nl}(F_i)$ of all non-leaf vertices of F_i. Then $|V_{nl}(F_i)| \geq |T_{F_i}^1|/\Delta \geq \eta n/\Delta^2 \geq \Delta^5(r+4)$ by (c1.2). Recall that R_{F_i} is a 5-independent set in F_i. So by Proposition 6.2 (I), there exists a 5-independent set $I \cup R_{F_i} \subseteq V_{nl}(F_i) \cup R_{F_i}$ of F_i with $|I| = r$ and $I \cap R_{F_i} = \emptyset$. Write $I = \{v_1, \ldots, v_r\}$ and note that no vertex in I is a leaf of F_i.

To ensure $(A5)_i$, we will construct τ_i in such a way that $\tau_i(I) = W_i'$. To see that this can be done, recall that the definition of W_i' gives that $d_{G_i, V(G_i) \setminus (V_0 \cup W_{F_i})}(w) \geq \Delta\varepsilon n + 4$ for all $w \in W_i'$. Thus there is a collection of vertex-disjoint stars $S(w_1), \ldots, S(w_r)$ in G_i, where $S(w_j)$ has $d_{F_i}(v_j)$ leaves which avoid $V_0 \cup W_{F_i} \cup \tau_i'(R_{F_i})$. Let X_i be the set of all these leaves of $S(w_1), \ldots, S(w_r)$. Let ϕ'' be the bijection between I and W_i' defined by $\phi''(v_j) := w_j$ for every $j \in [r]$. Let $J := N_{F_i}(I)$ and let $\phi' : J \cup R_{F_i} \to X_i \cup \tau_i'(R_{F_i})$ be a bijective function consistent with τ_i' such that for every $j \in [r]$,

$$\phi'(N_{F_i}(v_j)) = V(S(w_j)) \setminus \{w_j\}.$$

Let $F_i' := F_i - I$. Note that $J \cup R_{F_i}$ is a 3-independent set in F_i'. Moreover, $(A3)_i$ and (c1.2) imply that any two vertices $u, v \in V(G) \setminus V_0$ satisfy $d_{G_i, A_i \cup X_i}(u,v) \geq 2\eta^{3/4}n > |F_i'|$. So we can apply Lemma 3.19 with $|A_i \cup X_i|, |F_i'|$ playing the roles of n, k, to obtain a function ϕ_i embedding F_i' into $G_i[A_i \cup X_i]$ which is consistent with ϕ'. Let $\tau_i := \phi_i \cup \phi''$. Thus τ_i embeds F_i into $G_i[(V_0 \setminus W_{F_i}) \cup A_i \cup X_i]$ and satisfies $\tau_i^{-1}(W_i') = I$ and $X_i = N_{\tau_i(F_i)}(W_i')$. Hence $(A4)_i$–$(A6)_i$ hold.

Note that by definition of A', for any $u \in V(G) \setminus V_0$, we obtain that $p_A(u, i) \leq \eta^{1/2}n$. Since $E(\tau_j(F_j)) \cap E(\tau_{j'}(F_{j'})) = \emptyset$ whenever $1 \leq j \leq j' \leq i$, $(A6)_i$ implies $p_X(u, i) \leq d_{G, V_0}(u) \leq |V_0| \leq \varepsilon n$ for each $u \in V(G) \setminus V_0$. Thus $(A2)_i$ holds. This shows that we can construct τ_i such that $(A1)_i$–$(A6)_i$ hold.

Let $\tau := \bigcup_{i=1}^t \tau_i$. Then τ packs $\mathcal{F}$ into G and is consistent with $\{\tau_F'\}_{F \in \mathcal{F}}$. Clearly (C1.1) holds by $(A4)_1$–$(A4)_t$. Moreover, by $(A2)_t$, $(A4)_1$–$(A4)_t$ and (7.1), we have $d_{\tau(\mathcal{F})}(v) \leq \Delta(p_A(v, t) + p_X(v, t) + p_R(v, t)) \leq \eta^{1/3}n$ for all $v \in V(G) \setminus V_0$ and thus (C1.2) holds.

Note that $(A5)_i$ implies for every $w \in W_i'$ that

$$d_{G_i}(w) - d_{G_{i+1}}(w) \geq 2. \tag{7.4}$$

To show (C1.3), assume for a contradiction that there is a vertex $v \in V_0$ which does not satisfy (C1.3), i.e. that $d_{G_{t+1}}(v) \geq \varepsilon^{1/2} n$. Then

$$|\{i \in [t] : v \in W_i'\}| \geq t - |\{i \in [t] : v \in W_{F_i}\}| \overset{(c1.5)}{\geq} n/2 - 2\varepsilon n.$$

Together with (7.4) this yields that $d_{G_{t+1}}(v) \leq d_G(v) - n + 4\varepsilon n$, which is a contradiction to the assumption that $d_{G_{t+1}}(w) > \varepsilon^{1/2} n$. □

Given a collection of trees $\mathcal{T}$, a graph G with vertex partition (A, B) such that $G[B]$ is sparse, every vertex in B has many neighbours in A, and $G[A]$ is quasi-random, the following lemma allows us to find a packing of $\mathcal{T}$ into G which covers all edges of $G[B]$.

Lemma 7.2. *Suppose $n, \Delta \in \mathbb{N} \setminus \{1\}$ and $1/n \ll \varepsilon \ll \delta_2 \ll \delta_1 \ll \gamma \ll 1/\Delta, p \leq 1$. Let G be a graph on n vertices and A be a set of at least $\delta_1 n$ vertices of G. Let $B := V(G) \setminus A$, $H := G[A, B]$, and let $\mathcal{T}$ be a collection of trees with $\Delta(T) \leq \Delta$ for every $T \in \mathcal{T}$. Suppose*

(c2.1) $G[A]$ is $(\gamma^{1/10}, p)$-quasi-random,
(c2.2) $\Delta(G[B]) \leq 2\delta_2 n$,
(c2.3) $d_H(u) \geq \delta_1 |A|/20$ for every $u \in B$,
(c2.4) $\delta_2^{1/2} n \leq |\mathcal{T}| \leq \gamma |A|$,
(c2.5) every $T \in \mathcal{T}$ satisfies $\delta_1 n \leq |T| \leq p^2 |A|/2$ and has a root y_T; moreover, there is a set $W_T \subseteq V(G)$ with $|W_T| \leq \Delta^2$ and a function $\tau_T' : \{y_T\} \to A \setminus W_T$, and
(c2.6) $|\{T \in \mathcal{T} : u \in W_T\}| \leq \varepsilon n$ for every $u \in V(G)$.

Then there is a function τ packing $\mathcal{T}$ into G which is consistent with $\{\tau_T'\}_{T \in \mathcal{T}}$ such that

(C2.1) $E(G[B]) \subseteq E(\tau(\mathcal{T}))$ and
(C2.2) $W_T \cap \tau(T) = \emptyset$ for each $T \in \mathcal{T}$.

Proof. We write $\mathcal{T} = \{T_1, \ldots, T_t\}$ and let $\tau_i' := \tau_{T_i}'$ for every $i \in [t]$. Our proof strategy is as follows. We first decompose $G[B]$ into t edge-disjoint small matchings $M_1, \ldots, M_t$ such that $W_{T_i} \cap V(M_i) = \emptyset$ for all $i \in [t]$. We then choose a suitable matching N in $T_i - \{y_{T_i}\}$ and embed T_i into G in such a way that N is mapped to M_i and y_{T_i} is mapped to $\tau_i'(y_{T_i})$.

For every edge $e \in E(G)$, let $w(e) := |\{i \in [t] : W_{T_i} \cap V(e) \neq \emptyset\}|$. Hence $w(e) \leq 2\varepsilon n$ by (c2.6). For every set of edges $E \subseteq E(G)$, we define $w(E) := \sum_{e \in E} w(e)$. Observe that (c2.4) and (c2.5) imply that $w(E) \leq \Delta^2 \gamma n$ for any matching E.

By (c2.2), the graph $G[B]$ has an edge colouring using at most $3\delta_2 n$ colours. This gives rise to a partition of the edge set of $G[B]$ into matchings $M_1', \ldots, M_{3\delta_2 n}'$. We apply Proposition 3.18 to M_i' in order to split M_i' into $q := \delta_2^{-1/2}/3$ possibly empty matchings $M_1^i, \ldots, M_q^i$ such that $|M_j^i| \leq 3\delta_2^{1/2} n$ and $w(M_j^i) \leq 3\delta_2^{1/2} \cdot 2\Delta^2 \gamma n + 2\varepsilon n \leq \delta_2^{1/2} n/2$ for all $i \in [3\delta_2 n]$ and $j \in [q]$.

Note that $q \cdot 3\delta_2 n \leq t$ by (c2.4). So by adding empty matchings to $\{M_j^i\}_{i \in [3\delta_2 n], j \in [q]}$ if necessary, we obtain a collection $\mathcal{M} = \{M_1, \ldots, M_t\}$ of matchings partitioning $E(G[B])$ such that for all $i \in [t]$, we have

$$|M_i| \leq 3\delta_2^{1/2} n \quad \text{and} \quad w(M_i) \leq \delta_2^{1/2} n/2. \tag{7.5}$$

By (c2.2), observe that for every $v \in V(G)$

$$|\{i \in [t] : v \in M_i\}| \leq 2\delta_2 n. \tag{7.6}$$

Also recall $|\mathcal{M}| = |\mathcal{T}| = t$. Consider an auxiliary bipartite graph $\mathcal{H}$ with vertex partition $(\mathcal{T}, \mathcal{M})$ such that $T_i M_j \in E(\mathcal{H})$ if $W_{T_i} \cap V(M_j) = \emptyset$. Then, for every tree $T \in \mathcal{T}$, we conclude

$$d_{\mathcal{H}}(T) \geq |\mathcal{M}| - |\{i \in [t] : W_T \cap V(M_i) \neq \emptyset\}| \overset{(7.6)}{\geq} t - 2\delta_2 n |W_T| \overset{(c2.5)}{>} t/2.$$

Furthermore, $d_{\mathcal{H}}(M_i) = |\mathcal{T}| - w(M_i) \geq t/2$ for every $i \in [t]$ by (7.5). Thus $\mathcal{H}$ has a perfect matching. By relabelling, we may assume that $T_1 M_1, \ldots, T_t M_t$ is a perfect matching in $\mathcal{H}$ and hence $W_{T_i} \cap V(M_i) = \emptyset$ for every $i \in [t]$.

Next, we iteratively construct edge-disjoint embeddings τ_i so that for all $i \in [t]$ the following hold:

(P1)$_i$ τ_i embeds T_i into $G[(A \cup V(M_i)) \setminus W_{T_i}]$ and
(P2)$_i$ $E(\tau_i(T_i)) \cap E(G[B]) = M_i$.

Assume for some $i \in [t]$ we have defined $\tau_1, \ldots, \tau_{i-1}$ so that τ_j satisfies (P1)$_j$ and (P2)$_j$ for all $j < i$. Let $G_i := G - \bigcup_{j=1}^{i-1} E(\tau_j(T_j))$. Consider any $v \in B$. If $v \in \tau_j(T_j)$, then (P1)$_j$ implies that $v \in M_j$. Hence,

$$d_{G_i,A}(v) \overset{(7.6)}{\geq} d_{H,A}(v) - 2\delta_2 \Delta n \overset{(c2.3)}{\geq} \delta_1 |A|/25. \tag{7.7}$$

We write $m_i := |M_i|$ and $M_i = \{u_1 v_1, \ldots, u_{m_i} v_{m_i}\}$. By (7.5) and (7.7), there is a collection of stars $\{S(u_j), S(v_j)\}_{j \in [m_i]}$ in G_i so that all these stars are disjoint from each other and each star has Δ leaves in $A \setminus (W_{T_i} \cup \{\tau_i'(y_{T_i})\})$. Let $u_j^1, \ldots, u_j^\Delta$ be the leaves of $S(u_j)$ and let $v_j^1, \ldots, v_j^\Delta$ be the leaves of $S(v_j)$.

Let F_i be the forest obtained from T_i by first removing all vertices of distance at most 4 from y_{T_i} and secondly removing all isolated vertices. Note that $|F_i| \geq \delta_1 n - 2\Delta^5 \geq 4\Delta^5 m_i$ by (c2.5) and (7.5). Thus by Proposition 6.2, F_i has a 5-independent matching $N = \{x_1 z_1, \ldots, x_{m_i} z_{m_i}\}$. We will construct τ_i in such a way that edges $x_j z_j$ are mapped to $u_j v_j$ for each $j \in [m_i]$. To achieve this, let τ_i'' be an injective function mapping $N_{T_i}(x_j) \setminus \{z_j\}$ to the leaves of $S(u_j)$ and $N_{T_i}(z_j) \setminus \{x_j\}$ to the leaves of $S(v_j)$ for every $j \in [m_i]$ and so that $\tau_i''(y_{T_i}) := \tau_i'(y_{T_i})$. Let

$$J := \bigcup_{j=1}^{m_i} (N_{T_i}(x_j) \setminus \{z_j\}) \cup \bigcup_{j=1}^{m_i} (N_{T_i}(z_j) \setminus \{x_j\}.$$

Note that $J \cup \{y_{T_i}\}$ forms a 3-independent set in $F_i' := T_i - V(N)$ by the choice of N.

For each $v \in A$, by (c2.4), we obtain

$$d_G(v) - d_{G_i}(v) \leq i\Delta \leq t\Delta \leq \Delta \gamma |A|.$$

This together with (c2.1) and Proposition 3.9 implies that $G_i[A \setminus W_{T_i}]$ is $(\gamma^{1/11}, p)$-quasi-random. In particular, $d_{G_i, A \setminus W_{T_i}}(u, v) \geq p^2 |A|/2 \geq |F_i'|$ for any $u, v \in A \setminus W_{T_i}$ (where the final inequality follows from (c2.5)). Thus Lemma 3.19 implies that there exists a function ϕ_i embedding F_i' into $G_i[A \setminus W_{T_i}]$ consistent with τ_i''. Let $\tau_i : V(T_i) \to G_i$ be

$$\tau_i(z) := \begin{cases} u_j & \text{if } z = x_j \text{ for some } j \in [m_i], \\ v_j & \text{if } z = z_j \text{ for some } j \in [m_i], \\ \phi_i(z) & \text{otherwise.} \end{cases}$$

Then τ_i satisfies (P1)$_i$ and (P2)$_i$. Let $\tau := \bigcup_{i=1}^t \tau_i$. By construction, (C2.1) and (C2.2) hold. $\quad\square$

Given a collection $\mathcal{T}$ of trees and a graph G with a vertex partition (A, B) such that B is independent and $G[A, B]$ has very few edges, the next lemma guarantees a packing of $\mathcal{T}$ into G which covers all edges of G between A and B. The purpose of the lemma will be to adjust the parity of the leftover degrees of the vertices in B prior to applying Lemma 7.4.

Lemma 7.3. *Suppose $n, \Delta \in \mathbb{N} \setminus \{1\}$ and $1/n \ll \varepsilon \ll \delta_1 \ll \gamma \ll 1/\Delta, p \leq 1$. Let G be a graph on n vertices and A be a set of at least $\gamma^2 n$ vertices of G. Let $B := V(G) \setminus A$, $H := G[A, B]$, and let $\mathcal{T}$ be a collection of trees with $\Delta(T) \leq \Delta$ for every $T \in \mathcal{T}$. Suppose*

(c3.1) $G[A]$ is $(\gamma^{1/10}, p)$-quasi-random,
(c3.2) B is an independent set of G,
(c3.3) $|E(H)| \leq \delta_1 n$ and $|E(H)| + 2\varepsilon n \leq |\mathcal{T}| \leq \gamma |A|$,
(c3.4) every $T \in \mathcal{T}$ satisfies $2\Delta^4 \leq |T| \leq p^2 |A|/2$ and has a root y_T; moreover, there is a set $W_T \subseteq V(G)$ with $|W_T| \leq \Delta^2$ and a function $\tau_T' : \{y_T\} \to A \setminus W_T$, and
(c3.5) $|\{T \in \mathcal{T} : u \in W_T \cup \{\tau_T'(y_T)\}\}| \leq \varepsilon n$ for every $u \in V(G)$.

Then there is a function τ packing $\mathcal{T}$ into G which is consistent with $\{\tau_T'\}_{T \in \mathcal{T}}$ such that $E(H) \subseteq E(\tau(\mathcal{T}))$ and $W_T \cap \tau(T) = \emptyset$ for every $T \in \mathcal{T}$.

Proof. We write $\mathcal{T} = \{T_1, \ldots, T_t\}$ and $E(H) = \{u_1 v_1, \ldots, u_k v_k\}$ with $u_i \in A$ for all $i \in [k]$. Let $\tau_i' := \tau_{T_i}'$. For each $i \in [k]$, we choose $j_i \in [t]$ such that $v_i \notin W_{T_{j_i}}, u_i \notin W_{T_{j_i}} \cup \{\tau_{j_i}'(y_{T_{j_i}})\}$ and $j_1, \ldots, j_k$ are all distinct. This is possible using a greedy approach, since $t \geq k + \varepsilon n + \varepsilon n$ and

by (c3.5). By relabeling, we may assume that $v_i \notin W_{T_i}$ and $u_i \notin W_{T_i} \cup \{\tau_i'(y_{T_i})\}$. For $k < i \leq t$, let u_i be an arbitrary vertex in $A \setminus (W_{T_i} \cup \{\tau_{T_i}'(y_{T_i})\})$.

We sequentially construct $\tau_1, \ldots, \tau_t$ such that for all $i \in [t]$

(E1)$_i$ τ_i is consistent with τ_i' and embeds T_i into $G_i - W_{T_i}$ where $G_i := G - \bigcup_{j=1}^{i-1} E(\tau_j(T_j))$,

(E2)$_i$ if $i \in [k]$, then $E(\tau_i(T_i) \cap H) = \{u_i v_i\}$,

(E3)$_i$ if $i \in [k]$, then $\tau_i(T_i) \setminus \{v_i\} \subseteq A$, and if $i > k$, then $\tau_i(T_i) \subseteq A$.

Assume that for some $i \in [t]$ we have constructed $\tau_1, \ldots, \tau_{i-1}$ such that (E1)$_j$–(E3)$_j$ hold for all $j < i$. By (c3.3), $d_G(v) - d_{G_i}(v) \leq \Delta t \leq \Delta \gamma |A|$ for all $v \in V(G)$, and hence $G_i[A \setminus W_{T_i}]$ is $(\gamma^{1/11}, p)$-quasi-random by (c3.1) and Proposition 3.9. Let ℓ_i be a leaf of T_i which has distance at least 4 from y_{T_i}, and let x_i be the unique neighbour of ℓ_i in T_i. Such a leaf exists, since $|T_i| \geq 2\Delta^4$. Let

$$T_i' := \begin{cases} T_i - \{\ell_i\} & \text{if } i \in [k], \\ T_i & \text{if } i \in [t] \setminus [k]. \end{cases}$$

Let $\tau_i''(y_{T_i}) := \tau_i'(y_{T_i})$ and $\tau_i''(x_i) := u_i$.

Observe that $\{y_{T_i}, x_i\}$ forms a 3-independent set of T_i' and for any $u, v \in A \setminus W_{T_i}$, we have $d_{G_i, A \setminus W_{T_i}}(u, v) \geq p^2 |A|/2 \geq |T_i'|$ by (c3.4). Apply Lemma 3.19 to $T_i', G_i[A \setminus W_{T_i}]$ and τ_{T_i}'' to obtain a function ϕ_i embedding T_i' into $G_i[A \setminus W_{T_i}]$ which is consistent with τ_i''. Let $\tau_i : V(T_i) \to G_i$ be defined by

$$\tau_i(x) := \begin{cases} v_i & \text{if } x = \ell_i \text{ and } i \in [k], \\ \phi_i(x) & \text{otherwise.} \end{cases}$$

Then (E1)$_i$–(E3)$_i$ hold. Define $\tau := \bigcup_{i=1}^t \tau_i$, then τ is consistent with $\{\tau_T'\}_{T \in \mathcal{T}}$ and $E(H) \subseteq E(\tau(\mathcal{T}))$. □

The following lemma is a variant of Lemma 7.3 in which $G[A, B]$ is allowed to have more edges than in Lemma 7.3. It guarantees a packing of $\mathcal{T}$ into G which covers all edges between A and B apart from precisely one edge at every vertex of odd degree in B. In particular, once we have adjusted the parity of the leftover degrees of the vertices in B via Lemma 7.3, we can cover all remaining leftover edges via Lemma 7.4. The proof of Lemma 7.4 is more difficult than that of Lemma 7.3 since the covering is more "efficient": while in Lemma 7.3 we used one tree for each leftover edge, here each tree covers a linear number of leftover edges.

Lemma 7.4. *Suppose $n, \Delta \in \mathbb{N} \setminus \{1\}$ and $1/n \ll \varepsilon \ll \delta_1 \ll \gamma \ll 1/\Delta, p \leq 1$. Let G be a graph on n vertices and A be a set of at least $\gamma^2 n$ vertices of G. Let $B := V(G) \setminus A$, $H := G[A, B]$, and let $\mathcal{T}$ be a collection of trees with $\Delta(T) \leq \Delta$ for every $T \in \mathcal{T}$. Suppose*

(c4.1) $G[A]$ is $(\gamma^{1/10}, p)$-quasi-random,

(c4.2) B is an independent set of G,

(c4.3) $\Delta(H) \leq 2\delta_1 n$,

(c4.4) $\delta_1^{1/2} n \leq |\mathcal{T}| \leq \gamma |A|$,

(c4.5) every $T \in \mathcal{T}$ satisfies $\delta_1^{1/4} n \leq |T| \leq p^2 |A|/4$ and has a root y_T; moreover, there is a set $W_T \subseteq V(G)$ with $|W_T| \leq \Delta^2$ and a function $\tau_T' : \{y_T\} \to A \setminus W_T$, and

(c4.6) $|\{T \in \mathcal{T} : u \in W_T \cup \{\tau_T'(y_T)\}\}| \leq \varepsilon n$ for every $u \in V(G)$.

Then there is a function τ packing $\mathcal{T}$ into G which is consistent with $\{\tau_T'\}_{T \in \mathcal{T}}$ such that

(C4.1) writing $H^* := H - E(\tau(\mathcal{T}))$, we have $d_{H^*}(v) \leq 1$ for every $v \in B$,

(C4.2) $W_T \cap \tau(T) = \emptyset$ for every $T \in \mathcal{T}$, and

(C4.3) $d_{\tau(\mathcal{T})}(v)$ is even for all $v \in B$.

Proof. We define $B^{\text{odd}} := \{v \in B, d_H(v) \text{ is odd}\}$ and $t := |\mathcal{T}|$. For each $v \in B^{\text{odd}}$, we select an edge e_v joining v and A, and let $E^{\text{odd}} := \{e_v : v \in B^{\text{odd}}\}$. We define $G' := G - E^{\text{odd}}$. Our packing τ of $\mathcal{T}$ will cover all edges in $G'[A, B]$ (but no edges from E^{odd}). This will ensure that (C4.1) and (C4.3) are satisfied.

We let $\mathcal{T}^2 = \{\tilde{T}_1, T_1^{\diamond}, \tilde{T}_2, T_2^{\diamond}, \ldots, \tilde{T}_{t_2}, T_{t_2}^{\diamond}\}$ be a maximal collection of an even number of trees in $\mathcal{T}$ which have at least $\delta_1^{1/3} n$ leaves. Let $\mathcal{T}^1 = \{T_1, \ldots, T_{t_1}\} := \mathcal{T} \setminus \mathcal{T}^2$ such that T_i has at

most $\delta_1^{1/3}n$ leaves for all $i \leq t_1 - 1$. Let $f := t_1 + t_2 - \min\{1, t_1\}$. Next we define a collection $\mathcal{F} = \{F_1, \ldots, F_f\}$ of forests. For every $i \in [t_2]$, let

$$F_i := \tilde{T}_i \cup T_i^\diamond, \ \ W_i' := W_{\tilde{T}_i} \cup W_{T_i^\diamond}, \ \ W_i := W_i' \cup \{\tau_{\tilde{T}_i}'(y_{\tilde{T}_i}), \tau_{T_i^\diamond}'(y_{T_i^\diamond})\}.$$

For every $t_2 < i \leq f$, let

$$F_i := T_{i-t_2}, \ \ W_i' := W_{i-t_2}, \ \ W_i := W_i' \cup \{\tau_{T_{i-t_2}}'(y_{T_{i-t_2}})\}.$$

Thus if $t_1 \geq 1$, then T_{t_1} is the only tree which is not part of any forest in $\mathcal{F}$. The proof strategy now is as follows. We first decompose $G'[A, B]$ into a collection $\mathcal{S}$ of edge-disjoint flocks of seagulls (recall that these were defined in Section 3.4). For every flock of seagulls in $\mathcal{S}$, we use one forest $F \in \mathcal{F}$ to cover it. Note that either F has many vertices of degree 2 or F is the union of two trees both having many leaves. Accordingly we will cover the vertex of degree 2 of every seagull by a vertex of degree 2 in F or by two leaves from two distinct components in F.

For every seagull Z in G, let $w(Z) := |\{i \in [f] : W_i \cap V(Z) \neq \emptyset\}|$. By (c4.6), we conclude $w(Z) \leq 3\varepsilon n$. For every flock S of seagulls, let $w(S) := \sum_{Z \in S} w(Z)$. Recall that in G' every vertex in B has even degree. Thus by (c4.3) and Proposition 3.17, there is a partition of $E(G'[A, B])$ into $6\delta_1 n$ edge-disjoint flocks $S^1, \ldots, S^{6\delta_1 n}$ of seagulls with wings in A. For each $i \in [6\delta_1 n]$, by (c4.5), we have $w(S^i) \leq (\Delta^2 + 1)t$. Thus for each $i \in [6\delta_1 n]$, we can use Proposition 3.18 to partition S^i into $q := \delta_1^{-1/2}/60$ (possibly empty) disjoint flocks $S_1^i, \ldots, S_q^i$ such that $|S_j^i| \leq 40\delta_1^{1/2}n$ and $w(S_j^i) \leq 120\delta_1^{1/2}(\Delta^2 + 1)t + 3\varepsilon n \leq \delta_1^{1/2}n/100$ for all $j \in [q]$.

Note that $q \cdot 6\delta_1 n \leq t/10$ by (c4.4). By adding empty flocks to $\{S_j^i\}_{i \in [6\delta_1 n], j \in [q]}$ if necessary, we obtain a collection of edge-disjoint flocks $\mathcal{S} := \{S_1, \ldots, S_{t/10}\}$ partitioning $E(G'[A, B])$ with

$$|S_j| \leq 40\delta_1^{1/2}n, \text{ and } w(S_j) \leq \delta_1^{1/2}n/100 \tag{7.8}$$

for all $j \in [t/10]$.

Let $\mathcal{H}$ be an auxiliary bipartite graph with vertex partition $(\mathcal{F}, \mathcal{S})$ such that $F_i S_j \in E(\mathcal{H})$ if $W_i \cap V(S_j) = \emptyset$. Since $(t-1)/2 \leq f \leq t$, we have

$$d_{\mathcal{H}}(S_i) \geq |\mathcal{F}| - w(S_i) \geq (t-1)/2 - \delta_1^{1/2}n/100 \geq t/10 = |\mathcal{S}|.$$

Thus $\mathcal{H}$ contains a matching covering $\mathcal{S}$. By relabelling we may assume that $F_i S_i \in E(\mathcal{H})$ for all $i \in [t/10]$. Let $\hat{\mathcal{S}}$ arise from $\mathcal{S}$ by adding empty flocks $S_{t/10+1}, \ldots, S_f$.

We will greedily construct embeddings $\tau_1, \ldots, \tau_f$ such that for each $i \in [f]$ the following hold:

(P1)$_i$ τ_i packs the components of F_i into $G_i - W_i'$ where $G_i := G' - \bigcup_{j=1}^{i-1} E(\tau_j(F_j))$,

(P2)$_i$ $E(\tau_i(F_i)) \cap E(G'[A, B]) = E(S_i)$, and

(P3)$_i$ if $i \in [t_2]$, then τ_i is consistent with $\tau_{\tilde{T}_i}' \cup \tau_{T_i^\diamond}'$ and if $t_2 < i \leq f$, then τ_i is consistent with τ_{T_i}'.

Note that if $i \in [t_2]$, i.e. if F_i consists of two components $\tilde{T}_i, T_i^\diamond$, then $\tau_i(\tilde{T}_i)$ and $\tau_i(T_i^\diamond)$ are not necessarily vertex-disjoint, thus τ_i may not be an embedding of F_i.

Suppose for some $i \in [f]$, we have constructed $\tau_1, \ldots, \tau_{i-1}$ satisfying (P1)$_j$–(P3)$_j$ for all $j < i$. We will now construct τ_i. Since $d_{G,A \setminus W_i'}(v) - d_{G_i, A \setminus W_i'}(v) \leq \Delta i \leq \Delta t \leq \Delta \gamma |A|$ for each $v \in A$ by (c4.4) and $|W_i'| \leq 2\Delta^2$ by (c4.5), we can apply Proposition 3.9 to conclude from (c4.1) that $G_i[A \setminus W_i']$ is $(\gamma^{1/11}, p)$-quasi-random and so for any two vertices $u, v \in A \setminus W_i'$

$$d_{G_i, A \setminus W_i'}(u, v) \geq p^2 |A|/2 \geq |F_i|. \tag{7.9}$$

Let $s := |S_i|$ and write $S_i = \{u_1 v_1 w_1, \ldots, u_s v_s w_s\}$. Let us first consider the case when $s = 0$. Let $c \in [2]$ be the number of components in F_i and $T^1, \ldots, T^c$ be these components. By (7.9), we can apply Lemma 3.19 to each component T^j of F_i to find a function ϕ_{T^j} which embeds T^j into $G_i[A \setminus W_i']$ which is consistent with $\tau_{T^j}'(y_{T^j})$ for each $j \in [c]$ such that the sets $E(\phi_{T^j}(T^j))$ are pairwise edge-disjoint for different $j \in [c]$. (If $c = 2$, then in order to find ϕ_{T^2} we apply Lemma 3.19 to $G_i[A \setminus W_i'] - E(\phi_{T^1}(T^1))$.) Let $\tau_i := \phi_{T^1} \cup \cdots \cup \phi_{T^c}$. Then τ_i satisfies (P1)$_i$–(P3)$_i$.

Suppose next that $s > 0$ and F_i is a tree, i.e. $t_2 < i \leq f$. Thus F_i has at most $\delta_1^{1/3}n$ leaves. By (7.8) and (c4.5), we have $s \leq 40\delta_1^{1/2}n \leq (|F_i| - 2\delta_1^{1/3}n)/\Delta^5$. Thus Proposition 6.1 implies that

there exists a 5-independent set $I := \{y_{F_i}, y_1, \ldots, y_s\}$ in F_i such that $I \setminus \{y_{F_i}\}$ is a set of vertices of degree 2 in F_i. For each $j \in [s]$, let $\{x_j, z_j\} := N_{F_i}(y_j)$. Then $J := \{x_1, z_1, \ldots, x_s, z_s\} \cup \{y_{F_i}\}$ is a 3-independent set in $F_i - (I \setminus \{y_{F_i}\})$. Let

$$\phi(v) := \begin{cases} u_j & \text{if } v = x_j \text{ for some } j \in [s], \\ w_j & \text{if } v = z_j \text{ for some } j \in [s], \\ \tau'_{F_i}(y_{F_i}) & \text{if } v = y_{F_i}. \end{cases}$$

By (7.9), we can apply Lemma 3.19 to find a function ϕ' which is consistent with ϕ and embeds $F_i - (I \setminus \{y_{F_i}\})$ into $G_i[A \setminus W'_i]$. For every $v \in V(F_i)$, let

$$\tau_i(v) := \begin{cases} v_j & \text{if } v = y_j \text{ for some } j \in [s], \\ \phi'(v) & \text{if } v \in V(F_i) - (I \setminus \{y_{F_i}\}). \end{cases}$$

Then τ_i satisfies (P1)$_i$–(P3)$_i$.

Suppose next that $s > 0$ and F_i is a forest with two components T^1, T^2, i.e. $i \in [t_2]$ and $\{T^1, T^2\} = \{\tilde{T}_i, T_i^\diamond\}$. Thus T^1, T^2 both have at least $\delta_1^{1/3} n$ leaves. For each $c \in [2]$, let

$$Z_c := \{z \in V(T^c) : z \text{ is a neighbour of a leaf}\} \cup \{y_{T^c}\}.$$

Hence $|Z_c| \geq \delta_1^{1/3} n / \Delta \geq \Delta^3(s+1)$ by (7.8). Thus Proposition 6.2 implies that each Z_c contains a 3-independent set $I_c := \{y_{T^c}, y_1^c, \ldots, y_s^c\}$. For each $j \in [s]$, let x_j^c be a leaf of T^c adjacent to y_j^c and write $X^c := \{x_1^c, \ldots, x_s^c\}$. Let

$$\phi(v) := \begin{cases} u_j & \text{if } v = y_j^1 \text{ for some } j \in [s], \\ w_j & \text{if } v = y_j^2 \text{ for some } j \in [s], \\ \tau'_{T^c}(y_{T^c}) & \text{if } v = y_{T^c} \text{ for some } c \in [2]. \end{cases}$$

If $\tau'_{T^1}(y_{T^1}) = \tau'_{T^2}(y_{T^2})$, then let F'_i be the tree we obtain from F_i by identifying y_{T^1} and y_{T^2}. Otherwise let $F'_i := F_i$. By (7.9), we can apply Lemma 3.19 to find a function ϕ' which is consistent with ϕ and embeds $F'_i - (X^1 \cup X^2)$ into $G_i[A \setminus W'_i]$. Let

$$\tau_i(v) := \begin{cases} v_j & \text{if } v \in \{x_j^1, x_j^2\} \text{ for some } j \in [s], \\ \phi'(v) & \text{if } v \in V(F_i) - (X^1 \cup X^2). \end{cases}$$

Then τ_i satisfies (P1)$_i$–(P3)$_i$.

Suppose now that we have defined $\tau_1, \ldots, \tau_f$. If $t_1 = 0$, then $\tau := \bigcup_{i=1}^f \tau_i$ satisfies (C4.1)–(C4.3). If $t_1 \geq 1$, then there is a single tree $T \in \mathcal{T}$ which is not yet embedded. In this case let $G_{f+1} := G' - \bigcup_{j=1}^f E(\tau_j(F_j))$. We once again use Lemma 3.19 to find a function τ_{f+1} which embeds T into $G_{f+1}[A \setminus W_T]$ and is consistent with $\tau'_T(y_T)$. Then $\tau := \bigcup_{i=1}^{f+1} \tau_i$ satisfies (C4.1)–(C4.3). $\qquad\square$

8. Iteration Lemma

In this section we state and prove the key lemma of this paper. Given a suitable graph G^* on a vertex set V, a small set $A \subseteq V$, and a suitable set of forests, we can cover all edges of G^* incident to the vertices in $V \setminus A$ with these forests without using many edges of G^* inside A. In Section 10 we will apply this lemma iteratively to obtain our main result. More precisely, the above graph G^* will be the union of graphs G, G', H_1 and H_2, where G will be the "leftover" from the previous iteration step, and H_1 and H_2 are graphs we set aside at the start to ensure that each iteration can be carried out successfully. We will cover $G \cup H_1 \cup H_2$ entirely (see (Φ1)). Roughly speaking the leftover of G' from the current iteration step will play the role of G in the next iteration step, which is why we aim to use it as little as possible in the current iteration (see (Φ2)).

Lemma 8.1. *Suppose* $D, \Delta, n \in \mathbb{N}$ *with* $1/n \ll \varepsilon \ll \delta_2 \ll \delta_1 \ll \gamma_* \ll \gamma \ll \beta \ll \alpha, 1/D, 1/\Delta, p, d \leq 1$ *such that* $D \geq 6$ *is even and* $\Delta \geq 3$. *Suppose* G, G', H_1, H_2 *are four edge-disjoint graphs on a set* V *and* $A, R \subseteq V$ *satisfying the following:*

(g1) $|V| = n$, $|A| = \gamma n$, $|R| = \varepsilon n$, $A \cap R = \emptyset$, *and* A *is an independent set in* $G \cup H_1 \cup H_2$,

(g2) G is (β, α)-quasi-random with $V(G) = V$ and G' is $(\gamma^{1/3}, p)$-quasi-random with $V(G') = A$,

(g3) H_1 is a $(4\varepsilon, \delta_1)$-quasi-random bipartite graph with vertex partition $(A, V \setminus A)$, and

(g4) H_2 is a graph on V with $\Delta(H_2) \leq 3\delta_2 n/2$ such that $d_{H_2}(u,v) \geq 2\delta_2^2 n/3$ for any two distinct vertices $u, v \in V$.

Suppose $\mathcal{F}$ is a collection of rooted forests so that each $F \in \mathcal{F}$ consists of two components $(T_F^1, r_F^1), (T_F^2, r_F^2)$ and $\mathcal{F}$ satisfies the following:

(f1) $|\mathcal{F}| \geq (1/2 + d)n$,

(f2) $|F| \leq (1 - 4/D)n$, $\Delta(F) \leq \Delta$, and $|T_F^1|, |T_F^2| \geq \gamma n$ for all $F \in \mathcal{F}$,

(f3) $e(\mathcal{F}) = e(G) + e(H_1) + e(H_2) + (3\gamma_*^2 \pm \gamma_*^2)n^2$, and

(f4) there is a function $\phi' : \{r_F^c : F \in \mathcal{F}, c \in [2]\} \to R$ such that $|\phi'^{-1}(v)| \leq \varepsilon^{-2}$ for any $v \in R$ and $\phi'(r_F^1) \neq \phi'(r_F^2)$ for any $F \in \mathcal{F}$.

Then there exists a function ϕ which is consistent with ϕ' and which packs $\mathcal{F}$ into $G \cup G' \cup H_1 \cup H_2$ such that

(Φ1) $E(G) \cup E(H_1) \cup E(H_2) \subseteq E(\phi(\mathcal{F}))$ and

(Φ2) for every $v \in A$, we have $d_{G' \cap \phi(\mathcal{F})}(v) \leq \gamma_*^{1/2}|A|$.

We structure the proof into several steps. In the "preparation step", Step 1, we partition $\mathcal{F}$ into several subsets. We also apply Lemma 5.1 to obtain an approximate decomposition of G into suitable cycle blow-ups. Unfortunately, Lemma 5.1 also returns a small exceptional set V_0 of vertices which is not part of the cycle blow-ups. Furthermore, we partition each forest $F \in \mathcal{F}$ into a large subforest $\tilde{F}$ and a small subforest $F - V(\tilde{F})$.

In Step 2, we apply Lemma 6.9 to the cycle blow-ups obtained in Step 1 to approximately cover the edges in $E(G)$ with the set of large subforests. Crucially, the density of the leftover of G (i.e. the uncovered part) is insignificant compared to the density of H_2. In this step we will also use some edges of H_1 in order to satisfy the restrictions on the packing given by (f4). In the following steps we cover the remaining edges with the small subforests. In order to avoid an overlap between the embeddings of the small and large subforests, we equip V with an equitable partition $(U_1, \ldots, U_D)$ and make sure that for each $F \in \mathcal{F}$ the large subforest $\tilde{F}$ of F uses at most $D-3$ parts U_i while the remaining three parts are reserved for the small subforest $F - V(\tilde{F})$ of F. In order to maintain a symmetric setting, we actually partition $\mathcal{F}$ into $\binom{D}{3}$ subsets and carry out the above procedure for every choice of $D - 3$ sets in $(U_1, \ldots, U_D)$.

In Step 3, we prepare a link between the large and the small subforests by embedding the roots of each small subforest $F - V(\tilde{F})$ (i.e. those vertices of $F - V(\tilde{F})$ which attach to $\tilde{F}$) into some U_i which is not used by its corresponding large subforest $\tilde{F}$.

In the remaining steps we make use of the small subforests in the following way. In Step 4, we use Lemma 7.1 to cover almost all edges incident to the exceptional set V_0. In Step 5, we cover all the edges induced by $V \setminus A$ via Lemma 7.2. In Step 6, we make use of Lemma 7.3 to ensure that the number of uncovered edges incident to each vertex in $V \setminus A$ is even. In Step 7, we use Lemma 7.4 to cover all remaining edges between A and $V \setminus A$. In each of these four steps we make use of the fact that the leftover of G forms an insignificant part of the total current leftover.

Proof of Lemma 8.1. We start with the preparation step.

Step 1. Preparation. In this step, we partition G, G', H_1, H_2, the underlying vertex sets, and $\mathcal{F}$ into structures which are suitable for applications of the packing and embedding lemmas proved in Section 6 and 7. We additionally choose $M, M' \in \mathbb{N}$ and a constant η such that $1/n \ll 1/M \ll 1/M' \ll \varepsilon \ll \eta \ll \delta_2$. Let $\hat{D} := \binom{D}{3}$ and let $\mathcal{I} := \binom{[D]}{D-3} = \{S_1, \ldots, S_{\hat{D}}\}$. Thus $|\mathcal{I}| = \hat{D}$.

We select an equitable partition $(U_1, \ldots, U_D)$ of $V \setminus R$ satisfying the following:

(U1) $(U_1 \cap A, \ldots, U_D \cap A)$ is an equitable partition of A, and

(U2) for every $J \in \{G, G', H_1, H_2\}$, every $u, v \in V$ and $i, j \in [D]$, we have

(i) $|d_{J,U_i}(u) - d_{J,U_j}(u)| \leq \varepsilon n$,

(ii) $|d_{J,U_i \cap A}(u) - d_{J,U_j \cap A}(u)| \le \varepsilon n$, and
(iii) $|d_{J,U_i}(u,v) - d_{J,U_j}(u,v)| \le \varepsilon n$.

Such an equitable partition exists as an equitable partition of $V \setminus R$ chosen uniformly at random subject to (U1) satisfies (U2) with probability at least $1/2$ by Lemma 3.2. For a set $S \subseteq [D]$, we define

$$U_S := \bigcup_{i \in S} U_i.$$

For each $I \in [\hat{D}]$, let

$$A_I := A \cap U_{[D] \setminus S_I} \quad \text{and} \quad B_I := (U_{[D] \setminus S_I} \cup R) \setminus A. \tag{8.1}$$

We make the following observation (which follows from (g3), (U1), (U2) and the fact that $|R| = \varepsilon n$).

For all $i \in [D]$ and $I \in [\hat{D}]$, the bipartite graphs $H_1[U_i]$, $H_1[U_i \cup R]$, $H_1[A_I \cup B_I]$ and $H_1[A_I \cup (B_I \setminus R)] = H_1[U_{[D] \setminus S_I}]$ are $(\varepsilon^{1/2}, \delta_1)$-quasi-random. $\tag{8.2}$

Observe that $G - R$ is $(2\beta, \alpha)$-quasi-random by (g2), and thus $(\beta^{1/7}, \alpha/2)$-dense by Proposition 3.10. Thus we can apply Lemma 5.1 with the following graphs and parameters.

object/parameter	$G - R$	$\varepsilon/(1-\varepsilon)$	δ_2^{-4}	$\beta^{1/7}$	$\alpha/2$	α	D	$(U_1, \ldots, U_D)$	M	M'
playing the role of	G	ε	t	β	α	d	D	$(U_1, \ldots, U_D)$	M	M'

Therefore, there exist an exceptional set $V_0 \subseteq V \setminus R$, with

$$|V_0| \le 2\varepsilon n, \tag{8.3}$$

a positive integer Γ with $M' \le \Gamma \le M$, and collections of cycle blow-ups $\mathcal{C}_S$ (one for each $S \in \mathcal{I}$) such that the following properties hold, where we write $\mathcal{C} := \bigcup_{S \in \mathcal{I}} \mathcal{C}_S$:

(R'1) for every $S \in \mathcal{I}$, the set $\mathcal{C}_S$ is a set of $(3\varepsilon, \delta_2^4)$-super-regular cycle $(1 - 2\varepsilon)\frac{n}{\Gamma}$-blow-ups such that $C \subseteq G$ for each $C \in \mathcal{C}_S$, the length of each $C \in \mathcal{C}_S$ is odd and at least $(1 - 7/(2D))\Gamma$ and such that $V(C) \subseteq U_S \setminus V_0$ for each $C \in \mathcal{C}_S$,
(R'2) all cycle blow-ups in $\mathcal{C}$ are edge-disjoint from each other,
(R'3) $e(\mathcal{C}_S) = (1 \pm \beta^{1/14})\hat{D}^{-1}e(G)$ for all $S \in \mathcal{I}$,
(R'4) $d_G(v) - \sum_{C \in \mathcal{C}} d_C(v) \le 4\delta_2^2 n$ for $v \in V \setminus (V_0 \cup R)$. In particular, $e(\mathcal{C}) \ge e(G) - 2\delta_2^2 n^2$.

Because we split $G - R$ into $\hat{D}$ collections $\mathcal{C}_S$ of graphs, we also seek a corresponding partition of $\mathcal{F}$ into $\mathcal{F}_1, \ldots, \mathcal{F}_{\hat{D}}$ such that

(F1) $|\mathcal{F}_I| \ge (1/2 + d/2)\hat{D}^{-1}n$ for all $I \in [\hat{D}]$,
(F2) $e(\mathcal{F}_I) = e(\mathcal{C}_{S_I}) + (3\gamma_*^2 \pm 2\gamma_*^2)\hat{D}^{-1}n^2$ for all $I \in [\hat{D}]$.

It is not difficult so see that such a partition exists. Indeed, for all $I \in [\hat{D}]$, assign $F \in \mathcal{F}$ to $\mathcal{F}_I$ with probability $e(\mathcal{C}_{S_I})/e(\mathcal{C})$, independently of all other $F' \in \mathcal{F}$. Then straightforward applications of Theorem 3.1 show that this random partition satisfies (F1) and (F2) with probability at least $1/2$.

We will cover the edges of G, H_1, H_2 in several steps, mainly via applications of Lemma 6.9 and Lemma 7.1–7.4. For the applications of Lemma 7.1–7.4 we need to reserve an appropriate collection of forests. Accordingly, for every $I \in [\hat{D}]$, we partition $\mathcal{F}_I$ into sets $\mathcal{F}_I^\eta, \mathcal{F}_I^{\delta_2}, \mathcal{F}_I^{\delta_1}, \mathcal{F}_I^{\text{par}}, \mathcal{F}_I^0$ such that

$$|\mathcal{F}_I^\eta| = 3\left\lceil \frac{n}{6\hat{D}} \right\rceil, \quad |\mathcal{F}_I^{\delta_2}| = \delta_2^{1/2}n, \quad |\mathcal{F}_I^{\delta_1}| = |\mathcal{F}_I^{\text{par}}| = \gamma_* n \tag{8.4}$$

and let

$$\mathcal{F}_I^{\gamma_*} := \mathcal{F}_I^{\delta_2} \cup \mathcal{F}_I^{\delta_1} \cup \mathcal{F}_I^{\text{par}}, \quad \mathcal{F}_I^0 := \mathcal{F}_I \setminus (\mathcal{F}_I^\eta \cup \mathcal{F}_I^{\gamma_*}) \quad \text{and} \quad \mathcal{F}^\xi := \bigcup_{I=1}^{\hat{D}} \mathcal{F}_I^\xi \text{ for } \xi \in \{\eta, \delta_2, \delta_1, \text{par}, \gamma_*, 0\}.$$

Such a partition exists by (F1) and since

$$(2\gamma_* + \delta_2^{1/2})n + 3\left\lceil \frac{n}{6\hat{D}} \right\rceil \le (1/2 + d/2)\hat{D}^{-1}n.$$

Note that

$$2\gamma_* n \le |\mathcal{F}_I^{\gamma_*}| \le 3\gamma_* n. \tag{8.5}$$

We will use subforests of the forests in $\mathcal{F}^\eta$ to cover almost all the edges at the exceptional set $V_0 \cup R$ in Step 4, subforests of the forests in $\mathcal{F}^{\delta_2}$ to cover all remaining edges inside $V \setminus A$ in Step 5 (these edges will consist mainly of edges in H_2). We will use subforests of the forests in $\mathcal{F}^\eta \cup \mathcal{F}^{\mathrm{par}}$ to resolve parity issues in Step 6, and subforests of the forests in $\mathcal{F}^{\delta_1}$ to cover all remaining edges between A and $V \setminus A$ in Step 7 (these edges will consist mainly of edges in H_1). The forests in $\mathcal{F}^0$ are simply the forests which play no specific role in Step 4–7.

In Step 2, we will use Lemma 6.9 to embed the bulk of $\mathcal{F}$ into the cycle blow-ups in $\mathcal{C}$. Accordingly, for every $F \in \mathcal{F}$ we now define a "large" subforest $\tilde{F}$ of F which is embedded by Lemma 6.9. (The remaining subforests are used to cover the remaining edges by applications of Lemma 7.1–7.4, as described above.) For every $F \in \mathcal{F}$, we will also define a set X_F containing all those vertices of $\tilde{F}$ which attach to $F - V(\tilde{F})$.

More precisely, for each $I \in [\hat{D}]$, we proceed as follows. For each $F \in \mathcal{F}_I^0$, we define $\tilde{F} := F$ and $X_F := \emptyset$.

We now construct $\tilde{F}$ and X_F for all $F \in \mathcal{F}_I^{\gamma_*}$. Let

$$\beta_I := \frac{e(\mathcal{F}_I) - (1 - \delta_2^2)e(\mathcal{C}_{S_I})}{|\mathcal{F}_I^{\gamma_*}|n} \overset{(F2)}{=} \frac{(3\gamma_*^2 \pm 5\gamma_*^2/2)\hat{D}^{-1}n^2}{|\mathcal{F}_I^{\gamma_*}|n}. \tag{8.6}$$

Then (8.5) implies $2\Delta\gamma_*^2 \le \beta_I \le \gamma_*$. Recall from (f2) that each $F \in \mathcal{F}$ consists of two components $(T_F^1, r_F^1), (T_F^2, r_F^2)$ of order at least γn. Thus for all $F \in \mathcal{F}_I^{\gamma_*}$ we may apply Proposition 6.4 with $1, \beta_I, \gamma$ playing the roles of c, β, α to choose a subtree $T_F^1(y_F) =: T_F^{\gamma_*}$ of F such that $T_F^{\gamma_*}$ has distance at least 3 from r_F^1,

$$\gamma_*^2 n \le |T_F^{\gamma_*}| \le \Delta\gamma_* n, \tag{8.7}$$

and

$$\sum_{F \in \mathcal{F}_I^{\gamma_*}} e(T_F^{\gamma_*}) = e(\mathcal{F}_I) - (1 - \delta_2^2)e(\mathcal{C}_{S_I}) \pm n. \tag{8.8}$$

For each $F \in \mathcal{F}_I^{\gamma_*}$, we let $\tilde{F} := F - V(T_F^{\gamma_*})$, and let x_F be the unique vertex in $\tilde{F}$ which is adjacent to y_F in F. We view y_F as the root of $T_F^{\gamma_*}$ and let $X_F := \{x_F\}$.

Next we consider any forest $F \in \mathcal{F}_I^\eta$. By Proposition 6.4 (applied with $2, \eta, \gamma$ playing the roles of c, β, α), for every $c \in [2]$ we can choose a subtree $T_F^c(y_F^c)$ of T_F^c with

$$\eta n/(2\Delta) \le |T_F^c(y_F^c)| \le \Delta\eta n \tag{8.9}$$

such that the distance from r_F^c to y_F^c is at least 5. Again, we view y_F^c as the root of $T_F^c(y_F^c)$. Let $\tilde{F} := F - V(T_F^1(y_F^1) \cup T_F^2(y_F^2))$. For each $c \in [2]$, let x_F^c be the unique vertex in $\tilde{F}$ which is adjacent to y_F^c in F and define $X_F := \{x_F^1, x_F^2\}$. In addition, for all $c \in [2]$ and $F \in \mathcal{F}_I^\eta$, we choose one leaf ℓ_F^c of $T_F^c(y_F^c)$ at distance at least 6 from y_F^c and let z_F^c be the neighbour of ℓ_F^c. Hence

$$\{y_F^1, z_F^1, y_F^2, z_F^2\} \text{ is a 5-independent set in } F. \tag{8.10}$$

Embedding ℓ_F^c appropriately in a separate step will help us to change the parity of its image vertex in the leftover graph if required. For this it is essential that ℓ_F^c is in fact also a leaf of the original forests $F \in \mathcal{F}^\eta$. This is also the reason why we need to remove two subtrees in this case (rather than one): In total we need to reserve $2|\mathcal{F}^\eta|$ leaves in order to be able to deal with possible parity problems later on, but if F is a path, then we cannot find a single small subtree which contains both leaves of F.

Note that in all cases we have $|X_F| \le 2$ and that $X_F \cap (N_F[r_F^1] \cup N_F[r_F^2]) = \emptyset$. Recall that $\mathcal{F}_I^{\gamma_*} = \mathcal{F}_I^{\delta_2} \cup \mathcal{F}_I^{\delta_1} \cup \mathcal{F}_I^{\mathrm{par}}$. For all $\xi \in \{\delta_2, \delta_1, \mathrm{par}\}$ and $F \in \mathcal{F}_I^\xi$, let $T_F^\xi := T_F^{\gamma_*}$. For easier

referencing, given $\xi \in \{\eta, \gamma_*, \delta_2, \delta_1, \mathrm{par}\}$ and $F' \subseteq F \in \mathcal{F}_I^\xi$, we sometimes write $y_{F'}$ for y_F. For example, if $\xi \in \{\delta_2, \delta_1, \mathrm{par}\}$ and $F' = T_F^\xi$, then this allows us to directly refer to y_F as $y_{F'}$. Analogous conventions apply to $x_F, x_F^1, x_F^2, y_F^1, y_F^2, z_F^1, z_F^2, \ell_F^1, \ell_F^2$ accordingly.

We introduce the following families of forests for every $I \in [\hat{D}]$. Recall that the first three families refer to the "large" subforests and the other families to the "small" subtrees and forests.

- $\tilde{\mathcal{F}}_I^\xi := \{\tilde{F} : F \in \mathcal{F}_I^\xi\}$ for $\xi \in \{\eta, \gamma_*, \delta_2, \delta_1, \mathrm{par}, 0\}$,
- $\tilde{\mathcal{F}}_I := \{\tilde{F} : F \in \mathcal{F}_I\}$,
- $\tilde{\mathcal{F}} := \bigcup_{I=1}^{\hat{D}} \tilde{\mathcal{F}}_I = \{\tilde{F} : F \in \mathcal{F}\}$,
- $\mathscr{T}_I^\xi := \{T_F^\xi : F \in \mathcal{F}_I^\xi\}$ for $\xi \in \{\gamma_*, \delta_2, \delta_1, \mathrm{par}\}$,
- $\mathscr{F}_I^\eta := \{(T_F^1(y_F^1) - \ell_F^1) \cup (T_F^2(y_F^2) - \ell_F^2) : F \in \mathcal{F}_I^\eta\}$,
- $\mathscr{T}^\xi := \bigcup_{I=1}^{\hat{D}} \mathscr{T}_I^\xi$ for $\xi \in \{\gamma_*, \delta_2, \delta_1, \mathrm{par}\}$,
- $\mathscr{F}^\eta := \bigcup_{I=1}^{\hat{D}} \mathscr{F}_I^\eta$.

The set $\tilde{\mathcal{F}}$ contains all forests which we will embed in Step 2 using Lemma 6.9. The forests in $\tilde{\mathcal{F}}$ inherit their roots from their corresponding supergraphs in $\mathcal{F}$. As every component of $F \in \mathcal{F}$ has at least γn vertices (by (f2)) and $e(\mathcal{F}) \leq n^2$ (by (f3)), we also conclude

$$|\tilde{\mathcal{F}}| = |\mathcal{F}| \leq \gamma^{-1} n. \tag{8.11}$$

For later reference we note that

$$|\mathscr{F}_I^\eta| = |\mathcal{F}_I^\eta| \overset{(8.4)}{=} 3\left\lceil \frac{n}{6\hat{D}} \right\rceil = (1 \pm \varepsilon)\frac{n}{2\hat{D}}. \tag{8.12}$$

Moreover, (f2), (8.7), and (8.9) imply that

$$\begin{aligned}
\gamma n \leq |\tilde{F}| \leq (1 - 4/D)n \quad &\text{for all } \tilde{F} \in \tilde{\mathcal{F}}, \\
\gamma_*^2 n \leq |T| \leq \Delta\gamma_* n \quad &\text{for all } T \in \mathscr{T}^\xi \text{ and } \xi \in \{\gamma_*, \delta_2, \delta_1, \mathrm{par}\}, \\
\eta n/(3\Delta) \leq |T| \leq \Delta\eta n \quad &\text{for every component } T \text{ of each forest in } \mathscr{F}^\eta.
\end{aligned} \tag{8.13}$$

By (8.12) and (8.13), for all $I \in \hat{D}$, we have $e(\tilde{\mathcal{F}}_I^\eta) \geq e(\mathcal{F}_I^\eta) - 2\Delta\eta n|\mathcal{F}_I^\eta| \geq e(\mathcal{F}_I^\eta) - \eta^{2/3} n^2$. By (g2) and (R′3), we conclude $e(\mathcal{C}_{S_I}) \geq (1 - \beta^{1/14})(1 - \beta)\hat{D}^{-1}\alpha n^2/2 \geq \beta n^2$. Thus

$$e(\tilde{\mathcal{F}}_I) = \left(e(\mathcal{F}_I^{\gamma_*}) - \sum_{F \in \mathcal{F}_I^{\gamma_*}} (e(T_F^{\gamma_*}) + 1)\right) + (e(\mathcal{F}_I^\eta) \pm \eta^{2/3} n^2) + e(\mathcal{F}_I^0) \overset{(8.8)}{=} (1 - \delta_2^2 \pm \eta^{1/2})e(\mathcal{C}_{S_I}). \tag{8.14}$$

Next we need to decide which forests will be embedded into which cycle blow-up in $\mathcal{C}$. For this, for each $I \in [\hat{D}]$ we proceed as follows. We write $\mathcal{C}_{S_I} =: \{C_{I,1}, \ldots, C_{I,c_I'}\}$. Note that by (R′1) for every $i \in [c_I']$,

$$\frac{\delta_2^4 n^2}{3\Gamma} \leq e(C_{I,i}) \leq \frac{2\delta_2^4 n^2}{\Gamma}. \tag{8.15}$$

Together with (R′2), this implies that

$$|\mathcal{C}| \leq 2\delta_2^{-4}\Gamma. \tag{8.16}$$

Moreover, (8.14) and (8.15) imply that we can select a partition of $\tilde{\mathcal{F}}_I$ into $\tilde{\mathcal{F}}_{I,1}, \ldots, \tilde{\mathcal{F}}_{I,c_I'}$ such that for all $i \in [c_I']$

$$e(\tilde{\mathcal{F}}_{I,i}) = (1 - \delta_2^2 \pm 2\eta^{1/2})e(C_{I,i}). \tag{8.17}$$

Finally, (8.13), (8.15), and (8.17) allow us to conclude that for all $I \in [\hat{D}]$ and $i \in [c_I']$

$$|\tilde{\mathcal{F}}_{I,i}| \leq \frac{2\delta_2^4 n^2}{\Gamma} \cdot \frac{1}{\gamma n} \leq \frac{n}{\Gamma}. \tag{8.18}$$

Step 2. **Approximate covering**. In this step, we find an approximate covering of the edges of G. To make the leftover sufficiently small, we will actually do this by finding an approximate covering of the cycle blow-ups $C_{I,i}$ defined in Step 1 (which themselves cover almost all edges of G). Recall that assumption (f4) prescribes an embedding ϕ' of the roots of all $F \in \mathcal{F}$ and thus of the roots of all $\tilde{F} \in \tilde{\mathcal{F}}$. Ideally, we want to pack the forests $\tilde{F} \in \tilde{\mathcal{F}}_{I,i}$ without their roots into $C_{I,i}$ in such a way that the edges incident to their roots are present in H_1. However, one difficulty is that for a root r of F, the vertex $\phi'(r)$ might not have neighbours in $V(C_{I,i})$. Therefore, we introduce an intermediate step. We only pack the forests without their roots and without the neighbours of the roots into $C_{I,i}$ and embed the neighbours of the roots onto appropriate vertices to complete the packing. We arrange this so that all edges incident to a neighbour of a root are mapped to edges in H_1.

More precisely, recall that we denote the roots of each $F \in \mathcal{F}$ by r_F^1 and r_F^2. We will now extend the domain of the function ϕ' to

$$\{p_1, \ldots, p_m\} := \{x \in N_F(r_F^c) : c \in [2], F \in \mathcal{F}\}.$$

We will achieve this via Proposition 3.21. Note that by (8.11),

$$m \le 2\Delta\gamma^{-1}n. \tag{8.19}$$

For each $i' \in [m]$, let $F_{i'} \in \mathcal{F}$ be such that $p_{i'} \in F_{i'}$. Then $|\{i' \in [m] : F_{i'} = F\}| = d_F(r_F^1) + d_F(r_F^2)$ for each $F \in \mathcal{F}$. Let H^{index} be the graph on $[m]$ such that $i'j' \in H^{\text{index}}$ if $F_{i'} = F_{j'}$. Thus $\Delta(H^{\text{index}}) \le 2\Delta$. For each $i' \in [m]$, let

$$q_{i'} := \{r_{F_{i'}}^1, r_{F_{i'}}^2\} \cap N_{F_{i'}}(p_{i'}) \text{ and } u_{i'} := \phi'(q_{i'}).$$

For each $v \in R$, we obtain $|\{i' : v = u_{i'}\}| \le \Delta|\{F \in \mathcal{F} : v \in \phi'(\{r_F^1, r_F^2\})\}| \le \Delta\varepsilon^{-2}$ by (f4). Next, for each $i' \in [m]$, we define sets of vertices $W'_{i'}$ and $W''_{i'}$ which we want to exclude as possible images of $p_{i'}$. Let $I(i'), i(i')$ be the numbers defined by $\tilde{F}_{i'} \in \tilde{\mathcal{F}}_{I(i'),i(i')}$. We define

$$W'_{i'} := N_{H_1}(u_{i'}) \cap A \cap U_{S_{I(i')}},$$

$$\widehat{W}'_{i'} := N_{H_1}(u_{i'}) \cap A \cap U_{[D]\setminus S_{I(i')}}, \text{ and}$$

$$W''_{i'} := \{w \in \widehat{W}'_{i'} : d_{H_1, V(C_{I(i'),i(i')})\setminus A}(w) < \delta_1^2 n\}.$$

Note $u_{i'} \in R \subseteq V \setminus A$ by (f4). Thus (U2) and (g3) imply that

$$|\widehat{W}'_{i'}| \ge (\delta_1 - 4\varepsilon)|A| \cdot 3/D - \varepsilon n \ge \delta_1^2 n. \tag{8.20}$$

Theorem 3.8 and (g3) imply that $H_1[A, V \setminus A]$ is $(\varepsilon^{1/7}, \delta_1)$-super-regular. Observe that $|V(C_{I(i'),i(i')}) \setminus A| \ge (1 - 7/(2D))\Gamma \cdot (1 - 2\varepsilon)\frac{n}{\Gamma} - \gamma n \ge n/3$ by (R'1) and (g1). Together with Proposition 3.4 this implies that the graph $H_1[\widehat{W}'_{i'}, V(C_{I(i'),i(i')}) \setminus A]$ is $(\varepsilon^{1/8}, \delta_1)$-regular. Thus $|W''_{i'}| \le 2\varepsilon^{1/8}n$, by Proposition 3.6, and therefore

$$d_{H_1,A}(u_{i'}) - |W'_{i'} \cup W''_{i'}| \ge |\widehat{W}'_{i'}| - 2\varepsilon^{1/8}n \overset{(8.19),(8.20)}{\ge} 3\Delta\varepsilon^{-2} + m/\varepsilon^{-1} + \varepsilon^{-1}.$$

Thus we can apply Proposition 3.21 with the following graphs and parameters.

object/parameter	H_1	H^{index}	$\Delta\varepsilon^{-2}$	A	m	ε^{-1}	$W'_{i'} \cup W''_{i'}$	$u_{i'}$
playing the role of	G	H	Δ	A	m	s	$W_{i'}$	$u_{i'}$

We obtain $v_1, \ldots, v_m$ so that

(α1) $v_{i'} \in (A \cap U_{[D]\setminus S_{I(i')}}) \setminus W''_{i'}$,
(α2) $u_1v_1, \ldots, u_mv_m$ are distinct edges in H_1, and
(α3) $|\{i' : v = v_{i'}\}| \le \varepsilon^{-1}$ for all $v \in A$ and $v_{i'} \notin \{u_{j'}, v_{j'}\}$ if $F_{i'} = F_{j'}$.

Let $E' := \{u_{i'}v_{i'} : i' \in [m]\}$ and $H_1' := H_1 - E'$. Thus

$$\Delta(H_1 - E(H_1')) = \Delta(E') \le \varepsilon^{-1} + \Delta\varepsilon^{-2} \le 2\Delta\varepsilon^{-2}. \tag{8.21}$$

Given a vertex $x \in F$, recall that $N_F[x] = N_F(x) \cup \{x\}$. Moreover, we write $N_F^2[x]$ for the set of all vertices having distance at most 2 from x in F. Now let $\phi'(p_{i'}) := v_{i'}$ for all $i' \in [m]$. Note $(\alpha3)$ implies that ϕ' is injective on $N_F[r_F^1] \cup N_F[r_F^2]$ for any $F \in \mathcal{F}$. Also the domain of ϕ' is $\{x \in N_F[r_F^c] : c \in [2], F \in \mathcal{F}\}$ and for any $v \in V$ we obtain

$$|\phi'^{-1}(v)| \leq \Delta(E') \overset{(8.21)}{\leq} 2\Delta\varepsilon^{-2}. \tag{8.22}$$

For all $I \in [\hat{D}]$ and $i \in [c_I']$, let

$$R_{I,i} := \{\phi'(x) : x \in N_F(r_F^c) \text{ for some } \tilde{F} \in \tilde{\mathcal{F}}_{I,i} \text{ and } c \in [2]\}.$$

Note that for all $v \in R_{I,i}$, there exists $i' \in [m]$ such that $v = v_{i'}$, $I = I(i')$, and $i = i(i')$. Then

$$R_{I,i} \subseteq A \cap U_{[D] \setminus S_I} \tag{8.23}$$

by $(\alpha1)$. Thus $R_{I,i} \cap V(C_{I,i}) = \emptyset$ by $(R'1)$. Also for all $v \in R_{I,i}$, we have

$$d_{H_1', V(C_{I,i}) \setminus A}(v) \overset{(8.21)}{\geq} d_{H_1, V(C_{I,i}) \setminus A}(v) - 2\Delta\varepsilon^{-2} \overset{(\alpha1)}{\geq} \delta_1^2 n/2. \tag{8.24}$$

Recall that X_F was defined together with $\tilde{F}$ in Step 1. For each $F \in \mathcal{F}$, we let

$$X_F' := (N_F^2[r_F^1] \setminus N_F[r_F^1]) \cup (N_F^2[r_F^2] \setminus N_F[r_F^2]) \cup X_F.$$

Thus $|X_F'| \leq 2\Delta^2$. Now we sequentially pack the forests in $\tilde{\mathcal{F}}_{I,i}$ into $C_{I,i} \cup H_1$ according to the lexicographic order[1] on the set of tuples $\{(I, i) : I \in [\hat{D}], i \in [c_I']\}$.

More precisely, for each tuple (I, i) in turn we will construct functions $\tau_{\tilde{F}}$ consistent with ϕ' which embed each $\tilde{F} \in \tilde{\mathcal{F}}_{I,i}$ into $C_{I,i} \cup H_1$ such that the following properties hold:

(P1)$_{I,i}$ $\Delta(C_{I,i} - \bigcup_{\tilde{F} \in \tilde{\mathcal{F}}_{I,i}} E(\tau_{\tilde{F}}(\tilde{F}))) \leq 20\delta_2^6 n/\Gamma$,

(P2)$_{I,i}$ $\tau_{\tilde{F}}(\tilde{F} - (N_F[r_F^1] \cup N_F[r_F^2])) \subseteq C_{I,i}$ and thus $V(\tau_{\tilde{F}}(\tilde{F} - (N_F[r_F^1] \cup N_F[r_F^2]))) \subseteq U_{S_I} \setminus V_0$ for every $\tilde{F} \in \tilde{\mathcal{F}}_{I,i}$,

(P3)$_{I,i}$ for any vertex $v \in V$, we have

$$\left| \left\{ \tilde{F} \in \bigcup_{(I',i') \leq (I,i)} \tilde{\mathcal{F}}_{I',i'} : v \in \tau_{\tilde{F}}(X_F') \right\} \right| \leq \frac{2n}{\Gamma},$$

(P4)$_{I,i}$ $\tau_{\tilde{F}}(\tilde{F} - \{r_F^1, r_F^2\}) \subseteq C_{I,i} \cup H_{I,i}'$, where $E(\mathcal{F}, I, i) := \bigcup_{(I',i')<(I,i)} \bigcup_{\tilde{F} \in \tilde{\mathcal{F}}_{I',i'}} E(\tau_{\tilde{F}}(\tilde{F}))$ and $H_{I,i}' := H_1 - E(\mathcal{F}, I, i)$. Moreover, the graphs $\tau_{\tilde{F}}(\tilde{F})$ are pairwise edge-disjoint for different $\tilde{F} \in \tilde{\mathcal{F}}_{I,i}$.

Assume that for some (I, i) we have already constructed $\tau_{\tilde{F}}$ consistent with ϕ' for all $\tilde{F} \in \tilde{\mathcal{F}}_{I',i'}$ with $(I', i') < (I, i)$ such that (P1)$_{I',i'}$–(P4)$_{I',i'}$ hold. Let $U' \subseteq V(C_{I,i})$ be the set of vertices which lie in $\tau_{\tilde{F}}(X_F')$ for at least n/Γ forests $\tilde{F} \in \tilde{\mathcal{F}}_{I',i'}$ with $(I', i') < (I, i)$. Since $|X_F'| \leq 2\Delta^2$ for any $F \in \mathcal{F}$, (8.11) implies that

$$|U'| \leq \frac{2\Delta^2 \gamma^{-1} n}{n/\Gamma} \leq \Gamma^2. \tag{8.25}$$

Let ℓ be the length of $C_{I,i}$. Hence, by $(R'1)$, ℓ is odd and $\ell \geq (1 - 7/(2D))\Gamma$. In particular, together with $(R'1)$ and (8.13), this implies $(1 - \gamma^2)|C_{I,i}| \geq |\tilde{F}|$ for any $\tilde{F} \in \tilde{\mathcal{F}}_{I,i}$.

Let $\hat{F} := \tilde{F} - \{r_F^1, r_F^2\}$ and let $\hat{\mathcal{F}}_{I,i} := \{\hat{F} : \tilde{F} \in \tilde{\mathcal{F}}_{I,i}\}$. We wish to apply Lemma 6.9 with the following graphs and parameters.

object/parameter	$C_{I,i}$	$\hat{\mathcal{F}}_{I,i}$	$H_{I,i}'[V(C_{I,i}) \setminus A, R_{I,i}]$	ℓ	$\delta_2^2/2$	δ_2^4	$\delta_1^2/3$	γ^2
playing the role of	G	$\mathcal{F}$	G'	ℓ	α	d	d_0	η
object/parameter	ϕ'	$R_{I,i}$	$N_F(r_F^1) \cup N_F(r_F^2)$	U'	X_F'	Δ	3ε	
playing the role of	ϕ'	R	$r(F)$	U'	X_F'	Δ	ε	

[1]We use $\leq$ to equip tuples of reals with a lexicographic ordering; that is, $(I, i) \leq (I', i')$ if $I < I'$ or $I = I'$ and $i \leq i'$. For convenience, let $(I, 0) := (I - 1, c_{I-1}')$ and $(I, c_I' + 1) := (I + 1, 1)$.

So we need to verify conditions (a1)–(a7) of Lemma 6.9. Condition (a1) holds by (R′1). For any $v \in R_{I,i}$, by $(P2)_{I',i'}$ and $(P3)_{I',i'}$ for $(I',i') < (I,i)$, we obtain

$$d_{H'_{I,i},V(C_{I,i})\backslash A}(v) \geq d_{H'_1,V(C_{I,i})\backslash A}(v) - \Delta|\phi'^{-1}(v)| - \frac{2n}{\Gamma} \overset{(8.22),(8.24)}{\geq} \frac{\delta_1^2 n}{3} \geq \frac{\delta_1^2|C_{I,i}|}{3},$$

which verifies (a2). Condition (a3) follows from (f2), (8.13) as well as the fact that each $\tilde{F}$ has at most two components and $\hat{F} = \tilde{F} - \{r_F^1, r_F^2\}$, (a4) follows from (8.17), (a5) is trivial from the definition of $\hat{F}$ and r_F^c, and (a6) follows from (8.25) and the definition of X'_F. Also (8.22) implies that $|\phi'^{-1}(v)| \leq 2\Delta\varepsilon^{-2} \leq \ell^{1/2}$ for each $v \in R_{I,i}$, and as remarked before (8.22), ϕ' is injective on $N_F(r_F^1) \cup N_F(r_F^2)$ for any $F \in \mathcal{F}$, which verifies (a7).

Thus Lemma 6.9 gives a function $\tau_{\hat{\mathcal{F}}_{I,i}}$ which is consistent with ϕ' and which packs $\hat{\mathcal{F}}_{I,i}$ into $C_{I,i} \cup H'_{I,i}[V(C_{I,i}) \backslash A, R_{I,i}]$ so that

(A′1) $\Delta(C_{I,i} - E(\tau_{\hat{\mathcal{F}}_{I,i}}(\hat{\mathcal{F}}_{I,i}))) \leq 5\delta_2^6 n/\ell$,

(A′2) $\tau_{\hat{\mathcal{F}}_{I,i}}(V(\hat{F}) \backslash (N_F(r_F^1) \cup N_F(r_F^2))) \cap R_{I,i} = \emptyset$ for every $\hat{F} \in \hat{\mathcal{F}}_{I,i}$, and

(A′3) $\tau_{\hat{\mathcal{F}}_{I,i}}(X'_F) \cap U' = \emptyset$ for every $\hat{F} \in \hat{\mathcal{F}}_{I,i}$.

Define $\tau_{\hat{F}} := \tau_{\hat{\mathcal{F}}_{I,i}}|_{\hat{F}}$ and $\tau_{\tilde{F}} := \tau_{\hat{F}} \cup \phi'|_{\{r_F^1, r_F^2\}}$.

Next we verify $(P1)_{I,i}$–$(P4)_{I,i}$. Properties (A′1) and (A′2) directly imply $(P1)_{I,i}$ and $(P2)_{I,i}$. For any vertex $v \in U'$ and $\tilde{F} \in \tilde{\mathcal{F}}_{I,i}$, (A′3) implies that $v \notin \tau_{\tilde{F}}(X'_F)$. If $v \in V \backslash V(C_{I,i})$, then by $(P2)_{I,i}$ we also have that $v \notin \tau_{\tilde{F}}(X'_F)$. Thus for each $v \in U' \cup (V \backslash V(C_{I,i}))$, by $(P3)_{I',i'}$ with $(I',i') < (I,i)$, we have

$$\left|\left\{\tilde{F} \in \bigcup_{(I',i')\leq(I,i)} \tilde{\mathcal{F}}_{I',i'} : v \in \tau_{\tilde{F}}(X'_F)\right\}\right| = \left|\left\{\tilde{F} \in \bigcup_{(I',i')<(I,i)} \tilde{\mathcal{F}}_{I',i'} : v \in \tau_{\tilde{F}}(X'_F)\right\}\right| \leq \frac{2n}{\Gamma}.$$

On the other hand, for each vertex $v \in V(C_{I,i}) \backslash U'$, by definition of U' we have

$$\left|\left\{\tilde{F} \in \bigcup_{(I',i')\leq(I,i)} \tilde{\mathcal{F}}_{I',i'} : v \in \tau_{\tilde{F}}(X'_F)\right\}\right| \leq \frac{n}{\Gamma} + |\tilde{\mathcal{F}}_{I,i}| \overset{(8.18)}{\leq} \frac{2n}{\Gamma}.$$

So $(P3)_{I,i}$ holds. Observe that $(P4)_{I,i}$ holds by construction.

Let $\tilde{\tau} := \bigcup_{\tilde{F}\in\tilde{\mathcal{F}}} \tau_{\tilde{F}}$. Note the properties $(P2)_{I,i}$ and $(P4)_{I,i}$ imply that $\tilde{\tau}$ packs $\tilde{\mathcal{F}}$ into $H_1 \cup \bigcup \mathcal{C}$ and is consistent with ϕ'. Let us define the leftover graphs.

$$G^1 := G - E(\tilde{\tau}(\tilde{\mathcal{F}})) \text{ and } H_1^1 := H_1 - E(\tilde{\tau}(\tilde{\mathcal{F}})). \tag{8.26}$$

This with $(P1)_{I,i}$ and (R′4) imply that for every $v \in V \backslash (R \cup V_0)$, we have

$$d_{G^1}(v) \leq \frac{20\delta_2^6 n}{\Gamma} \cdot |\mathcal{C}| + 4\delta_2^2 n \overset{(8.16)}{\leq} 50\delta_2^2 n. \tag{8.27}$$

Hence $\tilde{\tau}$ forms a (very efficient) approximate cover of the edges of G. Note $(P2)_{I,i}$ implies that for any vertex $v \in V$ and $F \in \mathcal{F}$, we can only have $d_{\tilde{\tau}(\tilde{F})\cap H_1}(v) > 0$ if $v \in \tilde{\tau}(N_F^2[r_F^c])$ (for some $c \in [2]$) which in turn only holds if $v \in \phi'(N_F[r_F^c])$ or $v \in \tilde{\tau}(X'_F)$. By (8.22) the former holds for at most $2\Delta\varepsilon^{-2}$ forests $F \in \mathcal{F}$ and by $(P3)_{I,i}$ the latter holds for at most $2n/\Gamma$ forests $F \in \mathcal{F}$. Since $\Delta(F) \leq \Delta$, it follows that

$$\Delta(H_1 - E(H_1^1)) = \Delta(E(\tilde{\tau}(\tilde{\mathcal{F}})) \cap H_1) \leq \Delta(2\Delta\varepsilon^{-2} + 2n/\Gamma) \leq \varepsilon n. \tag{8.28}$$

Thus H_1^1 inherits all the relevant properties of H_1. We proceed with a few observations. Recall that for all $F \in \mathcal{F}^0$, we have $\tilde{F} = F$. Thus $\tilde{\tau}$ actually packs F into $H_1 \cup \bigcup \mathcal{C}$. On the other hand, for all forests $F \in \mathcal{F}^{\gamma*} \cup \mathcal{F}^\eta$, the function $\tilde{\tau}$ only packs a proper subgraph of F into $H_1 \cup \bigcup \mathcal{C}$. In the next few steps we will extend this by packing the forests in $\mathscr{T}^{\gamma*} \cup \mathscr{F}^\eta$. As a preliminary step towards this, we embed the roots of these forests via a function τ' in Step 3.

Step 3. **Embedding the roots of the forests in $\mathscr{T}^{\gamma_*} \cup \mathscr{F}^\eta$.** In this step, we define the function τ' which, for every $I \in [\hat{D}]$, embeds the roots $\{y_F^c : F \in \mathscr{F}_I^\eta, c \in [2]\} \cup \{y_F : F \in \mathscr{T}_I^{\gamma_*}\}$ into $A \cap U_i$ for some $i \notin S_I$ (recall that their neighbours x_F^c and x_F have already been embedded in the previous step as they belong to $\tilde{F}$). We stress here that one key point is the following. Given $I \in [\hat{D}]$ and $F \in \mathcal{F}_I^{\gamma_*}$, in Step 2 the subforest $\tilde{F}$ of F was embedded into U_{S_I} (apart from its roots r_F^1, r_F^2 which are mapped to R and the neighbours of those roots which are embedded into $A \cap U_{[D]\setminus S_I}$ by (8.23)), while the forest $F - V(\tilde{F}) \in \mathscr{T}_I^{\gamma_*}$ will be embedded into $U_{[D]\setminus S_I}$. So after we have chosen a suitable image $\tau'(y_F)$ of the root y_F of $F - V(\tilde{F})$ inside $U_{[D]\setminus S_I}$, any embedding of $F - V(\tilde{F})$ into $U_{[D]\setminus S_I}$ which is consistent with $\tau'(y_F)$ and which avoids $\tilde{\tau}(N(r_F^1) \cup N(r_F^2))$ yields an embedding of F. For $F \in \mathcal{F}_I^\eta$ the strategy is similar, but slightly more complicated.

Consider $F \in \mathcal{F}$. For any $F' \subseteq F$, we define

$$W_{F'} := \tilde{\tau}(N_F[r_F^1] \cup N_F[r_F^2]) = \phi'(N_F[r_F^1] \cup N_F[r_F^2]). \tag{8.29}$$

(So for example, if $F \in \mathcal{F}^{\gamma_*}$ and T is the subtree of F belonging to $\mathscr{T}^{\gamma_*}$, then this allows us to refer directly to W_F as W_T.) Note that by (f4), $(P2)_{I,i}$ and (8.23), for all $F \in \mathcal{F}_I$, we have

$$\tilde{\tau}(\tilde{F}) \cap (U_{[D]\setminus S_I} \cap V_0) = W_F \setminus \tilde{\tau}(\{r_F^1, r_F^2\}) \quad \text{and} \quad \tilde{\tau}(\tilde{F}) \cap R = \tilde{\tau}(\{r_F^1, r_F^2\}) \subseteq W_F. \tag{8.30}$$

As discussed above, we need to avoid W_F when defining τ' in the current step. Note

$$|W_F| \leq 2\Delta + 2 \leq \Delta^2. \tag{8.31}$$

Furthermore, (8.22) states that for all $v \in V$,

$$|\{F \in \mathcal{F} : v \in W_F\}| \leq 2\Delta\varepsilon^{-2} \leq \varepsilon^{-3}. \tag{8.32}$$

Consider any $I \in [\hat{D}]$. Recall from (8.12) that $|\mathscr{F}_I^\eta| = |\mathcal{F}_I^\eta| = 3\lceil n/(6\hat{D})\rceil$. Let $\{j, j', j''\}$ be such that $\{j, j', j''\} = [D] \setminus S_I$. Split the forests in $\mathscr{F}_I^\eta$ into three sets $\mathscr{F}_I^\eta(1), \mathscr{F}_I^\eta(2), \mathscr{F}_I^\eta(3)$ of size $\lceil n/(6\hat{D})\rceil$ and define

$$j(F) = \begin{cases} j & \text{if } F \in \mathscr{F}_I^\eta(1), \\ j' & \text{if } F \in \mathscr{F}_I^\eta(2), \\ j'' & \text{if } F \in \mathscr{F}_I^\eta(3). \end{cases}$$

In a similar way we define $j(T)$ for the trees T in the families $\mathscr{T}_I^{\delta_1}, \mathscr{T}_I^{\delta_2}, \mathscr{T}_I^{\text{par}}$ (again, assigning each of the three possible values in $[D] \setminus S_I$ to a third of the trees in each of these families). For each $i \in [D]$, let

- $\mathscr{F}^{i,\eta} := \{F \in \mathscr{F}^\eta : j(F) = i\}$ and
- $\mathscr{T}^{i,\xi} := \{T \in \mathscr{T}^\xi : j(T) = i\}$ for all $\xi \in \{\gamma_*, \delta_1, \delta_2, \text{par}\}$.

Thus (8.30) implies the following:

> For all $i \in [D]$ and $F \in \mathcal{F}^\eta \cup \mathcal{F}^{\gamma_*}$, let F' be the unique subforest of F belonging to $\mathscr{F}^\eta \cup \mathscr{T}^{\gamma_*}$. If $F' \in \mathscr{F}^{i,\eta} \cup \mathscr{T}^{i,\gamma_*}$, then $\tilde{\tau}(\tilde{F}) \cap (U_i \cup V_0 \cup R) = W_F$. (8.33)

Moreover, since $|R|, |V_0| \leq 2\varepsilon n$,

$$|\mathscr{F}^{i,\eta}| = \binom{D-1}{D-3} \left\lceil \frac{n}{6\hat{D}} \right\rceil = (1 \pm 5D\varepsilon)\frac{|U_i \cup V_0 \cup R|}{2}. \tag{8.34}$$

For each $i \in [D]$ in turn, we will now use Proposition 3.21 to define a function τ' packing $\{x_F y_F : F \in \mathscr{T}^{i,\gamma_*}\} \cup \{x_F^1 y_F^1, x_F^2 y_F^2 : F \in \mathscr{F}^{i,\eta}\}$ into $G' \cup H_1^1$ such that $y_F, y_F^1, y_F^2 \in A \cap U_i$ and τ' is consistent with $\tilde{\tau}$. Hence assume that for some $i \in [D]$ we have already determined τ' for all $F \in \bigcup_{j=1}^{i-1}(\mathscr{F}^{j,\eta} \cup \mathscr{T}^{j,\gamma_*})$ such that

$$\Delta(E_1^j) \leq \varepsilon^2 n \tag{8.35}$$

for all $j < i$, where

$$E_1^j := \bigcup_{F \in \mathscr{F}^{j,\eta}} \{\tau'(x_F^c)\tau'(y_F^c) : c \in [2]\} \cup \bigcup_{F \in \mathscr{T}^{j,\gamma_*}} \{\tau'(x_F)\tau'(y_F)\}.$$

Let $G_i^1 := (G' \cup H_1^1) - \bigcup_{j=1}^{i-1} E_1^j$. By (g2), (g3), (U2), (8.28) and (8.35), for every $v \in V$, we have

$$d_{G_i^1, U_i \cap A}(v) \geq \delta_1 \gamma n/(2D) - \varepsilon n - D\varepsilon^2 n \geq \delta_1^2 n. \qquad (8.36)$$

Write $\{F_1^\eta, \ldots, F_{m_1}^\eta\} := \mathscr{F}^{i,\eta}$ and $\{T_1^{\gamma*}, \ldots, T_{m_2}^{\gamma*}\} := \mathscr{T}^{i,\gamma*}$. Let $u_j := \tilde\tau(x_{F_j^\eta}^1)$, $u_{m_1+j} = \tilde\tau(x_{F_j^\eta}^2)$ for each $j \in [m_1]$ and $u_{2m_1+j} = \tilde\tau(x_{T_j^{\gamma*}})$ for each $j \in [m_2]$. We define $W_j := W_{m_1+j} := W_{F_j^\eta}$ if $j \in [m_1]$ and $W_{2m_1+j} := W_{T_j^{\gamma*}}$ if $j \in [m_2]$. By (P3)$_{I,i}$ and the definition of X_F, we conclude that for any $v \in V$, we have

$$|\{j \in [2m_1 + m_2] : v = u_j\}| = |\{F \in \mathcal{F}^{\gamma*} : v = \tilde\tau(x_F)\}| + |\{F \in \mathcal{F}^\eta : v \in \tilde\tau(\{x_F^1, x_F^2\})\}| \leq 2n/\Gamma. \qquad (8.37)$$

Let H_i^{index} be the graph on $[2m_1 + m_2]$ such that $jj' \in H^{\text{index}}$ if $j' = j + m_1$ and $j \leq m_1$. Note that for all $j \in [2m_1 + m_2]$, we have

$$d_{G_i^1, U_i \cap A}(u_j) - |W_j| - |V_0| \overset{(8.3),(8.31),(8.36)}{\geq} \delta_1^2 n - \Delta^2 - 2\varepsilon n \overset{(8.12)}{\geq} 6\Gamma^{-1}n + (2m_1 + m_2)/\varepsilon^{-1} + \varepsilon^{-1}.$$

Thus we can apply Proposition 3.21 with the following parameters and graphs.

object/parameter	G_i^1	H_i^{index}	$2n/\Gamma$	$A \cap U_i$	$2m_1 + m_2$	ε^{-1}	$W_j \cup V_0$	u_j
playing the role of	G	H	Δ	A	m	s	W_j	u_j

We obtain a sequence of vertices $v_1, \ldots, v_{2m_1+m_2}$ such that the following hold:

(β1) $u_1 v_1, \ldots, u_{2m_1+m_2} v_{2m_1+m_2}$ are distinct edges in G_i^1,

(β2) $v_j \notin \{u_{j'}, v_{j'}\}$ whenever $|j - j'| = m_1$ and $j, j' \in [2m_1]$,

(β3) $v_j, v_{m_1+j} \in (U_i \cap A) \setminus (W_{F_j^\eta} \cup V_0)$ for all $j \in [m_1]$ and $v_{2m_1+j} \in (U_i \cap A) \setminus (W_{T_j^{\gamma*}} \cup V_0)$ for all $j \in [m_2]$, and

(β4) every vertex $v \in V$ satisfies $|\{j \in [2m_1 + m_2] : v = v_j\}| \leq \varepsilon^{-1}$.

We define $\tau'(y_{F_j^\eta}^1) := v_j, \tau'(y_{F_j^\eta}^2) := v_{m_1+j}$ for all $j \in [m_1]$ and $\tau'(y_{T_j^{\gamma*}}) := v_{2m_1+j}$ for $j \in [m_2]$. Let

$$E_1^i := \{u_j v_j : j \in [2m_1 + m_2]\}.$$

Then (8.37) and (β4) imply that $\Delta(E_1^i) \leq 2n/\Gamma + \varepsilon^{-1} \leq \varepsilon^2 n$ as required in (8.35). By repeating this procedure for every $i \in [D]$, we define τ' as desired. We claim that τ' satisfies the following properties:

(γ1) for every $F \in \mathcal{F}^\eta \cup \mathcal{F}^{\gamma*}$ and every $y \in \{y_F, y_F^1, y_F^2\}$, we have $\tau'(y) \notin \tilde\tau(\tilde F)$; moreover, $\tau'(y_F^1) \neq \tau'(y_F^2)$ for each $F \in \mathcal{F}^\eta$,

(γ2) for any $v \in V$, there are at most ε^{-1} forests $F \in \mathscr{F}^\eta \cup \mathscr{T}^{\gamma*}$ such that $v = \tau'(y)$ for some $y \in \{y_F, y_F^1, y_F^2\}$,

(γ3) for all $j \in [D]$ and $F \in \mathscr{F}^{j,\eta} \cup \mathscr{T}^{j,\gamma*}$, every root y of a component of F satisfies $\tau'(y) \in (U_j \cap A) \setminus (W_F \cup V_0)$, and

(γ4) if $I \in [\hat D]$ and $F \in \mathscr{F}_I^\eta \cup \mathscr{T}_I^{\gamma*}$, then every root y of a component of F satisfies $\tau'(y) \in A_I \setminus W_F = (A \cap U_{[D] \setminus S_I}) \setminus W_F$.

(γ5) for all $T \in \mathscr{T}^{\gamma*}$, we have that $\tilde\tau(x_T)\tau'(y_T)$ is an edge of $G' \cup H_1^1$ and for all $F \in \mathscr{F}^\eta$ and $c \in [2]$, we have that $\tilde\tau(x_F^c)\tau'(y_F^c)$ is an edge of $G' \cup H_1^1$.

Indeed, (γ1) follows from (β2), (β3) and (8.33), (γ2) follows from (β4), while (γ3) and (γ4) both follow from (β3) and (γ5) follows from (β1).

Let $E_1 := \bigcup_{i=1}^D E_1^i$. We conclude from (8.35) that

$$\Delta(E_1) \leq D\varepsilon^2 n \leq \varepsilon n. \qquad (8.38)$$

Let

$$G'^2 := G' - E_1, \quad G^2 := G^1 \quad \text{and} \quad H_1^2 := H_1^1 - E_1. \qquad (8.39)$$

Following on from (8.26), these updates track the edges of G', G and H_1 which are still available after this step. We use (8.2), (8.28), (8.38) and Proposition 3.9 to conclude the following.

For all $j \in [D]$ and $I \in [\hat{D}]$, the bipartite graphs $H_1^2[U_j] = H_1^2[U_j \cap A, U_j \setminus A]$, $H_1^2[U_j \cup R] = H_1^2[U_j \cap A, (U_j \cup R) \setminus A]$ and $H_1^2[A_I \cup B_I]$ are $(\varepsilon^{1/3}, \delta_1)$-quasi-random. (8.40)

Step 4. Covering almost all the edges incident to the exceptional set. Property (8.27) shows that in Step 2, we have covered almost all edges of G incident to vertices in $V \setminus (V_0 \cup R)$. In the following step, we use the forests in $\mathscr{F}^\eta$ to cover almost all edges incident to $V_0 \cup R$. We will achieve this through several applications of Lemma 7.1.

For each $i \in [D]$, we consider the graph

$$G(i) := (G^2 \cup H_2)[U_i \cup R \cup V_0] - E((G^2 \cup H_2)[R \cup V_0]).$$

Observe that $G(1), \ldots, G(D)$ are pairwise edge-disjoint. Note that by (U2) and (g4), any two vertices $u, v \in U_i$ satisfy

$$d_{G(i)}(u, v) \geq 3\delta_2^2 n/(5D). \tag{8.41}$$

Recall that in Step 3 we partitioned $\mathscr{F}^\eta$ into $\mathscr{F}^{1,\eta}, \ldots, \mathscr{F}^{D,\eta}$. For each $F \in \mathscr{F}^{i,\eta}$, we choose two distinct vertices q_F^1, q_F^2 in $(U_i \cap A) \setminus (W_F \cup V_0 \cup \tau'(\{y_F^1, y_F^2\}))$ uniformly at random. Note that by (U1) and (8.31),

$$|(U_i \cap A) \setminus (W_F \cup V_0 \cup \tau'(\{y_F^1, y_F^2\}))| \overset{(8.3)}{\geq} D^{-1}\gamma n - 2\varepsilon n - 2\Delta^2 \geq \gamma^2 n$$

and recall that $|\mathscr{F}^{i,\eta}|$ is given by (8.34). Thus by straightforward applications of Lemma 3.2, with probability at least $1/2$, for each $v \in U_i \cap A$, we have

$$|\{F \in \mathscr{F}^{i,\eta} : v \in \{q_F^1, q_F^2\}\}| \leq n^{2/3}. \tag{8.42}$$

Thus there exists a choice such that (8.42) holds for all vertices $v \in A$. Recall from Step 1 that for all $F \in \mathscr{F}^{i,\eta}$ and $c \in [2]$, the vertex z_F^c is the unique neighbour of the leaf ℓ_F^c. We define $\tau'(z_F^c) := q_F^c$ for all $F \in \mathscr{F}^{i,\eta}$ and $c \in [2]$. For all $i \in [D]$, we now wish to apply Lemma 7.1 with the following graphs and parameters.

object/parameter	$G(i)$	$\mathscr{F}^{i,\eta}$	$|U_i \cup R \cup V_0|$	$V_0 \cup R$	$5D\varepsilon$	η
playing the role of	G	$\mathscr{F}$	n	V_0	ε	η
object/parameter	δ_2	Δ	W_F	$\tau'|_{\{y_F^1, y_F^2, z_F^1, z_F^2\}}$	$\{y_F^1, y_F^2\}$	$\{z_F^1, z_F^2\}$
playing the role of	δ_2	Δ	W_F	τ'_F	$\{y_F^1, y_F^2\}$	$\{z_F^1, z_F^2\}$

Assumptions (c1.1)–(c1.5) of Lemma 7.1 hold in the above set-up, as (8.41) implies (c1.1), (8.13) implies (c1.2), (8.10), (γ2), (γ3), and (8.42) imply (c1.3), (8.31) implies (c1.4), and (8.32) implies (c1.5). Also (8.34) ensures that $\mathscr{F}^{i,\eta}$ contains the appropriate number of forests.

From Lemma 7.1 we obtain a function τ_i^η packing $\mathscr{F}^{i,\eta}$ into $G(i)$ which is consistent with τ' such that the following properties hold:

(Q1.1)$_i$ $V(\tau_i^\eta(F)) \subseteq U_i \cup R \cup V_0$ and $\tau_i^\eta(F) \cap W_F = \emptyset$ for every $F \in \mathscr{F}^{i,\eta}$,

(Q1.2)$_i$ $d_{\tau_i^\eta(\mathscr{F}^{i,\eta})}(v) \leq \eta^{1/3}|U_i \cup R \cup V_0|$ for all $v \in U_i \setminus (V_0 \cup R)$, and

(Q1.3)$_i$ $d_{\tau_i^\eta(\mathscr{F}^{i,\eta})}(v) \geq d_{G(i)}(v) - \varepsilon^{1/3}|U_i \cup R \cup V_0|$ for every $v \in V_0 \cup R$.

For every $F' \in \mathscr{F}^{i,\eta}$ and $F \in \mathcal{F}^\eta$ such that F' is the unique subforest of F belonging to $\mathscr{F}^\eta$, (8.33) and (Q1.1)$_i$ imply that $\tilde{\tau}(\tilde{F}) \cap \tau_i^\eta(F') = \emptyset$. Together with ($\gamma$5) this means that $\tilde{\tau}(\tilde{F}) \cup \tau_i^\eta(F')$ yields an embedding of $F - \{\ell_F^1, \ell_F^2\}$.

We apply Lemma 7.1 for every $i \in [D]$ and let $\tau^\eta := \bigcup_{i=1}^D \tau_i^\eta$. Thus τ^η packs $\mathscr{F}^\eta$ into $G^2 \cup H_2$. Moreover, the function $\tilde{\tau} \cup \tau^\eta$ packs $\{F - \{\ell_F^1, \ell_F^2\} : F \in \mathcal{F}^\eta\}$ into $G \cup G' \cup H_1 \cup H_2$. Let

$$G^3 := G^2 - E(\tau^\eta(\mathscr{F}^\eta)), \quad G'^3 := G'^2, \quad H_1^3 := H_1^2 \quad \text{and} \quad H_2^3 := H_2 - E(\tau^\eta(\mathscr{F}^\eta)). \tag{8.43}$$

Following on from (8.39), these updates track the edges which are still available after this step. Let

$$G_3^* := G'^3 \cup G^3 \cup H_1^3 \cup H_2^3. \tag{8.44}$$

Thus G_3^* is the graph consisting of all the leftover edges. By $(Q1.2)_i$ for each $i \in [D]$, every $v \in V \setminus (V_0 \cup R)$ satisfies

$$d_{\tau^\eta(\mathscr{F}^\eta)}(v) \le \sum_{i=1}^{D} \eta^{1/3} |U_i \cup R \cup V_0| \le 2\eta^{1/3} n. \tag{8.45}$$

By $(Q1.3)_i$ for each $i \in [D]$, every $v \in R \cup V_0$ satisfies

$$d_{G^3 \cup H_2^3}(v) \le \sum_{i=1}^{D} \varepsilon^{1/3} |U_i \cup R \cup V_0| + |R \cup V_0| \overset{(8.3)}{\le} 2\varepsilon^{1/3} n. \tag{8.46}$$

Inequality (8.42) implies that for every vertex $v \in V$, we have

$$|\{F \in \mathscr{F}^\eta : v \in \tau^\eta(\{z_F^1, z_F^2\})\}| \le n^{2/3}. \tag{8.47}$$

Step 5. Covering the remaining edges in $V \setminus A$. Recall that $E(G_3^*)$ is precisely the total set of uncovered edges at this point. Next we pack the trees in $\mathscr{T}^{\delta_2}$ into G_3^* so that they cover all edges in $G_3^*[V \setminus A]$ and so that the packing is consistent with τ'; that is, τ' prescribes the image of the root y_T of every tree $T \in \mathscr{T}^{\delta_2}$. Lemma 7.2 is the main tool for this step.

More precisely, we will sequentially construct functions $\tau_1^{\delta_2}, \ldots, \tau_{\hat{D}}^{\delta_2}$ which are consistent with τ' and such that $\tau_I^{\delta_2}$ packs $\mathscr{T}_I^{\delta_2}$ into G_3^*. For $I \in [\hat{D}]$, let

$$G_{3,I}^* := G_3^*[A_I \cup B_I] - \bigcup_{I'=1}^{I-1} E(\tau_{I'}^{\delta_2}(\mathscr{T}_{I'}^{\delta_2})).$$

(Recall from (8.1) that $A_I = A \cap U_{[D] \setminus S_I}$ and $B_I = (U_{[D] \setminus S_I} \cup R) \setminus A$.) We will construct the functions $\tau_I^{\delta_2}$ such that they satisfy the following properties:

$(Q2.1)_I$ $\tau_I^{\delta_2}(\mathscr{T}^{\delta_2}) \subseteq G_{3,I}^*$,

$(Q2.2)_I$ $E(G_{3,I}^*[B_I]) \subseteq E(\tau_I^{\delta_2}(\mathscr{T}_I^{\delta_2}))$, and

$(Q2.3)_I$ $W_T \cap \tau_I^{\delta_2}(T) = \emptyset$ for all $T \in \mathscr{T}_I^{\delta_2}$.

Assume that for some $I \in [\hat{D}]$ we have already defined $\tau_1^{\delta_2}, \ldots, \tau_{I-1}^{\delta_2}$ which are consistent with τ' and so that $\tau_{I'}^{\delta_2}$ satisfies $(Q2.1)_{I'}$–$(Q2.3)_{I'}$ for each $I' < I$. Recall from (8.43) that $H_1^3 = H_1^2$. By (8.4) and (8.40), we obtain for every vertex $v \in B_I$ that

$$d_{G_{3,I}^*[A_I, B_I]}(v) \ge d_{H_1^3[A_I, B_I]}(v) - \sum_{I' < I} \Delta |\mathcal{F}_{I'}^{\delta_2}| \ge \delta_1 |A_I|/2 - \hat{D} \Delta \delta_2^{1/2} n \ge \delta_1 |A_I|/3. \tag{8.48}$$

Note that $G'[A_I] - E(G_{3,I}^*)$ consists of those edges of $G'[A_I]$ which have been used by the packing so far. More precisely, (8.39), (8.43) and the fact that $G_3^*[A_I] = G'^3[A_I]$ (see (g1)) together imply that $E(G'[A_I] - E(G_{3,I}^*))$ is the union of the edges in $G_3^*[A_I] - E(G_{3,I}^*[A_I]) \subseteq \bigcup_{I'=1}^{I-1} \tau_{I'}^{\delta_2}(\mathscr{T}_{I'}^{\delta_2})$ and of $E(G'[A_I] - E(G'^3[A_I])) \subseteq E_1$. Thus (8.4) and (8.38) imply that

$$\Delta(G'[A_I] - E(G_{3,I}^*)) \le \Delta \hat{D} \delta_2^{1/2} n + \varepsilon n \le \gamma^2 |A_I|.$$

In a similar way, it follows that for each $j \in [D]$, we have $\Delta(G'[A \cap U_j] - E(G_3^*[A \cap U_j])) \le \varepsilon n$ and that $\Delta(G'[A_I] - E(G_3^*[A_I])) \le \varepsilon n$. Using (g2), (U2) and Proposition 3.9, we conclude for each $j \in [D]$ that

$$G_3^*[A_I], \ G_{3,I}^*[A_I] \ \text{and} \ G_3^*[A \cap U_j] \ \text{are} \ (\gamma^{1/4}, p)\text{-quasi-random.} \tag{8.49}$$

Note that for each $v \in B_I \setminus (V_0 \cup R)$, we have

$$d_{G^*_{3,I},B_I}(v) \quad \leq \quad d_{G^*_3,B_I}(v) \overset{(g2),(g3)}{\leq} d_{G^3 \cup H^3_2, B_I}(v) \leq d_{G^1 \cup H_2, B_I}(v)$$
$$\overset{(8.27),(g4),(U2)}{\leq} \quad 50\delta_2^2 n + \frac{5}{3}\delta_2|U_{[D]\setminus S_I}| + |R|$$
$$\leq \quad 2\delta_2|A_I \cup B_I|.$$

Moreover, for each $v \in B_I \cap (V_0 \cup R)$, we have

$$d_{G^*_{3,I},B_I}(v) \leq d_{G^*_3,B_I}(v) \leq d_{G^3 \cup H^3_2}(v) \overset{(8.46)}{\leq} 2\varepsilon^{1/3}n \leq 2\delta_2|A_I \cup B_I|.$$

Thus

$$\Delta(G^*_{3,I}[B_I]) \leq 2\delta_2|A_I \cup B_I|. \tag{8.50}$$

We now wish to apply Lemma 7.2 with the following graphs and parameters.

object/parameter	$G^*_{3,I}$	$\mathcal{T}^{\delta_2}_I$	A_I	$|A_I \cup B_I|$	$D\varepsilon$	δ_2	δ_1	γ	Δ	p	y_T	$\tau'|_{\{y_T\}}$	W_T
playing the role of	G	$\mathcal{T}$	A	n	ε	δ_2	δ_1	γ	Δ	p	y_T	τ'_T	W_T

Observe that condition (c2.1) of Lemma 7.2 follows from (8.49), (c2.2) from (8.50), (c2.3) from (8.48), (c2.4) from (8.4), (c2.5) from (8.13), (8.31) and (γ4), and (c2.6) from (8.32). Thus Lemma 7.2 gives us a function $\tau^{\delta_2}_I$ which is consistent with τ', packs $\mathcal{T}^{\delta_2}_I$ into $G^*_{3,I}$, and satisfies (Q2.1)$_I$–(Q2.3)$_I$.

Assume now that for all $I \in [\hat{D}]$ we have defined such a function $\tau^{\delta_2}_I$. Let $\tau^{\delta_2} := \bigcup_{I=1}^{\hat{D}} \tau^{\delta_2}_I$. Note that for every edge $uv \in E(G^*_3[V \setminus A])$, there exists (a smallest) $I' \in [\hat{D}]$ such that $u, v \in B_{I'}$. Thus (Q2.2)$_{I'}$ implies that $uv \in \tau^{\delta_2}(\mathcal{T}^{\delta_2})$ and so

$$E(G^*_3[V \setminus A]) \subseteq \tau^{\delta_2}(\mathcal{T}^{\delta_2}). \tag{8.51}$$

Property (Q2.3)$_I$ for all $I \in [\hat{D}]$, (8.30) and the fact that $\tau^{\delta_2}(\mathcal{T}^{\delta_2}_I) \subseteq U_{[D]\setminus S_I} \cup R$ together ensure that for each $F \in \mathcal{F}^{\delta_2}$, we have $\tau^{\delta_2}(F - V(\tilde{F})) \cap \tilde{\tau}(\tilde{F}) = \emptyset$. Together with ($\gamma$5) this means that $\tilde{\tau} \cup \tau^{\delta_2}$ packs the forests in $\mathcal{F}^{\delta_2}$ into $G \cup G' \cup H_1 \cup H_2$. We define

$$G^4 := G^3 - E(\tau^{\delta_2}(\mathcal{T}^{\delta_2})), \quad G'^4 := G'^3 - E(\tau^{\delta_2}(\mathcal{T}^{\delta_2})),$$
$$H^4_1 := H^3_1 - E(\tau^{\delta_2}(\mathcal{T}^{\delta_2})) = H^2_1 - E(\tau^{\delta_2}(\mathcal{T}^{\delta_2})) \quad \text{and} \quad H^4_2 := H^3_2 - E(\tau^{\delta_2}(\mathcal{T}^{\delta_2})). \tag{8.52}$$

This follows on from the previous update in (8.43) and again tracks the current set(s) of leftover edges. To track the total set of leftover edges, let

$$G^*_4 := G^4 \cup G'^4 \cup H^4_1 \cup H^4_2 = G^*_3 - E(\tau^{\delta_2}(\mathcal{T}^{\delta_2})). \tag{8.53}$$

Note that

$$\Delta(\tau^{\delta_2}(\mathcal{T}^{\delta_2})) \leq \Delta|\mathcal{T}^{\delta_2}| \overset{(8.4)}{\leq} \hat{D}\Delta\delta_2^{1/2}n. \tag{8.54}$$

Thus by (8.49), (8.54), and Proposition 3.9 for every $I \in [\hat{D}]$ and $j \in [D]$, we obtain

$$G^*_4[A_I] \text{ and } G^*_4[A \cap U_j] \text{ are } (\gamma^{1/5}, p)\text{-quasi-random.} \tag{8.55}$$

By (g2), (g4), (8.27), (8.46), we have

$$\Delta((G^4 \cup G'^4 \cup H^4_2)[U_j \cap A, (U_j \cup R) \setminus A]) \leq \delta_2^{1/2}n.$$

Thus together with (8.40), (8.54), and Proposition 3.9, we obtain that

$$G^*_4[U_j \cap A, U_j \setminus A] \text{ and } G^*_4[U_j \cap A, (U_j \cup R) \setminus A] \text{ are } (\delta_2^{1/3}, \delta_1)\text{-quasi-random.} \tag{8.56}$$

Step 6. Resolving the parity. Note that by (8.51) and (8.53), in the current leftover graph G^*_4 every edge is incident to a vertex in A. We would like to cover the edges of $G^*_4[A, V \setminus A]$ via Lemma 7.4. For this, we first need to ensure for each $v \in V \setminus A$ that the number of edges incident to v which have not been covered yet is even. To achieve this, in this step we extend the

current packing by defining $\tau^\eta(\ell_F^c)$ for all $c \in [2]$ and $F \in \mathcal{F}^\eta$. (Recall that by the end of Step 4 we have already defined the packing of $\{F - \{\ell_F^1, \ell_F^2\} : F \in \mathcal{F}^\eta\}$.) This will already resolve most of the parity problems. Note that by (8.12), the total number of leaves of the form ℓ_F^c is close to n, so we do have sufficiently many of them for this purpose. We then define τ^{par} which packs $\mathscr{T}^{\mathrm{par}}$ into $G \cup G' \cup H_1 \cup H_2$. This then takes care of the remaining (comparatively small) set of parity problems. Recall that for each $F \in \mathcal{F}^\eta$, the vertex z_F^c denotes the neighbour of ℓ_F^c in F.

For all $i \in [D]$, let

$$V_i^{\mathrm{odd}} := \{v \in U_i \setminus A : d_{G_4^*}(v) \text{ is odd}\}.$$

For any $v \in V_i^{\mathrm{odd}}$, our aim is to find a tree $F \in \mathcal{F}^\eta$ so that v can play the role of ℓ_F^c in F. This will ensure that the degree of v in the resulting remaining subgraph of G_4^* is even. Recall that for all $i \in [D], c \in [2]$ and $F \in \mathscr{F}^{i,\eta}$, as described in Step 4, $\tau^\eta(z_F^c) \in U_i \cap A$ is the image of z_F^c in our current packing.

Consider the set

$$U_i^z := \{(u, F, c) : u \in U_i \cap A, u = \tau^\eta(z_F^c), c \in [2], F \in \mathscr{F}^{i,\eta}\}.$$

Thus $(1 + \gamma^{1/2})|U_i \setminus A| \geq 2|\mathscr{F}^{i,\eta}| = |U_i^z| \geq (1 - \varepsilon^{1/2})|U_i \setminus A|$ by (8.34). We define a bipartite graph H^i with vertex partition $(U_i^z, U_i \setminus A)$ and edge set

$$E(H^i) := \{(u, F, c)v : uv \in E(G_4^*), c \in [2], v \notin \tau^\eta(F) \cup W_F, F \in \mathscr{F}^{i,\eta}\}.$$

By (8.13) and (8.31), $|F| + |W_F| \leq \eta^{1/2}n$ for each $F \in \mathscr{F}^{i,\eta}$. This together with (8.56) implies that for all $(u, F, c), (u', F', c') \in U_i^z$ with $u \neq u'$, we have

$$d_{H^i}((u, F, c)) = (1 \pm \delta_2^{1/3})\delta_1|U_i \setminus A| \pm \eta^{1/2}n = (1 \pm 2\delta_2^{1/3})\delta_1|U_i \setminus A|, \qquad (8.57)$$

$$d_{H^i}((u, F, c), (u', F', c')) = (1 \pm \delta_2^{1/3})\delta_1^2|U_i \setminus A| \pm 2\eta^{1/2}n = (1 \pm 2\delta_2^{1/3})\delta_1^2|U_i \setminus A|. \qquad (8.58)$$

By (8.47), the number of pairs $(u, F, c), (u', F', c')$ with $u = u'$ is at most $2\delta_2^{1/3}|U_i^z|^2$. Thus (8.58) holds for all but at most $2\delta_2^{1/3}|U_i^z|^2$ pairs $(u, F, c), (u', F', c')$ and hence Theorem 3.8 implies that H^i is $(\delta_2^{1/20}, \delta_1)$-regular. Now Proposition 3.5 in turn implies that $H^i[U_i^z, V_i^{\mathrm{odd}}]$ contains a matching M_i of size at least $|V_i^{\mathrm{odd}}| - 2\delta_2^{1/20}n$.

Next we define $\tau^\eta(\ell_F^c)$ for all $F \in \mathscr{F}^{i,\eta}$. For each edge $(u, F, c)v$ in M_i, let $\tau^\eta(\ell_F^c) := v$. We will now embed the remaining "unused" leaves ℓ_F^c inside A using Proposition 3.21. In this way, they do not affect the parity of the vertices outside A. Let $E_4' := \{uv : (u, F, c)v \in M_i, i \in [D]\}$. Then

$$\Delta(E_4') \leq n^{2/3} \qquad (8.59)$$

by (8.47) and the fact that $d_{E_4'}(v) \leq 1$ for any vertex $v \in U_{[D]} \setminus A$. Note that

$$G_4^*[A \cap U_i] = (G_4^* - E_4')[A \cap U_i].$$

For each $i \in [D]$, let

$$\mathscr{F}_i^{\mathrm{match}} := \{(F, c) : (u, F, c) \in M_i \text{ for some } u \in A \cap U_i\},$$

$$\mathscr{F}_i^{\mathrm{unmatch}} := \{(F, c) : (F, c) \notin \mathscr{F}_i^{\mathrm{match}}, F \in \mathscr{F}^{i,\eta}, c \in [2]\}.$$

Thus we have already defined $\tau^\eta(\ell_F^c)$ for all $(F, c) \in \mathscr{F}_i^{\mathrm{match}}$ and $i \in [D]$. Let $\{(F_1^i, c_1), \ldots, (F_{m_i'}^i, c_{m_i'})\} := \mathscr{F}_i^{\mathrm{unmatch}}$ and let $u_{i,j} := \tau^\eta(z_{F_j^i}^{c_j})$ for all $j \in [m_i']$. Then

$$m_i' = |\mathscr{F}_i^{\mathrm{unmatch}}| \leq 2|\mathcal{F}^\eta| \overset{(8.4)}{\leq} 2n. \qquad (8.60)$$

Let $H_i'^{\mathrm{index}}$ be the graph on $[m_i']$ such that $jj' \in E(H_i'^{\mathrm{index}})$ if $F_j^i = F_{j'}^i$. Then $\Delta(H_i'^{\mathrm{index}}) \leq 1$. Let

$$W_j^i := \tau^\eta(F_j^i) \cup W_{F_j^i}. \qquad (8.61)$$

Note that by (8.13) and (8.31), $|W_j^i| \leq 2\Delta\eta n + \Delta^2 \leq \eta^{1/2}n$. Since $G_4^*[A \cap U_i]$ is $(\gamma^{1/5}, p)$-quasirandom, by (8.55), and since $u_{i,j} \in A \cap U_i$, this implies that

$$d_{G_4^*, A \cap U_i}(u_{i,j}) - |W_j^i| \geq \gamma^2 n - \eta^{1/2}n \overset{(8.60)}{\geq} 3n^{2/3} + m_i'/n^{2/3} + n^{2/3}.$$

We now apply Proposition 3.21 with the following parameters and graphs for each $i \in [D]$.

object/parameter	$G_4^*[A \cap U_i]$	$H_i'^{\text{index}}$	$n^{2/3}$	$A \cap U_i$	m_i'	$n^{2/3}$	W_j^i	$u_{i,j}$
playing the role of	G	H	Δ	A	m	s	W_j	u_j

(Condition (ii) of Proposition 3.21 follows from (8.47).) We obtain distinct edges $u_{i,1}v_{i,1}, \ldots, u_{i,m_i'}v_{i,m_i'}$ in $G_4^*[A \cap U_i]$ such that for each $j \in [m_i']$, $v_{i,j} \notin W_j^i$, and for each $v \in A$,

$$|\{(i,j) : v = v_{i,j}\}| \leq n^{2/3} \tag{8.62}$$

and such that $v_{i,j} \neq v_{i,j'}$ if $F_j^i = F_{j'}^i$. Since $A \cap U_i$ are disjoint for different $i \in [D]$, each application of Proposition 3.21 gives us distinct edges.

For each $i \in [D]$ and $j \in [m_i']$, let $\tau^\eta(\ell_{F_j^i}^{c_j}) := v_{i,j}$. Then $\tau^\eta(\ell_{F_j^i}^{c_j}) \notin \tau^\eta(F_j^i)$ by (8.61). In addition, $\tau^\eta(\ell_{F_j^i}^{c_j}) \neq \tau^\eta(\ell_{F_j^i}^{3-c_j})$. Altogether, this defines $\tau^\eta(\ell_F^c)$ for all $F \in \mathcal{F}^\eta$. Using (8.33) and $(Q1.1)_i$, it is easy to see that $\tau^\eta \cup \tilde{\tau}$ now packs $\mathcal{F}^\eta$ into $G \cup G' \cup H_1 \cup H_2$.

Let $E_4'' := \{u_{i,j}v_{i,j} : i \in [D], j \in m_i'\}$. Let $E_4 := E_4' \cup E_4''$, so (8.47), (8.59) and (8.62) imply that

$$\Delta(E_4) \leq 3n^{2/3}. \tag{8.63}$$

In what follows, we use Lemma 7.3 and the trees in $\mathcal{T}^{\text{par}}$ to adjust the parity of those vertices in $V \setminus A$ which were not involved in the above matching approach (in particular, we now also adjust the parity of the vertices in R). Let

$$G_5^* := G_4^* - E_4 \tag{8.64}$$

and for each $i \in [D]$, let

$$V_i^{\text{par}} := \begin{cases} \{u \in (U_i \cup R) \setminus A : d_{G_5^*}(u) \text{ is odd}\} & \text{if } i = 1, \\ \{u \in U_i \setminus A : d_{G_5^*}(u) \text{ is odd}\} & \text{if } i \neq 1. \end{cases}$$

As $|R| = \varepsilon n$ and $|V_i^{\text{odd}}| - |M_i| \leq 2\delta_2^{1/20}n$, this implies that $|V_i^{\text{par}}| \leq 3\delta_2^{1/20}n$. Let $G_1^5 := G_5^*[A \cap U_1, (U_1 \cup R) \setminus A]$ and $G_i^5 := G_5^*[A \cap U_i, U_i \setminus A]$ for $i \geq 2$. For each $i \in [D]$, we let $\{u_{i,1}', \ldots, u_{i,m_i''}'\} := V_i^{\text{par}}$.

Observe that (8.56) and (8.63) imply $d_{G_i^5, A \cap U_i}(u_{i,j}') \geq \delta_1^2 n \geq 3 + m_i''/n^{2/3} + n^{2/3}$. Let H consist of m_i'' isolated vertices. We can apply Proposition 3.21 with the following parameters and graphs.

object/parameter	G_i^5	H	1	$A \cap U_i$	m_i''	$n^{2/3}$	$\emptyset$	$u_{i,j}'$
playing the role of	G	H	Δ	A	m	s	W_i	u_j

We obtain a set of distinct edges $E_5^i := \{u_{i,j}'v_{i,j}' : j \in [m_i'']\}$ in G_i^5 such that $\Delta(E_5^i) \leq n^{2/3}$ and $d_{E_5^i}(u) = 1$ for all $u \in V_i^{\text{par}}$. Furthermore,

$$|E_5^i| = |V_i^{\text{par}}| \leq 3\delta_2^{1/20}n \leq \delta_1^2 n. \tag{8.65}$$

Now we wish to extend each $e \in E_5^i$ into a tree from $\mathcal{T}^{i,\text{par}}$ such that all other edges of this tree lie in A (note that it is not possible to proceed in this way for V_i^{odd} directly as $|V_i^{\text{odd}}|$ may be much larger than $|\mathcal{T}^{i,\text{par}}|$). To achieve this, for each $i \in [D]$, we apply Lemma 7.3 with the following graphs and parameters. Let $U_1' := U_1 \cup R$ and $U_i' := U_i$ for $i \geq 2$. (Recall that $\tau'(y_T)$ was defined in Step 3 and the domain of τ' on $T \in \mathcal{T}^{i,\text{par}}$ is exactly $\{y_T\}$.)

object/parameter	$G_5^*[A \cap U_i] \cup E_5^i$	$\mathscr{T}^{i,\mathrm{par}}$	γ	U_i'	ε	δ_1
playing the role of	G	$\mathscr{T}$	γ	$V(G)$	ε	δ_1
object/parameter	$A \cap U_i$	p	Δ	y_T	$\tau'|_{\{y_T\}}$	W_T
playing the role of	A	p	Δ	y_T	τ_T'	W_T

Note that (8.55) and (8.63) together with Proposition 3.9 imply condition (c3.1) of Lemma 7.3, and (8.51) implies (c3.2), (8.65) and (8.4) imply (c3.3), and (8.13), (8.31) and (γ3) imply (c3.4). Condition (c3.5) holds because of (8.32) and (γ2). Thus Lemma 7.3 gives a function τ_i^{par} packing $\mathscr{T}^{i,\mathrm{par}}$ into $G_5^*[A \cap U_i] \cup E_5^i$ which is consistent with τ' and satisfies $E_5^i \subseteq E(\tau_i^{\mathrm{par}}(\mathscr{T}^{i,\mathrm{par}}))$ and $\tau_i^{\mathrm{par}}(T) \cap W_T = \emptyset$ for every $T \in \mathscr{T}^{i,\mathrm{par}}$. We let $\tau^{\mathrm{par}} := \bigcup_{i=1}^D \tau_i^{\mathrm{par}}$. Note that

$$\Delta(\tau^{\mathrm{par}}(\mathscr{T}^{\mathrm{par}})) \le \Delta|\mathscr{T}^{\mathrm{par}}| \overset{(8.4)}{\le} \hat{D}\Delta\gamma_* n. \tag{8.66}$$

Moreover, using (8.33) and (γ5) it is easy to see that $\tilde{\tau} \cup \tau^{\mathrm{par}}$ packs $\mathcal{F}^{\mathrm{par}}$ into $G \cup G' \cup H_1 \cup H_2$. Let

$$G_6^* := G_5^* - E(\tau^{\mathrm{par}}(\mathscr{T}^{\mathrm{par}})) = G_4^* - E_4 - E(\tau^{\mathrm{par}}(\mathscr{T}^{\mathrm{par}})). \tag{8.67}$$

After the updates in (8.53) and (8.64), G_6^* consists of the current set of leftover edges in $G \cup G' \cup H_1 \cup H_2$. Note that by (8.55), (8.63), (8.66), and Proposition 3.9 for each $I \in [\hat{D}]$,

$$G_6^*[A_I] \text{ is } (\gamma^{1/6}, p)\text{-quasi-random.} \tag{8.68}$$

By construction every vertex in $V \setminus A$ has even degree in G_6^*.

Step 7. Covering the edges between A and $V \setminus A$. We finally complete the proof by covering the edges in $G_6^*[A, V \setminus A]$ (i.e. all remaining edges between A and $V \setminus A$) by the trees in $\mathscr{T}^{\delta_1}$. Again, we embed the trees sequentially according to the order $\mathscr{T}_1^{\delta_1}, \ldots, \mathscr{T}_{\hat{D}}^{\delta_1}$. Assume that for some $I \in [\hat{D}]$ we have already defined functions $\tau_1^{\delta_1}, \ldots, \tau_{I-1}^{\delta_1}$ such that for all $I' < I$, the function $\tau_{I'}^{\delta_1}$ packs $\mathscr{T}_{I'}^{\delta_1}$ into $G_{I'}^6$, where

$$G_{I'}^6 := G_6^* - \bigcup_{I''=1}^{I'-1} E(\tau_{I''}^{\delta_1}(\mathscr{T}_{I''}^{\delta_1})).$$

Recall from (8.1) that $B_I = (U_{[D] \setminus S_I} \cup R) \setminus A$. Note that (g2) implies $G_I^6[A_I, B_I] \subseteq G^1 \cup H_1 \cup H_2$. Thus (8.27), (8.46), (g3), (g4) and (U2) imply that

$$\Delta(G_I^6[A_I, B_I]) \le 2\delta_1|A_I \cup B_I|. \tag{8.69}$$

In addition, $G_I^6[A_I]$ is $(\gamma^{1/7}, p)$-quasi-random by (8.68), Proposition 3.9 and the fact that

$$\Delta\left(\bigcup_{I'=1}^{I-1} \tau_{I'}^{\delta_1}(\mathscr{T}_{I'}^{\delta_1})\right) \le \Delta|\mathcal{F}^{\delta_1}| \overset{(8.4)}{\le} \Delta\hat{D}\gamma_* n \le \gamma n.$$

Now for each $I \in [\hat{D}]$, we apply Lemma 7.4 with the following graphs and parameters.

object/parameter	$G_I^6[A_I \cup B_I]$	$\mathscr{T}_I^{\delta_1}$	A_I	δ_1	γ	Δ	p	y_T	$\tau'|_{\{y_T\}}$	W_T	ε
playing the role of	G	$\mathscr{T}$	A	δ_1	γ	Δ	p	y_T	τ_T'	W_T	ε

Observe that condition (c4.1) of Lemma 7.4 holds since $G_I^6[A_I]$ is $(\gamma^{1/7}, p)$-quasi-random, (8.51) implies (c4.2), (8.69) implies (c4.3), (8.4) implies (c4.4), (8.13), (8.31), and (γ4) imply (c4.5), and (8.32) and (γ2) give us (c4.6). Lemma 7.4 provides a function $\tau_I^{\delta_1}$ packing $\mathscr{T}_I^{\delta_1}$ into $G_I^6[A_I \cup B_I]$ which is consistent with τ' and satisfies

(Q4.1)$_I$ $d_{G_{I+1}^6, A_I}(v) \le 1$ for every $v \in B_I$,

(Q4.2)$_I$ $W_T \cap \tau_I^{\delta_1}(T) = \emptyset$ for every $T \in \mathscr{T}_I^{\delta_1}$, and

(Q4.3)$_I$ $d_{G_I^6, A}(v) - d_{G_{I+1}^6, A}(v)$ is even for all $v \in V \setminus A$.

Indeed, to check $(Q4.3)_I$ note that for all $v \in B_I$ we have $d_{G_I^6, A \setminus A_I}(v) = d_{G_{I+1}^6, A \setminus A_I}(v)$, that $d_{G_I^6, A_I}(v) - d_{G_{I+1}^6, A_I}(v)$ is even by (C4.3) of Lemma 7.4, and that for all $v \in (V \setminus A) \setminus B_I$ we have $d_{G_I^6, A_I}(v) = d_{G_{I+1}^6, A_I}(v)$.

Let $\tau^{\delta_1} := \bigcup_{I=1}^{\hat{D}} \tau_I^{\delta_1}$, then τ^{δ_1} packs $\mathscr{T}^{\delta_1}$ into G_6^* and is consistent with τ'. Moreover, recall that $A_I \cup B_I = U_{[D] \setminus S_I} \cup R$. This together with $(\gamma 5)$, (8.30), and $(Q4.2)_I$ implies that $\tilde{\tau} \cup \tau^{\delta_1}$ packs $\mathcal{F}^{\delta_1}$ into $G \cup G' \cup H_1 \cup H_2$. Let

$$G_7^* := G_6^* - E(\tau^{\delta_1}(\mathscr{T}^{\delta_1})) = G_{\hat{D}+1}^6. \tag{8.70}$$

After the previous update in (8.67), this is the (final) set of edges left over by the packing we defined so far. We claim that for all $v \in V \setminus A$, we have $d_{G_7^*}(v) = 0$. Assume for a contradiction that there is a vertex $v \in V \setminus A$, with $d_{G_7^*}(v) > 0$. Thus $d_{G_7^*, A}(v) > 0$ by (8.51). Moreover, $d_{G_7^*, A}(v) \geq 2$, since v has even degree in $G_6^6 = G_1^6$ and $(Q4.3)_I$ holds for all $I \in [\hat{D}]$. Let $i, i', i^* \in [D]$ be such that $v \in U_i \cup R$, and v has a neighbour in $U_{i'} \cap A$ and another neighbour in $U_{i^*} \cap A$ in G_7^*. Consider $I \in [\hat{D}]$ such that $S_I \subseteq [D] \setminus \{i, i', i^*\}$. Then $d_{G_{I+1}^6, A_I}(v) \geq d_{G_7^*, A_I}(v) \geq 2$, which is a contradiction to $(Q4.1)_I$ and proves the claim. (This is the point where we use that $|S_I| = D - 3$.)

We define $\phi := \tilde{\tau} \cup \tau^\eta \cup \tau^{\delta_1} \cup \tau^{\delta_2} \cup \tau^{\mathrm{par}}$. Thus ϕ packs $\mathcal{F}$ into $G \cup G' \cup H_1 \cup H_2$. The above claim and (g1) imply (Φ1). Since the sequence of all updates of the leftover edges is given by (8.26), (8.39), (8.43), (8.52), (8.53), (8.64), (8.67), and (8.70), we have $E(G' \cap \phi(\mathcal{F})) \subseteq E_1 \cup E_4 \cup (\tau^{\delta_1} \cup \tau^{\delta_2} \cup \tau^{\mathrm{par}})(\mathscr{T}^{\gamma_*})$. As $|\mathscr{T}^{\gamma_*}| \leq 3\hat{D}\gamma_* n$ by (8.5), we conclude by (8.38) and (8.63) that each $v \in A$ satisfies

$$d_{G' \cap \phi(\mathcal{F})}(v) \leq d_{E_1}(v) + d_{E_4}(v) + \Delta|\mathscr{T}^{\gamma_*}| \leq \varepsilon n + 3n^{2/3} + 3\hat{D}\Delta\gamma_* n \leq \gamma_*^{1/2}|A|.$$

This implies (Φ2). Note that here we make crucial use of the fact that when we packed the collection of forests which contain many trees (i.e. $\tilde{\mathcal{F}}$ and $\mathscr{F}^\eta$) we did not use any edges of G'. $\qquad\square$

Recall that the aim of Step 2 was to find a near-optimal packing of trees which cover most of the edges of G. Since G is quasi-random one could directly apply the results of [19] to achieve this. In other words, it would seem much more straightforward to apply Theorem 10.2 from [19] directly to obtain such a packing, rather than proving and applying the results in Section 6–8, which are based on more technical results from [19] and on Szemerédi's regularity lemma. However, the seemingly more straighforward approach would lead to the leftover density after Step 2 being too large (compared to that of H_1 and H_2) for the remaining steps to be feasible.

9. Orientations with regular outdegree

The following lemma states that every quasi-random graph has an "out-regular" orientation if the average degree is an even integer. We will need Lemma 9.1 at the very end of the proof of our main theorem where we need to embed only a single leaf for each tree in a given collection of trees. It will turn out that such an "out-regular" orientation of the remaining uncovered graph G will give rise to a valid embedding. We use $\bar{d}(G)$ to denote the average degree of a graph G.

Lemma 9.1. *Suppose $n, \bar{d} \in \mathbb{N}$ and $1/n \ll \beta \ll p \leq 1$. If G is a (β, p)-quasi-random graph on n vertices such that $\bar{d}(G) = 2\bar{d}$, then G has an orientation such that every vertex has exactly $\bar{d}$ outneighbours.*

Proof. We start by introducing for any graph H a function $Z(H)$ which measures the distance of the degree sequence of H to a $\bar{d}(H)$-regular degree sequence. We define

$$Z(H) := \sum_{v \in V(H)} |d(v) - \bar{d}(H)|.$$

Since G is (β, p)-quasi-random, we have $\Delta(G) - \delta(G) \leq 2p\beta n$ and thus $Z(G) \leq 2p\beta n^2$. We iteratively construct a sequence of graphs $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_s$ (for some $s \in \mathbb{N} \cup \{0\}$) such that G_s is regular and for all $j \in [s]$ the following hold:

(D1)$_j$ $\bar{d}(G_j) = 2(\bar{d} - j)$,
(D2)$_j$ $\Delta(G_j) - \delta(G_j) \leq \Delta(G_{j-1}) - \delta(G_{j-1})$ and $Z(G_j) \leq Z(G_{j-1})$,
(D3)$_j$ $\Delta(G_j) - \delta(G_j) \leq \Delta(G_{j-1}) - \delta(G_{j-1}) - 1$ or $Z(G_j) = Z(G_{j-1}) - p^2n/2$,
(D4)$_j$ $G_{j-1} - E(G_j)$ has an orientation such that every vertex has outdegree 1, and
(D5)$_j$ $\Delta(G_{j-1} - E(G_j)) \leq 3$.

Having defined such a sequence $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_s$, by (D1)$_s$ the graph G_s is Eulerian and so has an Eulerian circuit W. By orienting W consistently and orienting edges in $G_{j-1} - E(G_j)$ as in (D4)$_j$ for each $j \in [s]$, this leads to the desired orientation of G.

Note that (D1)$_0$ trivially holds and we view (D2)$_0$–(D5)$_0$ as being vacuously true. Suppose that for some $i \geq 0$ we have already defined graphs $G_0, \ldots, G_i$ such that (D1)$_j$–(D5)$_j$ hold for all $j \leq i$. By (D2)$_j$ and (D3)$_j$ for every $j \leq i$, we conclude that

$$i \leq (p^2n/2)^{-1}Z(G) + (\Delta(G) - \delta(G)) \leq 4\beta p^{-1}n + 2p\beta n \leq 6\beta p^{-1}n. \tag{9.1}$$

If G_i is regular, then set $s := i$. Suppose next that G_i is not regular. We show how to define G_{i+1}.

The fact that (D5)$_j$ holds for all $j \leq i$, together with (9.1) implies that $d_{G_i}(v) \geq d_G(v) - 3i \geq d_G(v) - 18\beta p^{-1}n$. Thus G_i is $(\beta^{1/2}, p)$-quasi-random by Proposition 3.9, and hence $(\beta^{1/12}, p/2)$-dense, by Proposition 3.10.

Let U be the set of vertices of maximum degree in G_i and let V be the set of vertices of minimum degree in G_i. Let $t := \min\{|U|, |V|, p^2n/4\}$. Let $U' := \{u_1, \ldots, u_t\} \subseteq U$ and $V' := \{v_1, \ldots, v_t\} \subseteq V$. Because G_i is $(\beta^{1/2}, p)$-quasi-random and thus $d_{G_i}(u, v) \geq (1 - \beta^{1/2})p^2n \geq 3t$ for all $u, v \in V(G_i)$, there is a set $W' := \{w_1, \ldots, w_t\} \subseteq V(G) \setminus (U' \cup V')$ such that $P_j := u_jw_jv_j$ is a path in G_i for all $j \in [t]$.

As G_i is $(\beta^{1/12}, p/2)$-dense, by Proposition 4.3, the graph $G' := G_i - (V' \cup W')$ is a robust $(p\beta^{1/12}/2, 4\beta^{1/12})$-expander and $\delta(G') \geq (1 - \beta^{1/2})pn - 2t \geq pn/3$. Thus, by Theorem 4.1, G' has a Hamilton cycle C. Let $G_{i+1} := G_i - E(C) - \bigcup_{j=1}^{t} E(P_j)$. It is easy to see that G_{i+1} satisfies (D1)$_{i+1}$, (D4)$_{i+1}$ and (D5)$_{i+1}$.

As, by (D1)$_i$, the average degree of G_i is an integer, we have $\delta(G_i) + 1 \leq \bar{d}(G_i) \leq \Delta(G_i) - 1$. Moreover, (D1)$_{i+1}$ implies $\bar{d}(G_{i+1}) = \bar{d}(G_i) - 2$. It is easy to check that together with our construction of G_{i+1} this implies that $|d_{G_i}(v) - \bar{d}(G_i)| \geq |d_{G_{i+1}}(v) - \bar{d}(G_{i+1})|$ for all $v \in V(G)$, and hence (D2)$_{i+1}$ follows.

If $t = \min\{|U|, |V|\}$, then $\Delta(G_{i+1}) - \delta(G_{i+1}) \leq \Delta(G_i) - \delta(G_i) - 1$ holds and if $t = p^2n/4$, then $Z(G_{i+1}) = Z(G_i) - p^2n/2$. Thus we have (D3)$_{i+1}$. This completes the construction of G_{i+1} satisfying (D1)$_{i+1}$–(D5)$_{i+1}$. $\square$

10. Proof of Theorem 1.7

In this section we prove Theorem 1.7. In Section 10.1 we first state and prove Theorem 10.1 (which is weaker than Theorem 1.7). We extend Theorem 10.1 to Theorem 1.7 in Section 10.2 by combining it with a result from [19].

10.1. Optimal tree packing.

Theorem 10.1. *For all $\Delta \in \mathbb{N}$ and $\delta > 0$, there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $n \geq N$ the following holds. Suppose G is an (ε, p)-quasi-random graph on n vertices and $\mathcal{T}$ a set of trees satisfying*

(i) $\Delta(T) \leq \Delta$ and $\delta n \leq |T| \leq (1 - \delta)n$ for all $T \in \mathcal{T}$,
(ii) $|\mathcal{T}| \geq (1/2 + \delta)n$, and
(iii) $e(\mathcal{T}) = e(G)$.

Then $\mathcal{T}$ packs into G.

Note that the conditions of the theorem imply that $p \geq \delta$.

We will use an iterative approach for the proof of Theorem 10.1. As explained earlier, in each iteration step, we will apply Lemma 8.1 to cover most of the current leftover graph with forests. To this end, we introduce a vortex of a quasi-random graph (a similar notion was already introduced in [13] to underpin the iterative absorption process carried out there). Suppose G is an (ε, p)-quasi-random graph on n vertices and $1/n \ll \varepsilon \ll \gamma \ll p$. A γ-*vortex* of G is a collection of vertex sets $A_0, \ldots, A_\Lambda, R_0, \ldots, R_{\Lambda-1}$ such that

(V1) $A_\Lambda \subseteq \ldots \subseteq A_0 = V(G)$,
(V2) $|A_i| = \lfloor \gamma^i n \rfloor =: n_i$,
(V3) $n_\Lambda \leq n^{1/3}$ and Λ is minimal with respect to this,
(V4) $R_i \subseteq A_i \setminus A_{i+1}$ and $|R_i| = \lfloor \varepsilon n_i \rfloor$ for every $i \in \{0, \ldots, \Lambda - 1\}$,
(V5) $d_{G,A_i}(u) = (1 \pm 3\varepsilon/2)p|A_i|$ and $d_{G,A_i}(u,v) = (1 \pm 3\varepsilon/2)p^2|A_i|$ for all $i \in \{0, \ldots, \Lambda\}$ and all distinct $u, v \in V(G)$, and
(V6) $d_{G,R_i}(u) = (1 \pm 2\varepsilon)p|R_i|$ and $d_{G,R_i}(u,v) = (1 \pm 2\varepsilon)p^2|R_i|$ for every $i \in \{0, \ldots, \Lambda - 1\}$ and all distinct $u, v \in V(G)$.

Is is not difficult to see that G has a γ-vortex, as the following random process produces a γ-vortex with probability, say, at least $1/2$. Let Λ and n_i be defined as in (V2) and (V3). Consider a random partition $(U_0, \ldots, U_\Lambda)$ such that $|U_\Lambda| = n_\Lambda$ and $|U_i| = n_i - n_{i+1}$ for every $i \in \{0, \ldots, \Lambda - 1\}$. Let $A_i := \bigcup_{j=i}^{\Lambda} U_j$. Then (V1)–(V3) hold. Lemma 3.2 shows that (V5) holds with probability at least $1/2$. Now it is easy to construct a γ-vortex of G, as we only need to find suitable sets R_i. Indeed, random sets $R_0, \ldots, R_{\Lambda-1}$ where $R_i \subseteq U_i$ and $|R_i| = \lfloor \varepsilon n_i \rfloor$ have the desired properties with probability at least $1/2$, by Lemma 3.2. In particular, such sets exist and (V6) holds.

We make the following observations, which follow directly from the definition of a vortex:

- for each $i \in [\Lambda]$, the bipartite graph $G[A_{i-1} \setminus A_i, A_i]$ is $(2\varepsilon, p)$-quasi-random, and
- $G[A_{i-1}] - E(G[A_i])$ is $(\gamma^{3/4}, p)$-quasi-random by Proposition 3.9.

When applying Lemma 8.1 in the proof of Theorem 10.1, A_{i+1} will play the role of A and R_i the role of R.

Proof of Theorem 10.1. First, we choose the following constants:

$$0 < 1/N \ll \varepsilon \ll \delta_2 \ll \delta_1 \ll \gamma_* \ll \gamma \ll \beta \ll \delta, 1/\Delta.$$

In view of the statement of Theorem 10.1, we may assume $1/\Delta \leq \delta/2$ by increasing the value of Δ if necessary. Let G be a graph satisfying the conditions of Theorem 10.1. As observed above, $p \geq \delta$ and G contains a γ-vortex $A_0, \ldots, A_\Lambda, R_0, \ldots, R_{\Lambda-1}$.

Note that for all distinct $i, j \in \{0, \ldots, \Lambda - 1\}$, the graphs $G[A_i] - E(G[A_{i+1}])$ and $G[A_j] - E(G[A_{j+1}])$ are edge-disjoint. For every $i \in \{0, \ldots, \Lambda - 1\}$, we decompose $G[A_i] - E(G[A_{i+1}])$ into three spanning subgraphs G_i, H_1^i, H_2^i such that

(H1)$_i$ H_1^i is a $(4\varepsilon, \delta_1)$-quasi-random bipartite graph with vertex partition $(A_i \setminus A_{i+1}, A_{i+1})$,
(H2)$_i$ $d_{H_2^i}(u) \leq 3\delta_2 n_i/2$ and $d_{H_2^i}(u,v) \geq 2\delta_2^2 n_i/3$ for any two vertices $u, v \in A_i$,
(H3)$_i$ G_i is $(\gamma^{1/2}, p)$-quasi-random and if $i \leq \Lambda - 2$, then $d_{G_i, R_{i+1}}(v) \geq p|R_{i+1}|/2$ for any vertex $v \in A_i \setminus A_{i+1}$, and in addition, if $i = 0$, then $d_{G_0, R_0}(v) \geq p\varepsilon n/2$ for all $v \in V(G)$, and
(H4)$_i$ A_{i+1} is an independent set in $H_1^i \cup H_2^i \cup G_i$.

It is not difficult to see that such a decomposition exists. Indeed, if we assign any edge in $G[A_i \setminus A_{i+1}, A_{i+1}]$ to H_1^i with probability δ_1/p, to H_2^i with probability δ_2/p, and to G_i with probability $1 - (\delta_1 + \delta_2)/p$ and any edge in $G[A_i \setminus A_{i+1}]$ to H_2^i with probability δ_2/p, and to G_i with probability $1 - \delta_2/p$, then we obtain graphs satisfying these conditions, with probability at least $1/2$.

Let $G_\Lambda := G[A_\Lambda]$. Before we proceed with an inductive argument, we carry out the following preparation step.

Preparation step. We will first remove a leaf from those trees which will play a role in the final absorbing step and embed the remainder of these trees into G_0 in a suitable way. For this,

let t_Λ be an integer such that

$$t_\Lambda n_\Lambda = e(G[A_\Lambda]) - 3\gamma_*^2 n_{\Lambda-1}^2 \pm n_{\Lambda-1}, \tag{10.1}$$

and consider an arbitrary collection of trees

$$\mathcal{T}^* := \{T_1^*, \ldots, T_{t_\Lambda n_\Lambda}^*\} \subseteq \mathcal{T}. \tag{10.2}$$

Let ℓ_i^* be a leaf of T_i^* and z_i^* be the unique neighbour of ℓ_i^* in T_i^*. For each $i \in [t_\Lambda n_\Lambda]$, let $T_{*,i} := T_i^* - \ell_i^*$ and $\mathcal{T}_* := \{T_{*,1}, \ldots, T_{*,t_\Lambda n_\Lambda}\}$. Let $v_1^*, \ldots, v_{t_\Lambda n_\Lambda}^*$ be a sequence of not necessarily distinct vertices such that every vertex in A_Λ appears in the sequence exactly t_Λ times. We sequentially construct functions $\phi_{T_{*,i}}$ which pack $T_{*,i}$ into $G_0[(A_0 \setminus A_1) \cup \{v_i^*\}]$ such that $\phi_{T_{*,i}}(z_i^*) = v_i^*$. (The embedding of $\ell_1^*, \ldots, \ell_{t_\Lambda n_\Lambda}^*$ will be deferred until the final step.) Assume that for some $i \in [t_\Lambda n_\Lambda]$ we already have constructed $\phi_{T_{*,1}}, \ldots, \phi_{T_{*,i-1}}$, and let

$$G(i) := G_0[(A_0 \setminus A_1) \cup \{v_i^*\}] - \bigcup_{j=1}^{i-1} E(\phi_{T_{*,j}}(T_{*,j})).$$

Note that $|T_{*,i}| \leq (1 - \delta)n \leq (1 - \gamma)n + 1 = |(A_0 \setminus A_1) \cup \{v_i^*\}|$. For any $v \in V(G(i))$, we have $d_{G_0}(v) - d_{G(i)}(v) \leq |A_1| + \Delta t_\Lambda n_\Lambda \leq 2\gamma n$ and hence $G(i)$ is $(\gamma^{2/5}, p)$-quasi-random by (H3)$_0$ and Proposition 3.9. Thus we can apply Lemma 3.20 to obtain $\phi_{T_{*,i}}$ such that

$$\phi_{T_{*,i}}(z_i^*) = v_i^* \quad \text{and} \quad \phi_{T_{*,i}}(T_{*,i}) \subseteq (A_0 \setminus A_1) \cup \{v_i^*\}. \tag{10.3}$$

Repeating this process for every $i \in [t_\Lambda n_\Lambda]$ gives rise to functions $\{\phi_{T_{*,i}}\}_{i \in [t_\Lambda n_\Lambda]}$. We define $\phi_{-1} := \phi_{T_{*,1}} \cup \ldots \cup \phi_{T_{*,t_\Lambda n_\Lambda}}$ and

$$G_{*,0} := G_0 - E(\phi_{-1}(\mathcal{T}_*)). \tag{10.4}$$

Then $G_{*,0}$ is $(\gamma^{1/3}, p)$-quasi-random as

$$d_{G_0}(v) - d_{G_{*,0}}(v) \leq \Delta t_\Lambda n_\Lambda \leq n^{3/4} \tag{10.5}$$

for all $v \in V(G_{*,0})$. Similarly, $G_{*,0}[A_0 \setminus A_1]$ is also $(\gamma^{1/3}, p)$-quasi-random.

For each tree $T \in \mathcal{T} \setminus \mathcal{T}^*$, we select an edge $e_T = r_T^1 r_T^2$ such that $T - e_T$ consists of two components of size at least $\delta n/(2\Delta) \geq n/\Delta^2$ (it is easy to see that such an edge exists). We also select $|\mathcal{T} \setminus \mathcal{T}^*|$ arbitrary edges $\{e_T' : T \in \mathcal{T} \setminus \mathcal{T}^*\}$ in $G_{*,0}[R_0]$ such that $\Delta(\{e_1', \ldots, e_{|\mathcal{T} \setminus \mathcal{T}^*|}'\}) \leq \varepsilon^{-2}$. By (10.5) and (H3)$_0$ we can greedily select these edges. Next we define ϕ_{-1}' by $\phi_{-1}'(e_T) := e_T'$ for all $T \in \mathcal{T} \setminus \mathcal{T}^*$. We also extend ϕ_{-1} be setting $\phi_{-1}(e_T) := \phi_{-1}'(e_T)$. Let $F_T^0 := T - e_T$. We view the endvertices of e_T as the roots of F_T^0. Define

$$\mathcal{F}^0 := \{F_T^0 : T \in \mathcal{T} \setminus \mathcal{T}^*\}, \quad G_0' := G_{*,0} - \{e_T' : T \in \mathcal{T} \setminus \mathcal{T}^*\} \quad \text{and} \quad G_0'' := G_0 - E(G_0'). \tag{10.6}$$

We point out that G_0' is (β^2, p)-quasi-random.

We say a function ψ *packs* a family $\mathcal{H}$ of graphs *perfectly* into a graph H if ψ packs $\mathcal{H}$ and every edge of H is covered. Thus

$$\phi_{-1} \text{ packs } \mathcal{T}_* \cup \{e_T : T \in \mathcal{T} \setminus \mathcal{T}^*\} \text{ perfectly into } G_0''. \tag{10.7}$$

Thus G_0'' consists of the edges of G_0 that have been covered in this step and G_0' denotes the (uncovered) leftover of G_0. Moreover, assumption (iii) of Theorem 10.1 implies that

$$
\begin{aligned}
e(\mathcal{F}^0) \quad &= \quad e(\mathcal{T}) - e(\mathcal{T}^*) - |\mathcal{T} \setminus \mathcal{T}^*| \\
&= \quad (e(G_0) + e(H_1^0) + e(H_2^0) + e(G[A_1])) - (e(\mathcal{T}_*) + t_\Lambda n_\Lambda) - |\mathcal{T} \setminus \mathcal{T}^*| \\
&\overset{(10.6),(10.7)}{=} \quad e(G_0') + e(H_1^0) + e(H_2^0) + e(G[A_1]) - t_\Lambda n_\Lambda.
\end{aligned}
$$

This completes the Preparation step.

We proceed with an inductive approach. We will define a packing of $\mathcal{F}^0$ step by step. In the tth step we will be given a collection $\mathcal{F}^t$ consisting of those subforests of the forests $\mathcal{F}^0$ which (apart from their roots) are not embedded yet. For each $F \in \mathcal{F}^t$ we will embed either a part of F or the entire forest F in such a way that all the edges of $G - E(G_0'')$ incident to a vertex in $A_t \setminus A_{t+1}$ are covered. The key tool for this is Lemma 8.1. In the $(t+1)$th step we will then

continue with the collection $\mathcal{F}^{t+1}$ consisting of all those subforests which are still not embedded yet. We will denote by ϕ_t the packing of the subforests defined by the end of Step t. The roots of each $F \in \mathcal{F}^{t+1}$ will already be embedded into $R_{t+1} \subseteq A_{t+1} \setminus A_{t+2}$ by ϕ_t (but as described above, no other vertex of F will be embedded yet).

More precisely, for each $0 \leq t \leq \Lambda$, we will define a packing ϕ_{t-1} and a collection $\mathcal{F}^t$ of rooted forests so that the following hold:

$(E1)_t$ There is a (β^2, p)-quasi-random spanning subgraph G'_t of G_t and if $t \leq \Lambda - 2$, then G'_t satisfies $d_{G'_t, R_{t+1}}(v) \geq p|R_{t+1}|/5$ for any vertex $v \in A_t \setminus A_{t+1}$.

$(E2)_t$ If $t = \Lambda$, then $\mathcal{F}^t = \emptyset$, and if $t \leq \Lambda - 1$, then $\mathcal{F}^t$ consists of rooted subforests of the forests in $\mathcal{F}^0$ and $|\mathcal{F}^t| \geq (1/2 + 1/\Delta)n_t$. Moreover, each $F \in \mathcal{F}^t$ satisfies $|F| \leq (1 - 1/\Delta)n_t$ and consists of two components (K_F^1, r_F^1), (K_F^2, r_F^2) with $|K_F^1|, |K_F^2| \geq p\Delta^{-3}n_t/5$.

$(E3)_t$ For each $F \in \mathcal{F}^t$, no vertex of $F - \{r_F^1, r_F^2\}$ is embedded by ϕ_{t-1} and $\phi_{t-1}(r_F^1), \phi_{t-1}(r_F^2) \in R_t$. Let $\phi'_{t-1} : \{r_F^c : c \in [2], F \in \mathcal{F}^t\} \to R_t$ be defined by $\phi'_{t-1}(r_F^c) := \phi_{t-1}(r_F^c)$. Then $|\phi'^{-1}_{t-1}(v)| \leq \varepsilon^{-2}$ for every $v \in R_t$.

$(E4)_t$ For each $F \in \mathcal{F}^t$, let F^{orig} be the unique forest in $\mathcal{F}^0$ with $F \subseteq F^{\text{orig}}$. Let $\mathcal{F}^{0,t}$ be the collection of all those $F' \in \mathcal{F}^0$ for which $\mathcal{F}^t$ does not contain a subforest of F'. If $t \geq 1$, then ϕ_{t-1} is consistent with ϕ_{t-2} and packs $\mathcal{F}^{0,t} \cup \{F^{\text{orig}} - (V(F) \setminus \{r_F^1, r_F^2\}) : F \in \mathcal{F}^t\}$ perfectly into

$$(G_t - E(G'_t)) \cup \bigcup_{i=1}^{t-1} G_i \cup \bigcup_{i=0}^{t-1} (H_1^i \cup H_2^i) \cup G'_0 = (G_t - E(G'_t)) \cup (G - E(G[A_t] \cup G''_0)).$$

Moreover, for each $F \in \mathcal{F}^t$ we have $A_t \cap \phi_{t-1}(F^{\text{orig}} - V(F)) = \emptyset$.

$(E5)_t$ If $t \leq \Lambda - 1$, then $e(\mathcal{F}^t) = e(G'_t) + e(H_1^t) + e(H_2^t) + e(G[A_{t+1}]) - t_\Lambda n_\Lambda$.

Note that $(E1)_0$–$(E5)_0$ hold. (To check $(E2)_0$, note that $|\mathcal{T}^*| = t_\Lambda n_\Lambda \leq n^{3/4}$ and recall that $1/\Delta \leq \delta/2$.) Hence suppose that $0 \leq t \leq \Lambda - 1$ and that we have defined ϕ_{t-1} and $\mathcal{F}^t$ satisfying $(E1)_t$–$(E5)_t$. Our aim of Step t is to show that there is a function ϕ_t consistent with ϕ_{t-1} and for all $F \in \mathcal{F}^t$ there is a subforest $F^* \subseteq F$ so that $(E1)_{t+1}$–$(E5)_{t+1}$ hold, where $\mathcal{F}^{t+1} := \{F - V(F^*) : F \in \mathcal{F}^t\}$. In particular, for each $F \in \mathcal{F}^t$, the function ϕ_t will embed F^* as well as the two roots of $F - V(F^*)$ into $G'_t \cup H_1^t \cup H_2^t \cup G_{t+1}$, but will embed no other vertices of $F - V(F^*)$. For some of the $F \in \mathcal{F}^t$ we will have $F = F^*$. Those forests F will be embedded in Step $t.2$, while the part F^* of all those F with $F \neq F^*$ will be embedded in Step $t.1$.

Note that $\mathcal{F}^{0,t}$ consists of those forests in $\mathcal{F}^0$ which have been completely embedded prior to Step $t.1$. Also we remark that G'_t is the subgraph of G_t which has not been covered prior to Step $t.1$. The graphs G'_t, H_1^t, and H_2^t will be covered entirely in Step t.

Step $t.1$. If $t = \Lambda - 1$, let $\mathcal{F} := \mathcal{F}^t$, $\mathcal{F}^\Lambda := \emptyset$ and $G^{\Lambda-1} := G'_{\Lambda-1}$. Then (10.1) and $(E5)_{\Lambda-1}$ imply

$$e(\mathcal{F}) = e(\mathcal{F}^{\Lambda-1}) = e(G^{\Lambda-1}) + e(H_1^{\Lambda-1}) + e(H_2^{\Lambda-1}) + 3\gamma_*^2 n_{\Lambda-1}^2 \pm n_{\Lambda-1}. \tag{10.8}$$

We proceed to Step $t.2$.

From now on we assume that $t < \Lambda - 1$. In what follows we prepare the set $\mathcal{F}^{t+1}$ for the next iteration step. For this, we choose an arbitrary sub-collection $\mathcal{F}' \subseteq \mathcal{F}^t$ of $\Delta^2 \gamma n_t = \Delta^2 n_{t+1}$ forests and let $\mathcal{F} := \mathcal{F}^t \setminus \mathcal{F}'$. Then

$$|\mathcal{F}| \geq (1/2 + 1/\Delta)n_t - \Delta^2 \gamma n_t \geq (1/2 + 1/(2\Delta))n_t. \tag{10.9}$$

The forests in $\mathcal{F}$ will be embedded in Step $t.2$. via Lemma 8.1, and $\mathcal{F}^{t+1}$ will consist of subforests of the forests in $\mathcal{F}'$. The set $\mathcal{F}^*$ of remaining subforests of the forests in $\mathcal{F}'$ will be embedded greedily in the current Step $t.1$. In particular, we will have $\mathcal{F}^{0,t+1} = \mathcal{F}^{0,t} \cup \mathcal{F}$, where $\mathcal{F}^{0,t}$ is as defined in $(E4)_t$. Since $t < \Lambda - 1$, (V5) and (10.1) give us

$$e(G[A_{t+1}]) - t_\Lambda n_\Lambda = (1 \pm \gamma^{1/2})pn_{t+1}^2/2 - t_\Lambda n_\Lambda = (1 \pm \gamma^{1/3})pn_{t+1}^2/2.$$

Thus

$$\beta' := \frac{e(G[A_{t+1}]) - t_\Lambda n_\Lambda - 3\gamma_*^2 n_t^2}{n_t |\mathcal{F}'|} = (1 \pm \gamma^{1/4})\frac{pn_{t+1}}{2\Delta^2 n_t} = (1 \pm \gamma^{1/4})\frac{\gamma p}{2\Delta^2}.$$

By Proposition 6.4 with and β', $p\Delta^{-3}/5$, n_t playing the roles of β, α, n, we can find a subforest $\tilde{F}$ of each $F \in \mathcal{F}'$ such that

(P1) both $\tilde{F}$ and $F^* := F - V(\tilde{F})$ consist of two components,

(P2) each component of $\tilde{F}$ has size between $\Delta^{-3}pn_{t+1}/5$ and $\Delta^{-1}pn_{t+1}$, and

(P3) $\tilde{F}$ does not contain r_F^1 or r_F^2.

(P4) Moreover, let $\mathcal{F}^{t+1} := \{\tilde{F} : F \in \mathcal{F}'\}$. Then $e(\mathcal{F}^{t+1}) = \sum_{F \in \mathcal{F}'} e(\tilde{F}) = e(G[A_{t+1}]) - t_\Lambda n_\Lambda - 3\gamma_* n_t^2 \pm n_t$.

Note that $(E2)_t$ implies that for all $F \in \mathcal{F}'$, we have

$$|F^*| \leq |F| \leq (1 - 1/\Delta)n_t. \tag{10.10}$$

For each $F \in \mathcal{F}'$ and $c \in [2]$, let $r_{\tilde{F}}^c$ be the unique vertex of $\tilde{F}$ in the component of F containing r_F^c which is adjacent to a vertex in $V(F^*)$ in F. Note that (P2) implies that $\mathcal{F}^{t+1}$ satisfies $(E2)_{t+1}$ with $r_{\tilde{F}}^c$ playing the role of r_F^c. For every $c \in [2]$, let x_F^c be the unique neighbour of $r_{\tilde{F}}^c$ in $V(F^*)$ in F.

Let $\{F_1, \ldots, F_m\} := \mathcal{F}'$ and $\mathcal{F}^* := \{F^* : F \in \mathcal{F}'\} = \{F_1^*, \ldots, F_m^*\}$. Note that

$$e(\mathcal{F}') = e(\mathcal{F}^*) + e(\mathcal{F}^{t+1}) + 2m. \tag{10.11}$$

Recall from $(E2)_t$ that $r_{F_i}^1$ and $r_{F_i}^2$ are the roots of F_i and thus, by (P3), the roots of F_i^*. In what follows, for all $i \in [m]$, we construct embeddings $\phi_{F_i^*}$ satisfying the following:

$(Q1)_i$ $\phi_{F_1^*} \cup \cdots \cup \phi_{F_i^*}$ packs $\{F_1^*, \ldots, F_i^*\}$ into $G_t'[A_t \setminus A_{t+1}]$ such that $\phi_{F_j^*}$ is consistent with ϕ_{t-1}' for all $j \in [i]$, and

$(Q2)_i$ $p(v, i) \leq \varepsilon^{-2}$ for any $v \in A_t \setminus A_{t+1}$, where $p(v, i) := |\{j \in [i] : v \in \phi_{F_j^*}(\{x_{F_j^*}^1, x_{F_j^*}^2\})\}|$.

$(Q3)_i$ $\phi_{F_i^*}(F_i^*) \cap R_t = \{\phi_{t-1}'(\{r_{F_i}^1, r_{F_i}^2\})\}$.

Assume for all $j < i$ we have constructed ϕ_j satisfying $(Q1)_j$–$(Q3)_j$. Now we construct ϕ_{F_i}. Recall that

$$m = |\mathcal{F}'| = \Delta^2 \gamma n_t = \Delta^2 n_{t+1}. \tag{10.12}$$

Let

$$B := \{v \in A_t \setminus (A_{t+1} \cup R_t) : p(v, i-1) > \varepsilon^{-2} - 1\}.$$

Hence $|B| \leq |\mathcal{F}'|/\varepsilon^{-1} = \varepsilon\Delta^2\gamma n_t \leq \varepsilon n_t$. Define

$$G_t'(i) := G_t'[A_t \setminus (A_{t+1} \cup B \cup (R_t \setminus \{\phi_{t-1}'(\{r_{F_i}^1, r_{F_i}^2\})\}))] - \bigcup_{j=1}^{i-1} E(\phi_{F_j^*}(F_j^*)).$$

Note that

$$\Delta(\bigcup_{j=1}^{i-1} E(\phi_{F_j^*}(F_j^*))) \leq \Delta m = \Delta^3 \gamma n_t. \tag{10.13}$$

Thus $G_t'(i)$ is $(\beta^{3/2}, p)$-quasi-random by $(E1)_t$ and Proposition 3.9. In addition, (V2), (V4) and (10.10) imply $|G_t'(i)| \geq (1 - \gamma - 2\varepsilon)n_t \geq |F_i^*|$. So we can apply Lemma 3.20 (with $\{r_{F_i}^1, r_{F_i}^2\}$ playing the role of I) to obtain a function $\phi_{F_i^*}$ packing F_i^* into $G_t'(i)$ which is consistent with ϕ_{t-1}'. Thus $(Q1)_i$ and $(Q3)_i$ hold, and $(Q2)_i$ follows from the definition of B and $(E3)_t$.

By repeating this procedure for every $i \in [m]$, we obtain a collection of functions $\{\phi_{F_i^*}\}_{i \in [m]}$ satisfying $(Q1)_i$–$(Q3)_i$ for every $i \in [m]$. We let $\phi_t^* := \phi_{F_1^*} \cup \cdots \cup \phi_{F_m^*}$. Let $G_*^t := G_t' - E(\phi_t^*(\mathcal{F}^*))$ and observe that G_*^t is $(\beta^{3/2}, p)$-quasi-random by (10.13), $(E1)_t$ and Proposition 3.9.

Now for each $F \in \mathcal{F}'$, we will embed the roots $r_{\tilde{F}}^1$ and $r_{\tilde{F}}^2$ of $\tilde{F}$ so that their embedding satisfies $(E3)_{t+1}$. Let $u_j := \phi_t^*(x_{F_j}^1)$ and $u_{m+j} := \phi_t^*(x_{F_j}^2)$ for each $j \in [m]$. Note that by $(Q2)_m$ no vertex occurs more than ε^{-2} times in the list $u_1, \ldots, u_{2m}$. Since $\phi_{F_j^*}(F_j^*) \cap R_{t+1} \subseteq \phi_{F_j^*}(F_j^*) \cap A_{t+1} = \emptyset$ for all $j \in [m]$, $(E1)_t$ implies that for each vertex u_j we have $d_{G_*^t, R_{t+1}}(u_j) = d_{G_t', R_{t+1}}(u_j) \geq p\varepsilon n_{t+1}/5 \geq 3\varepsilon^{-2} + 2m/\varepsilon^{-2} + \varepsilon^{-2}$.

Let H^{ind} be the graph on $[2m]$ such that $ij \in E(H^{\mathrm{ind}})$ if $i = j + m$. Hence we can apply Proposition 3.21 with the following parameters and graphs.

object/parameter	G_*^t	H^{ind}	ε^{-2}	R_{t+1}	$2m$	ε^{-2}	$\emptyset$	u_j
playing the role of	G	H	Δ	A	m	s	W_j	u_j

From Proposition 3.21, we obtain $v_1, \dots, v_{2m} \in R_{t+1}$ such that $E := \{u_i v_i : i \in [2m]\}$ is a collection of distinct edges in G_*^t, and we have $v_i \neq v_{i+m}$ for every $i \in [m]$. Moreover, for any $v \in R_{t+1}$, we have $|\{i \in [2m] : v = v_i\}| \leq \varepsilon^{-2}$. Thus $\Delta(E) \leq \varepsilon^{-2}$. We extend ϕ_t^* and define a new function ϕ_t' by setting for all $i \in [m]$

$$\phi_t^*(r_{\tilde{F}_i}^1) = \phi_t'(r_{\tilde{F}_i}^1) := v_i \text{ and } \phi_t^*(r_{\tilde{F}_i}^2) = \phi_t'(r_{\tilde{F}_i}^2) := v_{m+i}. \tag{10.14}$$

Thus $|\phi_t'^{-1}(v)| \leq \varepsilon^{-2}$ for every $v \in R_{t+1}$. As for each $i \in [m]$ we will not embed any vertex of $\tilde{F}_i - \{r_{\tilde{F}_i}^1, r_{\tilde{F}_i}^2\}$ in Step t, this will imply that $(\mathrm{E3})_{t+1}$ holds. Moreover, note that for every $i \in [m]$

$$A_{t+1} \cap \phi_t^*(F_i - V(\tilde{F}_i)) \overset{(\mathrm{P1})}{=} A_{t+1} \cap \phi_t^*(F_i^*) \overset{(\mathrm{Q1})_i}{=} \emptyset. \tag{10.15}$$

Let

$$G^t := G_*^t - E.$$

Since G_*^t is $(\beta^{3/2}, p)$-quasi-random and $\Delta(E) \leq \varepsilon^{-2}$,

$$G^t \text{ is } (\beta, p)\text{-quasi-random}. \tag{10.16}$$

Note that G^t is the current leftover of the graph G_t' given in $(\mathrm{E1})_t$ and $e(G_t') = e(G^t) + e(\mathcal{F}^*) + 2m$. Hence

$$
\begin{aligned}
e(\mathcal{F}) &= & e(\mathcal{F}^t) - e(\mathcal{F}') \\
&\overset{(\mathrm{E5})_t}{=} & e(G_t') + e(H_1^t) + e(H_2^t) + e(G[A_{t+1}]) - t_\Lambda n_\Lambda - e(\mathcal{F}') \\
&= & e(G^t) + e(\mathcal{F}^*) + 2m + e(H_1^t) + e(H_2^t) + e(G[A_{t+1}]) - t_\Lambda n_\Lambda - e(\mathcal{F}') \\
&\overset{(10.11),(\mathrm{P4})}{=} & e(G^t) + e(H_1^t) + e(H_2^t) + 3\gamma_* n_t^2 \pm n_t.
\end{aligned}
\tag{10.17}
$$

We proceed to Step $t.2.$, where we pack $\mathcal{F}$ into G^t so that the packing is consistent with ϕ_{t-1}'.

Step $t.2.$ Let $\hat{G}_{t+1}$ be a $(\gamma^{1/3}, p/2)$-quasi-random subgraph of the graph G_{t+1} (described in $(\mathrm{H3})_{t+1}$) such that, if $t \leq \Lambda - 3$, then for any vertex $v \in A_{t+1} \setminus A_{t+2}$, we have

$$d_{G_{t+1} - E(\hat{G}_{t+1}), R_{t+2}}(v) \geq p|R_{t+2}|/5. \tag{10.18}$$

Such a subgraph of G_{t+1} exists as a random subgraph chosen by including every edge of G_{t+1} independently with probability $1/2$ has these properties with probability at least $1/2$, by $(\mathrm{H3})_{t+1}$.

Now we apply Lemma 8.1 with the following parameters and graphs.

object/parameter	G^t	$\hat{G}_{t+1}$	H_1^t	H_2^t	A_{t+1}	R_t	$\mathcal{F}$	n_t	ε	δ_1
playing the role of	G	G'	H_1	H_2	A	R	$\mathcal{F}$	n	ε	δ_1
object/parameter	δ_2	γ_*	γ	β	p	10Δ	Δ	$p/2$	$1/(2\Delta)$	ϕ_{t-1}'
playing the role of	δ_2	γ_*	γ	β	α	D	Δ	p	d	ϕ'

Observe that (g1) holds by (V2), (V4) and $(\mathrm{H4})_t$, (g2) holds by (10.16) and our choice of $\hat{G}_{t+1}$, (g3) holds by $(\mathrm{H1})_t$, and (g4) holds by $(\mathrm{H2})_t$. Furthermore, (f1) holds by (10.9) if $t < \Lambda - 1$, or $(\mathrm{E2})_{\Lambda-1}$ if $t = \Lambda - 1$, (f2) holds by $(\mathrm{E2})_t$, (f3) holds by (10.17) or (10.8), and (f4) holds by $(\mathrm{E3})_t$.

We obtain a function $\hat{\phi}_t$ packing $\mathcal{F}$ into $G^t \cup \hat{G}_{t+1} \cup H_1^t \cup H_2^t$ consistent with ϕ_{t-1}' such that

$(\Phi1)_t$ $E(G^t) \cup E(H_1^t) \cup E(H_2^t) \subseteq E(\hat{\phi}_t(\mathcal{F}))$, and

$(\Phi2)_t$ for all $v \in A_{t+1}$, we have $d_{\hat{G}_{t+1} \cap \hat{\phi}_t(\mathcal{F})}(v) \leq \gamma_*^{1/2}|A_{t+1}|$.

Let $\phi_t := \phi_t^* \cup \hat{\phi}_t \cup \phi_{t-1}$. By (H3)$_{t+1}$ and $(\Phi 2)_t$, we conclude that $G'_{t+1} := G_{t+1} - E(\hat{\phi}_t(\mathcal{F}))$ is (β^2, p)-quasi-random. Thus (E1)$_{t+1}$ holds, by (10.18). We have already observed that (E2)$_{t+1}$ and (E3)$_{t+1}$ hold. Property (E4)$_{t+1}$ follows from (10.15), $(\Phi 1)_t$ and (E4)$_t$. In particular, note that in Step t we have now covered the (previous) leftover G'_t of G_t which was not covered prior to Step t. The graphs H_1^t and H_2^t are covered entirely in Step t and G'_{t+1} is the (new) leftover of G_{t+1}, which will be covered in Step $t+1$. To check (E5)$_{t+1}$, note that (E4)$_{t+1}$ implies that

$$e(\mathcal{F}^t) - e(\mathcal{F}^{t+1}) = e(G'_t) + e(H_1^t) + e(H_2^t) + e(G_{t+1}) - e(G'_{t+1}). \tag{10.19}$$

Thus if $t < \Lambda - 1$, then

$$
\begin{aligned}
e(\mathcal{F}^{t+1}) &= e(\mathcal{F}^t) - (e(\mathcal{F}^t) - e(\mathcal{F}^{t+1})) \\
&\overset{(\text{E5})_t, (10.19)}{=} e(G[A_{t+1}]) - e(G_{t+1}) + e(G'_{t+1}) - t_\Lambda n_\Lambda \\
&= e(G'_{t+1}) + e(H_1^{t+1}) + e(H_2^{t+1}) + e(G[A_{t+2}]) - t_\Lambda n_\Lambda.
\end{aligned}
$$

This verifies (E5)$_{t+1}$.

If $t = \Lambda - 1$, then we proceed to the Final step. Otherwise we proceed to Step $(t+1).1$.

Final step. Recall that $G_\Lambda = G[A_\Lambda]$. Since $\mathcal{F}^\Lambda = \emptyset$, properties (E4)$_1$–(E4)$_\Lambda$ imply that $\phi_{\Lambda-1}$ is consistent with ϕ_{-1} and packs $\mathcal{F}^{0,\Lambda} = \mathcal{F}^0$ perfectly into

$$(G_\Lambda - E(G'_\Lambda)) \cup \bigcup_{i=1}^{\Lambda-1} G_i \cup \bigcup_{i=0}^{\Lambda-1} (H_1^i \cup H_2^i) \cup G'_0 = G - (E(G'_\Lambda) \cup E(G''_0)).$$

Together with (10.7), this implies that $\phi_{\Lambda-1} \cup \phi_{-1}$ packs $\mathcal{T}_* \cup (\mathcal{T} \setminus \mathcal{T}^*)$ perfectly into $G - E(G'_\Lambda)$. In particular, the current set of leftover edges is given by G'_Λ. Recall that the trees in $\mathcal{T}_*$ were obtained from those in $\mathcal{T}^* = \{T_1^*, \ldots, T_{t_\Lambda n_\Lambda}^*\}$ by deleting the edges $z_i^* \ell_i^*$ for all $i \in [t_\Lambda n_\Lambda]$. In particular,

$$e(G'_\Lambda) = e(G) - e(\mathcal{T}_* \cup (\mathcal{T} \setminus \mathcal{T}^*)) = e(\mathcal{T}) - (e(\mathcal{T}^*) - t_\Lambda n_\Lambda) - e(\mathcal{T} \setminus \mathcal{T}^*) = t_\Lambda n_\Lambda. \tag{10.20}$$

We will now extend our current packing to a packing of $\mathcal{T}$ into G by embedding all these edges $z_i^* \ell_i^*$.

Since $|G'_\Lambda| = |A_\Lambda| = n_\Lambda$, by (V2), (10.20) implies that $\overline{d}(G'_\Lambda) = 2t_\Lambda$. Recall that G'_Λ is (β^2, p)-quasi-random, by (E1)$_\Lambda$. Thus by Lemma 9.1, G'_Λ has an orientation D such that every vertex in $V(G'_\Lambda) = A_\Lambda$ has exactly t_Λ out-neighbours. For a vertex $v \in V(G'_\Lambda)$, we denote by $N_D^+(v)$ the set of out-neighbours of v. By (10.3) and our choice of the vertices v_i^*, for each $v \in A_\Lambda$, we have $|\{i \in [t_\Lambda n_\Lambda] : \phi_{-1}(z_i^*) = v\}| = t_\Lambda = |N_D^+(v)|$. Thus there is a bijection $\tau_v : \{\ell_i^* : \phi_{-1}(z_i^*) = v\} \to N_D^+(v)$. Let

$$\phi_{-1}(\ell_i^*) := \tau_{\phi_{-1}(z_i^*)}(\ell_i^*).$$

Then $E(G'_\Lambda) = \{\phi_{-1}(z_i^*)\phi_{-1}(\ell_i^*) : i \in [t_\Lambda n_\Lambda]\}$ and this completes the partial embedding of T^* to a complete one for every $T^* \in \mathcal{T}^*$ (indeed, recall that $T_{*,i} = T_i^* - \ell_i^*$ and $\phi_{-1}(T_{*,i}) \cap A_\Lambda = \{\phi_{-1}(z_i^*)\}$ by (10.3), thus $\tau_{\phi_{-1}(z_i^*)}(\ell_i^*) \notin \phi_{-1}(T_{*,i})$).

Now, by construction, $\phi_{\Lambda-1} \cup \phi_{-1}$ packs $\mathcal{T}$ into G. $\qquad \square$

10.2. Proof of Theorem 1.7 and its consequences.

To prove Theorem 1.7 we need Theorem 1.2 from [19]. It allows us to pack a suitable collection of bounded degree graphs into a quasi-random graph G, and guarantees a near-optimal packing, i.e. almost all edges of G are used by the packing.

Theorem 10.2. *Suppose $n, \Delta \in \mathbb{N}$ with $1/n \ll \varepsilon \ll p, \alpha, 1/\Delta \leq 1$. Suppose $\mathcal{H}$ is a collection of graphs on n vertices with $\Delta(H) \leq \Delta$ for all $H \in \mathcal{H}$ and $(1 - 2\alpha)\binom{n}{2}p \leq e(\mathcal{H}) \leq (1 - \alpha)\binom{n}{2}p$. Suppose that G is an (ε, p)-quasi-random graph on n vertices. Then there exists a function ϕ which packs $\mathcal{H}$ into G such that $\Delta(G - E(\phi(\mathcal{H}))) \leq 4\alpha pn$.*

Proof of Theorem 1.7. We choose $\varepsilon, \varepsilon'$ and N such that

$$1/N \ll \varepsilon \ll \varepsilon' \ll \delta, 1/\Delta.$$

Let p' be a real number such that $e(\mathcal{H}) = p'\binom{n}{2}$. Note that by assumptions (ii)–(iv) of Theorem 1.7, we have $p' \leq p - \delta$. Let G' be a subgraph of G such that

(Z1) $G - E(G')$ is $(\varepsilon^{1/2}, p - p' - \varepsilon'^2)$-quasi-random, and

(Z2) G' is $(\varepsilon^{1/2}, p' + \varepsilon'^2)$ quasi-random.

Note that G' exists, as a random subgraph of G obtained by independently including each edge with probability $(p' + \varepsilon'^2)/p$ has these properties with probability at least $1/2$, by Lemma 3.2.

Since $\varepsilon \ll \varepsilon', 1/\Delta$, we can use Theorem 10.2 to find a function $\phi_{\mathcal{H}}$ packing $\mathcal{H}$ into G' such that

$$\Delta(G' - E(\phi_{\mathcal{H}}(\mathcal{H}))) \leq 4 \cdot \frac{\varepsilon'^2}{p' + \varepsilon'^2} \cdot (p' + \varepsilon'^2)n = 4\varepsilon'^2 n. \tag{10.21}$$

Define $G_1 := G - E(\phi_{\mathcal{H}}(\mathcal{H}))$. Then G_1 is $(\varepsilon', p - p')$-quasi-random by Proposition 3.9, (Z1), and (10.21). Note that $e(G_1) = e(\mathcal{T})$. Now we apply Theorem 10.1 to G_1 and $\mathcal{T}$ to complete the proof. □

Next we deduce Corollary 1.8 from Theorem 1.7.

Proof of Corollary 1.8. We may assume $\Delta \geq 2$ and $\alpha < 1/3$. Let $N \in \mathbb{N}$ and $\varepsilon > 0$ be the constants given by Theorem 1.7 for the parameters Δ and $\delta := p_0\alpha/2$. We may assume that $\varepsilon \ll \alpha, p_0$. We add arbitrary trees on at most $(1-\alpha)pn$ vertices with maximum degree at most Δ to $\mathcal{T}$ to obtain a set of trees $\mathcal{T}'$ so that $e(\mathcal{T}') = e(G)$. For two trees $T, T' \in \mathcal{T}'$ of order at most δn, we choose leaves ℓ of T and ℓ' of T', delete both T and T' from $\mathcal{T}'$ and add the tree obtained from T and T' by identifying ℓ and ℓ'. Then every tree in $\mathcal{T}'$ still has maximum degree at most Δ and its order is at most $\max\{2\delta n, (1-\alpha)pn\} \leq (1-\alpha)pn$, since $\alpha < 1/3$ and $p_0 \leq p$. By repeating this process, we may assume that every tree in $\mathcal{T}'$ but at most one has at least δn vertices and at most $(1-\alpha)pn \leq (1-\delta)n$ vertices. Let $\mathcal{T}'' \subseteq \mathcal{T}'$ consist of all trees having at least δn vertices. Let $\mathcal{H} := \mathcal{T}' \setminus \mathcal{T}''$. Then $|\mathcal{H}| \leq 1$. Note that $\mathcal{T}''$ and $\mathcal{H}$ satisfy conditions (i)–(iv) of Theorem 1.7. Thus $\mathcal{T}'' \cup \mathcal{H}$ packs into G, and a packing of $\mathcal{T}' = \mathcal{T}'' \cup \mathcal{H}$ into G naturally gives a packing of $\mathcal{T}$ into G. □

Finally we deduce Theorem 1.2 from Theorem 1.7.

Proof of Theorem 1.2. Let N and ε be such that the statement of Theorem 1.7 holds with Δ, $1/10$, 5ε playing the roles of Δ, δ, ε. We may assume that $\varepsilon \leq 1/100$. Next we iteratively pack the trees $T_1, \ldots, T_{\varepsilon n}$ into $G_0 := K_n$ in such a way that we cover less than $2\varepsilon n$ edges incident to each vertex.

Let $t \in \{0, \ldots, \varepsilon n - 1\}$. Suppose we have already packed $T_1, \ldots, T_t$ into G via a function τ_t such that $\delta(G_t) \geq (1 - 2\varepsilon)n$, where $G_t := G - \bigcup_{i=1}^{t} E(\tau_t(T_i))$. Let $X_t \subseteq V(G)$ be the set of all vertices with degree less than $(1-\varepsilon)n$ in G_t. Observe that $|X_t| \leq 2\sum_{i=1}^{\varepsilon n} e(T_i)/(\varepsilon n) \leq \varepsilon n$. Thus we can choose an arbitrary embedding of T_{t+1} into $G_t - X_t$ (as $\delta(G_t - X_t) \geq |T_{t+1}|$, such an embedding exists). Then $\delta(G_{t+1}) \geq (1 - 2\varepsilon)n$. This shows that we can pack $T_1, \ldots, T_{\varepsilon n}$ such that $\delta(G_{\varepsilon n}) \geq (1 - 2\varepsilon)n$.

In particular, $G_{\varepsilon n}$ is $(5\varepsilon, 1)$-quasi-random. Hence we can apply Theorem 1.7 with $\mathcal{T} := \{T_{n/10}, \ldots, T_{9n/10}\}$, $\mathcal{H} := \{T_{\varepsilon n+1}, \ldots, T_{n/10-1}, T_{9n/10+1}, \ldots, T_n\}$, $\delta = 1/10$, and obtain a packing of $\mathcal{H} \cup \mathcal{T}$ into $G_{\varepsilon n}$, which completes the proof. □

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School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom

E-mail address: `f.joos@bham.ac.uk, j.kim.3@bham.ac.uk, d.kuhn@bham.ac.uk, d.osthus@bham.ac.uk`