A CHARACTERIZATION OF TESTABLE HYPERGRAPH PROPERTIES

FELIX JOOS, JAEHOON KIM, DANIELA KÜHN, AND DERYK OSTHUS

Abstract. We provide a combinatorial characterization of all testable properties of k-graphs (i.e. k-uniform hypergraphs). Here, a k-graph property $\mathbf{P}$ is testable if there is a randomized algorithm which makes a bounded number of edge queries and distinguishes with probability $2/3$ between k-graphs that satisfy $\mathbf{P}$ and those that are far from satisfying $\mathbf{P}$. For the 2-graph case, such a combinatorial characterization was obtained by Alon, Fischer, Newman and Shapira. Our results for the k-graph setting are in contrast to those of Austin and Tao, who showed that for the somewhat stronger concept of local repairability, the testability results for graphs do not extend to the 3-graph setting.

Date: July 11, 2017.

The research leading to these results was partially supported by the EPSRC, grant nos. EP/M009408/1 (F. Joos, D. Kühn and D. Osthus), and EP/N019504/1 (D. Kühn). The research was also partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant 306349 (J. Kim and D. Osthus). An extended abstract of this paper appears in the Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2017).

Contents

1. Introduction 3
2. Proof sketch 5
  2.1. Testable properties are regular reducible 6
  2.2. Satisfying a regularity instance is testable 6
  2.3. The final step 7
3. Concepts and tools 7
  3.1. Hypergraphs 8
  3.2. Probabilistic tools 9
  3.3. Hypergraph regularity 9
  3.4. Partitions of hypergraphs and the regular approximation lemma 9
  3.5. The address space 12
    3.5.1. Basic properties of the address space 12
    3.5.2. Constructing families of partitions using the address space 16
    3.5.3. Density functions of address spaces 17
  3.6. Regularity instances 17
4. Hypergraph regularity: counting lemmas and approximation 18
  4.1. Simple hypergraph regularity results 18
  4.2. Counting lemmas 19
  4.3. Close hypergraphs with better regularity 23
  4.4. Refining a partition 27
  4.5. Small perturbations of partitions 28
  4.6. Distance of hypergraphs and density functions 30
5. Testable properties are regular reducible 33
6. Satisfying a regularity instance is testable 35
7. Testable hypergraph properties are estimable 36
8. Regular reducible hypergraph properties are testable 43
9. Regular approximations of partitions and hypergraphs 44
10. Sampling a regular partition 61
  10.1. Building a family of partitions from three others 61
  10.2. Random samples 72
11. Applications 75
  11.1. Testing the injective homomorphism density 75
  11.2. Testing the maximum cut size 77
References 80

1. Introduction

The universal question in the area of property testing is the following: By considering a small (random) sample S of a combinatorial object $\mathcal{O}$, can we distinguish (with high probability) whether $\mathcal{O}$ has a specific property $\mathbf{P}$ or whether it is far from satisfying $\mathbf{P}$? In this paper we answer this question for k-uniform hypergraphs, where a hypergraph H is *k-uniform* if all edges of H have size $k \in \mathbb{N}$. For brevity, we usually refer to k-uniform hypergraph as *k-graphs* (so 2-graphs are graphs).

We now formalize the notion of testability (throughout, we consider only properties $\mathbf{P}$ which are decidable). For this, we say that two k-graphs G and H on vertex set V with $|V| = n$ are *α-close* if $|G \triangle H| \leq \alpha\binom{n}{k}$, and *$\alpha$-far* otherwise[1]. We say that H is *α-close to satisfying a property* $\mathbf{P}$ if there exists a k-graph G that satisfies $\mathbf{P}$ and is α-close to H, and we say that H is *α-far from satisfying* $\mathbf{P}$ otherwise.

Definition 1.1 (Testability)**.** *Let $k \in \mathbb{N} \setminus \{1\}$ be fixed and let $q_k : (0,1) \to \mathbb{N}$ be a function. A k-graph property $\mathbf{P}$ is* testable with query complexity at most q_k *if for every $n \in \mathbb{N}$ and every $\alpha \in (0,1)$ there are an integer $q'_k = q'_k(n,\alpha) \leq q_k(\alpha)$ and a randomized algorithm $\mathbf{T} = \mathbf{T}(n,\alpha)$ that can distinguish with probability at least $2/3$ between n-vertex k-graphs satisfying $\mathbf{P}$ and n-vertex k-graphs that are α-far from satisfying $\mathbf{P}$, while making q'_k edge queries:*

(i) *if H satisfies $\mathbf{P}$, then $\mathbf{T}$ accepts H with probability at least $2/3$,*

(ii) *if H is α-far from satisfying $\mathbf{P}$, then $\mathbf{T}$ rejects H with probability at least $2/3$.*

In this case, we say $\mathbf{T}$ is a tester*, or* (n,α)-tester *for $\mathbf{P}$. We also say that $\mathbf{T}$ has query complexity q'_k. The property $\mathbf{P}$ is* testable *if it is testable with query complexity at most q_k for some function $q_k : (0,1) \to \mathbb{N}$.*

Property testing was introduced by Rubinfeld and Sudan [42]. In the graph setting, the earliest systematic results were obtained in a seminal paper of Goldreich, Goldwasser and Ron [23]. These included k-colourability, max-cut and more general graph partitioning problems. (In fact, these results are preceded by the famous triangle removal lemma of Ruzsa and Szemerédi [43], which can be rephrased in terms of testability of triangle-freeness.) This list of problems was greatly extended (e.g. via a description in terms of first order logic by Alon, Fischer, Krivelevich, and Szegedy [4]) and generalized first to monotone properties (which are closed under vertex and edge deletion) by Alon and Shapira [10] and then to hereditary properties (which are closed under vertex deletion), again by Alon and Shapira [9]. Examples of non-testable properties include some properties which are closed under edge deletion [24] and the property of being isomorphic to a given graph G [5, 18], provided the local structure of G is sufficiently 'complex' (e.g. G is obtained as a binomial random graph). This sequence of papers culminated in the result of Alon, Fischer, Newman and Shapira [5] who obtained a combinatorial characterization of all testable graph properties. This solved a problem posed already by [23], which was regarded as one of the main open problems in the area.

The characterization proved in [5] states that a 2-graph property $\mathbf{P}$ is testable if and only if it is 'regular reducible'. Roughly speaking, the latter means that $\mathbf{P}$ can be characterized by being close to one of a bounded number of (weighted) Szemerédi-partitions (which arise from an application of Szemerédi's regularity lemma). Our main theorem (Theorem 1.3) shows that this can be extended to hypergraphs of higher uniformity. Our characterization is based on the concept of (strong) hypergraph regularity, which was introduced in the ground-breaking work of Rödl et al. [21, 38, 39, 41], Gowers [26], see also Tao [44]. We defer the precise definition of regular reducibility for k-graphs to

[1]We identify hypergraphs with their edge set and for two sets A, B we denote by $A \triangle B$ the symmetric difference of A and B.

Section 3.6, as the concept of (strong) hypergraph regularity involves additional features compared to the graph setting (in particular, one needs to consider an entire (suitably nested) family of regular partitions, one for each $j \in [k]$). Accordingly, our argument relies on the so-called 'regular approximation lemma' due to Rödl and Schacht [39], which can be viewed as a powerful variant of the hypergraph regularity lemma. In turn, we derive a strengthening of this result which may have further applications.

Instead of testing whether H satisfies $\mathbf{P}$ or is α-far from $\mathbf{P}$, it is natural to consider the more general task of estimating the distance between H and $\mathbf{P}$: given $\alpha > \beta > 0$, is H $(\alpha - \beta)$-close to satisfying $\mathbf{P}$ or is H α-far from satisfying $\mathbf{P}$? In this case we refer to $\mathbf{P}$ as being *estimable*. The formal definition is as follows.

Definition 1.2 (Estimability). *Let $k \in \mathbb{N} \setminus \{1\}$ be fixed and let $q_k : (0,1)^2 \to \mathbb{N}$ be a function. A k-graph property $\mathbf{P}$ is* estimable with query complexity at most q_k *if for every $n \in \mathbb{N}$ and all $\alpha, \beta \in (0,1)$ with $0 < \beta < \alpha$ there are an integer $q_k' = q_k'(n, \alpha, \beta) \leq q_k(\alpha, \beta)$ and a randomized algorithm $\mathbf{T} = \mathbf{T}(n, \alpha, \beta)$ that can distinguish with probability $2/3$ between n-vertex k-graphs that are $(\alpha - \beta)$-close to satisfying $\mathbf{P}$ and n-vertex k-graphs that are α-far from satisfying $\mathbf{P}$ while making q_k' edge queries:*

- *if H is $(\alpha - \beta)$-close to satisfying $\mathbf{P}$, then $\mathbf{T}$ accepts H with probability at least $2/3$,*
- *if H is α-far from satisfying $\mathbf{P}$, then $\mathbf{T}$ rejects H with probability at least $2/3$.*

In this case, we say $\mathbf{T}$ is an estimator*, or* (n, α, β)-estimator *for $\mathbf{P}$. We also say that $\mathbf{T}$ has query complexity q_k'. The property $\mathbf{P}$ is* estimable *if it is estimable with query complexity at most q_k for some function $q_k : (0,1)^2 \to \mathbb{N}$.*

We show that testability and estimability are in fact equivalent. For graphs this goes back to Fischer and Newman [19].

Theorem 1.3. *Suppose $k \in \mathbb{N} \setminus \{1\}$ and suppose $\mathbf{P}$ is a k-graph property. Then the following three statements are equivalent:*

(a) *$\mathbf{P}$ is testable.*
(b) *$\mathbf{P}$ is estimable.*
(c) *$\mathbf{P}$ is regular reducible.*

In Section 11, we illustrate how Theorem 1.3 can be used to prove testability of a given property: firstly to test the injective homomorphism density of a given subgraph (which includes the classical example of H-freeness) and secondly to test the size of a maximum ℓ-way cut (which includes testing ℓ-colourability).

Previously, the most general result on hypergraph property testing was the testability of hereditary properties, which was proved by Rödl and Schacht [37, 40], based on deep results on hypergraph regularity. In fact, they showed that hereditary k-graph properties can be even tested with one-sided error (which means that the '2/3' is replaced by '1' in Definition 1.1(i)). This generalized earlier results in [12, 30].

The result of Alon and Shapira on the testability of hereditary graph properties was strengthened by Austin and Tao [11] in another direction: they showed that hereditary properties of graphs are not only testable with one-sided error, but they are also *locally repairable*[2] (one may think of this as a strengthening of testability). On the other hand, they showed that hereditary properties of 3-graphs are not necessarily locally repairable. Note that this is in contrast to Theorem 1.3.

[2] Suppose $\mathbf{P}$ is a hereditary graph property and $\varepsilon > 0$. We say that a graph G is locally δ-close to $\mathbf{P}$ if a random sample S satisfies $\mathbf{P}$ with probability at least $1 - \delta$. A result of Alon and Shapira [9] shows that whenever G is locally δ-close to $\mathbf{P}$ for some $\delta(\varepsilon) > 0$, then G is ε-close to $\mathbf{P}$. The concept of being locally repairable strengthens this by requiring a rule that generates $G' \in \mathbf{P}$ only based on S such that $|G \triangle G'| < \varepsilon n^2$ with probability at least $1 - \delta$.

An intimate connection between property testing and graph limits was established by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [15]. In particular, they showed that a graph property $\mathbf{P}$ is testable if and only if for all sequences (G_n) of graphs with $|V(G_n)| \to \infty$ and $\delta_\square(G_n, \mathbf{P}) \to 0$, we have $d_1(G_n, \mathbf{P}) \to 0$. Here $\delta_\square(G, \mathbf{P})$ denotes the cut-distance of G and the closest graph satisfying $\mathbf{P}$ and $d_1(G, \mathbf{P})$ is the normalized edit-distance between G and $\mathbf{P}$ (see also [33] for more background and discussion on this). Another characterization (in terms of localized samples) using the graph limit framework was given by Lovász and Szegedy [34]. Similarly, the result of Rödl and Schacht [37] on testing hereditary hypergraph properties was reproven via hypergraph limits by Elek and Szegedy [17] as well as Austin and Tao [11]. The latter further extended this to directed pre-coloured hypergraphs (none of these results however yield effective bounds on the query complexity).

Lovász and Vesztergombi [35] recently introduced the notion of 'non-deterministic' property testing, where the tester also has access to a 'certificate' for the property $\mathbf{P}$. By considering the graph limit setting, they proved the striking result that any non-deterministically testable graph property is also deterministically testable (one could think of their result as the graph property testing analogue of proving that P = NP). Karpinski and Markó [29] generalized the Lovász-Vesztergombi result to hypergraphs, also via the notion of (hyper-)graph limits. However, these proofs do not give an explicit bounds on the query complexity – this was achieved by Gishboliner and Shapira [32] for graphs and Karpinski and Markó [28] for hypergraphs.

Another direction of research concerns *easily testable properties*, where we require that the size of the sample is bounded from above by a polynomial in $1/\alpha$. (The bounds coming from Theorem 1.3 can be made explicit but are quite large, as the approach via the (hyper-)graph regularity lemma incurs at least a tower-type dependence on $1/\alpha$, see [25].) For k-graphs, Alon and Shapira [8] as well as Alon and Fox [6] obtained positive and negative results for the property of containing a given k-graph as an (induced) subgraph. For an approach via a 'polynomial' version of the regularity lemma see [20].

Recent progress on property testing includes many questions beyond the hypergraph setting. Instances include property testing of matrices [1], Boolean functions [2, 7], geometric objects [3], and algebraic structures [13, 20, 22]. Moreover, property testing in the sparse (graph) setting gives rise to many interesting results and questions (see e.g. [14, 36]). Little is known for hypergraphs in this case.

The paper is organized as follows. In the next section, we outline the main steps of the argument. In Section 3, we explain the relevant concepts of hypergraph regularity, in particular we introduce the regular approximation lemma of Rödl and Schacht (Theorem 3.8). In Section 4, we prove and derive a number of tools related to hypergraph regularity, in particular, we describe a suitable 'induced' version of the hypergraph counting lemma. In Section 5, we use this counting lemma to show that testable properties are regular reducible. In Section 6, we show how Lemma 6.1 implies that satisfying a given regularity instance is testable. In Section 7, we then show that estimability is equivalent to testability. In Section 8, we combine the previous results to show that regular reducible properties are testable. Sections 9 and 10 are then devoted to the proof of Lemma 6.1. Finally, in Section 11 we discuss applications of our main result and illustrate in detail how to apply Theorem 1.3.

2. Proof sketch

In the following, we describe the main steps leading to the proof of Theorem 1.3. While the general strategy emulates that of [5], the hypergraph setting leads to many additional challenges.

2.1. **Testable properties are regular reducible.** We first discuss the implication (a)⇒(c) in Theorem 1.3. (Note that the statement of (c) is formalized in Section 3.6.) The detailed proof is given in Section 5. The argument involves the following concepts. A regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ consists of a regularity parameter ε, a vector $\mathbf{a} \in \mathbb{N}^{k-1}$ determining the 'address space' of R, and a density function $d_{\mathbf{a},k}$ on the address space described by $\mathbf{a}$. (In the graph case, $\mathbf{a}$ equals the number of parts of the regularity partition and the address space consists of all pairs of parts.) We say a k-graph H satisfies R if there is a family of partitions $\mathscr{P} = \{\mathscr{P}^{(i)}\}_{i=1}^{k-1}$ (where $\mathscr{P}^{(1)}$ is a partition of $V(H)$ and $\mathscr{P}^{(i)}$ is a partition of all those i-sets which 'cross' $\mathscr{P}^{(1)}$) so that $\mathscr{P}$ is an ε-equitable partition of H with density function $d_{\mathbf{a},k}$. (In the graph case this means that $\mathscr{P} = \mathscr{P}^{(1)}$ is a vertex partition so that all pairs of partition classes induce ε-regular bipartite graphs.) Then a property $\mathbf{P}$ is regular reducible if there is a bounded size set $\mathcal{R}$ of regularity instances so that H is close to satisfying some $R \in \mathcal{R}$ if and only if H satisfies $\mathbf{P}$ (see Definition 3.15).

Goldreich and Trevisan [24] proved that every testable graph property is also testable in some canonical way (and their results translate to the hypergraph setting in a straightforward way). Thus we may restrict ourselves to such canonical testers. More precisely, an (n, α)-tester $\mathbf{T} = \mathbf{T}(n, \alpha)$ is *canonical* if, given an n-vertex k-graph H, it chooses a set Q of $q_k' = q_k'(n, \alpha)$ vertices of H uniformly at random, queries all k-sets in Q, and then accepts or rejects H (deterministically) according to (the isomorphism class of) $H[Q]$. In particular, $\mathbf{T}$ has query complexity $\binom{q_k'}{k}$. Moreover, every canonical tester is non-adaptive.

Let $\mathbf{P}$ be a testable k-graph property. Thus there exists a function $q_k : (0, 1) \to \mathbb{N}$ such that for every $n \in \mathbb{N}$ and $\alpha \in (0, 1)$, there exists a canonical (n, α)-tester $\mathbf{T} = \mathbf{T}(n, \alpha)$ for $\mathbf{P}$ with query complexity at most $q_k(\alpha)$. So $\mathbf{T}$ samples a set Q of $q \leq q_k(\alpha)$ vertices, considers $H[Q]$, and then deterministically accepts or rejects H based on $H[Q]$. Let $\mathcal{Q}$ be the set of all the k-graphs on q vertices such that $\mathbf{T}$ accepts H if and only if there is $Q' \in \mathcal{Q}$ that is isomorphic to $H[Q]$.

Now let $\mathbf{Pr}(\mathcal{Q}, H)$ denote the 'density' of copies of k-graphs $Q \in \mathcal{Q}$ in H (see Section 3.1). As $\mathbf{T}$ is an (n, α)-tester, $\mathbf{Pr}(\mathcal{Q}, H) \geq 2/3$ if H satisfies $\mathbf{P}$ and $\mathbf{Pr}(\mathcal{Q}, H) \leq 1/3$ if H is α-far from $\mathbf{P}$. The strategy is now to apply a suitable 'induced' version (Corollary 4.10) of the hypergraph counting lemma (Lemma 4.5). Corollary 4.10 shows that $\mathbf{Pr}(\mathcal{Q}, H)$ can be approximated by a function $IC(\mathcal{Q}, d_{\mathbf{a},k})$, where $d_{\mathbf{a},k}$ is the density function of an equitable partition $\mathscr{P}$ of H. Accordingly, for a suitable small $\varepsilon > 0$ and all $\mathbf{a} \in \mathbb{N}^{k-1}$ in a specified range (in terms of α, $q_k(\alpha)$ and k), we define a 'discretized' set $\mathbf{I}$ of regularity instances $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ such that $d_{\mathbf{a},k}(\cdot)$ only attains a bounded number of possible values. Now setting $\mathcal{R}(n, \alpha) := \{R \in \mathbf{I} : IC(\mathcal{Q}, d_{\mathbf{a},k}) \geq 1/2\}$ leads to the desired result, as Corollary 4.10 implies $IC(\mathcal{Q}, d_{\mathbf{a},k}) \sim \mathbf{Pr}(\mathcal{Q}, H)$ if H satisfies $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$. (In the actual argument, we consider some k-graph G obtained from the regular approximation lemma (Theorem 3.8) rather than H itself.)

2.2. **Satisfying a regularity instance is testable.** In this subsection we sketch how we prove that the property of satisfying a particular regularity instance is testable. This forms the main part of the proof of Theorem 1.3 and is described in Sections 6, 9 and 10. Suppose H is a k-graph and Q is a subset of the vertices chosen uniformly at random. First we show that if H satisfies a regularity instance R, then with high probability $H[Q]$ is close to satisfying R. Also the converse is true: if H is far from satisfying R, then with high probability $H[Q]$ is also far from satisfying R.

The main tool for this is Lemma 6.1 (which is proven in Sections 9 and 10). Roughly speaking, it states the following.

Suppose H is a k-graph and Q a random subset of $V(H)$. Then with high probability, the following hold (where $\delta \ll \varepsilon_0$).

- *If $\mathscr{O}_1$ is an ε_0-equitable partition of H with density function $d_{\mathbf{a},k}$, then there is an $(\varepsilon_0 + \delta)$-equitable partition of $H[Q]$ with the same density function $d_{\mathbf{a},k}$.*
- *If $\mathscr{O}_2$ is an ε_0-equitable partition of $H[Q]$ with density function $d_{\mathbf{a},k}$, then there is an $(\varepsilon_0 + \delta)$-equitable partition of H with the same density function $d_{\mathbf{a},k}$.*

The key point here is that the transfer between H and $H[Q]$ incurs only an additive increase in the regularity parameter ε_0. This additive increase can then be eliminated by slightly adjusting H (or $H[Q]$).

The key ingredient in the proof of Lemma 6.1 is Lemma 10.1. Roughly speaking, Lemma 10.1 states the following.

Suppose the following hold (where $\varepsilon \ll \delta \ll \varepsilon_0$).

- *H_1 is a k-graph on vertex set V_1 and $\mathscr{Q}_1$ is an ε-equitable partition of H_1 with density function $d_{\mathbf{a}^{\mathscr{Q}},k}$.*
- *H_2 is a k-graph on vertex set V_2 and $\mathscr{Q}_2$ is an ε-equitable partition of H_2 with the same density function $d_{\mathbf{a}^{\mathscr{Q}},k}$.*
- *$\mathscr{O}_1$ is an ε_0-equitable partition of H_1 with density function $d_{\mathbf{a}^{\mathscr{O}},k}$.*

Then there is an $(\varepsilon_0 + \delta)$-equitable partition $\mathscr{O}_2$ of H_2, also with density function $d_{\mathbf{a}^{\mathscr{O}},k}$.

In other words, if two k-graphs both have *some* 'high quality' regularity partition with the same parameters, then *any* 'low quality' regularity partition transfers from one to the other, with only a small additive increase in the regularity parameter. The proof of Lemma 10.1 relies on a strengthening of the regular approximation lemma (Lemma 9.1), which we derive in Section 9. Lemma 9.1 is also a useful tool in itself, for example, we apply it in the proof of Corollary 11.3.

To prove Lemma 6.1, we will apply Lemma 10.1 with H playing the role of H_1 and with the random sample $H[Q]$ playing the role of H_2 (and vice versa). It is not difficult to deduce from Lemma 6.1 that satisfying a given regularity instance is testable (Theorem 6.4).

2.3. **The final step.** We now aim to use Theorem 6.4 to show that (c)$\Rightarrow$(a) in Theorem 1.3, i.e. to prove that a regular reducible property $\mathbf{P}$ is also testable (see Section 8). As $\mathbf{P}$ is regular reducible, we can decide whether H satisfies $\mathbf{P}$ if we can test whether H is close to some regularity instance in a certain set $\mathcal{R}$. To achieve this, we strengthen Theorem 6.4 to show that the property of satisfying a given regularity instance R is actually estimable (the equivalence (a)$\Leftrightarrow$(b) is a by-product of this argument, see Section 7). Having proved this, it is straightforward to construct a tester for $\mathbf{P}$ by appropriately combining $|\mathcal{R}|$ estimators which estimate the distance of H and a given $R \in \mathcal{R}$.

3. Concepts and tools

In this section we introduce the main concepts and tools (mainly concerning hypergraph regularity partitions) which form the basis of our approach. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $1/n \ll a \ll b \leq 1$ (where $n \in \mathbb{N}$ is typically the number of vertices of a hypergraph), then this means that there are non-decreasing functions $f : (0,1] \to (0,1]$ and $g : (0,1] \to (0,1]$ such that the result holds for all

$0 < a, b \leq 1$ and all $n \in \mathbb{N}$ with $a \leq f(b)$ and $1/n \leq g(a)$. For a vector $\mathbf{x} = (\alpha_1, \ldots, \alpha_\ell)$, we let $\mathbf{x}_* := \{\alpha_1, \ldots, \alpha_\ell\}$ and write $\|\mathbf{x}\|_\infty = \max_{i \in [\ell]}\{\alpha_i\}$. We say a set E is an *i-set* if $|E| = i$. Unless stated otherwise, in the partitions considered in this paper, we allow some of the parts to be empty.

3.1. **Hypergraphs.** In the following we introduce several concepts about a hypergraph H. We typically refer to $V = V(H)$ as the vertex set of H and usually let $n := |V|$. Given a hypergraph H and a set $Q \subseteq V(H)$, we denote by $H[Q]$ the hypergraph induced on H by Q. For two k-graphs G, H on the same vertex set, we often refer to $|G \triangle H|$ as the *distance* between G and H. If the vertex set of H has a partition $\{V_1, \ldots, V_\ell\}$, we simply refer to H as a hypergraph on $\{V_1, \ldots, V_\ell\}$.

A partition $\{V_1, \ldots, V_\ell\}$ of V is an *equipartition* if $|V_i| = |V_j| \pm 1$ for all $i, j \in [\ell]$. For a partition $\{V_1, \ldots, V_\ell\}$ of V and $k \in [\ell]$, we denote by $K_\ell^{(k)}(V_1, \ldots, V_\ell)$ the *complete ℓ-partite* k-graph with vertex classes $V_1, \ldots, V_\ell$. Let $0 \leq \lambda < 1$. If $|V_i| = (1 \pm \lambda)m$ for every $i \in [\ell]$, then an *(m, ℓ, k, λ)-graph* H on $\{V_1, \ldots, V_\ell\}$ is a spanning subgraph of $K_\ell^{(k)}(V_1, \ldots, V_\ell)$. For notational convenience, we consider the vertex partition $\{V_1, \ldots, V_\ell\}$ as an $(m, \ell, 1, \lambda)$-graph. If $|V_i| \in \{m, m+1\}$, we drop λ and simply refer to (m, ℓ, k)-graphs. Similarly, if the value of λ is not relevant, then we say $H \subseteq K_\ell^{(k)}(V_1, \ldots, V_\ell)$ is an $(m, \ell, k, *)$-graph.

Given an $(m, \ell, k, *)$-graph H on $\{V_1, \ldots, V_\ell\}$, an integer $k \leq i \leq \ell$ and a set $\Lambda_i \in \binom{[\ell]}{i}$, we set $H[\Lambda_i] := H[\bigcup_{\lambda' \in \Lambda_i} V_{\lambda'}]$. If $2 \leq k \leq i \leq \ell$ and H is an $(m, \ell, k, *)$-graph, we denote by $\mathcal{K}_i(H)$ the family of all i-element subsets I of $V(H)$ for which $H[I] \cong K_i^{(k)}$, where $K_i^{(k)}$ denotes the complete k-graph on i vertices.

If $H^{(1)}$ is an $(m, \ell, 1, *)$-graph and $i \in [\ell]$, we denote by $\mathcal{K}_i(H^{(1)})$ the family of all i-element subsets I of $V(H^{(1)})$ which 'cross' the partition $\{V_1, \ldots, V_\ell\}$; that is, $I \in \mathcal{K}_i(H^{(1)})$ if and only if $|I \cap V_s| \leq 1$ for all $s \in [\ell]$.

We will consider hypergraphs of different uniformity on the same vertex set. Given an $(m, \ell, k-1, \lambda)$-graph $H^{(k-1)}$ and an (m, ℓ, k, λ)-graph $H^{(k)}$ on the same vertex set, we say $H^{(k-1)}$ *underlies* $H^{(k)}$ if $H^{(k)} \subseteq \mathcal{K}_k(H^{(k-1)})$; that is, for every edge $e \in H^{(k)}$ and every $(k-1)$-subset f of e, we have $f \in H^{(k-1)}$. If we have an entire cascade of underlying hypergraphs we refer to this as a complex. More precisely, let $m \geq 1$ and $\ell \geq k \geq 1$ be integers. An *(m, ℓ, k, λ)-complex* $\mathcal{H}$ on $\{V_1, \ldots, V_\ell\}$ is a collection of (m, ℓ, j, λ)-graphs $\{H^{(j)}\}_{j=1}^k$ on $\{V_1, \ldots, V_\ell\}$ such that $H^{(j-1)}$ underlies $H^{(j)}$ for all $i \in [k] \setminus \{1\}$, that is, $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$. Again, if $|V_i| \in \{m, m+1\}$, then we simply drop λ and refer to such a complex as an (m, ℓ, k)-complex. If the value of λ is not relevant, then we say that $\{H^{(j)}\}_{j=1}^k$ is an $(m, \ell, k, *)$-complex. A collection of hypergraphs is a *complex* if it is an $(m, \ell, k, *)$-complex for some integers m, ℓ, k.

When m is not of primary concern, we refer to (m, ℓ, k, λ)-graphs and (m, ℓ, k, λ)-complexes simply as (ℓ, k, λ)-graphs and (ℓ, k, λ)-complexes, respectively. Again, we also omit λ if $|V_i| \in \{m, m+1\}$ and refer to (ℓ, k)-graphs and (ℓ, k)-complexes and we write the symbol '$*$' instead of λ if λ is not relevant.

Note that there is no ambiguity between an (ℓ, k, λ)-graph and an (m, ℓ, k)-graph (and similarly for complexes) as $\lambda < 1$.

Suppose $n \geq \ell \geq k$ and suppose H is an n-vertex k-graph and F is an ℓ-vertex k-graph. We define $\mathbf{Pr}(F, H)$ such that $\mathbf{Pr}(F, H)\binom{n}{\ell}$ equals the number of induced copies of F in H. For a collection $\mathcal{F}$ of ℓ-vertex k-graphs, we define $\mathbf{Pr}(\mathcal{F}, H)$ such that $\mathbf{Pr}(\mathcal{F}, H)\binom{n}{\ell}$ equals the number of induced ℓ-vertex k-graphs F in H such that $F \in \mathcal{F}$. Note that the following proposition holds.

Proposition 3.1. *Suppose $n, k, q \in \mathbb{N}$ with $k \leq q \leq n$ and G and H are n-vertex k-graphs on vertex set V and $\mathcal{F}$ is a collection of q-vertex k-graphs. If $|G \triangle H| \leq \nu\binom{n}{k}$, then*

$$\mathbf{Pr}(\mathcal{F}, G) = \mathbf{Pr}(\mathcal{F}, H) \pm q^k \nu.$$

3.2. **Probabilistic tools.** For $m, n, N \in \mathbb{N}$ with $m, n < N$ the *hypergeometric distribution* with parameters N, n and m is the distribution of the random variable X defined as follows. Let S be a random subset of $\{1, 2, \dots, N\}$ of size n and let $X := |S \cap \{1, 2, \dots, m\}|$. We will use the following bound, which is a simple form of Chernoff-Hoeffding's inequality.

Lemma 3.2 (See [27, Remark 2.5, Theorem 2.8 and Theorem 2.10])**.** *Suppose $X_1, \dots, X_n$ are independent random variables such that $X_i \in \{0, 1\}$ for all $i \in [n]$. Let $X := X_1 + \cdots + X_n$. Then for all $t > 0$, $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2e^{-2t^2/n}$. Suppose Y has a hypergeometric distribution with parameters N, n, m, then $\mathbb{P}[|Y - \mathbb{E}[Y]| \geq t] \leq 2e^{-2t^2/n}$.*

The next lemma is easy to show, e.g. using Azuma's inequality. We omit the proof.

Lemma 3.3. *Suppose $0 < 1/n \leq 1/q \ll 1/k \leq 1/2$ and $1/q \ll \nu$. Let H be an n-vertex k-graph on vertex set V. Let $Q \in \binom{V}{q}$ be a q-vertex subset of V chosen uniformly at random. Then*

$$\mathbb{P}\left[|H[Q]| = \frac{q^k}{n^k}|H| \pm \nu\binom{q}{k}\right] \geq 1 - 2e^{\frac{-\nu^2 q}{8k^2}}.$$

3.3. **Hypergraph regularity.** In this subsection we introduce ε-regularity for hypergraphs. Suppose $\ell \geq k \geq 2$ and $V_1, \dots, V_\ell$ are pairwise disjoint vertex sets. Let $H^{(k)}$ be an $(\ell, k, *)$-graph on $\{V_1, \dots, V_\ell\}$, let $\{i_1, \dots, i_k\} \in \binom{[\ell]}{k}$, and let $H^{(k-1)}$ be a $(k, k-1, *)$-graph on $\{V_{i_1}, \dots, V_{i_k}\}$. We define the *density of $H^{(k)}$ with respect to $H^{(k-1)}$* as

$$d(H^{(k)} \mid H^{(k-1)}) := \begin{cases} \frac{|H^{(k)} \cap \mathcal{K}_k(H^{(k-1)})|}{|\mathcal{K}_k(H^{(k-1)})|} & \text{if } |\mathcal{K}_k(H^{(k-1)})| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\varepsilon > 0$ and $d \geq 0$. We say $H^{(k)}$ is *(ε, d)-regular with respect to $H^{(k-1)}$* if for all $Q^{(k-1)} \subseteq H^{(k-1)}$ with

$$|\mathcal{K}_k(Q^{(k-1)})| \geq \varepsilon|\mathcal{K}_k(H^{(k-1)})|, \text{ we have } |H^{(k)} \cap \mathcal{K}_k(Q^{(k-1)})| = (d \pm \varepsilon)|\mathcal{K}_k(Q^{(k-1)})|.$$

Note that if $H^{(k)}$ is (ε, d)-regular with respect to $H^{(k-1)}$ and $H^{(k-1)} \neq \emptyset$, then we have $d(H^{(k)} \mid H^{(k-1)}) = d \pm \varepsilon$. We say $H^{(k)}$ is *ε-regular with respect to $H^{(k-1)}$* if it is (ε, d)-regular with respect to $H^{(k-1)}$ for some $d \geq 0$.

We say an $(\ell, k, *)$-graph $H^{(k)}$ on $\{V_1, \dots, V_\ell\}$ is *(ε, d)-regular with respect to an $(\ell, k-1, *)$-graph* $H^{(k-1)}$ on $\{V_1, \dots, V_\ell\}$ if for every $\Lambda \in \binom{[\ell]}{k}$ $H^{(k)}$ is (ε, d)-regular with respect to the restriction $H^{(k-1)}[\Lambda]$.

Let $\mathbf{d} = (d_2, \dots, d_k) \in \mathbb{R}^{k-1}_{\geq 0}$. We say an $(\ell, k, *)$-complex $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ is *$(\varepsilon, \mathbf{d})$-regular* if $H^{(j)}$ is (ε, d_j)-regular with respect to $H^{(j-1)}$ for every $j \in [k] \setminus \{1\}$. We sometimes simply refer to a complex as being ε-regular if it is $(\varepsilon, \mathbf{d})$-regular for some vector $\mathbf{d}$.

3.4. **Partitions of hypergraphs and the regular approximation lemma.** The regular approximation lemma of Rödl and Schacht implies that for all k-graphs H, there exists a k-graph G which is very close to H and so that G has a very 'high quality' partition into ε-regular subgraphs. To state this formally we need to introduce further concepts involving partitions of hypergraphs.

Suppose $A \supseteq B$ are finite sets, $\mathscr{A}$ is a partition of A, and $\mathscr{B}$ is a partition of B. We say $\mathscr{A}$ *refines* $\mathscr{B}$ and write $\mathscr{A} \prec \mathscr{B}$ if for every $\mathcal{A} \in \mathscr{A}$ there either exists $\mathcal{B} \in \mathscr{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{A} \subseteq A \setminus B$. The following definition concerns 'approximate' refinements. Let $\nu \geq 0$. We say that $\mathscr{A}$ *ν-refines* $\mathscr{B}$ and write $\mathscr{A} \prec_\nu \mathscr{B}$ if there exists a function $f : \mathscr{A} \to \mathscr{B} \cup \{A \setminus B\}$ such that

$$\sum_{\mathcal{A} \in \mathscr{A}} |\mathcal{A} \setminus f(\mathcal{A})| \leq \nu |A|.$$

We make the following observations.

- *$\mathscr{A} \prec \mathscr{B}$ if and only if $\mathscr{A} \prec_0 \mathscr{B}$.*
- *Suppose $\mathscr{A}, \mathscr{A}', \mathscr{A}''$ are partitions of A, A', A'' respectively and $A'' \subseteq A' \subseteq A$. If $\mathscr{A} \prec_\nu \mathscr{A}'$ and $\mathscr{A}' \prec_{\nu'} \mathscr{A}''$, then $\mathscr{A} \prec_{\nu+\nu'} \mathscr{A}''$.* (3.1)

We now introduce the concept of a polyad. Roughly speaking, given a vertex partition $\mathscr{P}^{(1)}$, an i-polyad is an i-graph which arises from a partition $\mathscr{P}^{(i)}$ of the complete partite i-graph $\mathcal{K}_i(\mathscr{P}^{(1)})$. The $(i+1)$-cliques spanned by all the i-polyads give rise to a partition $\mathscr{P}^{(i+1)}$ of $\mathcal{K}_{i+1}(\mathscr{P}^{(1)})$ (see Definition 3.4). Such a 'family of partitions' then provides a suitable framework for describing a regularity partition (see Definition 3.6).

Suppose we have a vertex partition $\mathscr{P}^{(1)} = \{V_1, \dots, V_\ell\}$ and $\ell \geq k$. For integers $k \leq \ell' \leq \ell$, we say that a hypergraph H is an *$(\ell', k, *)$-graph with respect to $\mathscr{P}^{(1)}$* if it is an $(\ell', k, *)$-graph on $\{V_i : i \in \Lambda\}$ for some $\Lambda \in \binom{[\ell]}{\ell'}$.

Recall that $\mathcal{K}_j(\mathscr{P}^{(1)})$ is the family of all crossing j-sets with respect to $\mathscr{P}^{(1)}$. Suppose that for all $i \in [k-1] \setminus \{1\}$, we have partitions $\mathscr{P}^{(i)}$ of $\mathcal{K}_i(\mathscr{P}^{(1)})$ such that each part of $\mathscr{P}^{(i)}$ is an (i,i)-graph with respect to $\mathscr{P}^{(1)}$. By definition, for each i-set $I \in \mathcal{K}_i(\mathscr{P}^{(1)})$, there exists exactly one $P^{(i)} = P^{(i)}(I) \in \mathscr{P}^{(i)}$ so that $I \in P^{(i)}$. Consider $j \in [\ell]$ and any $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$. For each $i \in [\max\{j, k-1\}]$, the *i-polyad* $\hat{P}^{(i)}(J)$ of J is defined by

$$\hat{P}^{(i)}(J) := \bigcup \left\{ P^{(i)}(I) : I \in \binom{J}{i} \right\}. \tag{3.2}$$

Thus $\hat{P}^{(i)}(J)$ is a (j,i)-graph with respect to $\mathscr{P}^{(1)}$. Moreover, let

$$\hat{\mathcal{P}}(J) := \left\{ \hat{P}^{(i)}(J) \right\}_{i=1}^{\max\{j,k-1\}}, \tag{3.3}$$

and for $j \in [k-1]$, let

$$\hat{\mathscr{P}}^{(j)} := \left\{ \hat{P}^{(j)}(J) : J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)}) \right\}. \tag{3.4}$$

We note that $\hat{\mathscr{P}}^{(1)}$ is the set consisting of all $(2,1)$-graphs with vertex classes V_s, V_t (for all distinct $s, t \in [\ell]$). Moreover, note that if $\hat{P}^{(j)} \in \hat{\mathscr{P}}^{(j)}$, it follows that there is a set $J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)})$ such that $\hat{P}^{(j)} = \hat{P}^{(j)}(J)$. Since $J \in \mathcal{K}_{j+1}(\hat{P}^{(j)}(J))$, we obtain that $\mathcal{K}_{j+1}(\hat{P}^{(j)}) \neq \emptyset$ for any $\hat{P}^{(j)} \in \hat{\mathscr{P}}^{(j)}$.

The above definitions apply to arbitrary partitions $\mathscr{P}^{(i)}$ of $\mathcal{K}_i(\mathscr{P}^{(1)})$. However, it will be useful to consider partitions with more structure.

Definition 3.4 (Family of partitions)**.** *Suppose $k \in \mathbb{N} \setminus \{1\}$ and $\mathbf{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$. We say $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}) = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ is a* family of partitions on V *if it satisfies the following for each $j \in [k-1] \setminus \{1\}$:*

(i) *$\mathscr{P}^{(1)}$ is a partition of V into $a_1 \geq k$ nonempty classes,*
(ii) *$\mathscr{P}^{(j)}$ is a partition of $\mathcal{K}_j(\mathscr{P}^{(1)})$ into nonempty j-graphs such that*
- *$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{P}^{(j-1)}) : \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ and*
- *$|\{P^{(j)} \in \mathscr{P}^{(j)} : P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\}| = a_j$ for every $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$.*

We say $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ is *T-bounded* if $\|\mathbf{a}\|_\infty \le T$. For two families of partitions $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a}^{\mathscr{P}})$ and $\mathscr{Q} = \mathscr{Q}(k-1,\mathbf{a}^{\mathscr{Q}})$, we say $\mathscr{P} \prec \mathscr{Q}$ if $\mathscr{P}^{(j)} \prec \mathscr{Q}^{(j)}$ for all $j \in [k-1]$. We say $\mathscr{P} \prec_\nu \mathscr{Q}$ if $\mathscr{P}^{(j)} \prec_\nu \mathscr{Q}^{(j)}$ for all $j \in [k-1]$.

As the concept of polyads is central to this paper, we emphasize the following:

Proposition 3.5. *Let $k \in \mathbb{N} \setminus \{1\}$, $\mathbf{a} \in \mathbb{N}^{(k-1)}$ and $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ be a family of partitions. Then for all $i \in [k-1]$ and $j \in [a_1]$, the following hold.*

(i) *if $i > 1$, then $\mathscr{P}^{(i)}$ is a partition of $\mathcal{K}_i(\mathscr{P}^{(1)})$ into $(i,i,*)$-graphs with respect to $\mathscr{P}^{(1)}$,*
(ii) *each $\hat{P}^{(i)} \in \hat{\mathscr{P}}^{(i)}$ is an $(i+1,i,*)$-graph with respect to $\mathscr{P}^{(1)}$,*
(iii) *for each j-set $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$, $\hat{\mathcal{P}}(J)$ as defined in (3.3) is a complex.*

We now extend the concept of ε-regularity to families of partitions.

Definition 3.6 (Equitable family of partitions)**.** *Let $k \in \mathbb{N} \setminus \{1\}$. Suppose $\eta > 0$ and $\mathbf{a} = (a_1,\dots,a_{k-1}) \in \mathbb{N}^{k-1}$. Let V be a vertex set of size n. We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ on V is* $(\eta,\varepsilon,\mathbf{a},\lambda)$-equitable *if it satisfies the following:*

(i) *$a_1 \ge \eta^{-1}$,*
(ii) *$\mathscr{P}^{(1)} = \{V_i : i \in [a_1]\}$ satisfies $|V_i| = (1 \pm \lambda)n/a_1$ for all $i \in [a_1]$, and*
(iii) *if $k \ge 3$, then for every k-set $K \in \mathcal{K}_k(\mathscr{P}^{(1)})$ the collection $\hat{\mathcal{P}}(K) = \{\hat{P}^{(j)}(K)\}_{j=1}^{k-1}$ is an $(\varepsilon,\mathbf{d})$-regular $(k,k-1,*)$-complex, where $\mathbf{d} = (1/a_2,\dots,1/a_{k-1})$.*

As before we drop λ if $|V_i| \in \{\lfloor n/a_1 \rfloor, \lfloor n/a_1 \rfloor + 1\}$ and say $\mathscr{P}$ is $(\eta,\varepsilon,\mathbf{a})$-equitable. Note that for any $\lambda \le 1/3$, every $(\eta,\varepsilon,\mathbf{a},\lambda)$-equitable family of partitions $\mathscr{P}$ satisfies

$$\left| \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{P}^{(1)}) \right| \le k^2 \eta \binom{n}{k}. \tag{3.5}$$

We next introduce the concept of perfect ε-regularity with respect to a family of partitions.

Definition 3.7 (Perfectly regular)**.** *Suppose $\varepsilon > 0$ and $k \in \mathbb{N} \setminus \{1\}$. Let $H^{(k)}$ be a k-graph with vertex set V and let $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ be a family of partitions on V. We say $H^{(k)}$ is perfectly ε-regular with respect to $\mathscr{P}$ if for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ the graph $H^{(k)}$ is ε-regular with respect to $\hat{P}^{(k-1)}$.*

Having introduced the necessary notation, we are now ready to state the regular approximation lemma due to Rödl and Schacht. It states that for every k-graph H, there is a k-graph G that is close to H and that has very good regularity properties.

Theorem 3.8 (Regular approximation lemma [39])**.** *Let $k \in \mathbb{N} \setminus \{1\}$. For all $\eta,\nu > 0$ and every function $\varepsilon : \mathbb{N}^{k-1} \to (0,1]$, there are integers $t_0 := t_{3.8}(\eta,\nu,\varepsilon)$ and $n_0 := n_{3.8}(\eta,\nu,\varepsilon)$ so that the following holds:*

For every k-graph H on at least $n \ge n_0$ vertices, there exists a k-graph G on $V(H)$ and a family of partitions $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a}^{\mathscr{P}})$ on $V(H)$ so that

(i) *$\mathscr{P}$ is $(\eta,\varepsilon(\mathbf{a}^{\mathscr{P}}),\mathbf{a}^{\mathscr{P}})$-equitable and t_0-bounded,*
(ii) *G is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$, and*
(iii) *$|G \triangle H| \le \nu\binom{n}{k}$.*

The crucial point here is that in applications we may apply Theorem 3.8 with a function ε such that $\varepsilon(\mathbf{a}^{\mathscr{P}}) \ll \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-1}$. This is in contrast to other versions (see e.g. [26, 41, 44]) where (roughly speaking) in (iii) we have $G = H$ but in (ii) we have an error parameter ε' which may be large compared to $\|\mathbf{a}^{\mathscr{P}}\|_\infty^{-1}$.

We next state a generalization (Lemma 3.9) of the regular approximation lemma which was also proved by Rödl and Schacht (see Lemma 25 in [39]). Lemma 3.9 has two additional features in comparison to Theorem 3.8. Firstly, we can prescribe a family of partitions $\mathscr{Q}$ and obtain a refinement $\mathscr{P}$ of $\mathscr{Q}$, and secondly, we are not only given one k-graph H but a collection of k-graphs H_i that partitions the complete k-graph. Thus we may view Lemma 3.9 as a 'partition version' of Theorem 3.8.

Lemma 3.9 (Rödl and Schacht [39]). *For all $o, s \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$, all $\eta, \nu > 0$, and every function $\varepsilon : \mathbb{N}^{k-1} \to (0,1]$, there are $\mu = \mu_{3.9}(k, o, s, \eta, \nu, \varepsilon) > 0$ and $t = t_{3.9}(k, o, s, \eta, \nu, \varepsilon) \in \mathbb{N}$ and $n_0 = n_{3.9}(k, o, s, \eta, \nu, \varepsilon) \in \mathbb{N}$ such that the following hold. Suppose*

(O1)$_{3.9}$ *V is a set and $|V| = n \geq n_0$,*
(O2)$_{3.9}$ *$\mathscr{Q} = \mathscr{Q}(k, \mathbf{a}^{\mathscr{Q}})$ is a $(1/a_1^{\mathscr{Q}}, \mu, \mathbf{a}^{\mathscr{Q}})$-equitable o-bounded family of partitions on V,*
(O3)$_{3.9}$ *$\mathscr{H}^{(k)} = \{H_1^{(k)}, \dots, H_s^{(k)}\}$ is a partition of $\binom{V}{k}$ so that $\mathscr{H}^{(k)} \prec \mathscr{Q}^{(k)}$.*

Then there exist a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and a partition $\mathscr{G}^{(k)} = \{G_1^{(k)}, \dots, G_s^{(k)}\}$ of $\binom{V}{k}$ satisfying the following for every $i \in [s]$ and $j \in [k-1]$.

(P1)$_{3.9}$ *$\mathscr{P}$ is a t-bounded $(\eta, \varepsilon(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable family of partitions, and $a_j^{\mathscr{Q}}$ divides $a_j^{\mathscr{P}}$,*
(P2)$_{3.9}$ *$\mathscr{P} \prec \{\mathscr{Q}^{(j)}\}_{j=1}^{k-1}$,*
(P3)$_{3.9}$ *$G_i^{(k)}$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$,*
(P4)$_{3.9}$ *$\sum_{i=1}^{s} |G_i^{(k)} \triangle H_i^{(k)}| \leq \nu\binom{n}{k}$, and*
(P5)$_{3.9}$ *$\mathscr{G}^{(k)} \prec \mathscr{Q}^{(k)}$ and if $H_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$, then $G_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$.*

In Lemma 3.9 we may assume without loss of generality that $1/\mu, t, n_0$ are non-decreasing in k, o, s and non-increasing in η, ν.

3.5. **The address space.** Later on, we will need to explicitly refer to the densities arising, for example, in Theorem 3.8(ii). For this (and other reasons) it is convenient to consider the 'address space'. Roughly speaking the address space consists of a collection of vectors where each vector identifies a polyad.

For $a, s \in \mathbb{N}$, we recursively define $[a]^s$ by $[a]^s := [a]^{s-1} \times [a]$ and $[a]^1 := [a]$. To define the address space, let us write $\binom{[a_1]}{\ell}_< := \{(\alpha_1, \dots, \alpha_\ell) \in [a_1]^\ell : \alpha_1 < \cdots < \alpha_\ell\}$.

Suppose $k', \ell, p \in \mathbb{N}$, $\ell \geq k'$, and $p \geq \max\{k'-1, 1\}$, and $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{N}^p$. We define

$$\hat{A}(\ell, k'-1, \mathbf{a}) := \binom{[a_1]}{\ell}_< \times \prod_{j=2}^{k'-1} [a_j]^{\binom{\ell}{j}}$$

to be the (ℓ, k')-*address space*. Observe that $\hat{A}(1, 0, \mathbf{a}) = [a_1]$ and $\hat{A}(2, 1, \mathbf{a}) = \binom{[a_1]}{2}_<$. Recall that for a vector $\mathbf{x}$, the set $\mathbf{x}_*$ was defined at the beginning of Section 3. Note that if $k' > 1$, then each $\hat{\mathbf{x}} \in \hat{A}(\ell, k'-1, \mathbf{a})$ can be written as $\hat{\mathbf{x}} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k'-1)})$, where $\mathbf{x}^{(1)} \in \binom{[a_1]}{\ell}_<$ and $\mathbf{x}^{(j)} \in [a_j]^{\binom{\ell}{j}}$ for each $j \in [k'-1] \setminus \{1\}$. Thus each entry of the vector $\mathbf{x}^{(j)}$ corresponds to (i.e. is indexed by) a subset of $\binom{[\ell]}{j}$. We order the elements of both $\binom{[\ell]}{j}$ and $\binom{\mathbf{x}_*^{(1)}}{j}$ lexicographically and consider the bijection $g : \binom{\mathbf{x}_*^{(1)}}{j} \to \binom{[\ell]}{j}$ which preserves this ordering. For each $\Lambda \in \binom{\mathbf{x}_*^{(1)}}{j}$ and $j \in [k'-1]$, we denote by $\mathbf{x}_\Lambda^{(j)}$ the entry of $\mathbf{x}^{(j)}$ which corresponds to the set $g(\Lambda)$.

3.5.1. *Basic properties of the address space.* Let $k \in \mathbb{N} \setminus \{1\}$ and let V be a vertex set of size n. Let $\mathscr{P}(k-1, \mathbf{a})$ be a family of partitions on V. For each crossing ℓ-set $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$, the address space allows us to identify (and thus refer to) the set of polyads 'supporting' L. We will achieve this by defining a suitable operator $\hat{\mathbf{x}}(L)$ which maps L to the address space.

To do this, write $\mathscr{P}^{(1)} = \{V_i : i \in [a_1]\}$. Recall from Definition 3.4(ii) that for $j \in [k-1] \setminus \{1\}$, we partition $\mathcal{K}_j(\hat{P}^{(j-1)})$ of every $(j-1)$-polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ into a_j nonempty parts in such a way that $\mathscr{P}^{(j)}$ is the collection of all these parts. Thus, there is a labelling $\phi^{(j)} : \mathscr{P}^{(j)} \to [a_j]$ such that for every polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, the restriction of $\phi^{(j)}$ to $\{P^{(j)} \in \mathscr{P}^{(j)} : P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\}$ is injective. The set $\mathbf{\Phi} := \{\phi^{(2)}, \ldots, \phi^{(k-1)}\}$ is called an **a**-*labelling* of $\mathscr{P}(k-1, \mathbf{a})$. For a given set $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$, we denote $\mathrm{cl}(L) := \{i : V_i \cap L \neq \emptyset\}$.

Consider any $\ell \in [a_1]$. Let $j' := \min\{k-1, \ell-1\}$ and let $j'' := \max\{j', 1\}$. For every ℓ-set $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$ we define an integer vector $\hat{\mathbf{x}}(L) = (\mathbf{x}^{(1)}(L), \ldots, \mathbf{x}^{(j'')}(L))$ by

- $\mathbf{x}^{(1)}(L) := (\alpha_1, \ldots, \alpha_\ell)$, *where* $\alpha_1 < \ldots < \alpha_\ell$ *and* $L \cap V_{\alpha_i} = \{v_{\alpha_i}\}$,
- *and for* $i \in [j'] \setminus \{1\}$ *we set*

$$\mathbf{x}^{(i)}(L) := \Big(\phi^{(i)}(P^{(i)}) : \{v_\lambda : \lambda \in \Lambda\} \in P^{(i)}, P^{(i)} \in \mathscr{P}^{(i)}\Big)_{\Lambda \in \binom{\mathrm{cl}(L)}{i}}. \tag{3.6}$$

Here, we order $\binom{\mathrm{cl}(L)}{i}$ lexicographically. In particular, $\mathbf{x}^{(i)}(L)$ is a vector of length $\binom{\ell}{i}$.

By definition, $\hat{\mathbf{x}}(L) \in \hat{A}(\ell, j', \mathbf{a})$ for every $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$ with ℓ, j' as above. Our next aim is to define an operator $\hat{\mathbf{x}}(\cdot)$ which maps the set $\hat{\mathscr{P}}^{(j-1)}$ of $(j-1)$-polyads injectively into the address space $\hat{A}(j, j-1, \mathbf{a})$ (see (3.7)). We will then extend this further into a bijection between elements of the address spaces and their corresponding hypergraphs. However, before we can define $\hat{\mathbf{x}}(\cdot)$, we need to introduce some more notation.

Suppose $j \in [k'-1]$. For $\hat{\mathbf{x}} \in \hat{A}(\ell, k'-1, \mathbf{a})$ and $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ with $\mathrm{cl}(J) \subseteq \mathbf{x}_*^{(1)}$, we define $\mathbf{x}_J^{(j)} := \mathbf{x}_{\mathrm{cl}(J)}^{(j)}$. Thus from now on, we may refer to the entries of $\mathbf{x}^{(j)}$ either by an index set $\Lambda \in \binom{\mathbf{x}_*^{(1)}}{j}$ or by a set $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$.

Next we introduce a relation on the elements of (possibly different) address spaces. Consider $\hat{\mathbf{x}} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k'-1)}) \in \hat{A}(\ell, k'-1, \mathbf{a})$ with $\ell' \leq \ell$ and $k'' \leq k'$. We define $\hat{\mathbf{y}} \leq_{\ell', k''-1} \hat{\mathbf{x}}$ if

- $\hat{\mathbf{y}} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(k''-1)}) \in \hat{A}(\ell', k''-1, \mathbf{a})$,
- $\mathbf{y}_*^{(1)} \subseteq \mathbf{x}_*^{(1)}$ and
- $\mathbf{x}_\Lambda^{(j)} = \mathbf{y}_\Lambda^{(j)}$ for any $\Lambda \in \binom{\mathbf{y}_*^{(1)}}{j}$ and $j \in [k''-1] \setminus \{1\}$.

Thus any $\hat{\mathbf{y}} \in \hat{A}(\ell', k''-1, \mathbf{a})$ with $\hat{\mathbf{y}} \leq_{\ell', k''-1} \hat{\mathbf{x}}$ can be viewed as the restriction of $\hat{\mathbf{x}}$ to an ℓ'-subset of the ℓ-set $\mathbf{x}_*^{(1)}$. Hence for $\hat{\mathbf{x}} \in \hat{A}(\ell, k'-1, \mathbf{a})$, there are exactly $\binom{\ell}{\ell'}$ distinct integer vectors $\hat{\mathbf{y}} \in \hat{A}(\ell', k''-1, \mathbf{a})$ such that $\hat{\mathbf{y}} \leq_{\ell', k''-1} \hat{\mathbf{x}}$. Also it is easy to check the following properties.

Proposition 3.10. *Suppose* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *is a family of partitions,* $i \in [a_1]$ *and* $i' := \min\{i, k\}$.

(i) *Whenever* $I \in \mathcal{K}_i(\mathscr{P}^{(1)})$ *and* $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ *with* $I \subseteq J$, *then* $\hat{\mathbf{x}}(I) \leq_{i, i'-1} \hat{\mathbf{x}}(J)$.

(ii) *If* $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ *and* $\hat{\mathbf{y}} \leq_{i, i'-1} \hat{\mathbf{x}}(J)$, *then there exists a unique* $I \in \binom{J}{i}$ *such that* $\hat{\mathbf{y}} = \hat{\mathbf{x}}(I)$.

Now we are ready to introduce the promised bijection between the elements of address spaces and their corresponding hypergraphs.

Consider $j \in [k] \setminus \{1\}$. Recall that for every j-set $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$, we have $\hat{\mathbf{x}}(J) \in \hat{A}(j, j-1, \mathbf{a})$. Moreover, recall that $\mathcal{K}_j(\hat{P}^{(j-1)}) \neq \emptyset$ for any $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, and note that $\hat{\mathbf{x}}(J) = \hat{\mathbf{x}}(J')$ for all $J, J' \in \mathcal{K}_j(\hat{P}^{(j-1)})$ and all $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. Hence, for each

$\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ we can define

$$\hat{\mathbf{x}}(\hat{P}^{(j-1)}) := \hat{\mathbf{x}}(J) \text{ for some } J \in \mathcal{K}_j(\hat{P}^{(j-1)}). \tag{3.7}$$

Let

$$\begin{aligned}\hat{A}(j,j-1,\mathbf{a})_{\neq\emptyset} &:= \{\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a}) : \exists \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)} \text{ such that } \hat{\mathbf{x}}(\hat{P}^{(j-1)}) = \hat{\mathbf{x}}\}\\ &= \{\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a}) : \exists J \in \mathcal{K}_j(\mathscr{P}^{(1)}) \text{ such that } \hat{\mathbf{x}} = \hat{\mathbf{x}}(J)\},\\ \hat{A}(j,j-1,\mathbf{a})_{\emptyset} &:= \hat{A}(j,j-1,\mathbf{a}) \setminus \hat{A}(j,j-1,\mathbf{a})_{\neq\emptyset}.\end{aligned}$$

Clearly (3.7) gives rise to a bijection between $\hat{\mathscr{P}}^{(j-1)}$ and $\hat{A}(j,j-1,\mathbf{a})_{\neq\emptyset}$. Thus for each $\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a})_{\neq\emptyset}$, we can define the polyad $\hat{P}^{(j-1)}(\hat{\mathbf{x}})$ of $\hat{\mathbf{x}}$ by

$$\hat{P}^{(j-1)}(\hat{\mathbf{x}}) := \hat{P}^{(j-1)} \text{ such that } \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)} \text{ with } \hat{\mathbf{x}} = \hat{\mathbf{x}}(\hat{P}^{(j-1)}). \tag{3.8}$$

Note that for any $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$, we have $\hat{P}^{(j-1)}(\hat{\mathbf{x}}(J)) = \hat{P}^{(j-1)}(J)$.

We will frequently make use of an explicit description of a polyad in terms of the partition classes it contains (see (3.12)). For this, we proceed as follows. For each $b \in [a_1]$, let $P^{(1)}(b,b) := V_b$. For each $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j,j-1,\mathbf{a})_{\neq\emptyset} \times [a_j]$, we let

$$P^{(j)}(\hat{\mathbf{x}}, b) := P^{(j)} \in \mathscr{P}^{(j)} \text{ such that } \phi^{(j)}(P^{(j)}) = b \text{ and } P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})). \tag{3.9}$$

Using Definition 3.4(ii), we conclude that so far $P^{(j)}(\hat{\mathbf{x}}, b)$ is well-defined for each $(\hat{\mathbf{x}}, b) \in \hat{A}(j,j-1,\mathbf{a})_{\neq\emptyset} \times [a_j]$ and all $j \in [k-1] \setminus \{1\}$.

For convenience we now extend the domain of the above definitions to cover the 'trivial' cases. For $(\hat{\mathbf{x}}, b) \in \hat{A}(j,j-1,\mathbf{a})_{\emptyset} \times [a_j]$, we let

$$P^{(j)}(\hat{\mathbf{x}}, b) := \emptyset. \tag{3.10}$$

We also let $P^{(1)}(a,b) := \emptyset$ for all $a, b \in [a_1]$ with $a \neq b$. For all $j \in [k-1]$ and $\hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})_{\emptyset}$, we define

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) := \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}). \tag{3.11}$$

To summarize, given a family of partitions $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ and an $\mathbf{a}$-labelling $\mathbf{\Phi}$, for each $j \in [k-1]$, this defines $P^{(j)}(\hat{\mathbf{x}}, b)$ for $\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a})$ and $b \in [a_j]$ and $\hat{P}^{(j)}(\hat{\mathbf{x}})$ for all $\hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})$. For later reference, we collect the relevant properties of these objects below. For each $j \in [k-1] \setminus \{1\}$, it will be convenient to extend the domain of the $\mathbf{a}$-labelling $\phi^{(j)}$ of $\mathscr{P}^{(j)}$ to all j-sets $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ by setting $\phi^{(j)}(J) := \phi^{(j)}(P^{(j)})$, where $P^{(j)} \in \mathscr{P}^{(j)}$ is the unique j-graph that contains J.

Proposition 3.11. *For a given family of partitions $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ and an $\mathbf{a}$-labelling $\mathbf{\Phi}$, the following hold for all $j \in [k-1]$.*

(i) *$\hat{P}^{(j)}(\cdot) : \hat{A}(j+1,j,\mathbf{a})_{\neq\emptyset} \to \hat{\mathscr{P}}^{(j)}$ is a bijection.*
(ii) *For $j \geq 2$, the restriction of $P^{(j)}(\cdot,\cdot)$ onto $\hat{A}(j,j-1,\mathbf{a})_{\neq\emptyset} \times [a_j]$ is a bijection onto $\mathscr{P}^{(j)}$.*
(iii) *$\hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})_{\neq\emptyset}$ if and only if $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \neq \emptyset$.*
(iv) *Each $\hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})$ satisfies*

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}). \tag{3.12}$$

(v) *$\{P^{(j)}(\hat{\mathbf{x}}, b) : \hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a}), b \in [a_j]\}$ forms a partition of $\mathcal{K}_j(\mathscr{P}^{(1)})$.*
(vi) *$\{\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})\}$ forms a partition of $\mathcal{K}_{j+1}(\mathscr{P}^{(1)})$.*

(vii) $\{P^{(j+1)}(\hat{\mathbf{x}}, b) : \hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}), b \in [a_{j+1}]\} \prec \{\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})\}$.

(viii) *If $\mathscr{P}(k-1, \mathbf{a})$ is T-bounded, then $|\hat{\mathscr{P}}^{(j)}| \leq |\hat{A}(j+1, j, \mathbf{a})| \leq T^{2^{j+1}-1}$ and $|\mathscr{P}^{(j)}| \leq T^{2^j}$.*

(ix) *If $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \neq \emptyset$ for all $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$, then $\hat{P}^{(j)}(\cdot) : \hat{A}(j+1, j, \mathbf{a}) \to \hat{\mathscr{P}}^{(j)}$ is a bijection and, if in addition $j < k-1$, then $P^{(j+1)}(\cdot, \cdot) : \hat{A}(j+1, j, \mathbf{a}) \times [a_{j+1}] \to \mathscr{P}^{(j+1)}$ is also a bijection.*

(x) *$\hat{A}(j, j-1, \mathbf{a})_{\emptyset} = \emptyset$ for all $j \in [2]$ and thus $\hat{P}^{(1)}(\cdot)$ and $P^{(2)}(\cdot, \cdot)$ are always bijections.*

(xi) *If $\mathscr{P} \prec \mathscr{Q}(k-1, \mathbf{a}^{\mathscr{Q}})$, then $\{\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})\} \prec \{\mathcal{K}_{j+1}(\hat{Q}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})\}$.*

Proof. Observe that (i) and (ii) hold by definition. Note that $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})_{\neq\emptyset}$ if and only if $\hat{\mathbf{x}} = \hat{\mathbf{x}}(J)$ for some $J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)})$ if and only if there exists a set $J \in \mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}}))$. Thus (iii) holds. To show (iv), by (3.11), we may assume $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})_{\neq\emptyset}$. Thus we know that $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}}))$ contains at least one $(j+1)$-set J and $\hat{\mathbf{x}} = \hat{\mathbf{x}}(J)$. By (3.2), we have

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \hat{P}^{(j)}(J) = \bigcup_{I \in \binom{J}{j}} P^{(j)}(I).$$

By Proposition 3.10(ii), we know that $\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}$ if and only if $\hat{\mathbf{y}} = \hat{\mathbf{x}}(I)$ for some $I \in \binom{J}{j}$. Consider any j-set $I \subseteq J$. Recall that $\hat{P}^{(j-1)}(\hat{\mathbf{x}}(I)) = \hat{P}^{(j-1)}(I)$, and thus $I \in \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}}(I)))$. Together with (3.9) this implies that $P^{(j)}(I) = P^{(j)}(\hat{\mathbf{x}}(I), \phi^{(j)}(I))$, where $P^{(j)}(I)$ is the unique part of $\mathscr{P}^{(j)}$ that contains I. Since $\phi^{(j)}(I) = \phi^{(j)}(P^{(j)}(I)) = \mathbf{x}^{(j)}(J)_I$ holds by (3.6), we have

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \bigcup_{I \in \binom{J}{j}} P^{(j)}(I) = \bigcup_{I \in \binom{J}{j}} P^{(j)}(\hat{\mathbf{x}}(I), \mathbf{x}^{(j)}(J)_I) = \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}).$$

This shows that (iv) holds. It is easy to see that (i), (ii), (iii) and Definition 3.4(ii) together imply (v), (vi) and (vii). If $\mathscr{P}(k-1, \mathbf{a})$ is T-bounded, (i) implies that

$$|\hat{\mathscr{P}}^{(j)}| \leq |\hat{A}(j+1, j, \mathbf{a})| \leq \prod_{i=1}^{j} a_i^{\binom{j+1}{i}} \leq \prod_{i=1}^{j} T^{\binom{j+1}{i}} \leq T^{2^{j+1}-1}.$$

Thus for $j \in [k-1] \setminus \{1\}$, we have $|\mathscr{P}^{(j)}| \leq a_j |\hat{\mathscr{P}}^{(j-1)}| \leq T^{2^j}$. Also $|\mathscr{P}^{(1)}| = a_1 \leq T$, thus we have (viii). Statement (ix) follows from (i), (ii) and (iii). Property (x) is trivial from the definitions.

Finally we show (xi). Suppose $J \in \mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \cap \mathcal{K}_{j+1}(\hat{Q}^{(j)}(\hat{\mathbf{y}}))$ for some $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$ and $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$. Then (iii) implies that $\hat{P}^{(j)}(\hat{\mathbf{x}}) = \hat{P}^{(j)}(J)$ and $\hat{Q}^{(j)}(\hat{\mathbf{y}}) = \hat{Q}^{(j)}(J)$. Since $\mathscr{P} \prec \mathscr{Q}$, we have $P^{(j)}(I) \subseteq Q^{(j)}(I)$. Thus

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) \overset{(3.2)}{=} \bigcup_{I \in \binom{J}{j}} P^{(j)}(I) \subseteq \bigcup_{I \in \binom{J}{j}} Q^{(j)}(I) \overset{(3.2)}{=} \hat{Q}^{(j)}(\hat{\mathbf{y}}).$$

Thus we have $\mathcal{K}_{j+1}(\hat{P}^{(j)}) \subseteq \mathcal{K}_{j+1}(\hat{Q}^{(j)}(J))$. This implies (xi). $\square$

We remark that the counting lemma (see Lemma 4.5) will enable us to restrict our attention to families of partitions as in Proposition 3.11(ix). This is formalized in Lemma 4.6.

For $j \in [k-1]$, $\ell \geq j+1$ and for each $\hat{\mathbf{x}} \in \hat{A}(\ell, j, \mathbf{a})$, we define the *polyad of* $\hat{\mathbf{x}}$ by

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) := \bigcup_{\hat{\mathbf{y}} \leq_{j+1,j} \hat{\mathbf{x}}} \hat{P}^{(j)}(\hat{\mathbf{y}}) \overset{(3.12)}{=} \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{z}}, \mathbf{x}^{(j)}_{\mathbf{z}^{(1)}_*}). \tag{3.13}$$

(Note that this generalizes the definition made in (3.8) for the case $\ell = j+1$.) The following fact follows easily from the definition.

Proposition 3.12. *Let $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ be a family of partitions. Let $j \in [k-1]$ and $\ell \geq j+1$. Then for every $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$, there exists a unique $\hat{\mathbf{x}} \in \hat{A}(\ell, j, \mathbf{a})$ such that $L \in \mathcal{K}_\ell(\hat{P}^{(j)}(\hat{\mathbf{x}}))$.*

Note that (3.9) and (3.13) together imply that, for all $j \in [k-1]$ and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$,

$$\hat{\mathcal{P}}(\hat{\mathbf{x}}) := \left\{ \bigcup_{\hat{\mathbf{y}} \leq_{j+1,i} \hat{\mathbf{x}}} \hat{P}^{(i)}(\hat{\mathbf{y}}) \right\}_{i \in [j]} \tag{3.14}$$

is a $(j+1, j)$-complex. Moreover, using Proposition 3.11(iii) it is easy to check that for each $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$ with $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \neq \emptyset$, we have (for $\hat{\mathcal{P}}(J)$ as defined in (3.3))

$$\hat{\mathcal{P}}(\hat{\mathbf{x}}) = \hat{\mathcal{P}}(J) \text{ for some } J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)}). \tag{3.15}$$

3.5.2. *Constructing families of partitions using the address space.* On several occasions we will construct $P^{(j)}(\hat{\mathbf{x}}, b)$ and $\hat{P}^{(j)}(\hat{\mathbf{x}})$ first and then show that they actually give rise to a family of partitions for which we can use the properties listed in Proposition 3.11. The following lemma, which can easily be proved by induction, provides a criterion to show that this is indeed the case.

Lemma 3.13. *Suppose $k \in \mathbb{N} \setminus \{1\}$ and $\mathbf{a} \in \mathbb{N}^{k-1}$. Suppose $\mathscr{P}^{(1)} = \{V_1, \dots, V_{a_1}\}$ is a partition of a vertex set V. Suppose that for each $j \in [k-1] \setminus \{1\}$ and each $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, we are given a j-graph $P'^{(j)}(\hat{\mathbf{x}}, b)$, and for each $j \in [k] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, we are given a $(j-1)$-graph $\hat{P}'^{(j-1)}(\hat{\mathbf{x}})$. Let*

$$P'^{(1)}(b, b) := V_b \text{ for all } b \in [a_1], \text{ and}$$
$$\mathscr{P}^{(j)} := \{P'^{(j)}(\hat{\mathbf{x}}, b) : (\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]\} \text{ for all } j \in [k-1] \setminus \{1\}.$$

Suppose the following conditions hold:

- (FP1) *$P'^{(1)}(b, b) \neq \emptyset$ for each $b \in [a_1]$; moreover for each $j \in [k-1] \setminus \{1\}$ and each $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, we have $P'^{(j)}(\hat{\mathbf{x}}, b) \neq \emptyset$.*
- (FP2) *For each $j \in [k-1] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, the set $\{P'^{(j)}(\hat{\mathbf{x}}, b) : b \in [a_j]\}$ has size a_j and forms a partition of $\mathcal{K}_j(\hat{P}'^{(j-1)}(\hat{\mathbf{x}}))$.*
- (FP3) *For each $j \in [k-1]$ and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$, we have*

$$\hat{P}'^{(j)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P'^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}).$$

Then the $\mathbf{a}$-labelling $\mathbf{\Phi} = \{\phi^{(i)}\}_{i=2}^{k-1}$ given by $\phi^{(i)}(P'^{(i)}(\hat{\mathbf{x}}, b)) = b$ for each $(\hat{\mathbf{x}}, b) \in \hat{A}(i, i-1, \mathbf{a}) \times [a_i]$ is well-defined and satisfies the following:

- (FQ1) *$\mathscr{P} = \{\mathscr{P}^{(i)}\}_{i=1}^{k-1}$ is a family of partitions on V.*
- (FQ2) *The maps $P^{(j)}(\cdot, \cdot)$ and $\hat{P}^{(j)}(\cdot)$ defined in (3.8)–(3.11) for $\mathscr{P}$, $\mathbf{\Phi}$ satisfy that for each $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, we have*

$$P^{(j)}(\hat{\mathbf{x}}, b) = P'^{(j)}(\hat{\mathbf{x}}, b),$$

and for each $j \in [k-1]$ and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$ we have

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \hat{P'}^{(j)}(\hat{\mathbf{x}}).$$

3.5.3. *Density functions of address spaces.* For $k \in \mathbb{N} \setminus \{1\}$ and $\mathbf{a} \in \mathbb{N}^{k-1}$, we say a function $d_{\mathbf{a},k} : \hat{A}(k, k-1, \mathbf{a}) \to [0,1]$ is a *density function* of $\hat{A}(k, k-1, \mathbf{a})$. For two density functions $d^1_{\mathbf{a},k}$ and $d^2_{\mathbf{a},k}$, we define the *distance* between $d^1_{\mathbf{a},k}$ and $d^2_{\mathbf{a},k}$ by

$$\mathrm{dist}(d^1_{\mathbf{a},k}, d^2_{\mathbf{a},k}) := k! \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} |d^1_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^2_{\mathbf{a},k}(\hat{\mathbf{x}})|.$$

Since $|\hat{A}(k, k-1, \mathbf{a})| = \binom{a_1}{k} \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}$, we always have that $\mathrm{dist}(d^1_{\mathbf{a},k}, d^2_{\mathbf{a},k}) \le 1$. Suppose we are given a density function $d_{\mathbf{a},k}$, a real $\varepsilon > 0$, and a k-graph $H^{(k)}$. We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ on $V(H^{(k)})$ is an $(\varepsilon, d_{\mathbf{a},k})$-*partition* of $H^{(k)}$ if for every $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$ the k-graph $H^{(k)}$ is $(\varepsilon, d_{\mathbf{a},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$. If $\mathscr{P}$ is also $(1/a_1, \varepsilon, \mathbf{a})$-equitable (as specified in Definition 3.6), we say $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-*equitable partition* of $H^{(k)}$. Note that

if $\hat{P}^{(k-1)}(\cdot) : \hat{A}(k, k-1, \mathbf{a}) \to \hat{\mathscr{P}}^{(k-1)}$ is a bijection, then $H^{(k)}$ is perfectly ε-regular with respect to $\mathscr{P}$ if and only if there exists a density function $d_{\mathbf{a},k}$ such that $\mathscr{P}$ is an $(\varepsilon, d_{\mathbf{a},k})$-partition of $H^{(k)}$. (3.16)

3.6. **Regularity instances.** A regularity instance R encodes an address space, an associated density function and a regularity parameter. Roughly speaking, a regularity instance can be thought of as encoding a weighted 'reduced multihypergraph' obtained from an application of the regularity lemma for hypergraphs. To formalize this, let $\varepsilon_{3.14}(\cdot,\cdot) : \mathbb{N} \times \mathbb{N} \to (0,1]$ be a function which satisfies the following.

- $\varepsilon_{3.14}(\cdot, k)$ is a decreasing function for any fixed $k \in \mathbb{N}$ with $\lim_{t\to\infty} \varepsilon_{3.14}(t,k) = 0$,
- $\varepsilon_{3.14}(t, \cdot)$ is a decreasing function for any fixed $t \in \mathbb{N}$,
- $\varepsilon_{3.14}(t,k) < t^{-4^k} \varepsilon_{4.5}(1/t, 1/t, k-1, k)/4$, where $\varepsilon_{4.5}$ is defined in Lemma 4.5.

Definition 3.14 (Regularity instance)**.** *A regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ is a triple, where $\mathbf{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$ with $0 < \varepsilon \le \varepsilon_{3.14}(\|\mathbf{a}\|_\infty, k)$, and $d_{\mathbf{a},k}$ is a density function of $\hat{A}(k, k-1, \mathbf{a})$. A k-graph H satisfies the regularity instance R if there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ such that $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of H. The* complexity *of R is $1/\varepsilon$.*

Since $\varepsilon_{3.14}$ depends only on $\|\mathbf{a}\|_\infty$ and k, it follows that for given r and fixed k, the number of vectors $\mathbf{a}$ which could belong to a regularity instance R with complexity r is bounded by a function of r.

Definition 3.15 (Regular reducible)**.** *A k-graph property $\mathbf{P}$ is* regular reducible *if for any $\beta > 0$, there exists an $r = r_{3.15}(\beta, \mathbf{P})$ such that for any integer $n \ge k$, there is a family $\mathcal{R} = \mathcal{R}(n, \beta, \mathbf{P})$ of at most r regularity instances, each of complexity at most r, such that the following hold for every $\alpha > \beta$ and every n-vertex k-graph H:*

- *If H satisfies $\mathbf{P}$, then there exists $R \in \mathcal{R}$ such that H is β-close to satisfying R.*
- *If H is α-far from satisfying $\mathbf{P}$, then for any $R \in \mathcal{R}$ the k-graph H is $(\alpha-\beta)$-far from satisfying R.*

Thus a property is regular reducible if it can be (approximately) encoded by a bounded number of regularity instances of bounded complexity. We will often make use of the fact that if we apply the regular approximation lemma (Theorem 3.8) to a k-graph H to obtain G and $\mathscr{P}$, then $\mathbf{a}^{\mathscr{P}}$ together with the densities of G with respect to the polyads in $\hat{\mathscr{P}}^{(k-1)}$ naturally give rise to a regularity instance R where G satisfies R and H is close to satisfying R.

Note that different choices of $\varepsilon_{3.14}$ lead to a different definition of regularity instances and thus might lead to a different definition of being regular reducible. However, our main result implies that for *any* appropriate choice of $\varepsilon_{3.14}$, being regular reducible and testability are equivalent. In particular, if a property is regular reducible for an appropriate choice of $\varepsilon_{3.14}$, then it is regular reducible for all appropriate choices of $\varepsilon_{3.14}$, and so 'regular reducibility' is well defined.

4. Hypergraph regularity: counting lemmas and approximation

In this section we present several results about hypergraph regularity. The first few results are simple observations which follow either from the definition of ε-regularity or can be easily proved by standard probabilistic arguments. We omit the proofs. In Section 4.2 we then derive an induced version of the 'counting lemma' that is suitable for our needs (see Lemma 4.9).

In Section 4.3 we show that for every k-graph H, there is a k-graph G that is close to H and has better regularity parameters. As a qualitative statement of this is trivial, the crucial point of our statement is the exact relation of the parameters. In Section 4.4 we make two simple observations on refinements of partitions and in Section 4.5 we consider small perturbations of a given family of partitions. In Section 4.6 we relate the distance between two k-graphs and the distance of their density functions.

4.1. Simple hypergraph regularity results.

We will use the following results which follow easily from the definition of hypergraph regularity (see Section 3.3).

Lemma 4.1. *Suppose $m \in \mathbb{N}$, $0 < \varepsilon \leq \alpha^2 < 1$ and $d \in [0,1]$. Suppose $H^{(k)}$ is an $(m,k,k,1/2)$-graph which is (ε,d)-regular with respect to an $(m,k,k-1,1/2)$-graph $H^{(k-1)}$. Suppose $Q^{(k-1)} \subseteq H^{(k-1)}$ and $H'^{(k)} \subseteq H^{(k)}$ such that $|\mathcal{K}_k(Q^{(k-1)})| \geq \alpha|\mathcal{K}_k(H^{(k-1)})|$ and $H'^{(k)}$ is (ε,d')-regular with respect to $H^{(k-1)}$ for some $d' \leq d$. Then*

- (i) *$\mathcal{K}_k(H^{(k-1)}) \setminus H^{(k)}$ is $(\varepsilon, 1-d)$-regular with respect to $H^{(k-1)}$,*
- (ii) *$H^{(k)}$ is $(\varepsilon/\alpha, d)$-regular with respect to $Q^{(k-1)}$, and*
- (iii) *$H^{(k)} \setminus H'^{(k)}$ is $(2\varepsilon, d-d')$-regular with respect to $H^{(k-1)}$.*

Lemma 4.2. *Suppose $m \in \mathbb{N}$, $0 < \varepsilon \leq 1/100$, $d \in [0,1]$ and $\nu \leq \varepsilon^{10}$. Suppose $H^{(k)}$ and $G^{(k)}$ are $(m,k,k,1/2)$-graphs on $\{V_1,\dots,V_k\}$, and $H^{(k-1)}$ and $G^{(k-1)}$ are $(m,k,k-1,1/2)$-graphs on $\{V_1,\dots,V_k\}$. Suppose $H^{(k)}$ is (ε,d)-regular with respect to $H^{(k-1)}$. If $|\mathcal{K}_k(H^{(k-1)})| \geq \nu^{1/2}m^k$, $|H^{(k)} \triangle G^{(k)}| \leq \nu m^k$ and $|H^{(k-1)} \triangle G^{(k-1)}| \leq \nu m^{k-1}$, then $G^{(k)}$ is $(\varepsilon+\nu^{1/3}, d)$-regular with respect to $G^{(k-1)}$.*

Lemma 4.3. *Suppose $0 < \varepsilon \ll 1/k, 1/s$. Suppose that $H_1^{(k)}, \dots, H_s^{(k)}$ are edge-disjoint $(k,k,*)$-graphs such that each $H_i^{(k)}$ is ε-regular with respect to a $(k,k-1,*)$-graph $H^{(k-1)}$. Then $\bigcup_{i=1}^s H_i^{(k)}$ is $s\varepsilon$-regular with respect to $H^{(k-1)}$.*

We will also use the following observation (see for example [39]), which can be easily proved using Chernoff's inequality.

Lemma 4.4 (Slicing lemma [39])**.** *Suppose $0 < 1/m \ll d, \varepsilon, p_0, 1/s$ and $d \geq 2\varepsilon$. Suppose that*

- *$H^{(k)}$ is an (ε,d)-regular k-graph with respect to a $(k-1)$-graph $H^{(k-1)}$,*
- *$|\mathcal{K}_k(H^{(k-1)})| \geq m^k/\log m$,*
- *$p_1,\dots,p_s \geq p_0$ and $\sum_{i=1}^s p_i \leq 1$.*

Then there exists a partition $\{H_0^{(k)}, H_1^{(k)}, \dots, H_s^{(k)}\}$ *of* $H^{(k)}$ *such that* $H_i^{(k)}$ *is* $(3\varepsilon, p_i d)$*-regular with respect to* $H^{(k-1)}$ *for every* $i \in [s]$*, and* $H_0^{(k)}$ *is* $(3\varepsilon, (1-\sum p_i)d)$*-regular with respect to* $H^{(k-1)}$*.*

4.2. **Counting lemmas.** Kohayakawa, Rödl and Skokan proved the following 'counting lemma' (Theorem 6.5 in [31]), which asserts that the number of copies of a given $K_\ell^{(k)}$ in an $(\varepsilon, \mathbf{d})$-regular complex is close to what one could expect in a corresponding random complex. We will deduce several versions of this which suit our needs.

Lemma 4.5 (Counting lemma [31])**.** *For all* $\gamma, d_0 > 0$ *and* $k, \ell, m_0 \in \mathbb{N} \setminus \{1\}$ *with* $k \le \ell$*, there exist* $\varepsilon_0 := \varepsilon_{4.5}(\gamma, d_0, k, \ell) \le 1$ *and* $m_0 := n_{4.5}(\gamma, d_0, k, \ell)$ *such that the following holds: Suppose* $0 \le \lambda < 1/4$*. Suppose* $0 < \varepsilon \le \varepsilon_0$ *and* $m_0 \le m$ *and* $\mathbf{d} = (d_2, \dots, d_k) \in \mathbb{R}^{k-1}$ *such that* $d_j \ge d_0$ *for every* $j \in [k] \setminus \{1\}$*. Suppose that* $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ *is an* $(\varepsilon, \mathbf{d})$*-regular* (m, ℓ, k, λ)*-complex, and* $H^{(1)} = \{V_1, \dots, V_\ell\}$ *with* $m_i = |V_i|$ *for every* $i \in [\ell]$*. Then*

$$|\mathcal{K}_\ell(H^{(k)})| = (1 \pm \gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \cdot \prod_{i=1}^{\ell} m_i.$$

Recall that equitable families of partitions were defined in Section 3.4. Based on the counting lemma, it is easy to show that for an equitable family of partitions $\mathscr{P}$ and an $\mathbf{a}$-labelling $\mathbf{\Phi}$, the maps $\hat{P}^{(j-1)}(\cdot) : \hat{A}(j, j-1, \mathbf{a}) \to \hat{\mathscr{P}}^{(j-1)}$ and $P^{(j)}(\cdot, \cdot) : \hat{A}(j, j-1, \mathbf{a}) \times [a_j] \to \mathscr{P}^{(j)}$ defined in Section 3.5 are bijections. We will frequently make use of this fact in subsequent sections, often without referring to Lemma 4.6 explicitly.

Lemma 4.6. *Suppose that* $k, t \in \mathbb{N} \setminus \{1\}$*,* $0 \le \lambda < 1/4$ *and* $\varepsilon/3 \le \varepsilon_{3.14}(t, k)$ *and* $\mathbf{a} = (a_1, \dots, a_{k-1}) \in [t]^{k-1}$ *and* $|V| = n$ *with* $1/n \ll 1/t, 1/k$*. If* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *is a* $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$*-equitable family of partitions on* V*, and* $\mathscr{P}$ *with an* $\mathbf{a}$*-labelling* $\mathbf{\Phi}$ *defines maps* $\hat{P}^{(j-1)}(\cdot)$ *and* $P^{(j-1)}(\cdot, \cdot)$*, then the following hold.*

(i) *For each* $j \in [k-1]$*,* $\hat{P}^{(j)}(\cdot) : \hat{A}(j+1, j, \mathbf{a}) \to \hat{\mathscr{P}}^{(j)}$ *is a bijection and if* $j > 1$*, then* $P^{(j)}(\cdot, \cdot) : \hat{A}(j, j-1, \mathbf{a}) \times [a_j] \to \mathscr{P}^{(j)}$ *is also a bijection. In particular,* $\hat{A}(j, j-1, \mathbf{a}) = \hat{A}(j, j-1, \mathbf{a})_{\neq \emptyset}$*.*

(ii) *For each* $j \in [k-1] \setminus \{1\}$ *and* $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$*,* $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ *is an* $(\varepsilon, (1/a_2, \dots, 1/a_j))$*-regular* $(j+1, j, \lambda)$*-complex.*

Note that in Lemma 4.5 the graphs $H^{(k)}[\Lambda]$ for $\Lambda \in \binom{\ell}{k}$ are all (ε, d_k)-regular with respect to $H^{(k-1)}[\Lambda]$. In view of Lemma 4.4, we obtain the following corollary, which allows for varying densities at the k-th 'level'.

Corollary 4.7. *For all* $\gamma, d_0 > 0$ *and* $k, \ell \in \mathbb{N} \setminus \{1\}$ *with* $k \le \ell$*, there exist* $\varepsilon_0 := \varepsilon_{4.7}(\gamma, d_0, k, \ell)$ *and* $m_0 := n_{4.7}(\gamma, d_0, k, \ell)$ *such that the following holds: Suppose* $0 \le \lambda < 1/4$*. Suppose* $d' \ge d_0$*,* $0 < \varepsilon \le \varepsilon_0$ *and* $m \ge m_0$ *for each* $i \in [\ell]$*, and* $\mathbf{d} = (d_2, \dots, d_{k-1}) \in \mathbb{R}^{k-2}$ *such that* $d_j \ge d_0$ *for each* $j \in [k-1] \setminus \{1\}$*. Suppose* $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ *is an* (m, ℓ, k, λ)*-complex,* $H^{(1)} = \{V_1, \dots, V_\ell\}$ *with* $m_i = |V_i|$ *for every* $i \in [\ell]$*, and for every* $\Lambda \in \binom{\ell}{k}$*, the complex* $\mathcal{H}[\Lambda]$ *is* $(\varepsilon, (d_2, \dots, d_{k-1}, p_\Lambda))$*-regular, where* p_Λ *is a multiple of* d'*. Then*

$$|\mathcal{K}_\ell(H^{(k)})| = (1 \pm \gamma) \prod_{\Lambda \in \binom{\ell}{k}} p_\Lambda \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot \prod_{i=1}^{\ell} m_i.$$

Note that in the above lemma, some p_Λ are allowed to be zero.

Let F be an ℓ-vertex k-graph and $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ be a complex with $H^{(1)} = \{V_1, \dots, V_\ell\}$. For a bijection $\sigma : V(F) \to [\ell]$, we say an induced copy F' of F in $H^{(k)}$ is *σ-induced* if for each $v \in V(F)$ the vertex of F' corresponding to v lies in $V_{\sigma(v)}$. Let $IC_\sigma(F, \mathcal{H})$ be the number of σ-induced copies F' of F in $H^{(k)}$ such that F' is contained in an element of $\mathcal{K}_\ell(H^{(k-1)})$.

Lemma 4.8 (Induced counting lemma for many clusters)**.** *Suppose $0 < 1/m \ll \varepsilon \ll \gamma, d_0, 1/k, 1/\ell$ with $k \in \mathbb{N} \setminus \{1\}$ and suppose that*

- *F is an ℓ-vertex k-graph,*
- *$d_0 \le d_j \le 1 - d_0$ for every $j \in [k-1] \setminus \{1\}$,*
- *$\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ is an (m, ℓ, k)-complex with $H^{(1)} = \{V_1, \dots, V_\ell\}$,*
- *for each $\Lambda \in \binom{[\ell]}{k}$, the complex $\mathcal{H}[\Lambda]$ is an $(\varepsilon, (d_2, \dots, d_{k-1}, p_\Lambda))$-regular (m, k, k)-complex, and*
- *$\sigma : V(F) \to [\ell]$ is a bijection.*

Then

$$IC_\sigma(F, \mathcal{H}) = \left(\prod_{e \in F} p_{\sigma(e)} \prod_{e \notin F, |e|=k} (1 - p_{\sigma(e)}) \pm \gamma \right) \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot m^\ell.$$

Proof. We select $q \in \mathbb{N}$ such that $1/m \ll \varepsilon \ll 1/q \ll \gamma, d_0, 1/k, 1/\ell$ and define $\overline{H}^{(k)} := \mathcal{K}_k(H^{(k-1)}) \setminus H^{(k)}$. We also define an (m, ℓ, k)-graph H' on $\{V_1, \dots, V_\ell\}$ so that for each $e \in \binom{V(F)}{k}$, we have

$$H'[\sigma(e)] := \begin{cases} H^{(k)}[\sigma(e)] & \text{if } e \in F, \\ \overline{H}^{(k)}[\sigma(e)] & \text{otherwise,} \end{cases}$$

and let $H' := \bigcup_{e \in \binom{V(F)}{k}} H'[\sigma(e)]$. Note that $H^{(k-1)}$ underlies H'. Observe that there is a bijection between the set of all σ-induced copies F' of $F^{(k)}$ in $H^{(k)}$ such that F' is contained in an element of $\mathcal{K}_\ell(H^{(k-1)})$ and the set of copies of $K_\ell^{(k)}$ in H'. For $e \in \binom{V(F)}{k}$, we define

$$p'_{\sigma(e)} := \begin{cases} p_{\sigma(e)} & \text{if } e \in F, \\ 1 - p_{\sigma(e)} & \text{otherwise.} \end{cases}$$

By Lemma 4.1(i), for each $\Lambda \in \binom{[\ell]}{k}$, the set $\{H^{(j)}[\Lambda]\}_{j=1}^{k-1} \cup \{H'[\Lambda]\}$ is an $(\varepsilon, (d_2, \dots, d_{k-1}, p'_\Lambda))$-regular (m, k, k)-complex. It suffices to show that

$$|\mathcal{K}_\ell(H')| = \left(\prod_{\Lambda \in \binom{\ell}{k}} p'_\Lambda \pm \gamma \right) \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot m^\ell. \tag{4.1}$$

We apply the slicing lemma (Lemma 4.4) to find for each $\Lambda \in \binom{[\ell]}{k}$ a subgraph $H'_1[\Lambda]$ of $H'[\Lambda]$ which is $(3\varepsilon, \lfloor qp'_\Lambda \rfloor / q)$-regular with respect to $H^{(k-1)}[\Lambda]$. Similarly, for each $\Lambda \in \binom{[\ell]}{k}$, we apply Lemma 4.4 to the graph $\mathcal{K}_k(H^{(k-1)}[\Lambda]) \setminus H'[\Lambda]$. In combination with Lemma 4.3 this gives a supergraph $H'_2[\Lambda]$ of $H'[\Lambda]$ which is $(6\varepsilon, \lceil qp'_\Lambda \rceil / q)$-regular with respect to $H^{(k-1)}[\Lambda]$.

Let $H_i' := \bigcup_{\Lambda\in\binom{[\ell]}{k}} H_i'[\Lambda]$ for each $i \in [2]$. Observe that $|\mathcal{K}_\ell(H_1')| \le |\mathcal{K}_\ell(H')| \le |\mathcal{K}_\ell(H_2')|$. By Corollary 4.7 with $\gamma/2, 1/q, 1/q$ playing the roles of γ, d', d_0, respectively,

$$|\mathcal{K}_\ell(H_1')| \ge \left(1-\frac{\gamma}{2}\right)\prod_{\Lambda\in\binom{\ell}{k}} \lfloor qp_\Lambda'\rfloor/q \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}}\cdot m^\ell \quad \text{and}$$

$$|\mathcal{K}_\ell(H_2')| \le \left(1+\frac{\gamma}{2}\right)\prod_{\Lambda\in\binom{\ell}{k}} \lceil qp_\Lambda'\rceil/q \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}}\cdot m^\ell.$$

Note that for each $\Lambda \in \binom{\ell}{k}$, we have $p_\Lambda' - 1/q \le \lfloor qp_\Lambda'\rfloor/q$ and $\lceil qp_\Lambda'\rceil/q \le p_\Lambda' + 1/q$. Thus we obtain (4.1) as required. □

The previous lemma counts σ-induced copies of a k-graph F. However, ultimately, we want to count all induced copies of F. Let us introduce the necessary notation for this step.

Suppose $k, \ell \in \mathbb{N}\setminus\{1\}$ such that $\ell \ge k$ and suppose $\mathbf{a} \in \mathbb{N}^{k-1}$. Suppose that $d_{\mathbf{a},k} : \hat{A}(k, k-1, \mathbf{a}) \to [0,1]$ is a density function. Suppose F is a k-graph on ℓ vertices. Suppose $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$ and $\sigma : V(F) \to \mathbf{x}_*^{(1)}$ is a bijection. Let $A(F)$ be the size of the automorphism group of F. We now define three functions in terms of the parameters above that will estimate the number of induced copies of F in certain parts of an ε-regular k-graph. Let

$$IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma) := \prod_{\substack{\hat{\mathbf{y}}\le_{k,k-1}\hat{\mathbf{x}},\\ \mathbf{y}_*^{(1)}\in\sigma(F)}} d_{\mathbf{a},k}(\hat{\mathbf{y}}) \prod_{\substack{\hat{\mathbf{y}}\le_{k,k-1}\hat{\mathbf{x}},\\ \mathbf{y}_*^{(1)}\notin\sigma(F)}} (1-d_{\mathbf{a},k}(\hat{\mathbf{y}}))\prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}},$$

$$IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) := \frac{1}{A(F)}\sum_\sigma IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma),$$

$$IC(F, d_{\mathbf{a},k}) := \binom{a_1}{\ell}^{-1}\sum_{\hat{\mathbf{x}}\in\hat{A}(\ell,k-1,\mathbf{a})} IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}).$$

We will now show that for a k-graph H satisfying a suitable regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$, the value $IC(F, d_{\mathbf{a},k})$ is a very accurate estimate for $\mathbf{Pr}(F,H)$ (recall the latter was introduced in Section 3.1). The same is true if F is replaced by a finite family of k-graphs (see Corollary 4.10).

Lemma 4.9 (Induced counting lemma for general hypergraphs)**.** *Suppose $0 < 1/n \ll \varepsilon \ll 1/t, 1/a_1 \ll \gamma, 1/k, 1/\ell$ with $2 \le k \le \ell$. Suppose F is an ℓ-vertex k-graph and $\mathbf{a} \in [t]^{k-1}$. Suppose H is an n-vertex k-graph satisfying a regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$. Then*

$$\mathbf{Pr}(F,H) = IC(F, d_{\mathbf{a},k}) \pm \gamma.$$

Proof. Since H satisfies the regularity instance R, there exists a $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ of H (as defined in Section 3.5.3). Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{a_1}\}$ and $m := \lfloor n/a_1\rfloor$. We say an induced copy F' of F in H is *crossing-induced* if $V(F') \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$ and *non-crossing-induced* otherwise. Then by (3.5),

$$\text{there are at most } \tfrac{\gamma}{3}\tbinom{n}{\ell} \text{ non-crossing-induced copies of } F. \tag{4.2}$$

The strategy of the proof is as follows. We only consider crossing-induced copies of F, as the number of non-crossing-induced copies is negligible. For each $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$, we fix some bijection σ between $V(F)$ and $\mathbf{x}_*^{(1)}$. By Lemma 4.8, we can accurately estimate the number of σ-induced copies of F. By summing over all choices for $\hat{\mathbf{x}}$ and σ and taking in account which copies we counted multiple times, we can estimate the number of crossing-induced copies of F in H.

For each $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$, we consider the $(k-1)$-polyad $\hat{P}^{(k-1)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}} \hat{P}^{(k-1)}(\hat{\mathbf{y}})$ as defined in (3.13). By Proposition 3.12, for every crossing-induced copy F' of F in H, there is a unique $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$ such that F' is contained in some element of $\mathcal{K}_\ell(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$.

Consider any $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$ and a bijection $\sigma : V(F) \to \mathbf{x}_*^{(1)}$. Let

$$\mathcal{H}'(\hat{\mathbf{x}}) := \left\{ \bigcup_{\hat{\mathbf{z}} \leq_{k,i} \hat{\mathbf{x}}} \hat{P}^{(i)}(\hat{\mathbf{z}}) \right\}_{i \in [k-1]} \quad \text{and} \quad \mathcal{H}(\hat{\mathbf{x}}) := \mathcal{H}'(\hat{\mathbf{x}}) \cup \{H \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))\}.$$

Hence $\mathcal{H}(\hat{\mathbf{x}})$ is an (ℓ, k)-complex and $\mathcal{H}'(\hat{\mathbf{x}})$ is an $(\ell, k-1)$-complex. Note that $\mathcal{H}'(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}} \hat{\mathcal{P}}(\hat{\mathbf{y}})$, where $\hat{\mathcal{P}}(\hat{\mathbf{y}})$ is as defined in (3.14).

Lemma 4.6 implies that each $\hat{\mathcal{P}}(\hat{\mathbf{y}}) = \mathcal{H}'(\hat{\mathbf{x}})[\mathbf{y}_*^{(1)}]$ is $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}))$-regular (if $k \geq 3$). Furthermore, since $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of H, for each $e \in \binom{V(F)}{k}$, the k-graph $H[\sigma(e)]$ is $(\varepsilon, d_{\mathbf{a},k}(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{y}}) = \hat{P}^{(k-1)}(\hat{\mathbf{x}})[\bigcup_{i \in \mathbf{y}_*^{(1)}} V_i]$, where $\hat{\mathbf{y}}$ is the unique vector satisfying $\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}$ and $\mathbf{y}_*^{(1)} = \sigma(e)$.

Thus, by applying Lemma 4.8 with $\mathcal{H}(\hat{\mathbf{x}}), a_i^{-1}, \gamma/(3\ell!), d_{\mathbf{a},k}(\hat{\mathbf{y}})$ playing the roles of $\mathcal{H}, d_i, \gamma, p_{\mathbf{y}_*^{(1)}}$, we conclude that (with $IC_\sigma(F, \mathcal{H}(\hat{\mathbf{x}}))$ defined as in Lemma 4.8)

$$\begin{aligned} IC_\sigma(F, \mathcal{H}(\hat{\mathbf{x}})) &= \left(\prod_{\substack{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}, \\ \mathbf{y}_*^{(1)} \in \sigma(F)}} d_{\mathbf{a},k}(\hat{\mathbf{y}}) \prod_{\substack{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}, \\ \mathbf{y}_*^{(1)} \notin \sigma(F)}} (1 - d_{\mathbf{a},k}(\hat{\mathbf{y}})) \pm \frac{\gamma}{3\ell!} \right) \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \cdot m^\ell \\ &= \left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma) \pm \frac{\gamma}{3\ell!} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell. \end{aligned}$$

Next we want to estimate the number of all crossing-induced copies of F in H which lie in some element of $\mathcal{K}_\ell(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$. Observe that we count every copy of F exactly $A(F)$ times if we sum over all possible bijections σ. Therefore, the number of crossing-induced copies of F in H which lie in some element of $\mathcal{K}_\ell(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$ is

$$\begin{aligned} \frac{1}{A(F)} \sum_\sigma IC_\sigma(F, \mathcal{H}(\hat{\mathbf{x}})) &= \frac{1}{A(F)} \sum_\sigma \left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma) \pm \frac{\gamma}{3\ell!} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell \\ &= \left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) \pm \frac{\gamma}{3} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell. \end{aligned}$$

Note that $|\hat{A}(\ell, k-1, \mathbf{a})| = \binom{a_1}{\ell} \prod_{j=2}^{k-1} a_j^{\binom{\ell}{j}}$ and $\binom{a_1}{\ell} m^\ell = (1 \pm \gamma/10)\binom{n}{\ell}$, because $1/a_1 \ll \gamma, 1/\ell$. Hence the number of crossing-induced copies of F in H is

$$\begin{aligned}\sum_{\hat{\mathbf{x}} \in \hat{A}(\ell,k-1,\mathbf{a})} &\left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) \pm \frac{\gamma}{3} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell \\ &= \left(\sum_{\hat{\mathbf{x}} \in \hat{A}(\ell,k-1,\mathbf{a})} IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) \right) m^\ell \pm \frac{\gamma}{3} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} |\hat{A}(\ell, k-1, \mathbf{a})| m^\ell \\ &= (IC(F, d_{\mathbf{a},k}) \pm \gamma/2)\binom{n}{\ell}.\end{aligned}$$

This together with (4.2) implies the desired statement. □

In the previous lemma we counted the number of induced copies of a single k-graph F in H. It is not difficult to extend this approach to a finite family of k-graphs. For a finite family $\mathcal{F}$ of k-graphs, we define

$$IC(\mathcal{F}, d_{\mathbf{a},k}) := \sum_{F \in \mathcal{F}} IC(F, d_{\mathbf{a},k}). \tag{4.3}$$

Corollary 4.10. *Suppose $0 < 1/n \ll \varepsilon \ll 1/t, 1/a_1 \ll \gamma, 1/k, 1/\ell$ with $2 \le k \le \ell$. Let $\mathcal{F}$ be a collection of k-graphs on ℓ vertices. Suppose H is an n-vertex k-graph satisfying a regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ where $\mathbf{a} \in [t]^{k-1}$. Then*

$$\mathbf{Pr}(\mathcal{F}, H) = IC(\mathcal{F}, d_{\mathbf{a},k}) \pm \gamma.$$

Proof. For each $F \in \mathcal{F}$, we apply Lemma 4.9 with $\gamma/2^{\binom{\ell}{k}}$ playing the role of γ. As $|\mathcal{F}| \le 2^{\binom{\ell}{k}}$, this completes the proof. □

4.3. **Close hypergraphs with better regularity.** In this subsection we present several results that show in their simplest form that for an $(\varepsilon+\delta)$-regular k-graph H, there is a k-graph G on the same vertex set such that $|H \triangle G|$ is small and G is ε-regular. The key point is that we seek an additive improvement in the regularity parameter. Our approach in the following sections relies heavily on this (see the proof of Lemmas 6.2 and 6.3).

Lemma 4.11. *Suppose $0 < 1/m \ll \delta \ll \nu \ll \varepsilon, 1/k, 1/s, d_0 \le 1/2$. Suppose that $d_i \ge d_0$ for all $i \in [s]$ and $\sum_{i \in [s]} d_i = 1$. Suppose that*

- *$H^{(k-1)}$ is an $(m, k, k-1)$-graph such that $|\mathcal{K}_k(H^{(k-1)})| \ge \varepsilon m^k$,*
- *$H_1^{(k)}, \dots, H_s^{(k)}$ are (m, k, k)-graphs that form a partition of $\mathcal{K}_k(H^{(k-1)})$, and that*
- *$H_i^{(k)}$ is $(\varepsilon + \delta, d_i)$-regular with respect to $H^{(k-1)}$ for each $i \in [s]$.*

Then there exist (m, k, k)-graphs $G_1^{(k)}, \dots, G_s^{(k)}$ such that

(G1)4.11 *$G_1^{(k)}, \dots, G_s^{(k)}$ form a partition of $\mathcal{K}_k(H^{(k-1)})$,*
(G2)4.11 *$G_i^{(k)}$ is (ε, d_i)-regular with respect to $H^{(k-1)}$ for each $i \in [s]$, and*
(G3)4.11 *$|G_i^{(k)} \triangle H_i^{(k)}| \le \nu m^k$ for each $i \in [s]$.*

Roughly speaking, the idea of the proof is to construct $G_i^{(k)}$ from $H_i^{(k)}$ by randomly redistributing a small proportion of the k-edges of each $H_i^{(k)}$.

Proof of Lemma 4.11. First we claim that for $Q^{(k-1)} \subseteq H^{(k-1)}$ with $|\mathcal{K}_k(Q^{(k-1)})| \geq \varepsilon|\mathcal{K}_k(H^{(k-1)})|$, we have

$$d(H_i^{(k)} \mid Q^{(k-1)}) = d_i \pm (\varepsilon + \delta^{1/2}). \tag{4.4}$$

Observe that if $|\mathcal{K}_k(Q^{(k-1)})| \geq (\varepsilon+\delta)|\mathcal{K}_k(H^{(k-1)})|$, then this follows directly from the fact that $H_i^{(k)}$ is $(\varepsilon+\delta, d_i)$-regular with respect to $H^{(k-1)}$.

So suppose that $|\mathcal{K}_k(Q^{(k-1)})| < (\varepsilon+\delta)|\mathcal{K}_k(H^{(k-1)})|$. Choose a $(k-1)$-graph $Q'^{(k-1)}$ such that $Q^{(k-1)} \subseteq Q'^{(k-1)} \subseteq H^{(k-1)}$ as well as

$$|\mathcal{K}_k(Q'^{(k-1)})| \geq (\varepsilon+\delta)|\mathcal{K}_k(H^{(k-1)})| \text{ and } |\mathcal{K}_k(Q'^{(k-1)}) \setminus \mathcal{K}_k(Q^{(k-1)})| \leq 2\delta|\mathcal{K}_k(H^{(k-1)})|.$$

Then for each $i \in [s]$, we conclude

$$|H_i^{(k)} \cap \mathcal{K}_k(Q^{(k-1)})| = |H_i^{(k)} \cap \mathcal{K}_k(Q'^{(k-1)})| \pm 2\delta|\mathcal{K}_k(H^{(k-1)})|.$$

The fact that $H_i^{(k)}$ is $(\varepsilon+\delta, d_i)$-regular with respect to $H^{(k-1)}$ implies that

$$|H_i^{(k)} \cap \mathcal{K}_k(Q'^{(k-1)})| = (d_i \pm (\varepsilon+\delta))|\mathcal{K}_k(Q'^{(k-1)})|.$$

From this we obtain (4.4). Our next step is to construct suitable random sets $\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_s \subseteq \mathcal{K}_k(H^{(k-1)})$. We define $G_i^{(k)}$ based on these sets and show that (G1)$_{4.11}$–(G3)$_{4.11}$ hold with positive probability.

We assign each $e \in \mathcal{K}_k(H^{(k-1)})$ to be in $\mathcal{A} := \bigcup_{i\in[s]} \mathcal{B}_i$ independently with probability $\delta^{1/3}$ and assign every edge in $\mathcal{A}$ independently to one $\mathcal{B}_i$ with probability d_i. For each $i \in [s]$, we define

$$G_i^{(k)} := (H_i^{(k)} \setminus \mathcal{A}) \cup \mathcal{B}_i.$$

Observe that (G1)$_{4.11}$ holds by construction. Moreover,

$$\mathbb{E}[|H_i^{(k)} \cap \mathcal{A}|] = \delta^{1/3}|H_i^{(k)}| \quad \text{and} \quad \mathbb{E}[|\mathcal{B}_i|] = \delta^{1/3} d_i |\mathcal{K}_k(H^{(k-1)})|.$$

Thus Lemma 3.2 with the fact that $|H_i^{(k)}| \leq m^k$ implies that

$$\mathbb{P}[|H_i^{(k)} \cap \mathcal{A}| \leq 2\delta^{1/3} m^k] \geq 1 - 2e^{-\delta^{2/3} m^k} \geq 1 - 1/(6s),$$

and

$$\mathbb{P}[|\mathcal{B}_i| \leq 2\delta^{1/3} m^k] \geq 1 - 2e^{-\delta^{2/3} m^k} \geq 1 - 1/(6s).$$

Hence with probability at least $2/3$, we have $|H_i^{(k)} \cap \mathcal{A}| \leq 2\delta^{1/3} m^k$ for all $i \in [s]$ and $|\mathcal{B}_i| \leq 2\delta^{1/3} m^k$. This implies

$$|G_i^{(k)} \triangle H_i^{(k)}| \leq |H_i^{(k)} \cap \mathcal{A}| + |\mathcal{B}_i| \leq \nu m^k.$$

Thus (G3)$_{4.11}$ holds with probability at least $2/3$.

Furthermore, for each $i \in [s]$ and $Q^{(k-1)} \subseteq H^{(k-1)}$ with $|\mathcal{K}_k(Q^{(k-1)})| \geq \varepsilon|\mathcal{K}_k(H^{(k-1)})| \geq \varepsilon^2 m^k$, we obtain

$$\begin{aligned}
\mathbb{E}[|G_i^{(k)} \cap \mathcal{K}_k(Q^{(k-1)})|] &= \mathbb{E}[|(H_i^{(k)} \cap \mathcal{K}_k(Q^{(k-1)})) \setminus \mathcal{A}|] + \mathbb{E}[|\mathcal{B}_i \cap \mathcal{K}_k(Q^{(k-1)})|] \\
&\overset{(4.4)}{=} \Big((1-\delta^{1/3})(d_i \pm (\varepsilon+\delta^{1/2})) + \delta^{1/3} d_i\Big) |\mathcal{K}_k(Q^{(k-1)})| \\
&= (d_i \pm (\varepsilon - \delta^{1/2}))|\mathcal{K}_k(Q^{(k-1)})|. \qquad (4.5)
\end{aligned}$$

Thus Lemma 3.2 implies that

$$\mathbb{P}[|G_i^{(k)} \cap \mathcal{K}_k(Q^{(k-1)})| = (d_i \pm \varepsilon)|\mathcal{K}_k(Q^{(k-1)}|] \geq 1 - 2e^{-\delta|\mathcal{K}_k(Q^{(k-1)})|^2/m^k} \geq 1 - e^{-\delta^2 m^k}.$$

Since there are at most $2^{|H^{(k-1)}|} \leq 2^{m^{k-1}}$ distinct choices of $Q^{(k-1)}$, using a union bound, we conclude that with probability at least $1 - s2^{m^{k-1}}e^{-\delta^2 m^k} \geq 2/3$, for all $i \in [s]$ and $Q^{(k-1)} \subseteq H^{(k-1)}$ with $\mathcal{K}_k(Q^{(k-1)}) \geq \varepsilon|\mathcal{K}_k(H^{(k-1)})|$, we have

$$d(G_i^{(k)} \mid Q^{(k-1)}) = \frac{(d_i \pm \varepsilon)|\mathcal{K}_k(Q^{(k-1)})|}{|\mathcal{K}_k(Q^{(k-1)})|} = d_i \pm \varepsilon.$$

This implies that $G_i^{(k)}$ is (ε, d_i)-regular with respect to $H^{(k-1)}$, and so (G2)$_{4.11}$ holds with probability at least $2/3$. Hence $G_1^{(k)}, \dots, G_s^{(k)}$ satisfy (G1)$_{4.11}$–(G3)$_{4.11}$ with probability at least $1/3$. □

We now generalize Lemma 4.11 to the setting where we consider a family of partitions instead of only k-graphs with one common underlying $(k-1)$-graph.

Lemma 4.12. *Suppose $0 < 1/n \ll \delta \ll \nu \ll \varepsilon \leq 1/2$ and $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Suppose $H^{(k)}$ is an n-vertex k-graph on vertex set V. Suppose that $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ is an $(\varepsilon + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H^{(k)}$. Then there exists a family of partitions $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ and a k-graph $G^{(k)}$ on V such that*

(G1)$_{4.12}$ *$\mathscr{Q}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $G^{(k)}$ and*
(G2)$_{4.12}$ *$|G^{(k)} \triangle H^{(k)}| \leq \nu\binom{n}{k}$.*

Proof. We define $m := \lfloor n/a_1 \rfloor$. Consider $j \in [k] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$. We claim that

$$|\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}}))| \geq \frac{1}{2}\prod_{i=2}^{j-1} a_i^{-\binom{j}{i}} m^j \geq \varepsilon^{1/2} m^j. \tag{4.6}$$

Indeed, this holds if $j = 2$, so suppose that $j > 2$. As R is a regularity instance, we have $\varepsilon + \delta \leq \|\mathbf{a}\|_\infty^{-4^k}\varepsilon_{4.5}(\|\mathbf{a}\|_\infty^{-1}, \|\mathbf{a}\|_\infty^{-1}, j-1, j)$ and Lemma 4.6(ii) implies that $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ is an $(\varepsilon, (1/a_2, \dots, 1/a_{j-1}))$-regular $(m, j, j-1)$-complex. Thus we can apply Lemma 4.5 with $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ playing the role of $\mathcal{H}$ to show (4.6).

Note that Lemma 4.6(i) implies that $\hat{P}^{(j-1)}(\cdot) : \hat{A}(j, j-1, \mathbf{a}) \to \hat{\mathscr{P}}^{(j-1)}$ and $P^{(j)}(\cdot,\cdot) : \hat{A}(j, j-1, \mathbf{a}) \times [a_j] \to \mathscr{P}^{(j)}$ are bijections (except in the case when $j = k$ for the latter).

We choose $\delta \ll \nu_2 \ll \nu_3 \ll \cdots \ll \nu_{k+1} = \nu^2$. For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, let

$$P^{(k)}(\hat{\mathbf{x}}, 1) := H^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \text{ and } P^{(k)}(\hat{\mathbf{x}}, 2) := \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \setminus H^{(k)}. \tag{4.7}$$

In addition, we set $a_k := 2$.

We proceed in an inductive manner. Let $\mathscr{Q}^{(1)} := \mathscr{P}^{(1)}$, and for each $\hat{\mathbf{x}} \in \hat{A}(2, 1, \mathbf{a})$, let $\hat{Q}^{(1)}(\hat{\mathbf{x}}) := \hat{P}^{(1)}(\hat{\mathbf{x}})$. Assume that for $j \in [k] \setminus \{1\}$, we have defined $\{\mathscr{Q}^{(i)}\}_{i=1}^{j-1}$ such that

(Q1)$^j_{4.12}$ $\{\mathscr{Q}^{(i)}\}_{i=1}^{j-1}$ is a $(1/a_1, \varepsilon, (a_1, \dots, a_{j-1}))$-equitable family of partitions and
(Q2)$^j_{4.12}$ $|\hat{P}^{(i-1)}(\hat{\mathbf{x}}) \triangle \hat{Q}^{(i-1)}(\hat{\mathbf{x}})| \leq \nu_i^{1/2} m^{i-1}$ for all $i \in [j] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(i, i-1, \mathbf{a})$.

Note that this holds for $j = 2$. Suppose $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$. If $j < k$, then for each $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$ and $b \in [a_j]$, we define

$$Q'^{(j)}(\hat{\mathbf{x}}, b) := \begin{cases} P^{(j)}(\hat{\mathbf{x}}, b) \cap \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})) & \text{if } b \in [a_j - 1], \\ \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})) \setminus \bigcup_{b \in [a_j - 1]} Q'^{(j)}(\hat{\mathbf{x}}, b) & \text{if } b \in a_j. \end{cases}$$

If $j = k$, then we define

$$Q'^{(j)}(\hat{\mathbf{x}}, 1) := H^{(k)} \cap \mathcal{K}_k(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})), \text{ and } Q'^{(j)}(\hat{\mathbf{x}}, 2) := \mathcal{K}_k(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})) \setminus H^{(k)}.$$

So for each $j \in [k] \setminus \{1\}$ and $b \in [a_j]$, we obtain

$$|Q'^{(j)}(\hat{\mathbf{x}}, b) \triangle P^{(j)}(\hat{\mathbf{x}}, b)| \leq |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})) \triangle \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))| \overset{(\mathrm{Q2})^j_{4.12}}{\leq} 2\nu_j^{1/2} m^j. \tag{4.8}$$

Next, we will apply Lemma 4.11 to find a partition $\mathscr{Q}^{(j)}$ of $\mathcal{K}_j(\mathscr{Q}^{(1)})$. Fix $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$. If $j \in [k-1] \setminus \{1\}$, then for every $b \in [a_j]$ we define $d_b := 1/a_j$. If $j = k$, then let $d_1 := d_{\mathbf{a},k}(\hat{\mathbf{x}})$ and $d_2 := 1 - d_1$.

Since $\mathscr{P}$ is an $(\varepsilon + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H^{(k)}$, it follows that $P^{(j)}(\hat{\mathbf{x}}, b)$ is $(\varepsilon + \delta, d_b)$-regular with respect to $\hat{P}^{(j-1)}(\hat{\mathbf{x}})$ for each $b \in [a_j]$. (Here, we use Lemma 4.1(i) in the case when $j = k$.) Thus Lemma 4.2 together with (4.6), (4.8) and $(\mathrm{Q2})^j_{4.12}$ implies that $Q'^{(j)}(\hat{\mathbf{x}}, b)$ is $(\varepsilon + \delta + \nu_j^{1/9}, d_b)$-regular with respect to $\hat{Q}^{(j-1)}(\hat{\mathbf{x}})$.

By using Lemma 4.11 with $\hat{Q}^{(j-1)}(\hat{\mathbf{x}}), Q'^{(j)}(\hat{\mathbf{x}}, 1), \dots, Q'^{(j)}(\hat{\mathbf{x}}, a_j), a_j, d_b, \varepsilon, \delta + \nu_j^{1/9}, \nu_{j+1}/2$ playing the roles of $H^{(k-1)}, H_1^{(k)}, \dots, H_s^{(k)}, s, d_i, \varepsilon, \delta, \nu$, respectively, we obtain $Q^{(j)}(\hat{\mathbf{x}}, 1), \dots, Q^{(j)}(\hat{\mathbf{x}}, a_j)$ forming a partition of $\mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))$ such that $Q^{(j)}(\hat{\mathbf{x}}, b)$ is (ε, d_b)-regular with respect to $\hat{Q}^{(j-1)}(\hat{\mathbf{x}})$ and $|Q'^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu_{j+1} m^j/2$ for all $b \in [a_j]$. (A similar argument as for (4.6) shows that $\hat{Q}^{(j-1)}(\hat{\mathbf{x}})$ satisfies the requirements of Lemma 4.11.) By (4.8) we have that

$$|P^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| \leq |Q'^{(j)}(\hat{\mathbf{x}}, b) \triangle P^{(j)}(\hat{\mathbf{x}}, b)| + |Q'^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu_{j+1} m^j. \tag{4.9}$$

Suppose first that $j \in [k-1] \setminus \{1\}$. Define

$$\mathscr{Q}^{(j)} := \{Q^{(j)}(\hat{\mathbf{x}}, b) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}), b \in [a_j]\}.$$

Thus $\{\mathscr{Q}^{(i)}\}_{i=1}^j$ forms a $(1/a_1, \varepsilon, (a_1, \dots, a_j))$-equitable family of partitions since $d_b = 1/a_j$ for all $b \in [a_j]$. Hence $(\mathrm{Q1})^{j+1}_{4.12}$ holds.

Furthermore, for each $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$ we obtain a polyad

$$\hat{Q}^{(j)}(\hat{\mathbf{x}}) := \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} Q^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\hat{\mathbf{y}}^{(1)}_*}).$$

Then

$$|\hat{P}^{(j)}(\hat{\mathbf{x}}) \triangle \hat{Q}^{(j)}(\hat{\mathbf{x}})| \leq \sum_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} |P^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\hat{\mathbf{y}}^{(1)}_*}) \triangle Q^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\hat{\mathbf{y}}^{(1)}_*})| \overset{(4.9)}{\leq} (j+1)\nu_{j+1} m^j \leq \nu_{j+1}^{1/2} m^j,$$

and so $(\mathrm{Q2})^{j+1}_{4.12}$ holds.

Now suppose $j = k$. Let

$$G^{(k)} := \left(H^{(k)} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})\right) \cup \bigcup_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} Q^{(k)}(\hat{\mathbf{x}}, 1).$$

By definition of d_1, d_2, we obtain that $\{\mathscr{Q}^{(i)}\}_{i=1}^k$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $G^{(k)}$. Moreover, since $\mathscr{P}^{(1)} = \mathscr{Q}^{(1)}$, we have that

$$\begin{aligned} H^{(k)} &= \left(H^{(k)} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})\right) \cup \left(H^{(k)} \cap \bigcup_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))\right) \\ &\overset{(4.7)}{=} \left(H^{(k)} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})\right) \cup \bigcup_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} P^{(k)}(\hat{\mathbf{x}}, 1). \end{aligned}$$

Thus

$$|H^{(k)} \triangle G^{(k)}| \leq \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} |P^{(k)}(\hat{\mathbf{x}},1) \triangle Q^{(k)}(\hat{\mathbf{x}},1)| \overset{(4.9)}{\leq} |\hat{A}(k,k-1,\mathbf{a})| \nu_{k+1} m^k \leq \nu \binom{n}{k}.$$

Indeed, the final inequality holds since $\nu_{k+1} = \nu^2$, since $|\hat{A}(k,k-1,\mathbf{a})| \leq \|\mathbf{a}\|_\infty^{2^k}$ by Proposition 3.11(viii), and the definition of a regularity instance implies that $\nu \ll \varepsilon < \|\mathbf{a}\|_\infty^{-4^k}$. □

4.4. **Refining a partition.** In this subsection we make two simple observations regarding refinements of a given partition. The first one of these shows that we can refine a family of partitions without significantly affecting the regularity parameters.

Lemma 4.13. *Suppose $0 < 1/n \ll \varepsilon \ll 1/t, 1/k$ with $k, t \in \mathbb{N} \setminus \{1\}$, $0 < \eta < 1$, and $\mathbf{a} \in \mathbb{N}^{k-1}$. Suppose $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ is an $(\eta,\varepsilon,\mathbf{a})$-equitable family of partitions on V with $|V| = n$. Suppose $\mathbf{b} \in [t]^{k-1}$ and $a_i \mid b_i$ for all $i \in [k-1]$. Then there exists a family of partitions $\mathscr{Q} = \mathscr{Q}(k-1,\mathbf{b})$ on V which is $(\eta,\varepsilon^{1/3},\mathbf{b})$-equitable and $\mathscr{Q} \prec \mathscr{P}$.*

It is easy to prove this by induction on k via an appropriate application of the slicing lemma (Lemma 4.4). We omit the details.

The next observation shows that if $\mathscr{Q}$ is an equitable partition of H and $\mathscr{P} \prec \mathscr{Q}$, then we can modify H slightly to obtain G so that $\mathscr{P}$ is a equitable partition of G (where the relevant densities are inherited from $\mathscr{Q}$ and H).

Proposition 4.14. *Suppose that $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll 1/T, 1/a_1^{\mathscr{Q}} \ll \nu \ll 1/k$ with $k \in \mathbb{N} \setminus \{1\}$, and $\mathbf{a}^{\mathscr{P}} \in [T]^{k-1}$. Suppose that $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a}^{\mathscr{P}})$ is a $(1/a_1^{\mathscr{P}},\varepsilon',\mathbf{a}^{\mathscr{P}})$-equitable family of partitions on V, that $\mathscr{Q} = \mathscr{Q}(k-1,\mathbf{a}^{\mathscr{Q}})$ is an $(\varepsilon,\mathbf{a}^{\mathscr{Q}},d_{\mathbf{a}^{\mathscr{Q}},k})$-equitable partition of an n-vertex k-graph $H^{(k)}$ on V, and that $\mathscr{P} \prec \mathscr{Q}$. Let $d_{\mathbf{a}^{\mathscr{P}},k}$ be the density function defined by*

$$d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) := \begin{cases} d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}) & \text{if } \exists \hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}}) : \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})), \\ 0 & \text{if } \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) \cap \mathcal{K}_k(\mathscr{Q}^{(1)}) = \emptyset. \end{cases}$$

Let $G^{(k)} := H^{(k)} \cap \mathcal{K}_k(\mathscr{Q}^{(1)})$. Then

(i) *$\mathscr{P}$ is an $(\varepsilon',\mathbf{a}^{\mathscr{P}},d_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of $G^{(k)}$,*
(ii) *$|H^{(k)} \triangle G^{(k)}| \leq \nu\binom{n}{k}$.*

Proof. Note that (ii) follows from (3.5). We now verify (i). Since $\mathscr{P} \prec \mathscr{Q}$, by Proposition 3.11(vi) and (xi), for each $\hat{\mathbf{y}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{P}})$, either there exists a unique $\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}})$ such that $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}}))$ or $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) \cap \mathcal{K}_k(\mathscr{Q}^{(1)}) = \emptyset$. Suppose first that there exists a unique $\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}})$ such that $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}}))$. Then Lemma 4.5 implies that

$$|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}}))| \geq (1-1/4)\prod_{i=1}^{k-1}(a_i^{\mathscr{P}})^{-\binom{k}{i}} n^k \geq \varepsilon^{1/2}|\mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}}))|.$$

Together with Lemma 4.1(ii) and the fact that $d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) = d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}})$ and $G^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) = H^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}}))$, this implies that $G^{(k)}$ is $(\varepsilon^{1/2}, d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{y}})$.

Now suppose that $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) \cap \mathcal{K}_k(\mathscr{Q}^{(1)}) = \emptyset$. Since $G^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$, we have $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{y}})) \cap G^{(k)} = \emptyset$. Thus $G^{(k)}$ is also $(\varepsilon^{1/2}, d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{y}})$ since $d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) = 0$. Since $\varepsilon^{1/2} \leq \varepsilon'$, altogether this shows that $\mathscr{P}$ is an $(\varepsilon',\mathbf{a}^{\mathscr{P}},d_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of $G^{(k)}$. □

4.5. **Small perturbations of partitions.** Here we consider the effect of small changes in a partition on the resulting parameters. In particular, the next lemma implies that for any family of partitions $\mathscr{P}$, every family of partitions that is close to $\mathscr{P}$ in distance is a family of partitions with almost the same parameters.

Lemma 4.15. *Suppose $k \in \mathbb{N} \setminus \{1\}$, $0 < 1/n \ll \nu \ll \varepsilon$, and $0 \leq \lambda \leq 1/4$. Suppose $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Suppose V is a vertex set of size n and suppose $G^{(k)}, H^{(k)}$ are k-graphs on V with $|G^{(k)} \triangle H^{(k)}| \leq \nu\binom{n}{k}$. Suppose $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ is a $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$-equitable family of partitions on V which is an $(\varepsilon, d_{\mathbf{a},k})$-partition of $H^{(k)}$. Suppose $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ is a family of partitions on V such that for any $j \in [k-1]$, $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, and $b \in [a_j]$, we have*

$$|P^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu\binom{n}{j}. \tag{4.10}$$

Then $\mathscr{Q}$ is a $(1/a_1, \varepsilon + \nu^{1/6}, \mathbf{a}, \lambda + \nu^{1/6})$-equitable family of partitions which is an $(\varepsilon + \nu^{1/6}, d_{\mathbf{a},k})$-partition of $G^{(k)}$.

Proof. Let $t := \|\mathbf{a}\|_\infty$ and $m := \lfloor n/a_1 \rfloor$. Note that since $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance, we have $\varepsilon \leq 3\varepsilon_{3.14}(t,k) \leq t^{-4^k}\varepsilon_{4.5}(1/t, 1/t, k-1, k)$. Thus Lemma 4.5 (together with Lemma 4.6(ii)) implies for any $j \in [k-1] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$ that

$$|\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}}))| \geq (1 - 1/t)\prod_{i=2}^{j-1} a_i^{-\binom{j}{i}}((1-\lambda)m)^j \geq \varepsilon^{1/2}m^j. \tag{4.11}$$

Since $\mathscr{P}$ is a $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$-equitable family of partitions, for each $j \in [k-1] \setminus \{1\}$, $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, and $b \in [a_j]$, the j-graph $P^{(j)}(\hat{\mathbf{x}}, b)$ is $(\varepsilon, 1/a_j)$-regular with respect to $\hat{P}^{(j-1)}(\hat{\mathbf{x}})$. Hence

$$|P^{(j)}(\hat{\mathbf{x}}, b)| \geq (1/a_j - \varepsilon)|\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}}))| \geq \varepsilon^{2/3}m^j.$$

Observe that $\mathscr{P}$ does not provide a partition of $\mathcal{K}_k(\mathscr{P}^{(1)})$. We define such a partition with $a_k := 2$ by setting, for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$,

$$P^{(k)}(\hat{\mathbf{x}}, 1) := \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \cap H^{(k)} \text{ and } P^{(k)}(\hat{\mathbf{x}}, 2) := \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \setminus H^{(k)}.$$

Similarly, let

$$Q^{(k)}(\hat{\mathbf{x}}, 1) := \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})) \cap G^{(k)} \text{ and } Q^{(k)}(\hat{\mathbf{x}}, 2) := \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})) \setminus G^{(k)}.$$

Fix some $j \in [k] \setminus \{1\}$, $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, and $b \in [a_j]$. Let

$$d := \begin{cases} 1/a_j & \text{if } j \in [k-1], \\ d_{\mathbf{a},k}(\hat{\mathbf{x}}) & \text{if } j = k, b = 1, \text{ and} \\ 1 - d_{\mathbf{a},k}(\hat{\mathbf{x}}) & \text{if } j = k, b = 2. \end{cases}$$

Hence $P^{(j)}(\hat{\mathbf{x}}, b)$ is (ε, d)-regular with respect to $\hat{P}^{(j-1)}$. (Here we use Lemma 4.1(i) if $j = k$ and $b = 2$.) Recall that (3.12) holds for both $\mathscr{P}$ and $\mathscr{Q}$. Together with (4.10) this implies that

$$|\hat{P}^{(j-1)}(\hat{\mathbf{x}}) \triangle \hat{Q}^{(j-1)}(\hat{\mathbf{x}})| \leq \nu j\binom{n}{j-1} \leq \nu^{1/2}m^{j-1}.$$

Moreover, (4.10) also implies that $|P^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu\binom{n}{j} \leq \nu^{1/2}m^j$. Thus, by (4.11), we can use Lemma 4.2 with $P^{(j)}(\hat{\mathbf{x}}, b), \hat{P}^{(j-1)}(\hat{\mathbf{x}}), Q^{(j)}(\hat{\mathbf{x}}, b), \hat{Q}^{(j-1)}(\hat{\mathbf{x}}), \nu^{1/2}$ playing the roles of $H^{(k)}, H^{(k-1)}, G^{(k)}, G^{(k-1)}, \nu$ to conclude that $Q^{(j)}(\hat{\mathbf{x}}, b)$ is $(\varepsilon + \nu^{1/6}, d)$-regular with respect to $\hat{Q}^{(j)}(\hat{\mathbf{x}}, b)$. Furthermore, (4.10) for $j = 1$ implies that each part of $\mathscr{Q}^{(1)}$ has size $(1 \pm \lambda \pm \nu^{1/2})n/a_1$. Thus $\mathscr{Q}$ is an $(1/a_1, \varepsilon + \nu^{1/6}, \mathbf{a}, \lambda + \nu^{1/6})$-equitable family of partitions on V. Moreover, since $Q^{(k)}(\hat{\mathbf{x}},1) = \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})) \cap G^{(k)}$, we know that $\mathscr{Q}$ is also an $(\varepsilon+\nu^{1/6}, d_{\mathbf{a},k})$-partition of $G^{(k)}$. $\square$

The following lemma shows that for every equitable family of partitions $\mathscr{P}$ whose vertex partition $\mathscr{P}^{(1)}$ is an almost equipartition, there is an equitable family of partitions with almost the same parameters whose vertex partition is an equipartition.

Lemma 4.16. *Suppose $0 < 1/n \ll \lambda \ll \varepsilon \leq 1$, $k \in \mathbb{N} \setminus \{1\}$, and $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Suppose $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ is a $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$-equitable family of partitions and an $(\varepsilon, d_{\mathbf{a},k})$-partition of an n-vertex k-graph $H^{(k)}$. Then there exists a family of partitions $\mathscr{Q} = \mathscr{Q}(k-1,\mathbf{a})$ which is an $(\varepsilon+\lambda^{1/10}, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H^{(k)}$.*

Proof. Let $m := \lfloor n/a_1 \rfloor$. We write $\mathscr{P}^{(1)} = \{V_1,\dots,V_{a_1}\}$. Since $\mathscr{P}$ is a $(1/a_1,\varepsilon,\mathbf{a},\lambda)$-equitable family of partitions, we have $|V_i| = (1\pm\lambda)m$ for all $i \in [a_1]$, and Lemma 4.6 implies that for each $j \in [k-1]$, the function $\hat{P}^{(j)}(\cdot) : \hat{A}(j+1,j,\mathbf{a}) \to \hat{\mathscr{P}}^{(j)}$ is a bijection and for each $j \in [k-1]\setminus\{1\}$, the function $P^{(j)}(\cdot,\cdot) : \hat{A}(j,j-1,\mathbf{a}) \times [a_j] \to \mathscr{P}^{(j)}$ is also a bijection. Next, we fix the size of the parts in the new equitable partition $\mathscr{Q}$ of $V := V_1 \cup \dots \cup V_{a_1}$. For each $i \in [a_1]$, let $m_i := \lfloor (n+i-1)/a_1 \rfloor$. Thus $m_i \in \{m, m+1\}$. Choose $U_i' \subseteq V_i$ of size $\max\{|V_i|, m_i\}$ and let $U_0' := \bigcup_{i\in[a_1]} V_i \setminus U_i'$. We partition U_0' into $U_1'',\dots,U_{a_1}''$ in an arbitrary manner such that $|U_i''| = m_i - |U_i'|$. For each $i \in [a_1]$, let

$$U_i := U_i' \cup U_i'' \text{ and } \mathscr{Q}^{(1)} := \{U_1,\dots,U_{a_1}\}.$$

Moreover, let $Q^{(1)}(b,b) := U_b$ for each $b \in [a_1]$, and $Q^{(1)}(b,b') := \emptyset$ for all distinct $b,b' \in [a_1]$. For each $i \in [a_1]$, we have

$$|U_i \triangle V_i| \leq |(1\pm\lambda)m - m_i| \leq \lambda m + 1.$$

For each $\hat{\mathbf{x}} = (\alpha_1,\alpha_2) \in \hat{A}(2,1,\mathbf{a})$, let $\hat{Q}^{(1)}(\hat{\mathbf{x}}) := U_{\alpha_1} \cup U_{\alpha_2}$. Note that $\{\mathcal{K}_2(\hat{Q}^{(1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(2,1,\mathbf{a})\}$ forms a partition of $\mathcal{K}_2(\mathscr{Q}^{(1)})$.

Now, we inductively construct $\mathscr{Q}^{(2)},\dots,\mathscr{Q}^{(k-1)}$ in this order. Assume that for some $j \in [k]\setminus\{1\}$, we have already defined $\{\mathscr{Q}^{(i)}\}_{i=1}^{j-1}$ with $\mathscr{Q}^{(i)} = \{Q^{(i)}(\hat{\mathbf{x}},b) : \hat{\mathbf{x}} \in \hat{A}(i,i-1,\mathbf{a}), b \in [a_i]\}$ and $\hat{\mathscr{Q}}^{(i)} = \{\hat{Q}^{(i)}(\hat{\mathbf{x}}) : \hat{\mathbf{x}} \in \hat{A}(i+1,i,\mathbf{a})\}$ for each $i \in [j-1]$ such that the following hold.

- (Q1)$_{j-1}$ For each $i \in [j-1]$, $\hat{\mathbf{x}} \in \hat{A}(i,i-1,\mathbf{a})$ and $b \in [a_i]$, we have $|P^{(i)}(\hat{\mathbf{x}},b) \triangle Q^{(i)}(\hat{\mathbf{x}},b)| \leq 2^i i! \lambda n^i$.
- (Q2)$_{j-1}$ For each $i \in [j-1]\setminus\{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(i,i-1,\mathbf{a})$, the collection $\{Q^{(i)}(\hat{\mathbf{x}},b) : b \in [a_i]\}$ forms a partition of $\mathcal{K}_i(\hat{Q}^{(i-1)}(\hat{\mathbf{x}}))$.
- (Q3)$_{j-1}$ For each $i \in [j-1]$ and $\hat{\mathbf{x}} \in \hat{A}(i+1,i,\mathbf{a})$, we have $\hat{Q}^{(i)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{i,i-1} \hat{\mathbf{x}}} Q^{(i)}(\hat{\mathbf{y}}, \mathbf{x}^{(i)}_{\mathbf{y}^{(1)}_*})$.

Note that $\mathscr{Q}^{(1)}$ satisfies (Q1)$_1$–(Q3)$_1$. Suppose first that $j \leq k-1$. In this case we will define $\mathscr{Q}^{(j)}$ satisfying (Q1)$_j$–(Q3)$_j$. For each $\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a})$ and $b \in [a_j]$, we define

$$Q^{(j)}(\hat{\mathbf{x}},b) := \begin{cases} P^{(j)}(\hat{\mathbf{x}},b) \cap \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})) & \text{if } b \in [a_j-1], \\ \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})) \setminus \bigcup_{b\in[a_j-1]} P^{(j)}(\hat{\mathbf{x}},b) & \text{otherwise.} \end{cases}$$

Let

$$\mathscr{Q}^{(j)} := \{Q^{(j)}(\hat{\mathbf{x}},b) : \hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a}), b \in [a_j]\}.$$

Then for any fixed $\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a})$, it is obvious that $Q^{(j)}(\hat{\mathbf{x}},1),\dots,Q^{(j)}(\hat{\mathbf{x}},a_j)$ forms a partition of $\mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))$. Thus (Q2)$_j$ holds.

For each $\hat{\mathbf{z}} \in \hat{A}(j+1, j, \mathbf{a})$, let

$$\hat{Q}^{(j)}(\hat{\mathbf{z}}) := \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{z}}} Q^{(j)}(\hat{\mathbf{y}}, \mathbf{z}^{(j)}_{\mathbf{y}^{(1)}_*}).$$

Then $(\mathrm{Q3})_j$ also holds.

Note that for any fixed $(j-1)$-set $J' \in \hat{P}^{(j-1)}(\hat{\mathbf{x}}) \triangle \hat{Q}^{(j-1)}(\hat{\mathbf{x}})$, there are at most $(1+\lambda)m$ distinct j-sets in $\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})) \triangle \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))$ containing J'. Thus for $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$ and $b \in [a_j]$, we obtain

$$\begin{aligned}
|P^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| &\leq |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})) \triangle \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))| \\
&\leq (1+\lambda)m|\hat{P}^{(j-1)}(\hat{\mathbf{x}}) \triangle \hat{Q}^{(j-1)}(\hat{\mathbf{x}})| \\
&\overset{(3.12),(\mathrm{Q3})_{j-1}}{\leq} \sum_{\hat{\mathbf{y}} \leq_{j-1,j-2} \hat{\mathbf{x}}} 2m|P^{(j-1)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}) \triangle Q^{(j-1)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*})| \\
&\overset{(\mathrm{Q1})_{j-1}}{\leq} 2^j j! \lambda n^j.
\end{aligned}$$

Thus $(\mathrm{Q1})_j$ holds and we obtain $\{\mathscr{Q}^{(i)}\}_{i=1}^{j}$ satisfying $(\mathrm{Q1})_j$–$(\mathrm{Q3})_j$. Inductively we obtain $\mathscr{Q} = \{\mathscr{Q}^{(i)}\}_{i=1}^{k-1}$ satisfying $(\mathrm{Q1})_{k-1}$–$(\mathrm{Q3})_{k-1}$.

Note that since $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance, we have $\varepsilon \leq \|\mathbf{a}\|_\infty^{-4^k} \varepsilon_{4.5}(\|\mathbf{a}\|_\infty^{-1}, \|\mathbf{a}\|_\infty^{-1}, k-1, k)$. Thus Lemma 4.5 (together with Lemma 4.6(ii)) implies for any $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$ that $|P^{(j)}(\hat{\mathbf{x}}, b)| \geq \varepsilon^{1/2} n^j$. This with $(\mathrm{Q1})_{k-1}$ shows that for any $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, the j-graph $Q^{(j)}(\hat{\mathbf{x}}, b)$ is nonempty. Together with properties $(\mathrm{Q2})_{k-1}$ and $(\mathrm{Q3})_{k-1}$ this in turn ensures that we can apply Lemma 3.13 to show that $\mathscr{Q}$ is a family of partitions.

By $(\mathrm{Q1})_{k-1}$ and the assumption that $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance, we can apply Lemma 4.15 with $\mathscr{P}, \mathscr{Q}, H^{(k)}, H^{(k)}, \lambda, \lambda^{9/10}$ playing the roles of $\mathscr{P}, \mathscr{Q}, H^{(k)}, G^{(k)}, \lambda, \nu$ to obtain that $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ is an $(\varepsilon + \lambda^{1/10}, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H^{(k)}$. □

4.6. **Distance of hypergraphs and density functions.** Recall that the distance between two density functions was defined in Section 3.5.3. In this subsection we present two results that relate the distance between two hypergraphs to the distance between density functions of equitable families of partitions of these hypergraphs. The first result shows that the distance of the density functions provides a lower bound on the distance between the two hypergraphs (we will use this in the proof of Theorem 7.1). On the other hand, Lemma 4.18 shows that the lower bound given in Lemma 4.17 is essentially tight (this will also be applied in the proof of Theorem 7.1).

Lemma 4.17. *Suppose $0 < 1/n \ll \varepsilon \ll \nu, 1/t, 1/k$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose $\mathbf{a} \in [t]^{k-1}$. Suppose that $G^{(k)}$ and $H^{(k)}$ are k-graphs on vertex set V with $|V| = n$. Suppose that $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d^G_{\mathbf{a},k})$-equitable partition of $G^{(k)}$ and an $(\varepsilon, \mathbf{a}, d^H_{\mathbf{a},k})$-equitable partition of $H^{(k)}$. Then*

$$\mathrm{dist}(d^H_{\mathbf{a},k}, d^G_{\mathbf{a},k}) \leq |H^{(k)} \triangle G^{(k)}| \binom{n}{k}^{-1} + \nu.$$

Proof. For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, by definition, $G^{(k)}$ is $(\varepsilon, d^G_{k,\mathbf{a}}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$ and $H^{(k)}$ is $(\varepsilon, d^H_{k,\mathbf{a}}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$. Thus

$$|G^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| = (d^G_{k,\mathbf{a}}(\hat{\mathbf{x}}) \pm \varepsilon)|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| \text{ and}$$
$$|H^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| = (d^H_{k,\mathbf{a}}(\hat{\mathbf{x}}) \pm \varepsilon)|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))|.$$

Hence

$$\left(|d^G_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^H_{\mathbf{a},k}(\hat{\mathbf{x}})| - 2\varepsilon\right) |\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| \leq |(H^{(k)} \triangle G^{(k)}) \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))|. \quad (4.12)$$

Since $1/n \ll \varepsilon \ll \nu, 1/t, 1/k$, Lemma 4.5 (together with Lemma 4.6(ii)) implies that

$$|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| = \left(1 \pm \frac{\nu}{2}\right) \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} n^k. \quad (4.13)$$

As $\{\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})\}$ partitions $\mathcal{K}_k(\mathscr{P}^{(1)})$ (by Proposition 3.11(vi)), we obtain

$$\begin{aligned}
\mathrm{dist}(d^G_{\mathbf{a},k}, d^H_{\mathbf{a},k}) &= k! \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} |d^G_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^H_{\mathbf{a},k}(\hat{\mathbf{x}})| \\
&\overset{(4.12)}{\leq} k! \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} \frac{|(H^{(k)} \triangle G^{(k)}) \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| + 2\varepsilon|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))|}{|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))|} \\
&\overset{(4.13)}{\leq} \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} \frac{k!|(H^{(k)} \triangle G^{(k)}) \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| + \varepsilon^{1/2} n^k}{(1-\nu/2)n^k} \\
&\leq \frac{k!|H^{(k)} \triangle G^{(k)}|}{n^k} + \frac{2\nu}{3} \leq |H^{(k)} \triangle G^{(k)}| \binom{n}{k}^{-1} + \nu,
\end{aligned}$$

which completes the proof. □

Lemma 4.18. *Suppose $0 < 1/n \ll \varepsilon \ll \nu, 1/t, 1/k$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose $\mathbf{a} \in [t]^{k-1}$. Suppose that $H^{(k)}$ is an n-vertex k-graph and suppose that $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d^H_{\mathbf{a},k})$-equitable partition of $H^{(k)}$. Suppose $d^G_{\mathbf{a},k}$ is a density function of $\hat{A}(k, k-1, \mathbf{a})$. Then there exists a k-graph $G^{(k)}$ such that $\mathscr{P}$ is a $(3\varepsilon, \mathbf{a}, d^G_{\mathbf{a},k})$-equitable partition of $G^{(k)}$ and*

$$|H^{(k)} \triangle G^{(k)}| \leq (\mathrm{dist}(d^H_{\mathbf{a},k}, d^G_{\mathbf{a},k}) + \nu)\binom{n}{k}.$$

Proof. Suppose $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$. By Lemmas 4.5 and 4.6(ii), and the assumption $1/n, \varepsilon \ll \nu, 1/t, 1/k$, we obtain

$$|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| = \left(1 \pm \frac{\nu}{2}\right) \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} n^k. \quad (4.14)$$

We will distinguish the following cases depending on the values of $d^H_{\mathbf{a},k}(\hat{\mathbf{x}})$ and $d^G_{\mathbf{a},k}(\hat{\mathbf{x}})$.
Case 1: $|d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^G_{\mathbf{a},k}(\hat{\mathbf{x}})| \leq 2\varepsilon$. Let $G^{(k)}(\hat{\mathbf{x}}) := H^{(k)}(\hat{\mathbf{x}})$.
Case 2: $d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) > d^G_{\mathbf{a},k}(\hat{\mathbf{x}}) + 2\varepsilon$. Let

$$p_1 := \max\left\{\frac{d^G_{\mathbf{a},k}(\hat{\mathbf{x}})}{d^H_{\mathbf{a},k}(\hat{\mathbf{x}})}, 1 - \frac{d^G_{\mathbf{a},k}(\hat{\mathbf{x}})}{d^H_{\mathbf{a},k}(\hat{\mathbf{x}})}\right\}.$$

This definition implies that $1/2 \leq p_1 \leq 1$. Note $d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) \geq 2\varepsilon$. Because of this and (4.14), we can apply the slicing lemma (Lemma 4.4) to $H^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$ to obtain two k-graphs $F^{(k)}_0, F^{(k)}_1 \subseteq H^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$ such that

- $F^{(k)}_1$ is $(3\varepsilon, d^H_{\mathbf{a},k}(\hat{\mathbf{x}})p_1)$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$, and
- $F^{(k)}_0$ is $(3\varepsilon, d^H_{\mathbf{a},k}(\hat{\mathbf{x}})(1-p_1))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$.

Let

$$G^{(k)}(\hat{\mathbf{x}}) := \begin{cases} F_1^{(k)} & \text{if } \frac{d^G_{\mathbf{a},k}(\hat{\mathbf{x}})}{d^H_{\mathbf{a},k}(\hat{\mathbf{x}})} \geq 1/2, \\ F_0^{(k)} & \text{otherwise.} \end{cases}$$

By construction, we have $G^{(k)}(\hat{\mathbf{x}}) \subseteq H^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$.

Case 3: $d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) < d^G_{\mathbf{a},k}(\hat{\mathbf{x}}) - 2\varepsilon$. Let

$$p_1' := \max\left\{ \frac{1 - d^G_{\mathbf{a},k}(\hat{\mathbf{x}})}{1 - d^H_{\mathbf{a},k}(\hat{\mathbf{x}})}, 1 - \frac{1 - d^G_{\mathbf{a},k}(\hat{\mathbf{x}})}{1 - d^H_{\mathbf{a},k}(\hat{\mathbf{x}})} \right\}.$$

Similarly as before, $1/2 \leq p_1' \leq 1$. Note $1 - d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) \geq 2\varepsilon$. Because of this, Lemma 4.1(i) and (4.14), we can apply Lemma 4.4 with $\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \setminus H^{(k)}$, $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$, $1 - d^H_{\mathbf{a},k}(\hat{\mathbf{x}})$ and p_1' playing the roles of $H^{(k)}$, $H^{(k-1)}$, d and p_1. Then we obtain two k-graphs $F_0^{(k)}, F_1^{(k)} \subseteq \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \setminus H^{(k)}$ such that

- $F_1^{(k)}$ is $(3\varepsilon, (1 - d^H_{\mathbf{a},k}(\hat{\mathbf{x}}))p_1')$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$, and
- $F_0^{(k)}$ is $(3\varepsilon, (1 - d^H_{\mathbf{a},k}(\hat{\mathbf{x}}))(1 - p_1'))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$.

We define

$$G^{(k)}(\hat{\mathbf{x}}) := \begin{cases} \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \setminus F_1^{(k)} & \text{if } \frac{1-d^G_{\mathbf{a},k}(\hat{\mathbf{x}})}{1-d^H_{\mathbf{a},k}(\hat{\mathbf{x}})} \geq 1/2, \\ \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \setminus F_0^{(k)} & \text{otherwise.} \end{cases}$$

Hence

$$H^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})) \subseteq G^{(k)}(\hat{\mathbf{x}}) \subseteq \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})).$$

In addition, $G^{(k)}(\hat{\mathbf{x}})$ is $(3\varepsilon, d^G_{\mathbf{a},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$, by Lemma 4.1(i).

We can now combine the graphs from the above three cases and let

$$G^{(k)} := \left(\bigcup_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} G^{(k)}(\hat{\mathbf{x}}) \right) \cup (H^{(k)} \setminus \mathcal{K}_k(\mathscr{P}^{(1)})).$$

By construction, $\mathscr{P}$ is a $(3\varepsilon, \mathbf{a}, d^G_{\mathbf{a},k})$-equitable partition of $G^{(k)}$. In all three cases,

$$|(H^{(k)} \triangle G^{(k)}) \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| \leq (|d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^G_{\mathbf{a},k}(\hat{\mathbf{x}})| + 4\varepsilon)|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))|.$$

Therefore, we conclude

$$\begin{aligned} |H^{(k)} \triangle G^{(k)}| &= \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} |(H^{(k)} \triangle G^{(k)}) \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| \\ &\leq \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} (|d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^G_{\mathbf{a},k}(\hat{\mathbf{x}})| + 4\varepsilon)|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))| \\ &\overset{(4.14)}{\leq} \left(1 + \frac{\nu}{2}\right) \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} (|d^H_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^G_{\mathbf{a},k}(\hat{\mathbf{x}})| + 4\varepsilon) \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} n^k \\ &\leq (\mathrm{dist}(d^H_{\mathbf{a},k}, d^G_{\mathbf{a},k}) + \nu)\binom{n}{k}. \end{aligned}$$

□

5. Testable properties are regular reducible

In this section we show one direction of our main result (Lemma 5.1). It states that every testable k-graph property can be (approximately) described by suitable regularity instances. (Recall that the formal definition of being regular reducible is given in Definition 3.15.)

Lemma 5.1. *If a k-graph property is testable, then it is regular reducible.*

Goldreich and Trevisan [24] proved that every testable graph property is also testable in some canonical way. It is an easy exercise to translate their results to the hypergraph setting, as their arguments also work in this case. Thus in the proof of Lemma 5.1 we may restrict ourselves to such canonical testers.

To be precise, an (n,α)-tester $\mathbf{T} = \mathbf{T}(n,\alpha)$ is *canonical* if, given an n-vertex k-graph H, it chooses a set Q of $q_k = q_k(n,\alpha)$ vertices of H uniformly at random, queries all k-sets in Q, and then accepts or rejects H (deterministically) according to (the isomorphism class of) $H[Q]$. In particular, $\mathbf{T}$ has query complexity $\binom{q_k}{k}$. Moreover, every canonical tester is non-adaptive.

Lemma 5.2 (Goldreich and Trevisan [24])**.** *Suppose that $\mathbf{P}$ is a k-graph property which is testable with query complexity at most $q_k = q_k(\alpha)$. Then for all $n \in \mathbb{N}$ and $\alpha \in (0,1)$, there is a canonical (n,α)-tester $\mathbf{T} = \mathbf{T}(n,\alpha)$ for $\mathbf{P}$ with query complexity at most $\binom{9kq_k}{k}$.*

To prove Lemma 5.1, we let $\mathcal{Q}$ be the set of all k-graphs which are accepted by a canonical tester $\mathbf{T}$ for $\mathbf{P}$. We then construct a (bounded size) set of regularity instances $\mathcal{R}$ where the 'induced density' of $\mathcal{Q}$ is large for each $R \in \mathcal{R}$. We then apply the counting lemma for induced graphs (Corollary 4.10) to show that $\mathcal{R}$ satisfies the requirements of Definition 3.15.

Proof of Lemma 5.1. Let $\mathbf{P}$ be a testable k-graph property. Thus there exists a function $q'_k : (0,1) \to \mathbb{N}$ such that for every $n \in \mathbb{N}$ and $\beta \in (0,1)$, there exists an (n,β)-tester $\mathbf{T} = \mathbf{T}(n,\beta)$ for $\mathbf{P}$ with query complexity at most $q'_k(\beta)$. By Lemma 5.2, we may assume (by increasing $q'_k(\beta)$ if necessary) that $\mathbf{T}$ is canonical. Since $\mathbf{T}$ is canonical, there exists $q = q(n,\beta)$ with $q(n,\beta) \leq q'_k(\beta)$ such that given any k-graph H on n vertices $\mathbf{T}$ samples a set Q of q vertices, considers $H[Q]$, and then deterministically accepts or rejects H based on $H[Q]$.

Let $\mathcal{Q}$ be the set of all the k-graphs on q vertices such that $\mathbf{T}$ accepts H if and only if there is $Q' \in \mathcal{Q}$ that is isomorphic to $H[Q]$.

In order to show that $\mathbf{P}$ is regular reducible, it suffices to consider the case when $0 < \beta < 1/100$. We fix some function $\overline{\varepsilon} : \mathbb{N}^{k-1} \to (0,1)$ such that $\overline{\varepsilon}(\mathbf{a}) \ll \|\mathbf{a}\|_\infty^{-k}$ for all $(a_1,\dots,a_{k-1}) = \mathbf{a} \in \mathbb{N}^{k-1}$. We choose constants $\varepsilon_*, \varepsilon, \eta$, and $n_0, T \in \mathbb{N}$ such that $0 < \varepsilon_* \ll 1/n_0 \ll \varepsilon \ll 1/T \ll \eta \ll 1/q, \beta, 1/k$. In particular, we have $n_0 \geq n_{3.8}(\eta, \beta q^{-k}, \overline{\varepsilon})$, $T \geq t_{3.8}(\eta, \beta q^{-k}, \overline{\varepsilon})$ and $\varepsilon \ll \overline{\varepsilon}(\mathbf{a})$ for any $\mathbf{a} \in [T]^{k-1}$.

For each $\ell \in [n_0] \setminus [k-1]$, we let $\mathcal{Q}'(\ell)$ be the collection of ℓ-vertex k-graphs satisfying the property $\mathbf{P}$ and we let $\mathbf{I}'(\ell)$ be the collection of regularity instances $R = (\varepsilon_*, \mathbf{a}, d_{\mathbf{a},k})$ such that

(R0)$_{5.1}$ $\mathbf{a} = (\ell, 1, \dots, 1) \in \mathbb{N}^{k-1}$ and $d_{\mathbf{a},k}(\hat{\mathbf{x}}) \in \{0,1\}$ for every $\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})$.

Let $\mathbf{I}$ be the collection of regularity instances $R = (\varepsilon'', \mathbf{a}, d_{\mathbf{a},k})$ such that

(R1)$_{5.1}$ $\varepsilon'' \in \{\varepsilon, 2\varepsilon, \dots, \lceil (\overline{\varepsilon}(\mathbf{a}))^{1/2}\varepsilon^{-1} \rceil \varepsilon\}$,
(R2)$_{5.1}$ $\mathbf{a} \in [T]^{k-1}$ and $a_1 > \eta^{-1}$, and
(R3)$_{5.1}$ $d_{\mathbf{a},k}(\hat{\mathbf{x}}) \in \{0, \varepsilon^2, 2\varepsilon^2, \dots, 1\}$ for every $\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})$.

Observe that by construction $|\mathbf{I}|$ and $|\mathbf{I}'(\ell)|$ are bounded by a function of β, k and $q_k'(\beta)$ for every $\ell \in [n_0] \setminus [k-1]$. Recall that $IC(\mathcal{Q}, d_{\mathbf{a},k})$ was defined in (4.3). We define

$$\mathcal{R}(n,\beta) := \begin{cases} \{R \in \mathbf{I}'(n) : R = (\varepsilon'', \mathbf{a}, d_{\mathbf{a},k}) \text{ with } IC(\mathcal{Q}'(n), d_{\mathbf{a},k}) > 0\} & \text{if } n \leq n_0, \\ \{R \in \mathbf{I} : R = (\varepsilon'', \mathbf{a}, d_{\mathbf{a},k}) \text{ with } IC(\mathcal{Q}, d_{\mathbf{a},k}) \geq 1/2\} & \text{if } n > n_0. \end{cases}$$

In order to show that the property $\mathbf{P}$ is regular reducible, we need to show that for every $\alpha > \beta$ the following holds: every n-vertex k-graph H that satisfies $\mathbf{P}$ is β-close to satisfying R for at least one $R \in \mathcal{R}(n,\beta)$, and every n-vertex k-graph H that is α-far from satisfying $\mathbf{P}$ is $(\alpha-\beta)$-far from satisfying R for all $R \in \mathcal{R}(n,\beta)$.

First of all, we consider the case that $n \in [n_0] \setminus [k-1]$. Note that the only way for H to satisfy $R \in \mathcal{R}(n,\beta)$ is to have a partition $\mathscr{P}^{(1)}$ of $V(H)$ into n singleton clusters and the natural $(n,1,\ldots,1)$-equitable partition $\mathscr{P}$ arising from $\mathscr{P}^{(1)}$. In this case, it is easy to see that $IC(\mathcal{Q}'(n), d_{\mathbf{a},k}) > 0$ if and only if $H \in \mathcal{Q}'(n)$. Thus we conclude that every n-vertex k-graph H that satisfies $\mathbf{P}$ satisfies R for at least one $R \in \mathcal{R}(n,\beta)$, and every k-graph H that is α-far from satisfying $\mathbf{P}$ is α-far from satisfying R for all $R \in \mathcal{R}(n,\beta)$.

Now we suppose that $n > n_0$ and we let $\mathcal{R} := \mathcal{R}(n,\beta)$. First, suppose that a k-graph H satisfies $\mathbf{P}$, and thus $\mathbf{T}$ accepts H with probability at least $2/3$. Hence

$$\mathbf{Pr}(\mathcal{Q}, H) \geq 2/3. \tag{5.1}$$

Since $|V(H)| \geq n_0$, by applying the regular approximation lemma (Theorem 3.8) with $H, \eta, \beta q^{-k}, \overline{\varepsilon}$ playing the roles of $H, \eta, \nu, \varepsilon$, we obtain a k-graph G and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ such that

(A1)$_{5.1}$ $\mathscr{P}$ is $(\eta, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable for some $\mathbf{a}^{\mathscr{P}} \in [T]^{k-1}$,
(A2)$_{5.1}$ G is perfectly $\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$, and
(A3)$_{5.1}$ $|G \triangle H| \leq \beta q^{-k}\binom{n}{k}$.

Let $\varepsilon' := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$. By the choice of $\overline{\varepsilon}$ and η, we conclude that $0 < \varepsilon' \ll 1/\|\mathbf{a}^{\mathscr{P}}\|_\infty \leq 1/a_1^{\mathscr{P}} \ll 1/q, \beta, 1/k$ and by the choice of ε, we obtain $\varepsilon \ll \varepsilon'$. Note that if a k-graph F is (ε', d)-regular with respect to a $(k-1)$-graph F', then F is (ε'', d')-regular with respect to F' for some $d' \in \{0, \varepsilon^2, 2\varepsilon^2, \ldots, 1\}$ and $\varepsilon'' \in \{\varepsilon, 2\varepsilon, \ldots, \lceil \varepsilon'^{1/2}\varepsilon^{-1} \rceil \varepsilon\} \cap [2\varepsilon', 3\varepsilon']$. Thus there exists

$$R_G = (\varepsilon'', \mathbf{a}^{\mathscr{P}}, d^G_{\mathbf{a}^{\mathscr{P}},k}) \in \mathbf{I} \tag{5.2}$$

such that G satisfies R_G.

Proposition 3.1 with (A3)$_{5.1}$ implies that

$$\mathbf{Pr}(\mathcal{Q}, G) \geq \mathbf{Pr}(\mathcal{Q}, H) - \beta. \tag{5.3}$$

Since $0 < \varepsilon' \ll 1/\|\mathbf{a}^{\mathscr{P}}\|_\infty \leq 1/a_1^{\mathscr{P}} \ll 1/q, \beta, 1/k$, Corollary 4.10 implies that

$$IC(\mathcal{Q}, d^G_{\mathbf{a}^{\mathscr{P}},k}) \;\geq\; \mathbf{Pr}(\mathcal{Q}, G) - \beta \overset{(5.3)}{\geq} \mathbf{Pr}(\mathcal{Q}, H) - 2\beta \overset{(5.1)}{\geq} 2/3 - 2\beta > 1/2.$$

By the definition of $\mathcal{R}$ and (5.2), this implies that $R_G \in \mathcal{R}$ and so H is indeed β-close to a graph G satisfying R_G, one of the regularity instances of $\mathcal{R}$.

Suppose now that H is α-far from satisfying $\mathbf{P}$. Since $\mathbf{T}$ is a canonical (n,β)-tester, we conclude

$$\textit{if a } k\textit{-graph } H' \textit{ is } \beta\textit{-far from satisfying } \mathbf{P}\textit{, then } \mathbf{Pr}(\mathcal{Q}, H') \leq 1/3. \tag{5.4}$$

Assume for a contradiction that H is $(\alpha-\beta)$-close to satisfying some regularity instance $R_F = (\varepsilon'', \mathbf{b}, d^F_{\mathbf{b},k}) \in \mathcal{R}$. Thus there exists a k-graph F which satisfies R_F and which is $(\alpha-\beta)$-close to H. Since $R_F \in \mathcal{R}$, we obtain

$$IC(\mathcal{Q}, d^F_{\mathbf{b},k}) \geq 1/2.$$

Then (R1)$_{5.1}$ and (R2)$_{5.1}$ guarantee that we can apply Corollary 4.10 with $\mathcal{Q}, R_F, T, 1/10$ playing the roles of $\mathcal{F}, R, t, \gamma$ to conclude that

$$\mathbf{Pr}(\mathcal{Q}, F) \geq IC(\mathcal{Q}, d^F_{\mathbf{b},k}) - 1/10 \geq 1/2 - 1/10 > 1/3.$$

Together with (5.4) this implies that F is β-close to satisfying $\mathbf{P}$. Thus there exists a k-graph F' satisfying $\mathbf{P}$ which is β-close to F. Since H is $(\alpha-\beta)$-close to F, we conclude that F' is α-close to H, which contradicts the assumption that H is α-far from satisfying $\mathbf{P}$. This completes the proof that $\mathbf{P}$ is regular reducible. $\square$

6. Satisfying a regularity instance is testable

In this section we deduce from Lemma 6.1 that the property of satisfying a particular regularity instance is testable. Suppose H is a k-graph and Q is a subset of the vertices chosen uniformly at random. Lemma 6.2 shows that if H satisfies a regularity instance R, then with high probability $H[Q]$ is close to satisfying R. Lemma 6.3 gives the converse: if H is far from satisfying R, then with high probability $H[Q]$ is also far from satisfying R.

Lemmas 6.2 and 6.3 follow from Lemma 6.1. It implies that a family of partitions not only transfers from a hypergraph to its random samples with high probability, but also vice versa. Crucially, in both directions these transfer results allow only a small additive increase in the regularity parameters (which can then be eliminated via Lemma 4.12). We defer the proof of Lemma 6.1 to Sections 9 and 10.

Lemma 6.1. *Suppose $0 < 1/n < 1/q \ll c \ll \delta \ll \varepsilon_0 \leq 1$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose $R = (2\varepsilon_0/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Suppose H is a k-graph on vertex set V with $|V| = n$. Let $Q \in \binom{V}{q}$ be chosen uniformly at random. Then with probability at least $1 - e^{-cq}$ the following hold.*

(Q1)$_{6.1}$ *If there exists an $(\varepsilon_0, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_1$ of H, then there exists an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_2$ of $H[Q]$.*

(Q2)$_{6.1}$ *If there exists an $(\varepsilon_0, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_2$ of $H[Q]$, then there exists an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_1$ of H.*

Lemma 6.2. *Suppose that $0 < 1/n < 1/q \ll c \ll \nu \ll \varepsilon$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance and H is an n-vertex k-graph on vertex set V that satisfies R. Suppose $Q \in \binom{V}{q}$ is chosen uniformly at random. Then $H[Q]$ is ν-close to satisfying R with probability at least $1 - e^{-cq}$.*

Proof. Choose δ such that $c \ll \delta \ll \nu$. Since H satisfies R, there exists an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_1$ of H. Thus Lemma 6.1 implies that with probability at least $1 - e^{-cq}$ there exists an $(\varepsilon + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_2$ of $H[Q]$.

Thus Lemma 4.12, with $H[Q], Q, q$ playing the roles of $H^{(k)}, V, n$ respectively, implies that there exists a family of partitions $\mathscr{Q}$ and a k-graph G on vertex set Q such that $\mathscr{Q}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of G and $|G \triangle H[Q]| \leq \nu\binom{q}{k}$. Therefore, G satisfies R and $G[Q]$ is ν-close to $H[Q]$.

Thus with probability at least $1-e^{-cq}$ the k-graph $H[Q]$ is ν-close to satisfying R. $\square$

Lemma 6.3. *Suppose that $0 < 1/n < 1/q \ll c \ll \nu \ll \varepsilon, \alpha$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Suppose an n-vertex k-graph H on vertex set V is α-far from satisfying R, and $Q \in \binom{V}{q}$ is chosen uniformly at random. Then $H[Q]$ is ν-far from satisfying R with probability at least $1 - e^{-cq}$.*

We say that $R = (\varepsilon', \mathbf{a}, d_{\mathbf{a},k})$ is a *relaxed regularity instance* if $(2\varepsilon'/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance.

Proof of Lemma 6.3. Choose δ and α^* such that $c \ll \delta \ll \nu \ll \alpha^* \ll \varepsilon, \alpha$. Note that $2(\varepsilon + \nu^{1/6} + \delta)/3 \leq \varepsilon$, thus $(2(\varepsilon + \nu^{1/6} + \delta)/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Let $\mathcal{E}$ be the event that the following holds:

If $H[Q]$ satisfies the relaxed regularity instance $R' := (\varepsilon + \nu^{1/6}, \mathbf{a}, d_{\mathbf{a},k})$, then H satisfies the relaxed regularity instance $R'' := (\varepsilon + \nu^{1/6} + \delta, \mathbf{a}, d_{\mathbf{a},k})$. (6.1)

By Lemma 6.1, we have $\mathbb{P}[\mathcal{E}] \geq 1 - e^{-cq}$. We claim that if $\mathcal{E}$ occurs and H is α-far from satisfying R, then $H[Q]$ is ν-far from satisfying R. This clearly implies the lemma.

Suppose $\mathcal{E}$ holds and $H[Q]$ is ν-close to satisfying R. Then there exists a k-graph L on vertex set Q which satisfies R and $|H[Q] \triangle L| \leq \nu\binom{q}{k}$. This implies that there exists an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_1$ of L. Thus Lemma 4.15 with $\mathscr{O}_1$ playing the roles of both $\mathscr{P}$ and $\mathscr{Q}$ as well as $L, H[Q]$ playing the roles of $H^{(k)}$, $G^{(k)}$, respectively implies that $H[Q]$ satisfies the relaxed regularity instance $R' = (\varepsilon + \nu^{1/6}, \mathbf{a}, d_{\mathbf{a},k})$.

By our assumption (6.1), this means that H satisfies the relaxed regularity instance $R'' = (\varepsilon + \nu^{1/6} + \delta, \mathbf{a}, d_{\mathbf{a},k})$. Thus Lemma 4.12 with $\nu^{1/6} + \delta, \alpha^*$ playing the roles of δ, ν respectively implies that there exists a family of partitions $\mathscr{Q}$ and a k-graph F on vertex set V such that $\mathscr{Q}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of F and $|F \triangle H| \leq \alpha^*\binom{n}{k} \leq \alpha\binom{n}{k}$. So F satisfies the regularity instance R and F is α-close to H. Thus H is α-close to satisfying R, a contradiction to our assumption. Therefore, $H[Q]$ is ν-far from satisfying R with probability at least $\mathbb{P}[\mathcal{E}] \geq 1 - e^{-cq}$. $\square$

Theorem 6.4. *For all $k \in \mathbb{N} \setminus \{1\}$ and all regularity instances $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$, the property of satisfying R is testable.*

Proof. Consider $\alpha \in (0, 1)$ and $n \in \mathbb{N}$. We choose $q = q(\alpha, \varepsilon) \in \mathbb{N}$ and constants $\nu = \nu(\alpha, \varepsilon), c = c(\alpha, \varepsilon) > 0$ such that $1/q \ll c \ll \nu \ll \varepsilon, \alpha$. Let H be a k-graph on n vertices. Without loss of generality, we assume that $n > q$. We choose a set Q of q vertices of H uniformly at random and accept H if and only if $H[Q]$ is ν-close to satisfying R. Then Lemmas 6.2 and 6.3 imply that with probability at least $1 - e^{-cq} \geq 2/3$, this algorithm distinguishes between the case that H satisfies R and that H is α-far from satisfying R. This completes the proof. $\square$

7. Testable hypergraph properties are estimable

The notion of testability is similar to the notion of estimability. In fact, we prove that for k-graphs these notions are equivalent. This generalizes a result in [19] where this is proved for graph properties. The special case when $\mathbf{P}$ is the property of satisfying a given regularity instance R will be an important ingredient (together with Theorem 6.4) in the proof of Theorem 1.3 to show that every regular reducible property is testable.

Theorem 7.1. *For every $k \in \mathbb{N} \setminus \{1\}$, a k-graph property $\mathbf{P}$ is testable if and only if it is estimable.*

To prove Theorem 7.1, it suffices to show that for k-graphs every testable property $\mathbf{P}$ is estimable. As in Section 5 we can assume the existence of a family $\mathcal{F}$ of bounded size and a canonical tester $\mathbf{T}$ for $\mathbf{P}$ which accepts its input H if and only if $\mathbf{T}$ samples some k-graph in $\mathcal{F}$.

We then consider the set $\mathbf{F}$ of regularity instances which correspond to a high density of copies of the k-graphs in $\mathcal{F}$. Our estimator $\mathbf{T}_E$ accepts H if a random sample $H[Q]$ is sufficiently close to satisfying some $R \in \mathbf{F}$ (see (7.6)).

Now suppose H is close to satisfying $\mathbf{P}$. Then there is a k-graph F which satisfies $\mathbf{P}$ and is close to H. Via the partition version of the regular approximation lemma (Lemma 3.9) we can associate a high quality regularity partition $\mathscr{P}$ with F (actually we consider some F' which is close to F) together with a suitable density function $d^F_{\mathbf{a}^{\mathscr{P}},k}$. Since F satisfies $\mathbf{P}$, $d^F_{\mathbf{a}^{\mathscr{P}},k}$ will give rise to a regularity instance $R_F \in \mathbf{F}$ which is satisfied by F. Via the regular approximation lemma and some additional arguments, it also turns out that instead of H we can actually consider some G_* to which we can associate the same regularity partition $\mathscr{P}$ but with a density function $d^G_{\mathbf{a}^{\mathscr{P}},k}$ (which will also give rise to a regularity instance R'_{G_*}). By Lemma 6.1(i), $\mathscr{P}$ and $d^G_{\mathbf{a}^{\mathscr{P}},k}$ are inherited by $G_*[Q]$ with high probability. We can also construct J on Q which is close to $G_*[Q]$ and inherits $\mathscr{P}$ and $d^F_{\mathbf{a}^{\mathscr{P}},k}$ from F, and also satisfies $R_F \in \mathbf{F}$. But this means that the estimator $\mathbf{T}_E$ will indeed accept H. See Figure 1 for an illustration.

If H is far from satisfying $\mathbf{P}$, then we argue via Lemma 6.1(ii) rather than Lemma 6.1(i).

Proof of Theorem 7.1. Clearly, every estimable property is also testable. Thus it suffices to show that given any $n \in \mathbb{N}$ and $\alpha > \beta > 0$ and any testable k-graph property $\mathbf{P}$, we can construct an (n, α, β)-estimator.

Assume that $\mathbf{P}$ is a testable k-graph property. By Lemma 5.2, there exists a function $q_k : (0, 1) \to \mathbb{N}$ such that the following holds: For any $n \in \mathbb{N}$, there exist a canonical $(n, \beta/4)$-tester $\mathbf{T}$ and an integer $q' = q'(n, \beta) \leq q_k(\beta/4)$ such that, given any n-vertex k-graph H, the tester $\mathbf{T}$ chooses a random subset Q' of q' vertices of H and (deterministically) accepts or rejects H based on the isomorphism class of $H[Q']$. Let $\mathcal{F}$ be the collection of q'-vertex k-graphs such that $\mathbf{T}$ accepts H if and only if $H[Q']$ induces one of the k-graphs in $\mathcal{F}$. Since $\mathbf{T}$ is a $(n, \beta/4)$-tester, we conclude the following:

$$\begin{aligned}&\textit{If } H \textit{ satisfies } \mathbf{P}, \textit{ then } \mathbf{Pr}(\mathcal{F}, H) \geq 2/3, \textit{ and}\\ &\textit{if } H \textit{ is } (\beta/4)\textit{-far from satisfying } \mathbf{P}, \textit{ then } \mathbf{Pr}(\mathcal{F}, H) \leq 1/3.\end{aligned} \tag{7.1}$$

Let $\overline{\varepsilon} : \mathbb{N}^{k-1} \to (0, 1]$ be a function such that $\overline{\varepsilon}(\mathbf{a}) \ll \|\mathbf{a}\|_\infty^{-k}$ for every $\mathbf{a} \in \mathbb{N}^{k-1}$. We choose constants η, ν such that

$$0 < \eta \ll \nu \ll 1/q', \beta, 1/k. \tag{7.2}$$

Let $\mu : \mathbb{N}^{k-1} \to (0, 1]$ be a function such that for any $\mathbf{a} \in \mathbb{N}^{k-1}$, we have $\mu(\mathbf{a}) \ll \|\mathbf{a}\|_\infty^{-k}, \eta, \nu$ and

$$\mu(\mathbf{a}) \ll \mu_{3.9}(k, \|\mathbf{a}\|_\infty, \|\mathbf{a}\|_\infty^{4^k}, \eta, \nu, \overline{\varepsilon}), 1/t_{3.9}(k, \|\mathbf{a}\|_\infty, \|\mathbf{a}\|_\infty^{4^k}, \eta, \nu, \overline{\varepsilon}), \varepsilon_{3.14}(\|\mathbf{a}\|_\infty, k). \tag{7.3}$$

In particular, we assume that

$$\mu(\mathbf{a}) \ll \overline{\varepsilon}(\mathbf{b}) \text{ for all } \mathbf{a} \in \mathbb{N}^{k-1} \text{ and } \mathbf{b} \in [t_{3.9}(k, \|\mathbf{a}\|_\infty, \|\mathbf{a}\|_\infty^{4^k}, \eta, \nu, \overline{\varepsilon})]^{k-1}. \tag{7.4}$$

Let T, μ_*, c, q, n_0 be numbers such that

$$\begin{aligned}&T := t_{3.8}(\eta, \nu/2, \mu), \quad \mu_* := \min_{\mathbf{a} \in [T]^{k-1}} \mu(\mathbf{a}), \\ &1/n_0 \ll 1/q \ll c \ll \mu_*, \nu \quad \text{and} \quad n_0 \geq n_{3.8}(\eta, \nu/2, \mu), n_{3.9}(k, T, T^{4^k}, \eta, \nu, \overline{\varepsilon}).\end{aligned} \tag{7.5}$$

Let $\mathbf{R}$ be the collection of all regularity instances $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ satisfying the following.

(R1)$_{7.1}$ $\varepsilon \in \{\mu_*, 2\mu_*, \ldots, 1\}$ and $\varepsilon \leq \overline{\varepsilon}(\mathbf{a})^{1/2}$,
(R2)$_{7.1}$ $d_{\mathbf{a},k}(\hat{\mathbf{x}}) \in \{0, \mu_*^2, 2\mu_*^2, \ldots, 1\}$ for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$.

Recall that $IC(\mathcal{F}, d_{\mathbf{a},k})$ was defined in (4.3). Let

$$\mathbf{F} := \{R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k}) \in \mathbf{R} : IC(\mathcal{F}, d_{\mathbf{a},k}) \geq 1/2 \text{ and } a_1 \geq \eta^{-1}\}.$$

Note that $|\mathbf{F}|$ is bounded by a function only depending on β, k, and $\mathbf{P}$.

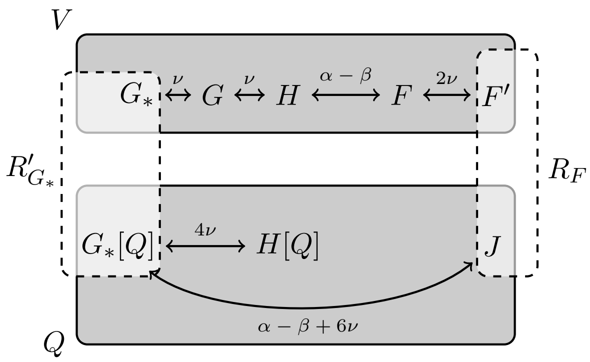

FIGURE 1. The proof strategy for Theorem 7.1 (Case 1).

Fix $\alpha > \beta$. As a next step we describe the algorithm $\mathbf{T}_E(n, \alpha, \beta)$ which receives an n-vertex k-graph H as an input:

> *If $n < n_0$, then $\mathbf{T}_E(n, \alpha, \beta)$ considers the entire k-graph H and determines how close H is to satisfying $\mathbf{P}$. If $n \geq n_0$, then $\mathbf{T}_E(n, \alpha, \beta)$ chooses a subset Q of q vertices of H uniformly at random. If $H[Q]$ is $(\alpha - \beta/2)$-close to some k-graph which satisfies a regularity instance $R \in \mathbf{F}$, then $\mathbf{T}_E(n, \alpha, \beta)$ accepts H, otherwise it rejects H.* (7.6)

Recall that q and $|\mathbf{F}|$ are both bounded by a function of β, k, and $\mathbf{P}$. and so the query complexity of $\mathbf{T}_E(n, \alpha, \beta)$ is also bounded by a function depending only on α, β and k.

In the following we verify that $\mathbf{T}_E(n, \alpha, \beta)$ distinguishes k-graphs which are $(\alpha - \beta)$-close to satisfying $\mathbf{P}$ and k-graphs which are α-far from satisfying $\mathbf{P}$ with probability at least $2/3$. If $n < n_0$, then this is trivial to verify since we consider entire k-graph H. Thus we assume that $n \geq n_0$.

Let us fix an n-vertex k-graph H. By (7.5), we can apply the regular approximation lemma (Theorem 3.8) with $H, \eta, \nu/2, \mu, T$ playing the roles of $H, \eta, \nu, \varepsilon, t_0$. Hence there exists a k-graph G on $V(H)$, $\mathbf{a}^{\mathscr{Q}} \in \mathbb{N}^{k-1}$ and a family of partitions $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a}^{\mathscr{Q}})$ satisfying the following.

(GA) $\mathscr{Q}$ is $(\eta, \mu(\mathbf{a}^{\mathscr{Q}}), \mathbf{a}^{\mathscr{Q}})$-equitable and T-bounded,
(GB) G is perfectly $\mu(\mathbf{a}^{\mathscr{Q}})$-regular with respect to $\mathscr{Q}$, and
(GC) $|G \triangle H| \leq \nu\binom{n}{k}$.

Property (GB) implies that there exists a density function $d^G_{\mathbf{a}^{\mathscr{Q}},k}$ such that for all $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})$ the k-graph G is $(\mu(\mathbf{a}^{\mathscr{Q}}), d^G_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{Q}^{(k-1)}(\hat{\mathbf{x}})$. Thus

$$\{\mathscr{Q}^{(j)}\}_{j=1}^{k-1} \text{ is a } (\mu(\mathbf{a}^{\mathscr{Q}}), \mathbf{a}^{\mathscr{Q}}, d^G_{\mathbf{a}^{\mathscr{Q}},k})\text{-equitable partition of } G. \tag{7.7}$$

Case 1: *H is $(\alpha - \beta)$-close to satisfying* $\mathbf{P}$.

Hence there exists a k-graph F which satisfies $\mathbf{P}$ and which is $(\alpha - \beta)$-close to H. By (7.1), we have the following:

$$F \text{ is } (\alpha - \beta)\text{-close to } H \text{ and } \mathbf{Pr}(\mathcal{F}, F) \geq 2/3. \tag{7.8}$$

We choose a suitable partition of the edges of F and its complement by setting

$$\{F_1^{(k)}, \dots, F_s^{(k)}\} := \{F \cap \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})\} \setminus \{\emptyset\},$$

$$\{F_{s+1}^{(k)}, \dots, F_{s+s'}^{(k)}\} := \left(\left\{\mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})) \setminus F : \hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})\right\}\right.$$
$$\left.\cup \left\{F \setminus \mathcal{K}_k(\mathscr{Q}^{(1)}),\ \binom{V}{k} \setminus (\mathcal{K}_k(\mathscr{Q}^{(1)}) \cup F)\right\}\right) \setminus \{\emptyset\}.$$

Note that by Proposition 3.11(viii), $s + s' \leq \|\mathbf{a}^{\mathscr{Q}}\|_\infty^{4^k}$. Let

$$T' := t_{3.9}(k, \|\mathbf{a}^{\mathscr{Q}}\|_\infty, \|\mathbf{a}^{\mathscr{Q}}\|_\infty^{4^k}, \eta, \nu, \overline{\varepsilon}). \tag{7.9}$$

Let

$$\mathscr{Q}^{(k)} := \{\mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})\}.$$

Clearly, $\{F_1^{(k)}, \dots, F_{s+s'}^{(k)}\} \prec \mathscr{Q}^{(k)}$. We apply the 'partition version' of the regular approximation lemma (Lemma 3.9) with the following objects and parameters. Indeed this is possible by (GA) and (7.3).

object/parameter	$\{\mathscr{Q}^{(j)}\}_{j=1}^k$	$\{F_1^{(k)}, \dots, F_{s+s'}^{(k)}\}$	k	$s+s'$	η	ν	$\overline{\varepsilon}$	T'	$\|\mathbf{a}^{\mathscr{Q}}\|_\infty$
playing the role of	$\{\mathscr{Q}^{(j)}\}_{j=1}^k$	$\mathscr{H}^{(k)}$	k	s	η	ν	ε	t	o

We obtain a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and k-graphs $\{F_i'^{(k)}\}_{i=1}^{s+s'}$ such that

(P1)$_{7.1}$ $\mathscr{P}$ is $(\eta, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable and T'-bounded,
(P2)$_{7.1}$ $\mathscr{P} \prec \{\mathscr{Q}^{(j)}\}_{j=1}^{k-1}$,
(P3)$_{7.1}$ $F_i'^{(k)}$ is perfectly $\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$, and
(P4)$_{7.1}$ $\sum_{i=1}^s |F_i^{(k)} \triangle F_i'^{(k)}| \leq \nu\binom{n}{k}$.

Note that by (7.2)–(7.4)

$$\mu(\mathbf{a}^{\mathscr{Q}}) \ll 1/T', \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}) \quad \text{and} \quad \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}) \ll \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-1} \leq 1/a_1^{\mathscr{P}} \leq 1/a_1^{\mathscr{Q}} \leq \eta \ll \nu. \tag{7.10}$$

Let $F' := \bigcup_{i=1}^s F_i'^{(k)}$. Then by (P4)$_{7.1}$

$$|F \triangle F'| \leq \nu\binom{n}{k} + |F \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})| \stackrel{(3.5)}{\leq} (\nu + k^2\eta)\binom{n}{k} \leq 2\nu\binom{n}{k}. \tag{7.11}$$

By (P3)$_{7.1}$ and Lemma 4.3

$$F' \textit{ is perfectly } s\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})\textit{-regular with respect to } \mathscr{P}. \tag{7.12}$$

Together with (7.8) and (7.11), Proposition 3.1 implies that

$$\mathbf{Pr}(\mathcal{F}, F') \geq 2/3 - 2q'^k\nu \geq 2/3 - \nu^{1/2}. \tag{7.13}$$

Observe that so far we introduced a k-graph G that is very close to H and a k-graph F' that is very close to F, where G and F' satisfy very strong regularity conditions. We now modify G slightly to obtain G_* so that $\mathscr{P}$ is an equitable partition of G_*.

By (7.7), (7.10), (P1)$_{7.1}$ and (P2)$_{7.1}$, we can apply Proposition 4.14 with $\mu(\mathbf{a}^{\mathscr{Q}}), \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \nu$, $\|\mathbf{a}^{\mathscr{P}}\|_\infty, \mathscr{P}$, $\{\mathscr{Q}^{(j)}\}_{j=1}^{k-1}, G$ and $d^G_{\mathbf{a}^{\mathscr{Q}},k}$ playing the roles of $\varepsilon, \varepsilon', \nu, T, \mathscr{P}, \mathscr{Q}, H^{(k)}$ and $d_{\mathbf{a}^{\mathscr{Q}},k}$ to obtain a density function $d^G_{\mathbf{a}^{\mathscr{P}},k} : \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}}) \to [0,1]$ and an n-vertex k-graph G_* such that

(G$_*$1) $\mathscr{P}$ is an $(\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}}, d^G_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of G_*,
(G$_*$2) $|G \triangle G_*| \leq \nu\binom{n}{k}$.

Let $\varepsilon_* := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$. Since $\|\mathbf{a}^{\mathscr{Q}}\|_\infty \leq T$ by (GA), it follows from (7.5) and (7.10) that $\mu_* \leq \mu(\mathbf{a}^{\mathscr{Q}}) \ll \varepsilon_*$. For each $\hat{\mathbf{y}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}})$, let $d^F_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) \in \{0, \mu_*^2, 2\mu_*^2, \ldots, 1\}$ such that

$$d^F_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) = d(F' \mid \hat{P}^{(k-1)}(\hat{\mathbf{y}})) \pm \mu_*^2.$$

Thus $d^F_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}})$ satisfies $(\mathrm{R2})_{7.1}$.

Let ε^* be a number satisfying $(\mathrm{R1})_{7.1}$ with $\mathbf{a}^{\mathscr{P}}$ playing the role of $\mathbf{a}$ and such that $6s\varepsilon_* \leq \varepsilon^* \leq 7s\varepsilon_*$. Then by (7.12), $\mathscr{P}$ is an $(\varepsilon^*, \mathbf{a}^{\mathscr{P}}, d^F_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of F'. So

$$F' \textit{ satisfies the regularity instance } R_F := (\varepsilon^*, \mathbf{a}^{\mathscr{P}}, d^F_{\mathbf{a}^{\mathscr{P}},k}) \textit{ and } R_F \in \mathbf{R}. \tag{7.14}$$

Also, (G_*1) implies that G_* satisfies the regularity instance $R_{G_*} = (\varepsilon_*, \mathbf{a}^{\mathscr{P}}, d^G_{\mathbf{a}^{\mathscr{P}},k})$.

Note that (7.8), (7.11), (GC) and (G_*2) together imply that G_* and F' are $(\alpha-\beta+4\nu)$-close. Thus Lemma 4.17 implies that

$$\mathrm{dist}(d^G_{\mathbf{a}^{\mathscr{P}},k}, d^F_{\mathbf{a}^{\mathscr{P}},k}) \leq \alpha - \beta + 5\nu. \tag{7.15}$$

Recall that the algorithm $\mathbf{T}_E(n, \alpha, \beta)$ chooses a random q-set Q of $V(H)$. Define events $\mathcal{E}_0$ and $\mathcal{E}_1$ as follows:

($\mathcal{E}_0$) $|G_*[Q] \triangle H[Q]| = \frac{q^k}{n^k}|G_* \triangle H| \pm \nu\binom{q}{k}$,

($\mathcal{E}_1$) $G_*[Q]$ satisfies the regularity instance $R'_{G_*} := (2\varepsilon_*, \mathbf{a}^{\mathscr{P}}, d^G_{\mathbf{a}^{\mathscr{P}},k})$.

Lemma 3.3 implies $\mathbb{P}[\mathcal{E}_0] \geq 1 - e^{-\nu^3 q}$, and Lemma 6.1 implies $\mathbb{P}[\mathcal{E}_1] \geq 1 - e^{-cq}$. Hence

$$\mathbb{P}[\mathcal{E}_0 \wedge \mathcal{E}_1] \geq 1 - 2e^{-cq} \geq \frac{2}{3}. \tag{7.16}$$

It suffices to show that if $\mathcal{E}_0 \wedge \mathcal{E}_1$ holds, then $\mathbf{T}_E(n, \alpha, \beta)$ always accepts H. Suppose in the following that $\mathcal{E}_0 \wedge \mathcal{E}_1$ holds. Then ($\mathcal{E}_0$), (7.15) and Lemma 4.18 with $q, G_*[Q], d^G_{\mathbf{a}^{\mathscr{P}},k}, d^F_{\mathbf{a}^{\mathscr{P}},k}$ playing the roles of $n, H^{(k)}, d^H_{\mathbf{a}^{\mathscr{P}},k}, d^G_{\mathbf{a}^{\mathscr{P}},k}$ imply that there exists a q-vertex k-graph J on vertex set Q such that

$(\mathrm{J1})_{7.1}$ J is $(\alpha - \beta + 6\nu)$-close to $G_*[Q]$, and

$(\mathrm{J2})_{7.1}$ J satisfies the regularity instance $R_F = (\varepsilon^*, \mathbf{a}^{\mathscr{P}}, d^F_{\mathbf{a}^{\mathscr{P}},k})$.

To obtain $(\mathrm{J2})_{7.1}$, we use that $6s\varepsilon_* \leq \varepsilon^* \leq 7s\varepsilon_*$. Note that ($\mathcal{E}_0$), (GC) and (G_*2) together imply

$$|G_*[Q] \triangle H[Q]| \leq 4\nu\binom{q}{k}. \tag{7.17}$$

Together with $(\mathrm{J1})_{7.1}$ and (7.2), this gives

J *is* $(\alpha - \beta/2)$*-close to* $H[Q]$.

Moreover, note $\varepsilon^* \ll \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-1} \leq 1/a_1^{\mathscr{P}} \ll \nu, 1/q', 1/k$ by (7.2) and (7.10). Thus (7.14) together with Corollary 4.10 implies that

$$IC(\mathcal{F}, d^F_{\mathbf{a}^{\mathscr{P}},k}) \geq \mathbf{Pr}(\mathcal{F}, F') - \nu \overset{(7.13)}{\geq} 2/3 - 2\nu^{1/2} \geq 1/2.$$

Since we also have $R_F \in \mathbf{R}$ by (7.14) and $a_1^{\mathscr{P}} \geq 1/\eta$ by $(\mathrm{P1})_{7.1}$, it follows that $R_F \in \mathbf{F}$. Thus there exists J which is $(\alpha - \beta/2)$-close to $H[Q]$ and which satisfies $R_F \in \mathbf{F}$. Hence (7.6) implies that $\mathbf{T}_E(n, \alpha, \beta)$ accepts H. Together with (7.16) this shows that whenever H is $(\alpha - \beta)$-close to satisfying $\mathbf{P}$, then $\mathbf{T}_E(n, \alpha, \beta)$ accepts H with probability at least $2/3$.

Case 2: H *is* α*-far from satisfying* $\mathbf{P}$.

Again, we may use Theorem 3.8 to show the existence of a family of partitions $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a}^{\mathscr{Q}})$ and a k-graph G on V satisfying (GA)–(GC). Then G satisfies the regularity instance $R_G := (\mu(\mathbf{a}^{\mathscr{Q}}), \mathbf{a}^{\mathscr{Q}}, d^G_{\mathbf{a}^{\mathscr{Q}},k})$ for some density function $d^G_{\mathbf{a}^{\mathscr{Q}},k}$ and (7.7) holds. (By (7.3), R_G is indeed a regularity instance.) Recall that $\mathbf{T}_E(n,\alpha,\beta)$ chooses a random q-set Q of $V(H)$. Define events $\mathcal{E}'_0$ and $\mathcal{E}'_1$ as follows.

($\mathcal{E}'_0$) $|G[Q]\triangle H[Q]| = \frac{q^k}{n^k}|G\triangle H| \pm \nu\binom{q}{k}$,
($\mathcal{E}'_1$) $G[Q]$ satisfies the regularity instance $(2\mu(\mathbf{a}^{\mathscr{Q}}), \mathbf{a}^{\mathscr{Q}}, d^G_{\mathbf{a}^{\mathscr{Q}},k})$.

Thus, if $\mathcal{E}'_1$ occurs, then there exists a family of partitions $\mathscr{Q}' = \mathscr{Q}'(k-1,\mathbf{a}^{\mathscr{Q}})$ on Q which is a $(2\mu(\mathbf{a}^{\mathscr{Q}}), \mathbf{a}^{\mathscr{Q}}, d^G_{\mathbf{a}^{\mathscr{Q}},k})$-equitable partition of $G[Q]$. Note that $\mathscr{Q}'$ is $\|\mathbf{a}^{\mathscr{Q}}\|_\infty$-bounded. Lemma 3.3 implies $\mathbb{P}[\mathcal{E}'_0] \geq 1 - e^{-\nu^3 q}$, while Lemma 6.1 and (7.5) imply that $\mathbb{P}[\mathcal{E}'_1] \geq 1 - e^{-cq}$. Thus

$$\mathbb{P}[\mathcal{E}'_0 \wedge \mathcal{E}'_1] \geq 1 - 2e^{-cq} \geq 2/3. \tag{7.18}$$

So it suffices to show that if $\mathcal{E}'_0 \wedge \mathcal{E}'_1$ holds, then $\mathbf{T}_E(n,\alpha,\beta)$ rejects H. Assume for a contradiction that $\mathbf{T}_E(n,\alpha,\beta)$ accepts H. This implies that there exists a k-graph L on Q so that

L is $(\alpha-\beta/2)$-close to $H[Q]$ and L satisfies some regularity instance $R_L \in \mathbf{F}$. (7.19)

Together with (R1)$_{7.1}$, (7.2), and the definition of $\mathbf{F}$, Corollary 4.10 implies that

$$\mathbf{Pr}(\mathcal{F}, L) \geq 1/2 - \nu. \tag{7.20}$$

Our strategy is as follows. We aim to construct a k-graph M on V whose structure is very similar to L. However, this is hard to achieve directly as L may not satisfy sufficiently strong regularity assumptions. Thus we first approximate L by a suitable k-graph L'. Similarly, we also approximate G (and thus H) by suitable k-graphs G' and G^*. Based on this, we construct M which is $(\alpha-\beta/2+10\nu)$-close to H; that is, almost as close as L and $H[Q]$. Based on (7.19) and (7.20), we verify that M is $(\beta/4)$-close to satisfying $\mathbf{P}$, yielding a contradiction to the assumption that H is α-far from satisfying $\mathbf{P}$.

Let

$$\begin{aligned}\{L_1,\dots,L_s\} &:= \{L\cap \mathcal{K}_k(\hat{Q}'^{(k-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}})\}\setminus\{\emptyset\},\\ \{L_{s+1},\dots,L_{s+s'}\} &:= \Big(\Big\{\mathcal{K}_k(\hat{Q}'^{(k-1)}(\hat{\mathbf{x}}))\setminus L : \hat{\mathbf{x}}\in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}})\Big\}\\ &\qquad \cup \Big\{L\setminus \mathcal{K}_k(\mathscr{Q}'^{(1)}),\ \binom{Q}{k}\setminus(\mathcal{K}_k(\mathscr{Q}'^{(1)})\cup L)\Big\}\Big)\setminus\{\emptyset\}.\end{aligned}$$

Thus $s+s' \leq \|\mathbf{a}^{\mathscr{Q}}\|_\infty^{4^k}$ by Proposition 3.11(viii). Let $\mathscr{Q}'^{(k)} := \{\mathcal{K}_k(\hat{Q}'^{(k-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}})\}$. It follows that $\{L_1,\dots,L_{s+s'}\} \prec \mathscr{Q}'^{(k)}$. Again, by (7.3) and (7.5) we can apply the 'partition version' of the regular approximation lemma (Lemma 3.9) with the following objects and parameters.

object/parameter	$\{\mathscr{Q}'^{(j)}\}_{j=1}^k$	$\{L_1,\dots,L_{s+s'}\}$	k	q	$s+s'$	η	ν	$\overline{\varepsilon}$	T'	$\|\mathbf{a}^{\mathscr{Q}}\|_\infty$
playing the role of	$\{\mathscr{Q}^{(j)}\}_{j=1}^k$	$\mathscr{H}^{(k)}$	k	n	s	η	ν	ε	t	o

We obtain a family of partitions $\mathscr{P}' = \mathscr{P}'(k-1,\mathbf{a}^{\mathscr{P}'})$ and k-graphs $\{L'_i\}_{i=1}^{s+s'}$ such that

(P$'$1)$_{7.1}$ $\mathscr{P}'$ is $(\eta, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'}), \mathbf{a}^{\mathscr{P}'})$-equitable and T'-bounded, and $a_i^{\mathscr{Q}}$ divides $a_i^{\mathscr{P}'}$ for each $i\in[k-1]$,
(P$'$2)$_{7.1}$ $\mathscr{P}' \prec \{\mathscr{Q}'^{(j)}\}_{j=1}^{k-1}$,
(P$'$3)$_{7.1}$ L'_i is perfectly $\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'})$-regular with respect to $\mathscr{P}'$,

(P′4)$_{7.1}$ $\sum_{i=1}^{s}|L_i \triangle L_i'| \leq \nu\binom{q}{k}$.

Let $L' := \bigcup_{i=1}^{s} L_i'$. Then the argument for (7.11) shows that

$$L' \textit{ is } 2\nu\textit{-close to } L. \tag{7.21}$$

By (P′3)$_{7.1}$ and Lemma 4.3 we have

$$L' \textit{ is perfectly } s\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'})\textit{-regular with respect to } \mathscr{P}'. \tag{7.22}$$

Thus there exists a density function $d^{L'}_{\mathbf{a}^{\mathscr{P}'},k} : \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}'}) \to [0,1]$ such that $\mathscr{P}'$ is an $(s\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'}), d^{L'}_{\mathbf{a}^{\mathscr{P}'},k})$-partition of L'. Hence

$$L' \textit{ satisfies the regularity instance } R_{L'} := (s\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'}), \mathbf{a}^{\mathscr{P}'}, d^{L'}_{\mathbf{a}^{\mathscr{P}'},k}). \tag{7.23}$$

From Proposition 3.1, (7.20) and (7.21) we can conclude that

$$\mathbf{Pr}(\mathcal{F}, L') \geq \mathbf{Pr}(\mathcal{F}, L) - 2q'^k\nu \geq 1/2 - \nu^{1/2} > 1/3 + 2\nu. \tag{7.24}$$

By $(\mathcal{E}_1')$, (P′1)$_{7.1}$, (P′2)$_{7.1}$, and the analogue of (7.10), we can apply Proposition 4.14 with $2\mu(\mathbf{a}^{\mathscr{Q}})$, $\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'})$, ν, $\mathscr{P}'$, $\{\mathscr{Q}'^{(j)}\}_{j=1}^{k-1}$, $G[Q]$, q and $d^{G}_{\mathbf{a}^{\mathscr{Q}},k}$ playing the roles of ε, ε', ν, $\mathscr{P}$, $\mathscr{Q}$, $H^{(k)}$, n and $d_{\mathbf{a}^{\mathscr{Q}},k}$ to obtain a k-graph G' on Q and a density function $d^{G'}_{\mathbf{a}^{\mathscr{P}'},k}$: $\hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}'}) \to [0,1]$ such that

(G′1) $\mathscr{P}'$ is an $(\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'}), d^{G'}_{\mathbf{a}^{\mathscr{P}'},k})$-partition of G',

(G′2) $|G' \triangle G[Q]| \leq \nu\binom{q}{k}$.

(G′3) $$d^{G'}_{\mathbf{a}^{\mathscr{P}'},k}(\hat{\mathbf{y}}) = \begin{cases} d^{G}_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}) & \text{if } \exists \hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}}) : \mathcal{K}_k(\hat{P'}^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{Q'}^{(k-1)}(\hat{\mathbf{x}})), \\ 0 & \text{if } \mathcal{K}_k(\hat{P'}^{(k-1)}(\hat{\mathbf{y}})) \cap \mathcal{K}_k(\mathscr{Q}'^{(1)}) = \emptyset. \end{cases}$$

For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})$, let

$$\hat{B}(\hat{\mathbf{x}}) := \{\hat{\mathbf{y}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}'}) : \mathcal{K}_k(\hat{P'}^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{Q'}^{(k-1)}(\hat{\mathbf{x}}))\}. \tag{7.25}$$

Note that (GC), $(\mathcal{E}_0')$, (7.19), (7.21) and (G′2) imply that L' is $(\alpha - \beta/2 + 6\nu)$-close to G'. Since we also have (G′1) we can apply Lemma 4.17 to see that

$$\text{dist}(d^{G'}_{\mathbf{a}^{\mathscr{P}'},k}, d^{L'}_{\mathbf{a}^{\mathscr{P}'},k}) \leq \alpha - \beta/2 + 7\nu. \tag{7.26}$$

Recall that by (P′1)$_{7.1}$, $a_i^{\mathscr{Q}}$ divides $a_i^{\mathscr{P}'}$ for all $i \in [k-1]$, and that $\mathscr{Q}$ is an $(\eta, \mu(\mathbf{a}^{\mathscr{Q}}), \mathbf{a}^{\mathscr{Q}})$-equitable family of partitions on V by (GA). So (recalling (7.3)) we can apply Lemma 4.13 with $\mathscr{Q}, \mathbf{a}^{\mathscr{Q}}, \mathbf{a}^{\mathscr{P}'}, \mu(\mathbf{a}^{\mathscr{Q}}), V$ playing the roles of $\mathscr{P}, \mathbf{a}, \mathbf{b}, \varepsilon, V$ to obtain a family of partitions $\mathscr{P}^*$ on V satisfying the following.

(P*1)$_{7.1}$ $\mathscr{P}^* = \mathscr{P}^*(k-1, \mathbf{a}^{\mathscr{P}'})$ is a $(1/a_1^{\mathscr{P}'}, \mu(\mathbf{a}^{\mathscr{Q}})^{1/3}, \mathbf{a}^{\mathscr{P}'})$-equitable family of partitions on V.

(P*2)$_{7.1}$ $\mathscr{P}^* \prec \mathscr{Q}$.

Moreover, we choose an appropriate $\mathbf{a}^{\mathscr{P}'}$-labelling so that for all $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})$ and $\hat{\mathbf{y}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}'})$, we have

$$\mathcal{K}_k(\hat{P^*}^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})) \text{ if and only if } \hat{\mathbf{y}} \in \hat{B}(\hat{\mathbf{x}}). \tag{7.27}$$

By (7.7), (P*1)$_{7.1}$ and (P*2)$_{7.1}$, we can apply Proposition 4.14 with $\mu(\mathbf{a}^{\mathscr{Q}})$, $\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'})$, $\mathscr{P}^*$, $\mathscr{Q}$, G and $d^{G}_{\mathbf{a}^{\mathscr{Q}},k}$ playing the roles of ε, ε', $\mathscr{P}$, $\mathscr{Q}$, H and $d_{\mathbf{a}^{\mathscr{Q}},k}$ to obtain an n-vertex k-graph G^* on V and density function $d^{G^*}_{\mathbf{a}^{\mathscr{P}^*},k} : \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}'}) \to [0,1]$ such that

(G*1) $\mathscr{P}^*$ is an $(\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'}), \mathbf{a}^{\mathscr{P}'}, d^{G^*}_{\mathbf{a}^{\mathscr{P}'},k})$-equitable partition of G^*,

(G*2) $|G \triangle G^*| \leq \nu\binom{n}{k}$.

$$\text{(G}^*3\text{)}\ \ d^{G^*}_{\mathbf{a}^{\mathscr{P}'},k}(\hat{\mathbf{y}}) = \begin{cases} d^{G}_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}) & \text{if } \exists \hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}^{\mathscr{Q}}) : \mathcal{K}_k(\hat{P^*}^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{x}})), \\ 0 & \text{if } \mathcal{K}_k(\hat{P^*}^{(k-1)}(\hat{\mathbf{y}})) \cap \mathcal{K}_k(\mathscr{Q}^{(1)}) = \emptyset. \end{cases}$$

This together with (G′3), (7.25) and (7.27) implies that $d^{G^*}_{\mathbf{a}^{\mathscr{P}'},k} = d^{G'}_{\mathbf{a}^{\mathscr{P}'},k}$. Using this with (7.26), (G*1), and Lemma 4.18 (applied with $G^*, \mathscr{P}^*, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'}), \nu, d^{L'}_{\mathbf{a}^{\mathscr{P}'},k}$ playing the roles of $H^{(k)}, \mathscr{P}, \varepsilon, \nu, d^{G}_{\mathbf{a},k}$), we conclude that there exists an n-vertex k-graph M on vertex set V such that

(M1)$_{7.1}$ M is $(\alpha - \beta/2 + 8\nu)$-close to G^*, and

(M2)$_{7.1}$ M satisfies the regularity instance $R_M := (3\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}'}), \mathbf{a}^{\mathscr{P}'}, d^{L'}_{\mathbf{a}^{\mathscr{P}'},k})$.

By (GC), (G*2) and (M1)$_{7.1}$, we conclude that

$$M \textit{ is } (\alpha - \beta/2 + 10\nu)\textit{-close to } H. \tag{7.28}$$

On the other hand, by (7.23), (M2)$_{7.1}$, and two applications of Corollary 4.10we have

$$\mathbf{Pr}(\mathcal{F}, M) \geq IC(\mathcal{F}, d^{L'}_{\mathbf{a}^{\mathscr{P}'},k}) - \nu \geq \mathbf{Pr}(\mathcal{F}, L') - 2\nu \overset{(7.24)}{>} 1/3.$$

Thus (7.1) implies that M is not $(\beta/4)$-far from satisfying $\mathbf{P}$; that is, M is $(\beta/4)$-close to satisfying $\mathbf{P}$. Together with (7.28), this implies that H is $(\alpha-\beta/2+10\nu+\beta/4)$-close to satisfying $\mathbf{P}$. But $\alpha - \beta/2 + 10\nu + \beta/4 < \alpha$ by (7.2), so H is α-close to satisfying $\mathbf{P}$, a contradiction. Thus as long as $\mathcal{E}_0' \wedge \mathcal{E}_1'$ holds (which, by (7.18), happens with probability at least 2/3), $\mathbf{T}_E(n, \alpha, \beta)$ rejects H. This completes the proof that $\mathbf{P}$ is (n, α, β)-estimable. □

8. Regular reducible hypergraph properties are testable

In this section we derive our main theorem from results stated and proved so far.

Proof of Theorem 1.3. By Lemma 5.1, (a) implies (c) and Theorem 7.1 implies that (a) and (b) are equivalent. It remains to show (c) implies (a).

Suppose that a k-graph property $\mathbf{P}$ is regular reducible. Fix $0 < \alpha < 1$ and $n \geq k$. We will now introduce an algorithm which distinguishes n-vertex k-graphs satisfying $\mathbf{P}$ from n-vertex k-graphs which are α-far from satisfying $\mathbf{P}$. Since $\mathbf{P}$ is a regular reducible property, by Definition 3.15, there exists $r := r_{3.15}(\alpha/4, \mathbf{P})$ and a collection $\mathcal{R} = \mathcal{R}(n, \alpha/4, \mathbf{P})$ of at most r regularity instances each of complexity at most r satisfying the following for every n-vertex k-graph H. (Note that we may assume that $r \geq 100$.)

(R1)$_{1.3}$ If H satisfies $\mathbf{P}$, then H is $\alpha/4$-close to satisfying R for some $R \in \mathcal{R}$,

(R2)$_{1.3}$ If H is α-far from satisfying $\mathbf{P}$, then H is $3\alpha/4$-far from satisfying R for all $R \in \mathcal{R}$.

By Theorems 6.4 and 7.1, for any $R \in \mathcal{R}$, there exist a function $q_k : (0,1) \to \mathbb{N}$ and an algorithm $\mathbf{T}_R = \mathbf{T}(n, \alpha)$ which distinguishes n-vertex k-graphs which are $\alpha/4$-close to satisfying R from n-vertex k-graphs which are $3\alpha/4$-far from satisfying R with probability at least 2/3, by making at most $g_k(\alpha)$ queries.

Now we let $\mathbf{T}'_R$ be an algorithm which independently applies the algorithm $\mathbf{T}_R$ exactly $6r + 1$ times on an input n-vertex k-graph H and accepts or rejects depending on the majority vote. Let $\mathbf{T}_1, \dots, \mathbf{T}_{6r+1}$ denote these independent repetitions. Let

$$X_i := \begin{cases} 1 & \text{if } \mathbf{T}_i \text{ accepts,} \\ 0 & \text{if } \mathbf{T}_i \text{ rejects.} \end{cases}$$

Let $X := \sum_{i=1}^{6r+1} X_i$. Suppose first that H is $\alpha/4$-close to satisfying R. Then $\mathbb{P}[X_i] \geq 2/3$ by the definition of $\mathbf{T}_R$, and so $\mathbb{E}[X] \geq 4r$. Thus by Lemma 3.2 and the fact that $r \geq 100$, we obtain

$$\mathbb{P}[\mathbf{T}'_R \text{ accepts } H] = \mathbb{P}\left[X \geq 3r+1\right] \geq 1 - 2e^{-\frac{2r^2}{6r+1}} \geq 1 - \frac{1}{3r}.$$

Similarly, if H is $3\alpha/4$-far from satisfying R, then $\mathbb{E}[X] \leq 2r+1$ and

$$\mathbb{P}[\mathbf{T}'_R \text{ rejects } H] = \mathbb{P}[X \leq 3r] \geq 1 - 2e^{-\frac{2r^2}{6r+1}} \geq 1 - \frac{1}{3r}.$$

Observe that $\mathbf{T}'_R$ makes at most $(6r+1)g_k(\alpha)$ queries. We now describe our tester $\mathbf{T} = \mathbf{T}(n,\alpha)$ which receives as an input an integer $n \geq k$, a real $\alpha > 0$ and an n-vertex k-graph H.

Run $\mathbf{T}'_R$ on the input (n, H) for every $R \in \mathcal{R}$. If there exists $R \in \mathcal{R}$ such that $\mathbf{T}'_R$ accepts H, then $\mathbf{T}$ also accepts H, and if $\mathbf{T}'_R$ rejects H for all $R \in \mathcal{R}$, then $\mathbf{T}$ also rejects H. (8.1)

Let us show that $\mathbf{T}$ is indeed an (n,α)-tester for $\mathbf{P}$. First, assume that H satisfies $\mathbf{P}$. By (R1)$_{1.3}$, there exists $R \in \mathcal{R}$ such that H is $\alpha/4$-close to R. So $\mathbf{T}'_R$ accepts H with probability at least $1 - 1/(3r)$ and hence $\mathbf{T}$ accepts H with probability at least $1 - 1/(3r) \geq 2/3$.

Now assume that H is α-far from satisfying $\mathbf{P}$. By (R2)$_{1.3}$, the k-graph H is $3\alpha/4$-far from satisfying R for every $R \in \mathcal{R}$. Thus for every $R \in \mathcal{R}$, the tester $\mathbf{T}'_R$ accepts H with probability at most $1/(3r)$. This in turn implies that $\mathbf{T}$ accepts H with probability at most $1/3$.

Therefore, $\mathbf{T}$ is an (n,α)-tester for $\mathbf{P}$, which in particular implies that $\mathbf{P}$ is testable. $\square$

9. Regular approximations of partitions and hypergraphs

The main aim of this section is to prove a strengthening of the partition version (Lemma 3.9) of the regular approximation lemma. As described in Section 3.4, Lemma 3.9 outputs for a given equitable family of partitions $\mathscr{Q}$ another family of partitions $\mathscr{P}$ that refines $\mathscr{Q}$. In Lemma 9.1 $\mathscr{P}$ has the additional feature that it almost refines a further given (arbitrary) family of partitions $\mathscr{O}$. Observe that we cannot hope to refine $\mathscr{O}$ itself, as for example some sets in $\mathscr{O}^{(1)}$ may be very small. We also prove two further tools: Lemma 9.2 allows us to transfer the large scale structure of a hypergraph to another one on a different vertex set and Lemma 9.3 concerns suitable perturbations of a given partition. Lemmas 9.1–9.3 will all be used in the proof of Lemma 10.1.

Lemma 9.1. *For all $k, o \in \mathbb{N} \setminus \{1\}$, $s \in \mathbb{N}$, all $\eta, \nu > 0$, and every function $\varepsilon : \mathbb{N}^{k-1} \to (0,1]$, there are $\mu = \mu_{9.1}(k,o,s,\eta,\nu,\varepsilon) > 0$ and $t = t_{9.1}(k,o,s,\eta,\nu,\varepsilon) \in \mathbb{N}$ and $n_0 = n_{9.1}(k,o,s,\eta,\nu,\varepsilon) \in \mathbb{N}$ such that the following hold. Suppose*

(O1)$_{9.1}$ *V is a set and $|V| = n \geq n_0$,*

(O2)$_{9.1}$ *$\mathscr{O}(k-1, \mathbf{a}^{\mathscr{O}}) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$ is an o-bounded family of partitions on V,*

(O3)$_{9.1}$ *$\mathscr{Q} = \mathscr{Q}(k, \mathbf{a}^{\mathscr{Q}})$ is a $(1/a_1^{\mathscr{Q}}, \mu, \mathbf{a}^{\mathscr{Q}})$-equitable o-bounded family of partitions on V, and*

(O4)$_{9.1}$ *$\mathscr{H}^{(k)} = \{H_1^{(k)}, \dots, H_s^{(k)}\}$ is a partition of $\binom{V}{k}$ so that $\mathscr{H}^{(k)} \prec \mathscr{Q}^{(k)}$.*

Then there exist a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and a partition $\mathscr{G}^{(k)} = \{G_1^{(k)}, \dots, G_s^{(k)}\}$ of $\binom{V}{k}$ satisfying the following for each $j \in [k-1]$ and $i \in [s]$.

(P1)$_{9.1}$ *$\mathscr{P}$ is a t-bounded $(\eta, \varepsilon(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable family of partitions, and $a_j^{\mathscr{Q}}$ divides $a_j^{\mathscr{P}}$,*

(P2)$_{9.1}$ *$\mathscr{P}^{(j)} \prec \mathscr{Q}^{(j)}$ and $\mathscr{P}^{(j)} \prec_\nu \mathscr{O}^{(j)}$,*

(G1)$_{9.1}$ *$G_i^{(k)}$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$,*

(G2)$_{9.1}$ $\sum_{i=1}^s |G_i^{(k)} \triangle H_i^{(k)}| \le \nu\binom{n}{k}$, *and*
(G3)$_{9.1}$ $\mathscr{G}^{(k)} \prec \mathscr{Q}^{(k)}$ *and if* $H_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$ *then* $G_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$.

We believe that Lemma 9.1 is of independent interest and will have additional applications (for example, in addition to the proof of Theorem 1.3 we also apply it to derive Corollary 11.3).

In Lemma 9.1 we may assume without loss of generality that $1/\mu, t, n_0$ are non-decreasing in k, o, s and non-increasing in η, ν.

To prove Lemma 9.1 we proceed by induction on k. In the induction step, we first construct an 'intermediate' family of partitions $\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ which satisfies (P1)$_{9.1}$ and (P2)$_{9.1}$. The partitions $\mathscr{L}^{(1)}, \dots, \mathscr{L}^{(k-1)}$ are constructed via the inductive assumption of Lemma 9.1 (see Claim 2). We then construct a partition $\mathscr{L}^{(k)}$ via appropriate applications of the slicing lemma (see Claim 3). Finally, we apply Lemma 3.9 with $\mathscr{L}_* = \{\mathscr{L}^{(i)}\}_{i=1}^k$ playing the role of $\mathscr{Q}$ to obtain our desired family of partitions $\mathscr{P}$ and construct $G_i^{(k)}$ based on the k-graphs guaranteed by Lemma 3.9.

Proof of Lemma 9.1. First of all, by decreasing the value of η if necessary, we may assume that $\eta < 1/(10k!)$. We may also assume that $\nu \le \eta$.

We use induction on k. For each $k \in \mathbb{N} \setminus \{1\}$, let $L_{9.1}(k)$ be the statement of the lemma. Let $L_{9.1}(1)$ be the following statement (Claim 1).

Claim 1 ($L_{9.1}(1)$)**.** *For all $o, s \in \mathbb{N}$, all $\eta, \nu > 0$, there are $t = t_{9.1}(1, o, s, \nu) := so\lceil 2\nu^{-2}\rceil$ and $n_0 = n_{9.1}(1, o, s, \nu) \in \mathbb{N}$ such that the following hold. Suppose*
(O1)$^1_{9.1}$ *V is a set and $|V| = n \ge n_0$,*
(O2)$^1_{9.1}$ *$\mathscr{Q}^{(1)}$ is an equipartition of V into $a_1^{\mathscr{Q}} \le o$ parts,*
(O3)$^1_{9.1}$ *$\mathscr{H}^{(1)} = \{H_1^{(1)}, \dots, H_s^{(1)}\}$ is a partition of V so that $\mathscr{H}^{(1)} \prec \mathscr{Q}^{(1)}$.*
Then there exists a partition $\mathscr{P}^{(1)}$ of V satisfying the following.
(P1)$^1_{9.1}$ *$\mathscr{P}^{(1)}$ is an equipartition of V into $a_1^{\mathscr{P}} \le t$ parts, and $a_1^{\mathscr{Q}}$ divides $a_1^{\mathscr{P}}$,*
(P2)$^1_{9.1}$ *$\mathscr{P}^{(1)} \prec \mathscr{Q}^{(1)}$ and $\mathscr{P}^{(1)} \prec_{\nu^2} \mathscr{H}^{(1)}$.*

Proof. Write $\mathscr{Q}^{(1)} = \{Q_i^{(1)} : i \in [a_1^{\mathscr{Q}}]\}$. Let $a_1^{\mathscr{P}} := sa_1^{\mathscr{Q}}\lceil 2\nu^{-2}\rceil$, let $m := \min\{|Q_i^{(1)}| : i \in [a_1^{\mathscr{Q}}]\}$, and let $m' := \lfloor |V|/a_1^{\mathscr{P}} \rfloor$. Thus $|Q_i^{(1)}| \in \{m, m+1\}$ for each $i \in [a_1^{\mathscr{Q}}]$. The sets in $\mathscr{P}^{(1)}$ will have size m' or $m'+1$. Note that

$$m' \cdot \frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}} \le \left\lfloor \frac{|V|}{a_1^{\mathscr{P}}} \cdot \frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}} \right\rfloor = m < (m'+1) \cdot \frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}}. \tag{9.1}$$

To obtain $\mathscr{P}^{(1)}$ we further (almost) refine $\mathscr{H}^{(1)}$. For each $i \in [s]$, we define $\ell_i := \lfloor |H_i^{(1)}|/m' \rfloor$. We arbitrarily partition $H_i^{(1)}$ into $\mathscr{L}(i,0), \dots, \mathscr{L}(i,\ell_i)$ such that $|\mathscr{L}(i,r)| = m'$ for all $r \in [\ell_i]$ and $|\mathscr{L}(i,0)| < m'$. For each $j \in [a_1^{\mathscr{Q}}]$, let $\mathscr{L}'(j,0) := \bigcup_{H_i^{(1)} \subseteq Q_j^{(1)}} \mathscr{L}(i,0)$. Let $\ell'_j := \lfloor |\mathscr{L}'(j,0)|/m' \rfloor$. We arbitrarily partition $\mathscr{L}'(j,0)$ into $\mathscr{L}''(j,0), \mathscr{L}'(j,1), \dots, \mathscr{L}'(j,\ell'_j)$ such that $|\mathscr{L}'(j,r)| = m'$ for all $r \in [\ell'_j]$ and $|\mathscr{L}''(j,0)| < m'$. Note that since $\mathscr{H}^{(1)} \prec \mathscr{Q}^{(1)}$, for all $j \in [a_1^{\mathscr{Q}}]$, we have

$$Q_j^{(1)} = \bigcup_{i\colon H_i^{(1)} \subseteq Q_j^{(1)}} H_i^{(1)} = \mathscr{L}''(j,0) \cup \bigcup_{r \in [\ell'_j]} \mathscr{L}'(j,r) \cup \bigcup_{i\colon H_i^{(1)} \subseteq Q_j^{(1)}, r \in [\ell_i]} \mathscr{L}(i,r).$$

Using (9.1) and the fact that $|\mathscr{L}''(j,0)| < m'$, it is easy to see that $|\mathscr{L}''(j,0)| \le a_1^{\mathscr{P}}/a_1^{\mathscr{Q}} = \ell'_j + \sum_{i\colon H_i^{(1)} \subseteq Q_j^{(1)}} \ell_i$.

Hence, by distributing at most one vertex from $\mathscr{L}''(j,0)$ into each of the sets in $\{\mathscr{L}'(j,r) : r \in [\ell'_j]\} \cup \{\mathscr{L}(i,r) : H_i^{(1)} \subseteq Q_j^{(1)}, r \in [\ell_i]\}$, we can obtain a collection

$\{L^{(1)}(j,1),\dots,L^{(1)}(j,a_1^{\mathscr{P}}/a_1^{\mathscr{Q}})\}$ of sets of size m' or $m'+1$, which forms an equipartition of $Q_j^{(1)}$. Let

$$\mathscr{P}^{(1)} := \{L^{(1)}(j,r) : j\in[a_1^{\mathscr{Q}}], r\in[a_1^{\mathscr{P}}/a_1^{\mathscr{Q}}]\}.$$

Then (P1)$^1_{9.1}$ holds. By construction, for each $L^{(1)}\in\mathscr{P}^{(1)}$, either there exists $(i,r)\in[s]\times[\ell_i]$ such that $|L^{(1)}\setminus\mathscr{L}(i,r)|\le 1$ or there exists $(j,r)\in[a_1^{\mathscr{Q}}]\times[\ell'_j]$ such that $|L^{(1)}\setminus\mathscr{L}'(j,r)|\le 1$. In the former case, let $f(L^{(1)}) := H_i^{(1)}$ and the latter case, let $f(L^{(1)})$ be an arbitrary set in $\mathscr{H}^{(1)}$. Then

$$\begin{aligned}\sum_{L^{(1)}\in\mathscr{P}^{(1)}}|L^{(1)}\setminus f(L^{(1)})| &\le |\mathscr{P}^{(1)}| + \sum_{(j,r)\in[a_1^{\mathscr{Q}}]\times[\ell'_j]}|\mathscr{L}'(j,r)| \le a_1^{\mathscr{P}} + \sum_{j\in[a_1^{\mathscr{Q}}],H_i^{(1)}\subseteq Q_j^{(1)}}|\mathscr{L}(i,0)|\\ &\le a_1^{\mathscr{P}} + sa_1^{\mathscr{Q}}m' \le \nu^2|V|.\end{aligned}$$

The final inequality follows since $n\ge n_0$. This and the construction of $\mathscr{P}^{(1)}$ shows that $\mathscr{P}^{(1)}\prec_{\nu^2}\mathscr{H}^{(1)}$ and $\mathscr{P}^{(1)}\prec\mathscr{Q}^{(1)}$. This shows that (P2)$^1_{9.1}$ holds and thus completes the proof of Claim 1. □

So assume that $k\ge 2$ and $L_{9.1}(k-1)$ holds. Let $\mu_{3.9}, t_{3.9}, n_{3.9}$ be the functions defined in Lemma 3.9. By decreasing the value of $\varepsilon(\mathbf{a})$ if necessary, we may assume that for all $\mathbf{a}\in\mathbb{N}^{k-1}$, we have

$$\varepsilon(\mathbf{a}) \ll 1/s, 1/k, 1/\|\mathbf{a}\|_\infty. \tag{9.2}$$

If $k=2$, let $T := o^{4^k+1}\lceil 2\nu^{-2}\rceil$. If $k\ge 3$, for each $\mathbf{a}\in\mathbb{N}^{k-2}$, let $T = T(\mathbf{a},o,\nu) = \max\{\|\mathbf{a}\|_\infty, o^{4^k+1}\lceil 2\nu^{-2}\rceil\}$. If $k\ge 3$, then we also let $\mu' : \mathbb{N}^{k-2}\to(0,1]$ be a function such that for any $\mathbf{a}\in\mathbb{N}^{k-2}$, we have

$$\mu'(\mathbf{a}) \ll \nu, 1/k, 1/o, 1/\|\mathbf{a}\|_\infty \quad\text{and}\quad \mu'(\mathbf{a}) < (\mu_{3.9}(k,T,2sT^{2^k},\eta,\nu/3,\varepsilon^2))^2. \tag{9.3}$$

For all $k\ge 2$, let

$$t_{k-1} := \begin{cases} so^{4^k+1}\lceil 2\nu^{-2}\rceil & \text{if } k=2,\\ \max\{t_{9.1}(k-1,o,o^{4^k},\eta,\nu/3,\mu'), o^{4^k+1}\lceil 2\nu^{-2}\rceil\} & \text{if } k\ge 3,\end{cases} \tag{9.4}$$

which exists by the induction hypothesis. Choose an integer t such that

$$1/t \ll 1/t_{3.9}(k,t_{k-1},2st_{k-1}^{2^k},\eta,\nu/3,\varepsilon^2), 1/t_{k-1}, \tag{9.5}$$

and choose $\mu>0$ such that

$$\mu \ll \begin{cases} 1/t, \mu_{3.9}(k,T,2sT^{2^k},\eta,\nu/3,\varepsilon^2) & \text{if } k=2,\\ 1/t, \mu'(\mathbf{a}), \varepsilon(\mathbf{a}'), \mu_{9.1}(k-1,o,o^{4^k},\eta,\nu/3,\mu') & \text{if } k\ge 3.\\ \text{for any } \mathbf{a}\in[t]^{k-2}, \mathbf{a}'\in[t]^{k-1} & \end{cases} \tag{9.6}$$

Finally, choose an integer n_0 such that

$$1/n_0 \ll \begin{cases} 1/n_{3.9}(k,t_{k-1},2st_{k-1}^{2^k},\eta,\nu/3,\varepsilon^2), 1/n_{9.1}(k-1,o,o^{4^k},\nu), 1/\mu. & \text{if } k=2,\\ 1/n_{3.9}(k,t_{k-1},2st_{k-1}^{2^k},\eta,\nu/3,\varepsilon^2), 1/n_{9.1}(k-1,o,o^{4^k},\eta,\nu/3,\mu'), 1/\mu & \text{if } k\ge 3.\end{cases} \tag{9.7}$$

Suppose $\mathscr{O}(k-1,\mathbf{a}^{\mathscr{O}})$, $\mathscr{Q}(k,\mathbf{a}^{\mathscr{Q}})$ and $\mathscr{H}^{(k)}$ are given (families of) partitions satisfying (O1)$_{9.1}$–(O4)$_{9.1}$ with μ,t,n_0 as defined above. Write

$$\mathscr{O}^{(k-1)} = \{O_1^{(k-1)},\dots,O_{s_O}^{(k-1)}\} \quad\text{and}\quad \mathscr{Q}^{(k-1)} = \{Q_1^{(k-1)},\dots,Q_{s_Q}^{(k-1)}\}.$$

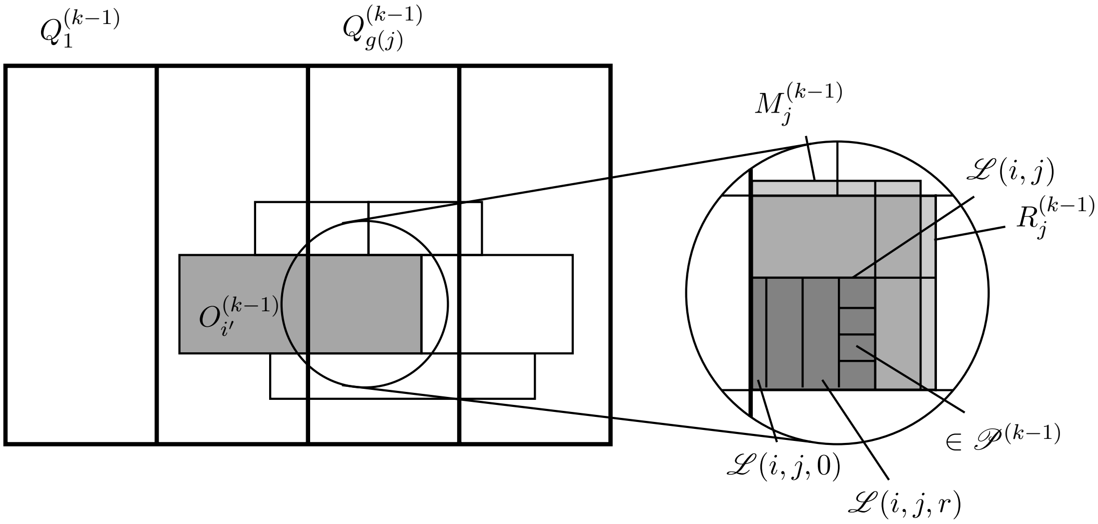

FIGURE 2. An illustration of the cascade of partitions in the proof of Lemma 9.1.

Let

$$O^{(k-1)}_{s_O+1} := \binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{O}^{(1)}), \quad Q^{(k-1)}_{s_Q+1} := \binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{Q}^{(1)}), \text{ and} \tag{9.8}$$
$$\mathscr{R}^{(k-1)} := \{O^{(k-1)}_i \cap Q^{(k-1)}_j : i \in [s_O+1], j \in [s_Q+1]\} \setminus \{\emptyset\}.$$

We also write $\mathscr{R}^{(k-1)} = \{R^{(k-1)}_1, \dots, R^{(k-1)}_{s''}\}$. See Figure 2 for an illustration of the relationship of the different partitions defined in the proof.

Since $\mathscr{O}$ and $\mathscr{Q}$ are both o-bounded, Proposition 3.11(viii) implies that

$$s'' \le \left(o^{2^k}\right)^2 \le o^{4^k}. \tag{9.9}$$

Now our aim is to construct a family of partitions $\mathscr{L}$ as follows.

Claim 2. *There exist* $\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ *and* $\mathbf{a}^{\mathscr{L}}_{\text{long}} = (a^{\mathscr{L}}_1, \dots, a^{\mathscr{L}}_{k-1}) \in [t_{k-1}]^{k-1}$ *satisfying the following for all* $j \in [k-1]$*, where* $\mathbf{a}^{\mathscr{L}} := (a^{\mathscr{L}}_1, \dots, a^{\mathscr{L}}_{k-2})$.

(L*1) $\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ *forms an* $(\eta, \mu'(\mathbf{a}^{\mathscr{L}})^{1/2}, \mathbf{a}^{\mathscr{L}}_{\text{long}})$*-equitable* t_{k-1}*-bounded family of partitions, and* $a^{\mathscr{Q}}_j$ *divides* $a^{\mathscr{L}}_j$*,*
(L*2) $\mathscr{L}^{(j)} \prec \mathscr{Q}^{(j)}$.
(L*3) $\mathscr{L}^{(j)} \prec_{\nu/2} \mathscr{O}^{(j)}$.

Note that if $k = 2$, then the function μ' is not defined, but in this case, $\mu'(\mathbf{a}^{\mathscr{L}})^{1/2}$ plays no role in the definition of an equitable family of partitions (Definition 3.6) since Definition 3.6(iii) is vacuously true.

Proof. First we prove the claim for $k = 2$. We apply $L_{9.1}(1)$ with $\mathscr{Q}^{(1)}, \mathscr{R}^{(1)}, s'', \nu$ playing the roles of $\mathscr{Q}^{(1)}, \mathscr{H}^{(1)}, s, \nu$. (This is possible by (9.7).) Then we obtain a partition $\mathscr{L}^{(1)}$ of V which satisfies $(\text{P1})^1_{9.1}$ and $(\text{P2})^1_{9.1}$. Moreover, (9.4) implies that $\mathscr{L}^{(1)}$ is t_1-bounded, i.e. $\mathbf{a}^{\mathscr{L}}_{\text{long}} \in [t_{k-1}]$. Since $\mathscr{R}^{(1)} \prec \mathscr{O}^{(1)}$, this in turn implies (L*1)–(L*3).

Now we assume $k \ge 3$. First, we apply $L_{9.1}(k-1)$ with the following objects and parameters. (This is possible by (9.4) and (9.6)–(9.9).)

object/parameter	V	$\{\mathscr{O}^{(j)}\}_{j=1}^{k-2}$	$\{\mathscr{Q}^{(i)}\}_{i=1}^{k-1}$	$\mathscr{R}^{(k-1)}$	o	s''	η	$\nu/3$	μ'	t_{k-1}
playing the role of	V	$\mathscr{O}$	$\mathscr{Q}$	$\mathscr{H}^{(k)}$	o	s	η	ν	ε	t

Then we obtain a family of partitions $\mathscr{L} = \mathscr{L}(k-2, \mathbf{a}^{\mathscr{L}})$ and a partition $\mathscr{M}^{(k-1)} = \{M_1^{(k-1)}, \dots, M_{s''}^{(k-1)}\}$ of $\binom{V}{k-1}$ which satisfy the following for each $i \in [s'']$ and $j \in [k-2]$.

(L′1) $\mathscr{L}$ is $(\eta, \mu'(\mathbf{a}^{\mathscr{L}}), \mathbf{a}^{\mathscr{L}})$-equitable and t_{k-1}-bounded, and $a_j^{\mathscr{Q}}$ divides $a_j^{\mathscr{L}}$,
(L′2) $\mathscr{L}^{(j)} \prec \mathscr{Q}^{(j)}$ and $\mathscr{L}^{(j)} \prec_{\nu/3} \mathscr{O}^{(j)}$,
(M′1) $M_i^{(k-1)}$ is perfectly $\mu'(\mathbf{a}^{\mathscr{L}})$-regular with respect to $\mathscr{L}$,
(M′2) $\sum_{i=1}^{s''} |M_i^{(k-1)} \triangle R_i^{(k-1)}| \leq (\nu/3)\binom{n}{k-1}$, and
(M′3) $\mathscr{M}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$ and if $R_i^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$, then $M_i^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$.

Thus $\{\mathscr{L}^{(i)}\}_{i=1}^{k-2}$ satisfies (L*1)–(L*3) for $j \in [k-2]$ and it only remains to construct $\mathscr{L}^{(k-1)}$. Let

$$t' := \max\{\|\mathbf{a}^{\mathscr{L}}\|_\infty, a_{k-1}^{\mathscr{Q}} o^{4^k} \lceil 2\nu^{-2} \rceil\}. \tag{9.10}$$

Thus $\mathscr{L}$ is t'-bounded and $t' \leq \min\{t_{k-1}, T(\mathbf{a}^{\mathscr{L}}, o, \nu)\}$ by (9.4). Write $\hat{\mathscr{L}}^{(k-2)} = \{\hat{L}_1^{(k-2)}, \dots, \hat{L}_{s'}^{(k-2)}\}$. Since $\mathscr{M}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$ by (M′3), for each $j \in [s'']$ there exists a unique $g(j) \in [s_Q + 1]$ such that $M_j^{(k-1)} \subseteq Q_{g(j)}^{(k-1)}$. For each $i \in [s']$, $j \in [s'']$, we define

$$\mathscr{L}(i,j) := \mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap M_j^{(k-1)}. \tag{9.11}$$

For each $i \in [s']$, we define $J(i) := \{j \in [s''] : \mathscr{L}(i,j) \neq \emptyset\}$. Note that $\mathscr{L}(i,j) \subseteq Q_{g(j)}^{(k-1)}$ for all $i \in [s']$.

Subclaim 1. *For each $i \in [s']$ and for each $j \in J(i)$, the $(k-1)$-graph $Q_{g(j)}^{(k-1)}$ is $(\mu'(\mathbf{a}^{\mathscr{L}}), d'_{g(j)})$-regular with respect to $\hat{L}_i^{(k-2)}$, where $d'_{g(j)} \in \{1/a_{k-1}^{\mathscr{Q}}, 1\}$.*

Proof. First, note that since $\mathscr{L} \prec \{\mathscr{Q}^{(j)}\}_{j=1}^{k-2}$, one of the following holds.

(LL1) There exists $\hat{Q}^{(k-2)} \in \hat{\mathscr{Q}}^{(k-2)}$ such that $\hat{L}_i^{(k-2)} \subseteq \hat{Q}^{(k-2)}$.
(LL2) $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \subseteq \binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$.

If (LL1) holds, then by (L′1) and the fact that $\mu \ll \mu'(\mathbf{a}^{\mathscr{L}}) \ll \|\mathbf{a}^{\mathscr{L}}\|_\infty^{-k}$ we can apply Lemma 4.5 twice to conclude that

$$|\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)})| \geq t_{k-1}^{-2^k} |\mathcal{K}_{k-1}(\hat{Q}^{(k-2)})| \overset{(9.5),(9.6)}{\geq} \mu^{1/3} |\mathcal{K}_{k-1}(\hat{Q}^{(k-2)})|.$$

Note that, for each $j \in J(i)$, $Q_{g(j)}^{(k-1)}$ is $(\mu, 1/a_{k-1}^{\mathscr{Q}})$-regular with respect to $\hat{Q}^{(k-2)}$. Together with Lemma 4.1(ii) this implies that $Q_{g(j)}^{(k-1)}$ is $(\mu^{2/3}, 1/a_{k-1}^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-2)}$ for each $j \in J(i)$.

If (LL2) holds, $j \in J(i)$ implies that $M_j^{(k-1)} \nsubseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$. Thus (9.8) means that $M_j^{(k-1)} \cap Q_{s_Q+1}^{(k-1)} \neq \emptyset$, which implies $g(j) = s_Q + 1$. Also (LL2) implies that $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \subseteq Q_{s_Q+1}^{(k-1)}$. Thus $Q_{s_Q+1}^{(k-1)}$ is $(0,1)$-regular with respect to $\hat{L}_i^{(k-2)}$. This completes the proof of Subclaim 1. □

Moreover, (M$'$1) implies that for each $i \in [s']$ and $j \in [s'']$, the $(k-1)$-graph $\mathscr{L}(i,j)$ is $\mu'(\mathbf{a}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$, and thus it is $(2\mu'(\mathbf{a}^{\mathscr{L}}), d(\mathscr{L}(i,j) \mid \hat{L}_i^{(k-2)}))$-regular with respect to $\hat{L}_i^{(k-2)}$. Let

$$a_{k-1}^{\mathscr{L}} := a_{k-1}^{\mathscr{Q}} o^{4^k} \lceil 2\nu^{-2} \rceil, \tag{9.12}$$

and $\mathbf{a}_{\text{long}}^{\mathscr{L}} := (a_1^{\mathscr{L}}, \dots, a_{k-1}^{\mathscr{L}})$. By (9.10), we conclude

$$\|\mathbf{a}_{\text{long}}^{\mathscr{L}}\|_\infty = t' \le \min\{t_{k-1}, T(\mathbf{a}^{\mathscr{L}}, o, \nu)\}. \tag{9.13}$$

Let $\ell_{i,j} := \lfloor d(\mathscr{L}(i,j) \mid \hat{L}_i^{(k-2)}) a_{k-1}^{\mathscr{L}} \rfloor$, so $\ell_{i,j} = 0$ if $j \notin J(i)$. We now apply the slicing lemma (Lemma 4.4) to $\mathscr{L}(i,j)$ for each $i \in [s']$ and $j \in [s'']$. We obtain edge-disjoint $(k-1)$-graphs $\mathscr{L}(i,j,0), \dots, \mathscr{L}(i,j,\ell_{i,j})$ such that

- ($\mathscr{L}$1) $\mathscr{L}(i,j) = \mathscr{L}(i,j,0) \cup \bigcup_{r=1}^{\ell_{i,j}} \mathscr{L}(i,j,r)$,
- ($\mathscr{L}$2) $\mathscr{L}(i,j,r)$ is $(6\mu'(\mathbf{a}^{\mathscr{L}}), 1/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$ for each $r \in [\ell_{i,j}]$, and
- ($\mathscr{L}$3) $\mathscr{L}(i,j,0)$ is $(6\mu'(\mathbf{a}^{\mathscr{L}}), d'_{i,j})$-regular with respect to $\hat{L}_i^{(k-2)}$, where $d'_{i,j} \le 1/a_{k-1}^{\mathscr{L}}$.

Observe that $\mathscr{L}(i,j,0)$ may not have density $1/a_{k-1}^{\mathscr{L}}$. Since we would like to achieve this density for all classes, we now take the union of all these $(k-1)$-graphs and split this union into suitable pieces. For all $i \in [s']$ and $p \in [s_Q + 1]$, let

$$\mathscr{L}'(i,p) := \bigcup_{j\colon g(j)=p} \mathscr{L}(i,j,0) = \left(\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}\right) \setminus \left(\bigcup_{j,r\colon g(j)=p, r \in [\ell_{i,j}]} \mathscr{L}(i,j,r)\right).$$

Note that if $p \notin g(J(i))$, then $\mathscr{L}'(i,p) = \emptyset$. So suppose that $p \in g(J(i))$. Then

$$\text{$\mathscr{L}'(i,p)$ is $(\mu'(\mathbf{a}^{\mathscr{L}})^{2/3}, \ell'_{i,p}/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$ for some $\ell'_{i,p} \in \mathbb{N}$.} \tag{9.14}$$

Indeed Lemma 4.3 (applied with $\sum_{j\colon g(j)=p} \ell_{i,j} \le s'' a_{k-1}^{\mathscr{L}} \le o^{4^k} a_{k-1}^{\mathscr{L}}$ playing the role of s) implies that $\bigcup_{j,r\colon g(j)=p, r\in[\ell_{i,j}]} \mathscr{L}(i,j,r)$ is $(\mu'(\mathbf{a}^{\mathscr{L}})^{3/4}, \sum_{j\colon g(j)=p} \ell_{i,j}/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$. In addition, by Subclaim 1, $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}$ is $(\mu'(\mathbf{a}^{\mathscr{L}}), d'_p)$-regular with respect to $\hat{L}_i^{(k-2)}$ for some $d'_p \in \{1/a_{k-1}^{\mathscr{Q}}, 1\}$. Note that (9.11) implies

$$\bigcup_{j,r\colon g(j)=p, r\in[\ell_{i,j}]} \mathscr{L}(i,j,r) \subseteq \mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}.$$

So Lemma 4.1(iii) implies that (9.14) holds where $\ell'_{i,p} := a_{k-1}^{\mathscr{L}} d'_p - \sum_{j: g(j)=p} \ell_{i,j}$. (Note $\ell'_{i,p} \in \mathbb{N}$ since $d'_p \in \{1/a_{k-1}^{\mathscr{Q}}, 1\}$ and $a_{k-1}^{\mathscr{Q}} \mid a_{k-1}^{\mathscr{L}}$.)

In addition, for all $i \in [s']$ and $p \in g(J(i))$, we have

$$|\mathscr{L}'(i,p)| \overset{(\mathscr{L}3)}{\le} |g^{-1}(p)| \cdot \frac{5}{4a_{k-1}^{\mathscr{L}}} \cdot |\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)})| \overset{(9.9),(9.12)}{\le} \nu^2 |\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}|. \tag{9.15}$$

Again, we apply the slicing lemma (Lemma 4.4), this time to $\mathscr{L}'(i,p)$. By (9.14), we obtain edge-disjoint $(k-1)$-graphs $\mathscr{L}'(i,p,1), \dots, \mathscr{L}'(i,p,\ell'_{i,p})$ such that

- ($\mathscr{L}'$1) $\mathscr{L}'(i,p) = \bigcup_{\ell=1}^{\ell'_{i,p}} \mathscr{L}'(i,p,\ell)$,
- ($\mathscr{L}'$2) $\mathscr{L}'(i,p,\ell)$ is $(\mu'(\mathbf{a}^{\mathscr{L}})^{1/2}, 1/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$ for each $\ell \in [\ell'_{i,p}]$.

Thus for each $i \in [s']$, (9.11), $(\mathscr{L}1)$ and $(\mathscr{L}'1)$ imply that

$$\bigcup_{p \in g(J(i))} \left(\{\mathscr{L}(i,j,r) : j, r \text{ with } g(j) = p, r \in [\ell_{i,j}]\} \cup \{\mathscr{L}'(i,p,1), \dots, \mathscr{L}'(i,p,\ell'_{i,p})\} \right)$$

forms a partition of $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)})$ into $a_{k-1}^{\mathscr{L}}$ edge-disjoint $(k-1)$-graphs, each of which is $(\mu'(\mathbf{a}^{\mathscr{L}})^{1/2}, 1/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$, where the latter follows from $(\mathscr{L}2)$ and $(\mathscr{L}'2)$. We define

$$\mathscr{L}^{(k-1)} := \bigcup_{i \in [s'], j \in J(i)} \{\mathscr{L}(i,j,r) : r \in [\ell_{i,j}]\} \cup \bigcup_{i \in [s'], p \in g(J(i))} \{\mathscr{L}'(i,p,\ell) : \ell \in [\ell'_{i,p}]\}.$$

Then (L*1) follows from (L′1) and the construction of $\mathscr{L}^{(k-1)}$ (t_{k-1}-boundedness follows by (9.13)). Note that for all $i \in [s']$, $j \in [s'']$, $r \in [\ell_{i,j}]$, $p \in [s_Q + 1]$, $\ell \in [\ell'_{i,p}]$, we have $\mathscr{L}(i,j,r) \subseteq Q_{g(j)}^{(k-1)}$ and $\mathscr{L}'(i,p,\ell) \subseteq Q_p^{(k-1)}$, and so (L*2) holds.

Subclaim 2. $\mathscr{L}^{(k-1)} \prec_{\nu/2} \mathscr{O}^{(k-1)}$.

Proof. To prove the subclaim, we define a suitable function $f_{k-1} : \mathscr{L}^{(k-1)} \to \mathscr{O}^{(k-1)}$. For each $j \in [s'']$, let $h(j) \in [s_O + 1]$ be the index such that $R_j^{(k-1)} = O_{h(j)}^{(k-1)} \cap Q_p^{(k-1)}$ for some $p \in [s_Q + 1]$. For each $i \in [s']$, $j \in J(i)$, $r \in [\ell_{i,j}]$, $\ell \in [\ell'_{i,g(j)}]$, let

$$f_{k-1}(\mathscr{L}(i,j,r)) := O_{h(j)}^{(k-1)} \text{ and } f_{k-1}(\mathscr{L}'(i,g(j),\ell)) := O_{h(j)}^{(k-1)}.$$

For fixed $j \in [s'']$, (9.11) and $(\mathscr{L}1)$ imply that

$$\bigcup_{i \in [s'], r \in [\ell_{i,j}]} \mathscr{L}(i,j,r) \subseteq M_j^{(k-1)}. \tag{9.16}$$

Hence

$$\begin{aligned}
\sum_{L^{(k-1)} \in \mathscr{L}^{(k-1)}} |L^{(k-1)} \setminus f_{k-1}(L^{(k-1)})| &\leq \sum_{i,j,r} |\mathscr{L}(i,j,r) \setminus f_{k-1}(\mathscr{L}(i,j,r))| + \sum_{i,p,\ell} |\mathscr{L}'(i,p,\ell)| \\
&\overset{(9.16),(\mathscr{L}'1)}{\leq} \sum_{j \in [s'']} |M_j^{(k-1)} \setminus O_{h(j)}^{(k-1)}| + \sum_{i,p} |\mathscr{L}'(i,p)| \\
&\overset{(9.15)}{\leq} \sum_{j \in [s'']} |M_j^{(k-1)} \setminus R_j^{(k-1)}| + \nu^2 \sum_{i,p} |\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}| \\
&\overset{(\mathrm{M}'2)}{\leq} \frac{\nu}{3}\binom{n}{k-1} + \nu^2 \binom{n}{k-1} \leq \frac{2\nu}{5}\binom{n}{k-1}.
\end{aligned}$$

The fact that $a_1^{\mathscr{L}} \geq \eta^{-1}$ and (3.5) together imply that $|\mathcal{K}_{k-1}(\mathscr{L}^{(1)})| \geq \frac{4}{5}\binom{n}{k-1}$, so the subclaim follows. □

This shows that (L*3) holds and completes the proof of Claim 2. □

Note that $\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ obtained in Claim 2 naturally defines $\hat{\mathscr{L}}^{(k-1)}$. Write $\hat{\mathscr{L}}^{(k-1)} = \{\hat{L}_1^{(k-1)}, \dots, \hat{L}_{\hat{s}_{\mathscr{L}}}^{(k-1)}\}$. We now construct $\mathscr{L}^{(k)}$ by refining $\mathcal{K}_k(\hat{L}_i^{(k-1)})$ for all $i \in [\hat{s}_{\mathscr{L}}]$.

Claim 3. *For each $i \in [\hat{s}_{\mathscr{L}}]$, there is a partition $\{L^{(k)}(i,1), \dots, L^{(k)}(i, a_k^{\mathscr{Q}})\}$ of $\mathcal{K}_k(\hat{L}_i^{(k-1)})$ such that $L^{(k)}(i,r)$ is $(\mu^{1/2}, 1/a_k^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-1)}$ for each $r \in [a_k^{\mathscr{Q}}]$. Moreover, we can ensure that $\{L(i,r) : i \in [\hat{s}_{\mathscr{L}}], r \in [a_k^{\mathscr{Q}}]\} \prec \mathscr{Q}^{(k)}$.*

Proof. Since $\mathscr{L}^{(1)} \prec \mathscr{Q}^{(1)}$, for each $\hat{L}_i^{(k-1)} \in \hat{\mathscr{L}}^{(k-1)}$, either $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$ or $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$.

Suppose first that $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$. As $\mathscr{L}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$, there exists a (unique) $\hat{Q}_j^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$ such that $\hat{L}_i^{(k-1)} \subseteq \hat{Q}_j^{(k-1)}$. In addition, there are exactly $a_k^{\mathscr{Q}}$ many k-graphs $Q^{(k)}(j,1), \dots, Q^{(k)}(j, a_k^{\mathscr{Q}})$ in $\mathscr{Q}^{(k)}$ that partition $\mathcal{K}_k(\hat{Q}_j^{(k-1)})$. For each $r \in [a_k^{\mathscr{Q}}]$, let

$$L^{(k)}(i,r) := Q^{(k)}(j,r) \cap \mathcal{K}_k(\hat{L}_i^{(k-1)}).$$

Hence $\{L^{(k)}(i,1), \dots, L^{(k)}(i, a_k^{\mathscr{Q}})\}$ forms a partition of $\mathcal{K}_k(\hat{L}_i^{(k-1)})$. We can now apply Lemma 4.5 twice and use (L*1) as well as $(\mathrm{O3})_{9.1}$ to obtain that

$$|\mathcal{K}_k(\hat{L}_i^{(k-1)})| \geq t_{k-1}^{-2^k} |\mathcal{K}_k(\hat{Q}_j^{(k-1)})|.$$

Thus Lemma 4.1(ii), $(\mathrm{O3})_{9.1}$ and (9.6) imply that $L^{(k)}(i,r)$ is $(\mu^{1/2}, 1/a_k^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-1)}$ for each $r \in [a_k^{\mathscr{Q}}]$.

Suppose next that we have $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$. We apply the slicing lemma (Lemma 4.4) with $\mathcal{K}_k(\hat{L}_i^{(k-1)}), \hat{L}_i^{(k-1)}, 1, 1/a_k^{\mathscr{Q}}$ playing the roles of $H^{(k)}, H^{(k-1)}, d, p_i$ respectively. We obtain a partition $\{L^{(k)}(i,1), \dots, L^{(k)}(i, a_k^{\mathscr{Q}})\}$ of $\mathcal{K}_k(\hat{L}_i^{(k-1)})$ such that $L^{(k)}(i,r)$ is $(\mu^{1/2}, 1/a_k^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-1)}$ for each $r \in [a_k^{\mathscr{Q}}]$.

The moreover part of Claim 3 is immediate from the construction in both cases. $\square$

Let

$$\mathscr{L}^{(k)} := \{L^{(k)}(i,r) : i \in [\hat{s}_{\mathscr{L}}], r \in [a_k^{\mathscr{Q}}]\}, \ \ \mathbf{a}^{\mathscr{L}_*} := (a_1^{\mathscr{L}}, \dots, a_{k-1}^{\mathscr{L}}, a_k^{\mathscr{Q}}) \ \text{ and } \ \mathscr{L}_* := \{\mathscr{L}^{(i)}\}_{i=1}^k,$$

$$\mathscr{J}^{(k)} := \Big(\{H_i^{(k)} \cap L_*^{(k)} : i \in [s], L_*^{(k)} \in \mathscr{L}^{(k)}\} \cup \{H_i^{(k)} \setminus \mathcal{K}_k(\mathscr{L}^{(1)}) : i \in [s]\} \Big) \setminus \{\emptyset\}.$$

$$\{J_1^{(k)}, \dots, J_{s_J}^{(k)}\} := \mathscr{J}^{(k)}, \text{ and } j'(i) := \{j' \in [s_J] : J_{j'}^{(k)} \subseteq H_i^{(k)}\} \text{ for each } i \in [s]. \quad (9.17)$$

Then $\mathscr{L}^{(k)} \prec \mathscr{Q}^{(k)}$. Let $\mu_* := \mu^{1/2}$ if $k = 2$ and $\mu_* := \mu'(\mathbf{a}^{\mathscr{L}})^{1/2} > \mu^{1/2}$ if $k \geq 3$. Then by (L*1), Claim 3, and (9.13), we have

$$\mathscr{L}_* \textit{ is a } (1/a_1^{\mathscr{L}}, \mu_*, \mathbf{a}^{\mathscr{L}_*})\textit{-equitable } t'\textit{-bounded family of partitions.} \quad (9.18)$$

Moreover, $s_J \leq 2st'^{2^k}$ by (9.13) and Proposition 3.11(viii). Also $\mathscr{J}^{(k)} \prec \mathscr{H}^{(k)}$ and $\{j'(1), \dots, j'(s)\}$ forms a partition of $[s_J]$.

Our next aim is to apply Lemma 3.9 with the following objects and parameters.

object/parameter	$\mathscr{L}_*$	$\mathscr{J}^{(k)}$	t	t'	s_J	η	$\nu/3$	ε^2	μ^*
playing the role of	$\mathscr{Q}$	$\mathscr{H}^{(k)}$	t	o	s	η	ν	ε	μ

Indeed, we can apply Lemma 3.9: (9.7) ensures that $(\mathrm{O1})_{3.9}$ holds; by (9.3), (9.6), (9.13) and (9.18), $\mathscr{L}_*$ satisfies $(\mathrm{O2})_{3.9}$. By construction, $\mathscr{J}^{(k)} \prec \mathscr{L}_*^{(k)}$, thus $(\mathrm{O3})_{3.9}$ also holds. We obtain $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and $\mathscr{G}'^{(k)} = \{G_1'^{(k)}, \dots, G_{s_J}'^{(k)}\}$ satisfying the following.

- (P′1) $\mathscr{P}$ is $(\eta, \varepsilon(\mathbf{a}^{\mathscr{P}})^2, \mathbf{a}^{\mathscr{P}})$-equitable and t-bounded, and $a_j^{\mathscr{L}_*}$ divides $a_j^{\mathscr{P}}$ for all $j \in [k-1]$,
- (P′2) for each $j \in [k-1]$, $\mathscr{P}^{(j)} \prec \mathscr{L}^{(j)}$,
- (P′3) $G_i'^{(k)}$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})^2$-regular with respect to $\mathscr{P}$ for all $i \in [s_J]$,
- (P′4) $\sum_{i \in [s_J]} |G_i'^{(k)} \triangle J_i^{(k)}| \leq (\nu/3)\binom{n}{k}$, and
- (P′5) $\mathscr{G}'^{(k)} \prec \mathscr{L}^{(k)}$ and if $J_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{L}^{(1)})$, then $G_i'^{(k)} \subseteq \mathcal{K}_k(\mathscr{L}^{(1)})$.

Here we obtain (P′1) from (9.5). In addition, we also have the following.

$$Q^{(k)}_{i'} \textit{ is perfectly } \varepsilon(\mathbf{a}^{\mathscr{P}})^2\textit{-regular with respect to } \mathscr{P} \textit{ for all } i' \in [s_Q+1]. \tag{9.19}$$

Indeed, $\mathscr{P}^{(k-1)} \prec \mathscr{L}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$. Thus for each $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$, either there exists unique $\hat{Q}^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$ such that $\mathcal{K}_k(\hat{P}^{(k-1)}) \subseteq \mathcal{K}_k(\hat{Q}^{(k-1)})$, or $\mathcal{K}_k(\hat{P}^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$.

In the former case, by two applications of Lemma 4.5 and (P′1), it is easy to see that

$$|\mathcal{K}_k(\hat{P}^{(k-1)})| \geq t^{-2^k}|\mathcal{K}_k(\hat{Q}^{(k-1)})|.$$

Thus $(\mathrm{O3})_{9.1}$ with Lemma 4.1(ii) and (9.6) implies that $Q^{(k)}_{i'}$ is $\varepsilon(\mathbf{a}^{\mathscr{P}})^2$-regular with respect to $\hat{P}^{(k-1)}$.

Now suppose that $\mathcal{K}_k(\hat{P}^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$. If $i' \in [s_Q]$, then we have $Q^{(k)}_{i'} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$. Thus $Q^{(k)}_{i'} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) = \emptyset$, and $Q^{(k)}_{i'}$ is $(\varepsilon(\mathbf{a}^{\mathscr{P}})^2, 0)$-regular with respect to $\hat{P}^{(k-1)}$. If $i' = s_Q+1$, then $Q^{(k)}_{i'} = \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$, thus $Q^{(k)}_{i'}$ is $(\varepsilon(\mathbf{a}^{\mathscr{P}})^2, 1)$-regular with respect to $\hat{P}^{(k-1)}$. Thus we have (9.19).

It is easy to see that (P′1) and (L*1) imply $(\mathrm{P1})_{9.1}$. The statements (P′2), (3.1) together with (L*2) and (L*3) imply $(\mathrm{P2})_{9.1}$.

As $\mathscr{L}^{(k)} \prec \mathscr{Q}^{(k)}$ and (P′5) holds, we obtain $\mathscr{G}'^{(k)} \prec \mathscr{Q}^{(k)}$. For each $i' \in [s_Q+1]$, let

$$g'(i') := \{j' \in [s_J] : G'^{(k)}_{j'} \subseteq Q^{(k)}_{i'}\}, \text{ and } \; h'(i') := \{i \in [s] : H^{(k)}_i \subseteq Q^{(k)}_{i'}\}.$$

Note that $\{g'(1), \dots g'(s_Q+1)\}$ forms a partition of $[s_J]$. Also by $(\mathrm{O4})_{9.1}$, $\{h'(1), \dots h'(s_Q+1)\}$ forms a partition of $[s]$. Moreover, both $g'(i')$ and $h'(i')$ are non-empty sets. For each $i' \in [s_Q+1]$, we arbitrarily choose a representative $h'_{i'} \in h'(i')$.

Recall that $j'(i)$ was defined in (9.17). For each $i' \in [s_Q+1]$ and $i \in h'(i') \setminus \{h'_{i'}\}$, we define

$$G^{(k)}_i := \bigcup_{j' \in j'(i) \cap g'(i')} G'^{(k)}_{j'} \quad \text{and} \quad G^{(k)}_{h'_{i'}} := Q^{(k)}_{i'} \setminus \bigcup_{\ell \in h'(i') \setminus h'_{i'}} G^{(k)}_\ell.$$

Let

$$\mathscr{G}^{(k)} := \{G^{(k)}_i : i \in [s]\}.$$

By the construction, $\mathscr{G}^{(k)}$ forms a partition of $\binom{V}{k}$. Moreover, we have the following:

$$\textit{Suppose that } i' \in [s_Q+1],\ i \in h'(i') \textit{ and } j' \in j'(i) \cap g'(i'). \textit{ Then } G'^{(k)}_{j'} \subseteq G^{(k)}_i. \tag{9.20}$$

Note that the construction of $\mathscr{G}^{(k)}$, (P′3), Lemma 4.3, (9.2) and (9.13) together imply that for each $i' \in [s_Q+1]$ and $i \in h'(i') \setminus \{h'_{i'}\}$, $G^{(k)}_i$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})^{3/2}$-regular with respect to $\mathscr{P}$. In particular, together with (9.19), Lemma 4.3 and Lemma 4.1(iii), this implies that for each $i' \in [s_Q+1]$, $G^{(k)}_{h'_{i'}}$ is also perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$. Thus we obtain $(\mathrm{G1})_{9.1}$.

By the definition of $G^{(k)}_i$, we conclude that for every $i \in [s]$, there exists $i' \in [s_Q+1]$ such that $G^{(k)}_i \subseteq Q^{(k)}_{i'}$. Thus $\mathscr{G}^{(k)} \prec \mathscr{Q}^{(k)}$. Moreover, if $H^{(k)}_i \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$, then $i \in h'(i')$ for some $i' \neq s_Q+1$. Hence in this case $G^{(k)}_i \subseteq Q^{(k)}_{i'}$ with $i' \neq s_Q+1$, and so $G^{(k)}_i \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$. Thus $(\mathrm{G3})_{9.1}$ holds.

We now verify $(\mathrm{G2})_{9.1}$. Consider any edge $e \in G^{(k)}_i \setminus H^{(k)}_i$ for some $i \in [s]$. We claim that

$$e \in \bigcup_{j' \in [s_J]} J^{(k)}_{j'} \setminus G'^{(k)}_{j'}. \tag{9.21}$$

To prove (9.21) note that since $\mathscr{J}^{(k)}$ is a partition of $\binom{V}{k}$, there exists $j' \in [s_J]$ such that $e \in J^{(k)}_{j'}$. So (9.21) holds if $e \notin G'^{(k)}_{j'}$. Thus assume for a contradiction that $e \in G'^{(k)}_{j'}$. Let $i_* \in [s]$ be the index such that $j' \in j'(i_*)$. Then $J^{(k)}_{j'} \subseteq H^{(k)}_{i_*}$.

Since $\{h'(1), \dots, h'(s_Q+1)\}$ forms a partition of $[s]$, there exists $i' \in [s_Q+1]$ such that $i_* \in h'(i')$. Thus $e \in J^{(k)}_{j'} \subseteq H^{(k)}_{i_*} \subseteq Q^{(k)}_{i'}$. Hence $Q^{(k)}_{i'} \cap G'^{(k)}_{j'} \neq \emptyset$. Since $\mathscr{G}'^{(k)} \prec \mathscr{Q}^{(k)}$, this implies that $G'^{(k)}_{j'} \subseteq Q^{(k)}_{i'}$ and so $j' \in g'(i')$. Consequently, we have $i_* \in h'(i')$ and $j' \in j'(i_*) \cap g'(i')$. This together with (9.20) implies that $e \in G'^{(k)}_{j'} \subseteq G^{(k)}_{i_*}$. Since $\mathscr{G}^{(k)}$ is a partition of $\binom{V}{k}$, this implies that $i = i_*$. But then $e \in H^{(k)}_{i_*} = H^{(k)}_i$, a contradiction. This proves (9.21).

Then

$$\sum_{i \in [s]} |G^{(k)}_i \setminus H^{(k)}_i| \overset{(9.21)}{\leq} \sum_{j' \in [s_J]} |J'^{(k)}_{j'} \setminus G'^{(k)}_{j'}|. \tag{9.22}$$

Since all of $\mathscr{H}^{(k)}, \mathscr{G}^{(k)}$, $\mathscr{J}^{(k)}$ and $\mathscr{G}'^{(k)}$ are partitions of $\binom{V}{k}$, we obtain

$$\sum_{i \in [s]} |G^{(k)}_i \triangle H^{(k)}_i| = 2 \sum_{i \in [s]} |G^{(k)}_i \setminus H^{(k)}_i|, \text{ and } \sum_{i \in [s_J]} |G'^{(k)}_i \triangle J^{(k)}_i| = 2 \sum_{i \in [s_J]} |J^{(k)}_i \setminus G'^{(k)}_i|.$$

Thus we conclude

$$\sum_{i \in [s]} |G^{(k)}_i \triangle H^{(k)}_i| = 2 \sum_{i \in [s]} |G^{(k)}_i \setminus H^{(k)}_i| \overset{(9.22)}{\leq} 2 \sum_{j' \in [s_J]} |J'^{(k)}_{j'} \setminus G'^{(k)}_{j'}| \overset{(\mathrm{P}'4)}{\leq} \nu \binom{n}{k}.$$

Thus (G2)$_{9.1}$ holds. □

Suppose we are given a $(k-1)$-graph H on a vertex set V. In the next lemma we apply Lemma 3.9 to show that, given a different vertex set V', there exists another $(k-1)$-graph F on V' whose large scale structure is very close to that of H.

Lemma 9.2. *Suppose* $0 < 1/m, 1/n \ll \varepsilon \ll \nu, 1/o, 1/k \leq 1$ *and* $k, o \in \mathbb{N} \setminus \{1\}$. *Suppose* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *and* $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ *are both o-bounded* $(1/a_1, \varepsilon, \mathbf{a})$*-equitable families of partitions of* V *and* V' *respectively with* $|V| = n$ *and* $|V'| = m$. *Suppose that* $H^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$. *Then there exists a* $(k-1)$*-graph* $G^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$ *on* V *and a* $(k-1)$*-graph* $F^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$ *on* V' *such that*

(F1)$_{9.2}$ $|H^{(k-1)} \triangle G^{(k-1)}| \leq \nu \binom{n}{k-1}$,

(F2)$_{9.2}$ $d(\mathcal{K}_k(G^{(k-1)}) \mid \hat{P}^{(k-1)}(\hat{\mathbf{z}})) = d(\mathcal{K}_k(F^{(k-1)}) \mid \hat{Q}^{(k-1)}(\hat{\mathbf{z}})) \pm \nu$ *for each* $\hat{\mathbf{z}} \in \hat{A}(k, k-1, \mathbf{a})$.

To prove Lemma 9.2, we first apply Lemma 3.9 to obtain a family of partitions $\mathscr{R} = \mathscr{R}(k-2, \mathbf{a}^{\mathscr{R}})$ and a k-graph $G^{(k-1)}$ as in (F1)$_{9.2}$. We then 'project' $\mathscr{R}$ onto V' (so that it refines $\mathscr{Q}$). This results in a partition $\mathscr{L}$. We then apply the slicing lemma to construct $F^{(k-1)}$ which respects $\mathscr{L}$ (and in particular has the appropriate densities).

Proof of Lemma 9.2. First suppose $k = 2$. Then $H^{(1)} \subseteq V$. Let $G^{(1)} := H^{(1)}$. Thus (F1)$_{9.2}$ holds. Recall that for each $b \in [a_1]$, the vertex sets $P^{(1)}(b, b)$ and $Q^{(1)}(b, b)$ denote the b-th parts in $\mathscr{P}^{(1)}$ and $\mathscr{Q}^{(1)}$, respectively. For each $b \in [a_1]$, let $F^{(1)}(b, b)$ be a subset of $Q^{(1)}(b, b)$ with

$$|F^{(1)}(b, b)| = \left\lfloor \frac{m}{n} |H^{(1)} \cap P^{(1)}(b, b)| \right\rfloor$$

and let $F^{(1)} := \bigcup_{b\in[a_1]} F^{(1)}(b,b)$. For each $\mathbf{z} = (\alpha_1,\alpha_2) \in \hat{A}(2,1,\mathbf{a})$, we have

$$\begin{aligned} d(\mathcal{K}_2(F^{(1)}) \mid \hat{Q}^{(1)}(\hat{\mathbf{z}})) &= \frac{|F^{(1)}(\alpha_1,\alpha_1)||F^{(1)}(\alpha_2,\alpha_2)|}{(m/a_1 \pm 1)^2} \\ &= \frac{(|H^{(1)} \cap P^{(1)}(\alpha_1,\alpha_1)| \pm n/m)(|H^{(1)} \cap P^{(1)}(\alpha_2,\alpha_2)| \pm n/m)}{(n/a_1 \pm n/m)^2} \\ &= d(\mathcal{K}_2(G^{(1)}) \mid \hat{P}^{(1)}(\hat{\mathbf{z}})) \pm \nu. \end{aligned}$$

Thus (F2)$_{9.2}$ holds.

Now we show the lemma for $k \geq 3$. Let η' be a constant such that $\varepsilon \ll \eta' \ll \nu, 1/o, 1/k$. Let $\varepsilon' : \mathbb{N}^{k-2} \to (0,1]$ be a function such that

$$\varepsilon'(\mathbf{b}) \ll \nu, 1/o, 1/k, 1/\|\mathbf{b}\|_\infty \text{ for all } \mathbf{b} \in \mathbb{N}^{k-2}. \tag{9.23}$$

Let $t := t_{3.9}(k-1, o, o^{4^k}, \eta', \nu, \varepsilon')$. Since $\varepsilon \ll \nu, 1/o, 1/k, \eta'$, we may assume that

$$0 < \varepsilon \ll \mu_{3.9}(k-1, o, o^{4^k}, \eta', \nu, \varepsilon'), 1/t, \min\{\varepsilon'(\mathbf{b}) : \mathbf{b} \in [t]^{k-2}\}, \tag{9.24}$$

and we may assume that $n, m > n_0 := n_{3.9}(k-1, o, o^{4^k}, \eta', \nu, \varepsilon')$. Let

$$\begin{aligned} \{H_1^{(k-1)}, \dots, H_s^{(k-1)}\} &:= \{P^{(k-1)} \cap H^{(k-1)} : P^{(k-1)} \in \mathscr{P}^{(k-1)}\} \setminus \{\emptyset\}, \\ \{H_{s+1}^{(k-1)}, \dots, H_{s+s'}^{(k-1)}\} &:= \left(\{P^{(k-1)} \setminus H^{(k-1)} : P^{(k-1)} \in \mathscr{P}^{(k-1)}\} \cup \left\{\binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{P}^{(1)})\right\}\right) \setminus \{\emptyset\}, \\ \mathscr{H}^{(k-1)} &:= \{H_1^{(k-1)}, \dots, H_{s+s'}^{(k-1)}\}. \end{aligned}$$

Hence $\mathscr{H}^{(k-1)}$ is a partition of $\binom{V}{k-1}$ such that $\mathscr{H}^{(k-1)} \prec \mathscr{P}^{(k-1)}$ and $s+s' \leq 2o^{2^k}+1 \leq o^{4^k}$ by Proposition 3.11(viii). We first construct $G^{(k-1)}$. By (9.24), we may apply Lemma 3.9 with the following objects and parameters.

object/parameter	$\mathscr{P}$	$\mathscr{H}^{(k-1)}$	o	$s+s'$	η'	ν	ε'	$k-1$	t
playing the role of	$\mathscr{Q}$	$\mathscr{H}^{(k)}$	o	s	η	ν	ε	k	t

We obtain $\mathscr{R} = \mathscr{R}(k-2, \mathbf{a}^{\mathscr{R}})$ and $\mathscr{G}^{(k-1)} = \{G_1^{(k-1)}, \dots, G_{s+s'}^{(k-1)}\}$ satisfying the following.

(R1)$_{9.2}$ $\mathscr{R}$ is $(\eta', \varepsilon'(\mathbf{a}^{\mathscr{R}}), \mathbf{a}^{\mathscr{R}})$-equitable and t-bounded and for each $j \in [k-2]$, a_j divides $a_j^{\mathscr{R}}$,

(R2)$_{9.2}$ $\{\mathscr{R}^{(j)}\}_{j=1}^{k-2} = \mathscr{R} \prec \{\mathscr{P}^{(j)}\}_{j=1}^{k-2}$,

(R3)$_{9.2}$ for each $i \in [s+s']$, $G_i^{(k-1)}$ is perfectly $\varepsilon'(\mathbf{a}^{\mathscr{R}})$-regular with respect to $\mathscr{R}$,

(R4)$_{9.2}$ $\sum_{i=1}^{s+s'} |G_i^{(k-1)} \triangle H_i^{(k-1)}| \leq \nu\binom{n}{k-1}$, and

(R5)$_{9.2}$ $\mathscr{G}^{(k-1)} \prec \mathscr{P}^{(k-1)}$, and if $H_i^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$, then $G_i^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$.

Observe that $a_1^{\mathscr{R}} > \eta'^{-1}$ by (R1)$_{9.2}$. Thus

$$1/a_1^{\mathscr{R}} \ll \nu, 1/o, 1/k. \tag{9.25}$$

Let $G^{(k-1)} := \bigcup_{i=1}^{s} G_i^{(k-1)}$. Then (F1)$_{9.2}$ holds and $G^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$.

Next we show how to construct $F^{(k-1)}$. To this end we define a family of partitions $\mathscr{L}$ on V' which has the same number of parts as $\mathscr{R}$. We apply Lemma 4.13 with $\{\mathscr{Q}^{(j)}\}_{j=1}^{k-2}, \varepsilon, \mathbf{a}^{\mathscr{R}}$ playing the roles of $\mathscr{P}, \varepsilon, \mathbf{b}$ to obtain $\mathscr{L}$ so that

$\mathscr{L} = \mathscr{L}(k-2, \mathbf{a}^{\mathscr{R}})$ is an $(\eta', \varepsilon^{1/3}, \mathbf{a}^{\mathscr{R}})$-equitable family of partitions such that $\mathscr{L} \prec \{\mathscr{Q}\}_{j=1}^{k-2}$.

Let $\mathbf{a}' := (a_1, \dots, a_{k-2})$, where $\mathbf{a} := (a_1, \dots, a_{k-1})$. By taking an appropriate $\mathbf{a}^{\mathscr{R}}$-labelling for $\mathscr{L}$, we may also assume that for each $\hat{\mathbf{x}} \in \hat{A}(k-1, k-2, \mathbf{a}')$,

$$\begin{aligned} A(\hat{\mathbf{x}}) &:= \{\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}}) : \hat{R}^{(k-2)}(\hat{\mathbf{y}}) \subseteq \hat{P}^{(k-2)}(\hat{\mathbf{x}})\} \\ &= \{\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}}) : \hat{L}^{(k-2)}(\hat{\mathbf{y}}) \subseteq \hat{Q}^{(k-2)}(\hat{\mathbf{x}})\}. \end{aligned} \tag{9.26}$$

For each $\hat{\mathbf{x}} \in \hat{A}(k-1, k-2, \mathbf{a}')$ and $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$, Lemma 4.5 implies that

$$\begin{aligned} |\mathcal{K}_{k-1}(\hat{L}^{(k-2)}(\hat{\mathbf{y}}))| &\geq (1-\nu) \prod_{j=1}^{k-1} (a_j^{\mathscr{R}})^{-\binom{k-1}{j}} m^{k-1} \overset{(9.24)}{\geq} \varepsilon^{1/3}(1+\nu) \prod_{j=1}^{k-1} (a_j^{\mathscr{Q}})^{-\binom{k-1}{j}} m^{k-1} \\ &\geq \varepsilon^{1/3} |\mathcal{K}_{k-1}(\hat{Q}^{(k-2)}(\hat{\mathbf{x}}))|. \end{aligned} \tag{9.27}$$

We would like the relative densities of $F^{(k-1)}$ (with respect to the polyads of $\mathscr{L}$) to reflect the relative densities of $G^{(k-1)}$ (with respect to the polyads of $\mathscr{R}$). For this, we first determine the relative densities of $G^{(k-1)}$ (see (9.32)). For each $\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}})$, $b \in [a_{k-1}]$, and the unique vector $\hat{\mathbf{x}}$ with $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$, we define

$$\begin{aligned} Q_*^{(k-1)}(\hat{\mathbf{y}}, b) &:= Q^{(k-1)}(\hat{\mathbf{x}}, b) \cap \mathcal{K}_{k-1}(\hat{L}^{(k-2)}(\hat{\mathbf{y}})), \\ P_*^{(k-1)}(\hat{\mathbf{y}}, b) &:= P^{(k-1)}(\hat{\mathbf{x}}, b) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})). \end{aligned} \tag{9.28}$$

Since each $Q^{(k-1)}(\hat{\mathbf{x}}, b) \in \mathscr{Q}^{(k-1)}$ is $(\varepsilon, 1/a_{k-1})$-regular with respect to $\hat{Q}^{(k-2)}(\hat{\mathbf{x}})$ for each $b \in [a_{k-1}]$, Lemma 4.1(ii) and (9.27) with the definition of $A(\hat{\mathbf{x}})$ imply that

$$Q_*^{(k-1)}(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon^{2/3}, 1/a_{k-1})\textit{-regular with respect to } \hat{L}^{(k-2)}(\hat{\mathbf{y}}). \tag{9.29}$$

Similarly,

$$P_*^{(k-1)}(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon^{2/3}, 1/a_{k-1})\textit{-regular with respect to } \hat{R}^{(k-2)}(\hat{\mathbf{y}}). \tag{9.30}$$

For each $\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}})$ and $b \in [a_{k-1}]$, let

$$G(\hat{\mathbf{y}}, b) := G^{(k-1)} \cap P_*^{(k-1)}(\hat{\mathbf{y}}, b). \tag{9.31}$$

Thus $G(\hat{\mathbf{y}}, b) \subseteq \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}}))$ by (9.28). Since $\mathscr{G}^{(k-1)} \prec \mathscr{P}^{(k-1)}$ by $(\mathrm{R5})_{9.2}$, we know that $G(\hat{\mathbf{y}}, b)$ is the union of some $(k-1)$-graphs in $\{G_i^{(k-1)} \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) : i \in [s]\}$. Thus $(\mathrm{R3})_{9.2}$ and Lemma 4.3 with the fact that $\varepsilon'(\mathbf{a}^{\mathscr{R}}) \ll 1/s$ imply that $G(\hat{\mathbf{y}}, b)$ is $\varepsilon'(\mathbf{a}^{\mathscr{R}})^{2/3}$-regular with respect to $\hat{R}^{(k-2)}(\hat{\mathbf{y}})$. As $G(\hat{\mathbf{y}}, b) \subseteq P_*^{(k-1)}(\hat{\mathbf{y}}, b)$ and (9.30) holds, there exists a number $d(\hat{\mathbf{y}}, b) \in [0, 1/a_{k-1}]$ such that

$$G(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d(\hat{\mathbf{y}}, b))\textit{-regular with respect to } \hat{R}^{(k-2)}(\hat{\mathbf{y}}). \tag{9.32}$$

Now we use the values $d(\hat{\mathbf{y}}, b)$ to construct $F^{(k-1)}$. We apply the slicing lemma (Lemma 4.4) with the following objects and parameters.

object/parameter	$Q_*^{(k-1)}(\hat{\mathbf{y}}, b)$	$\hat{L}^{(k-2)}(\hat{\mathbf{y}})$	$1/a_{k-1}$	$\max\{d(\hat{\mathbf{y}}, b)a_{k-1}, 1 - d(\hat{\mathbf{y}}, b)a_{k-1}\}$	1
playing the role of	$H^{(k)}$	$H^{(k-1)}$	d	p_1	s

By (9.29) we obtain a partition of $Q_*^{(k-1)}(\hat{\mathbf{y}}, b)$ into two $(k-1)$-graphs such that for one of these, say $F(\hat{\mathbf{y}}, b)$, we have that

$$\begin{aligned} &F(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d(\hat{\mathbf{y}}, b))\textit{-regular with respect to } \hat{L}^{(k-2)}(\hat{\mathbf{y}}) \textit{ and} \\ &F(\hat{\mathbf{y}}, b) \subseteq Q_*^{(k-1)}(\hat{\mathbf{y}}, b). \end{aligned} \tag{9.33}$$

Let

$$F^{(k-1)} := \bigcup_{\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}}), b \in [a_{k-1}]} F(\hat{\mathbf{y}}, b).$$

Thus $F^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$.

Now we have defined $F^{(k-1)}$ and $G^{(k-1)}$. It only remains to show that these two $(k-1)$-graphs satisfy (F2)$_{9.2}$. Fix any vector $\hat{\mathbf{z}} \in \hat{A}(k, k-1, \mathbf{a})$. Consider $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ such that $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$ and $\hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}}$. By (3.12) we have

$$\hat{P}^{(k-1)}(\hat{\mathbf{z}}) = \bigcup_{\hat{\mathbf{w}} \leq_{k-1,k-2} \hat{\mathbf{z}}} P^{(k-1)}(\hat{\mathbf{w}}, \mathbf{z}^{(k-1)}_{\mathbf{w}^{(1)}_*}). \tag{9.34}$$

By (9.26), $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$ implies that

$$\begin{gathered}\mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_{k-1}(\hat{P}^{(k-2)}(\hat{\mathbf{x}})) \quad \text{and} \\ \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \cap \mathcal{K}_{k-1}(\hat{P}^{(k-2)}(\hat{\mathbf{w}})) = \emptyset \text{ for } \hat{\mathbf{w}} \neq \hat{\mathbf{x}}.\end{gathered}$$

Also $P^{(k-1)}(\hat{\mathbf{w}}, \mathbf{z}^{(k-1)}_{\mathbf{w}^{(1)}_*}) \subseteq \mathcal{K}_{k-1}(\hat{P}^{(k-2)}(\hat{\mathbf{w}}))$ whenever $\hat{\mathbf{w}} \leq_{k-1,k-2} \hat{\mathbf{z}}$. Together this implies

$$\hat{P}^{(k-1)}(\hat{\mathbf{z}}) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \overset{(9.34)}{=} P^{(k-1)}(\hat{\mathbf{x}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \overset{(9.28)}{=} P^{(k-1)}_*(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}).$$

Thus

$$G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}}) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) = G^{(k-1)} \cap P^{(k-1)}_*(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}) \overset{(9.31)}{=} G(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}). \tag{9.35}$$

Together with (R2)$_{9.2}$ this implies that

$$G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}}) = \bigcup_{\hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}}} \bigcup_{\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})} G(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}).$$

Similarly

$$F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}}) = \bigcup_{\hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}}} \bigcup_{\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})} F(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}).$$

For each $\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}})$ let

$$d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1}(\hat{\mathbf{y}}) := \begin{cases} d(\hat{\mathbf{y}}, b) & \text{if } \hat{\mathbf{y}} \in A(\hat{\mathbf{x}}) \text{ for some } \hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}} \text{ and } b = \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}, \\ 0 & \text{otherwise.} \end{cases}$$

The properties (9.32) and (9.35) together imply that for each $\hat{R}^{(k-2)}(\hat{\mathbf{y}}) \in \hat{\mathscr{R}}^{(k-2)}$,

> $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$ *is* $(\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1}(\hat{\mathbf{y}}))$*-regular with respect to* $\hat{R}^{(k-2)}(\hat{\mathbf{y}})$.

Analogously using (9.33), we obtain that for each $\hat{L}^{(k-2)}(\hat{\mathbf{y}}) \in \hat{\mathscr{L}}^{(k-2)}$,

> $F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}})$ *is* $(\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1}(\hat{\mathbf{y}}))$*-regular with respect to* $\hat{L}^{(k-2)}(\hat{\mathbf{y}})$.

In other words, $\mathscr{R}$ is an $(\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1})$-partition of $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$. From (9.23) and (9.25), we know

$$\varepsilon'(\mathbf{a}^{\mathscr{R}}) \ll 1/\|\mathbf{a}^{\mathscr{R}}\|_\infty \leq 1/a_1^{\mathscr{R}} \ll \nu, 1/o, 1/k.$$

In particular, this means that $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$ satisfies the regularity instance $R := (\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, \mathbf{a}^{\mathscr{R}}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1})$. Similarly, $F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}})$ also satisfies the regularity instance R. Thus we can apply Lemma 4.9 twice with the following objects and parameters, once with $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$ playing the role of H and once more with $F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}})$ playing the role of H.

object/parameter	$\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}$	$\|a^{\mathscr{R}}\|_\infty$	$a_1^{\mathscr{R}}$	$\nu^2 o^{-4^k}$	$\mathbf{a}^{\mathscr{R}}$	$k-1$	$K_k^{(k-1)}$
playing the role of	ε	t	a_1	γ	$\mathbf{a}$	k	F

Thus we obtain

$$\frac{|\mathcal{K}_k(G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}}))|}{\binom{n}{k}} = IC(K_k^{(k-1)}, d_{\mathbf{a}^{\mathscr{R}},\hat{\mathbf{z}},k-1}) \pm \nu^2 o^{-4^k} \quad \text{and} \tag{9.36}$$

$$\frac{|\mathcal{K}_k(F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}}))|}{\binom{m}{k}} = IC(K_k^{(k-1)}, d_{\mathbf{a}^{\mathscr{R}},\hat{\mathbf{z}},k-1}) \pm \nu^2 o^{-4^k}. \tag{9.37}$$

On the other hand, we can apply Lemma 4.5 to show that for every $\hat{\mathbf{z}} \in \hat{A}(k, k-1, \mathbf{a})$

$$|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{z}}))| = (1 \pm \nu^2) \prod_{j=1}^{k-1} a_j^{-\binom{k}{j}} n^k \geq o^{-4^k} \binom{n}{k} \quad \text{and}$$

$$|\mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{z}}))| = (1 \pm \nu^2) \prod_{j=1}^{k-1} a_j^{-\binom{k}{j}} m^k \geq o^{-4^k} \binom{m}{k}.$$

This together with (9.36) and (9.37) implies that

$$d(\mathcal{K}_k(G^{(k-1)}) \mid \hat{P}^{(k-1)}(\hat{\mathbf{z}})) = d(\mathcal{K}_k(F^{(k-1)}) \mid \hat{Q}^{(k-1)}(\hat{\mathbf{z}})) \pm \nu.$$

□

Suppose we are given two families of partitions $\mathscr{P}, \mathscr{O}$ such that $\mathscr{P}$ almost refines $\mathscr{O}$ and such that $\mathscr{O}$ is an equitable partition of some k-graph H. Roughly speaking, the next lemma shows that there is a family of partitions $\mathscr{O}'$ such that $\mathscr{P} \prec \mathscr{O}'$ and such that $\mathscr{O}'$ is still an equitable partition of H (with a somewhat larger regularity constant).

Lemma 9.3. *Suppose $0 < 1/m, 1/n \ll \varepsilon \ll \nu \ll \varepsilon_0 \leq 1$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose V is a vertex set of size n. Suppose $R = (\varepsilon_0/2, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance and $\mathscr{O} = \mathscr{O}(k-1, \mathbf{a}^{\mathscr{O}})$ is an $(\varepsilon_0, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition of a k-graph $H^{(k)}$ on V. Suppose there exists an $(\eta, \varepsilon, \mathbf{a}^{\mathscr{P}})$-equitable family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ on V such that $\mathscr{P} \prec_\nu \mathscr{O}$. Then there exists a family of partitions $\mathscr{O}'$ on V such that*

(O′1)$_{9.3}$ *$\mathscr{P} \prec \mathscr{O}'$,*

(O′2)$_{9.3}$ *$\mathscr{O}'$ is a $(1/a_1^{\mathscr{O}}, \varepsilon_0 + \nu^{1/20}, \mathbf{a}^{\mathscr{O}}, \nu^{1/20})$-equitable family of partitions and an $(\varepsilon_0 + \nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $H^{(k)}$,*

(O′3)$_{9.3}$ *for each $j \in [k-1]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have $|O'^{(j)}(\hat{\mathbf{x}}, b) \triangle O^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu^{1/2}\binom{n}{j}$.*

We construct $\mathscr{O}'$ by induction on $j \in [k-1]$. When constructing $\mathscr{O}'^{(j-1)}$ a natural approach is a follows. For a given class $O^{(j)}(\hat{\mathbf{x}}, b)$ of $\mathscr{O}^{(j)}$ we can let $O'^{(j)}(\hat{\mathbf{x}}, b)$ consist e.g. of all classes of $\mathscr{P}^{(j)}$ which lie (mostly) in $O^{(j)}(\hat{\mathbf{x}}, b)$. This is formalized by the function f_j in (9.38). However, this construction may not fit with the existing polyads of $\hat{P}^{(j-1)}$ (i.e. it may violate Definition 3.4(ii)). This issue requires some adjustments, whose overall effect can be shown to be negligible.

Proof of Lemma 9.3. For any function $f : \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}] \to \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, let

$$d(f) := \sum_{(\hat{\mathbf{x}},b) \in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{x}}, b) \setminus O^{(j)}(f(\hat{\mathbf{x}}, b))|.$$

Note that $\mathscr{P} \prec_\nu \mathscr{O}$ implies that for each $j \in [k-1]$, there exists a function $f_j : \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}] \to \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$ such that

$$d(f_j) \leq \nu \binom{n}{j}. \tag{9.38}$$

Moreover, note that since R is a regularity instance (see Definition 3.14), we have $\varepsilon_0 \leq \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-4^k} \varepsilon_{4.5}(\|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, k-1, k)$. Thus Lemma 4.5 and the definition of an equitable family of partitions (see Definition 3.6) imply that for any $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have

$$|O^{(j)}(\hat{\mathbf{x}}, b)| \geq \frac{1}{2\|\mathbf{a}^{\mathscr{O}}\|_\infty^{2^k}} n^j \geq \varepsilon_0^{1/2} n^j. \tag{9.39}$$

For each $i \in [a_1^{\mathscr{O}}]$, let

$$O'^{(1)}(i, i) := \bigcup_{s \in [a_1^{\mathscr{P}}], f_1(s) = i} P^{(1)}(s, s) \quad \text{and let} \quad \mathscr{O}'^{(1)} := \{O'^{(1)}(i, i) : i \in [a_1^{\mathscr{O}}]\}.$$

Note that (9.38) implies that $|O'^{(1)}(i,i)| = (1 \pm a_1^{\mathscr{O}}\nu)n/a_1^{\mathscr{O}} = (1 \pm \nu^{1/2})n/a_1^{\mathscr{O}}$. For all distinct $i, i' \in [a_1^{\mathscr{O}}]$, let $O'^{(1)}(i, i') := \emptyset$. Hence $\mathscr{O}'^{(1)}$ satisfies properties $(\mathrm{O}'1)_1$–$(\mathrm{O}'4)_1$ below. (Here, $(\mathrm{O}'2)_1$ and $(\mathrm{O}'4)_1$ are vacuous.) Assume for some $j \in [k-1] \setminus \{1\}$ we have defined $\mathscr{O}'^{(1)}, \ldots, \mathscr{O}'^{(j-1)}$ satisfying the following for each $i \in [j-1]$:

$(\mathrm{O}'1)_i$ $\mathscr{O}'^{(i)}$ forms a partition of $\mathcal{K}_i(\mathscr{O}'^{(1)})$ and $\mathscr{P}^{(i)} \prec \mathscr{O}'^{(i)}$,

$(\mathrm{O}'2)_i$ if $i > 1$, then $\mathscr{O}'^{(i)} = \{O'^{(i)}(\hat{\mathbf{x}}, b) : (\hat{\mathbf{x}}, b) \in \hat{A}(i, i-1, \mathbf{a}^{\mathscr{O}}) \times [a_i^{\mathscr{O}}]\}$,

$(\mathrm{O}'3)_i$

$$\sum_{(\hat{\mathbf{x}}, b) \in \hat{A}(i, i-1, \mathbf{a}^{\mathscr{O}}) \times [a_i^{\mathscr{O}}]} |O'^{(i)}(\hat{\mathbf{x}}, b) \setminus O^{(i)}(\hat{\mathbf{x}}, b)| \leq i\nu n^i,$$

$(\mathrm{O}'4)_i$ if $i > 1$, then for each $\hat{\mathbf{x}} \in \hat{A}(i, i-1, \mathbf{a}^{\mathscr{O}})$, the collection $\{O'^{(i)}(\hat{\mathbf{x}}, 1), \ldots, O'^{(i)}(\hat{\mathbf{x}}, a_i^{\mathscr{O}})\}$ forms a partition of $\mathcal{K}_i(\hat{O'}^{(i-1)}(\hat{\mathbf{x}}))$, where

$$\hat{O'}^{(i-1)}(\hat{\mathbf{x}}) := \bigcup_{\hat{\mathbf{y}} \leq_{i-1, i-2} \hat{\mathbf{x}}} O'^{(i-1)}(\hat{\mathbf{y}}, \mathbf{x}^{(i-1)}_{\mathbf{y}_*^{(1)}}).$$

We will now construct $\mathscr{O}'^{(j)}$ satisfying $(\mathrm{O}'1)_j$–$(\mathrm{O}'4)_j$. So assume that $k \geq 3$. Note that $(\mathrm{O}'3)_1$–$(\mathrm{O}'3)_{j-1}$ with (9.39) shows that for any $i \in [j-1]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(i, i-1, \mathbf{a}^{\mathscr{O}}) \times [a_i^{\mathscr{O}}]$, the i-graph $O'^{(i)}(\hat{\mathbf{x}}, b)$ is nonempty. Together with $(\mathrm{O}'1)_1$–$(\mathrm{O}'1)_{j-1}$, $(\mathrm{O}'2)_1$–$(\mathrm{O}'2)_{j-1}$, $(\mathrm{O}'3)_1$–$(\mathrm{O}'3)_{j-1}$, $(\mathrm{O}'4)_1$–$(\mathrm{O}'4)_{j-1}$, (9.39), and Lemma 3.13 this implies that $\{\mathscr{O}'^{(i)}\}_{i=1}^{j-1}$ forms a family of partitions. Let $\hat{\mathscr{O}'}^{(j-1)}$ be the collection of all the $\hat{O'}^{(j-1)}(\hat{\mathbf{x}})$ with $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$. Note that Proposition 3.11(iv) and (vi) implies that

$$\{\mathcal{K}_j(\hat{O}'^{(j-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})\} \textit{ forms a partition of } \mathcal{K}_j(\mathscr{O}'^{(1)}). \tag{9.40}$$

By Proposition 3.11(xi)

$$\{\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}})\} \prec \{\mathcal{K}_j(\hat{O}'^{(j-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})\}. \tag{9.41}$$

Let

$$A := \{\hat{\mathbf{y}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) : \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_j(\mathscr{O}'^{(1)})\}.$$

Then (9.41) implies that

$$\bigcup_{\hat{\mathbf{y}} \in A} \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) = \mathcal{K}_j(\mathscr{O}'^{(1)}). \tag{9.42}$$

By (9.40) and (9.41), for each $\hat{\mathbf{y}} \in A$, there exists $g(\hat{\mathbf{y}}) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ such that $\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_j(\hat{O'}^{(j-1)}(g(\hat{\mathbf{y}})))$.

Claim 1.

$$\sum_{\hat{\mathbf{y}} \in A} |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}})))| \leq (j-1)\nu n^j.$$

Proof. Observe that by (9.41) for each $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$, we have

$$\bigcup_{\hat{\mathbf{y}}\colon g(\hat{\mathbf{y}}) = \hat{\mathbf{x}}} \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) = \mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}})).$$

This implies that

$$\sum_{\hat{\mathbf{y}} \in A} |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}})))| = \sum_{\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})} |\mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(\hat{\mathbf{x}}))|.$$

Any j-set counted on the right hand side lies in $\mathcal{K}_j(\mathscr{O}'^{(1)})$ and contains a $(j-1)$-set $J \in O'^{(j-1)}(\hat{\mathbf{z}}, b) \setminus O^{(j-1)}(\hat{\mathbf{z}}, b)$ for some $\hat{\mathbf{z}} \leq_{j-1,j-2} \hat{\mathbf{x}}$ and $b = \mathbf{x}^{(j-1)}_{\mathbf{z}^{(1)}_*}$. Note that $(\mathrm{O}'3)_{j-1}$ implies that there are at most $(j-1)\nu n^{j-1}$ such sets J. For such a fixed $(j-1)$-set J, there are at most n j-sets in $\mathcal{K}_j(\mathscr{O}'^{(1)})$ containing J. Thus at most $(j-1)\nu n^{j-1} \cdot n = (j-1)\nu n^j$ j-sets are counted in the above expression. This proves the claim. $\square$

Ideally, for every $\mathbf{x} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_j^{\mathscr{O}}]$, we would like to define $O'^{(j)}(\mathbf{x}, b)$ as the union of all $P^{(j)}(\hat{\mathbf{y}}, b')$ for which $f_j(\hat{\mathbf{y}}, b') = (\hat{\mathbf{x}}, b)$ holds. However, we may have $f_j(\hat{\mathbf{y}}, b') \neq (g(\hat{\mathbf{y}}), b)$ for all $b \in [a_j^{\mathscr{O}}]$. This leads to difficulties when attempting to prove $(\mathrm{O}'4)_j$. We resolve this problem by defining a function f'_j, which is a slight modification of f_j. To this end, let

$$W := \{(\hat{\mathbf{y}}, b') \in A \times [a_j^{\mathscr{P}}] : f_j(\hat{\mathbf{y}}, b') \neq (g(\hat{\mathbf{y}}), b) \text{ for all } b \in [a_j^{\mathscr{O}}]\}.$$

Thus if $(\hat{\mathbf{y}}, b') \in W$, then $O^{(j)}(f_j(\hat{\mathbf{y}}, b')) \cap \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}}))) = \emptyset$. This and the fact that $P^{(j)}(\hat{\mathbf{y}}, b') \subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}}))$ imply that for $(\hat{\mathbf{y}}, b') \in W$

$$P^{(j)}(\hat{\mathbf{y}}, b') \cap O^{(j)}(f_j(\hat{\mathbf{y}}, b')) \subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}}))). \tag{9.43}$$

We define a function $f'_j : A \times [a_j^{\mathscr{P}}] \to \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$ by

$$f'_j(\hat{\mathbf{y}}, b') := \begin{cases} (g(\hat{\mathbf{y}}), b) \text{ for an arbitrary } b \in [a_j^{\mathscr{O}}] & \text{if } (\hat{\mathbf{y}}, b') \in W, \\ f_j(\hat{\mathbf{y}}, b') & \text{otherwise.} \end{cases}$$

For each $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_j^{\mathscr{O}}]$, let

$$O'^{(j)}(\hat{\mathbf{x}}, b) := \bigcup_{\substack{(\hat{\mathbf{y}}, b') \in A \times [a_j^{\mathscr{P}}]: \\ f'_j(\hat{\mathbf{y}}, b') = (\hat{\mathbf{x}}, b)}} P^{(j)}(\hat{\mathbf{y}}, b'). \tag{9.44}$$

Let $\mathscr{O}'^{(j)}$ be as described in $(\mathrm{O}'2)_j$. By (9.40), (9.41), and the fact that f'_j is defined for all $A \times [a_j^{\mathscr{P}}]$, we obtain $(\mathrm{O}'1)_j$.

We now verify $(\mathrm{O}'3)_j$. For this, we estimate $d(f'_j)$, namely

$$\begin{aligned}
d(f'_j) &= \sum_{(\hat{\mathbf{y}},b')\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{P}})\times[a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{y}},b')\setminus O^{(j)}(f'_j(\hat{\mathbf{y}},b'))| \\
&\leq \sum_{(\hat{\mathbf{y}},b')\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{P}})\times[a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{y}},b')\setminus O^{(j)}(f_j(\hat{\mathbf{y}},b'))| \\
&\qquad + \sum_{(\hat{\mathbf{y}},b')\in W} |P^{(j)}(\hat{\mathbf{y}},b')\cap O^{(j)}(f_j(\hat{\mathbf{y}},b'))| \\
&\overset{(9.43)}{\leq} d(f_j) + \sum_{\hat{\mathbf{y}}\in A} |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}}))\setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}})))| \\
&\overset{\text{Claim 1}}{\leq} d(f_j) + (j-1)\nu n^j \overset{(9.38)}{\leq} j\nu n^j.
\end{aligned} \tag{9.45}$$

This in turn implies that

$$\begin{aligned}
&\sum_{(\hat{\mathbf{x}},b)\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})\times[a_j^{\mathscr{O}}]} |O'^{(j)}(\hat{\mathbf{x}},b)\setminus O^{(j)}(\hat{\mathbf{x}},b)| \\
&\overset{(9.44)}{=} \sum_{(\hat{\mathbf{x}},b)\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})\times[a_j^{\mathscr{O}}]} \sum_{\substack{(\hat{\mathbf{y}},b')\in A\times[a_j^{\mathscr{P}}]:\\ f'_j(\hat{\mathbf{y}},b')=(\hat{\mathbf{x}},b)}} |P^{(j)}(\hat{\mathbf{y}},b')\setminus O^{(j)}(\hat{\mathbf{x}},b)| \\
&= \sum_{(\hat{\mathbf{y}},b')\in A\times[a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{y}},b')\setminus O^{(j)}(f'_j(\hat{\mathbf{y}},b'))| \leq d(f'_j) \overset{(9.45)}{\leq} j\nu n^j.
\end{aligned}$$

Thus $(\mathrm{O}'3)_j$ holds.

Suppose $\hat{\mathbf{x}}\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})$ and $b\in[a_j^{\mathscr{O}}]$. Note that $P^{(j)}(\hat{\mathbf{y}},b')\subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}}))\subseteq \mathcal{K}_j(\hat{O'}^{(j-1)}(g(\hat{\mathbf{y}})))$ for each $\hat{\mathbf{y}}\in A$ and $b'\in[a_j^{\mathscr{P}}]$. Together with (9.44) and the definition of f'_j, we obtain that $O'^{(j)}(\hat{\mathbf{x}},b)\subseteq \mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}}))$. By this and (9.40)–(9.42), the collection $\{O'^{(j)}(\hat{\mathbf{x}},1),\dots,O'^{(j)}(\hat{\mathbf{x}},a_i^{\mathscr{O}})\}$ forms a partition of $\mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}}))$. Thus $(\mathrm{O}'4)_j$ holds.

By repeating this procedure, we obtain $\mathscr{O}'^{(1)},\dots,\mathscr{O}'^{(k-1)}$. Let $\mathscr{O}' := \{\mathscr{O}'^{(j)}\}_{j=1}^{k-1}$. As observed before (9.40), $\mathscr{O}'$ is a family of partitions. Properties $(\mathrm{O}'1)_1$–$(\mathrm{O}'1)_{k-1}$ and imply $(\mathrm{O}'1)_{9.3}$.

Note that $(\mathrm{O}'3)_1$ implies that for each $j\in[k-1]$ we have $|\mathcal{K}_j(\mathscr{O}^{(1)})\triangle\mathcal{K}_j(\mathscr{O}'^{(1)})|\leq 2\nu n^j$. Thus for each $j\in[k-1]$ and $(\hat{\mathbf{x}},b)\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})\times[a_j^{\mathscr{O}}]$, this implies that

$$\begin{aligned}
|O'^{(j)}(\hat{\mathbf{x}},b)\triangle O^{(j)}(\hat{\mathbf{x}},b)| &\leq |\mathcal{K}_j(\mathscr{O}^{(1)})\triangle\mathcal{K}_j(\mathscr{O}'^{(1)})| + \sum_{\hat{\mathbf{x}}\in\hat{A}(j,j-1,\mathbf{a}),b\in[a_j^{\mathscr{O}}]} |O'^{(j)}(\hat{\mathbf{x}},b)\setminus O^{(j)}(\hat{\mathbf{x}},b)| \\
&\overset{(\mathrm{O}'3)_j}{\leq} (j+2)\nu n^j \leq \nu^{1/2}\binom{n}{j}.
\end{aligned}$$

Thus we have $(\mathrm{O}'3)_{9.3}$. Finally, since R is a regularity instance, $(\mathrm{O}'3)_{9.3}$ enables us to apply Lemma 4.15 with the following objects and parameters.

object/parameter	$\mathscr{O}$	$\mathscr{O}'$	$\nu^{1/2}$	0	ε_0	$d_{\mathbf{a}^{\mathscr{O}},k}$	$H^{(k)}$	$H^{(k)}$
playing the role of	$\mathscr{P}$	$\mathscr{Q}$	ν	λ	ε	$d_{\mathbf{a},k}$	$H^{(k)}$	$G^{(k)}$

This implies $(\mathrm{O}'2)_{9.3}$. $\square$

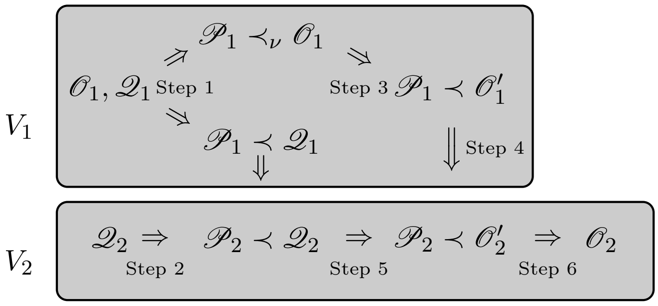

FIGURE 3. The proof strategy for Lemma 10.1.

10. Sampling a regular partition

In this section we prove Lemma 6.1. In Section 10.1 we provide the main tool (Lemma 10.1) for this result and in Section 10.2 we deduce Lemma 6.1.

10.1. **Building a family of partitions from three others.** In this subsection we prove our key tool (Lemma 10.1) for the proof of Lemma 6.1. Roughly speaking Lemma 10.1 states the following. Suppose there are two k-graphs H_1, H_2 with vertex set V_1, V_2, respectively, and there are two ε-equitable families of partitions of these k-graphs which have the same parameters. Suppose further that there is another ε_0-equitable family of partitions $\mathscr{O}_1$ for H_1. Then there is an equitable family of partitions $\mathscr{O}_2$ of H_2 which has the (roughly) same parameters as $\mathscr{O}_1$ provided $\varepsilon \ll \varepsilon_0$. Even more loosely, the result says that if two hypergraphs share a single 'high-quality' regularity partition, then they share any 'low-quality' regularity partition.

Lemma 10.1. *Suppose $0 < 1/n, 1/m \ll \varepsilon \ll 1/T, 1/a_1^{\mathscr{Q}} \ll \delta \ll \varepsilon_0 \leq 1$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose $\mathbf{a}^{\mathscr{Q}} \in [T]^{k-1}$. Suppose that $R = (\varepsilon_0/2, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance. Suppose V_1, V_2 are sets of size n, m, and $H_1^{(k)}, H_2^{(k)}$ are k-graphs on V_1, V_2, respectively. Suppose*

(P1)$_{10.1}$ *$\mathscr{Q}_1 = \mathscr{Q}_1(k-1, \mathbf{a}^{\mathscr{Q}})$ is an $(\varepsilon, \mathbf{a}^{\mathscr{Q}}, d_{\mathbf{a}^{\mathscr{Q}},k})$-equitable partition of $H_1^{(k)}$,*
(P2)$_{10.1}$ *$\mathscr{Q}_2 = \mathscr{Q}_2(k-1, \mathbf{a}^{\mathscr{Q}})$ is an $(\varepsilon, \mathbf{a}^{\mathscr{Q}}, d_{\mathbf{a}^{\mathscr{Q}},k})$-equitable partition of $H_2^{(k)}$, and*
(P3)$_{10.1}$ *$\mathscr{O}_1 = \mathscr{O}_1(k-1, \mathbf{a}^{\mathscr{O}})$ is an $(\varepsilon_0, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition of $H_1^{(k)}$.*

Then there exists an $(\varepsilon_0 + \delta, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition $\mathscr{O}_2$ of $H_2^{(k)}$.

A crucial point here is that the construction of $\mathscr{O}_2$ incurs only an additive increase (by δ) of the regularity parameter of $\mathscr{O}_1$.

For an illustration of the proof strategy of Lemma 10.1 see Figure 3. Our strategy is first to apply Lemma 9.1 to $\mathscr{Q}_1, \mathscr{O}_1$ to obtain a family of partitions $\mathscr{P}_1$ that refines $\mathscr{Q}_1$ and almost refines $\mathscr{O}_1$ (see Step 1). Moreover, we refine $\mathscr{Q}_2$ and obtain $\mathscr{P}_2$ in such a way that $\mathscr{P}_2$ has the same number of partition classes as $\mathscr{P}_1$ (see Step 2). We then apply Lemma 9.3 to construct a family $\mathscr{O}'_1$ of partitions that is very similar to $\mathscr{O}_1$ and satisfies $\mathscr{P}_1 \prec \mathscr{O}'_1$ (see Step 3). Then we analyse how $\mathscr{P}_1$ refines $\mathscr{O}'_1$ (see Step 4). We then use Lemma 9.2 to mimic this structure in order to build $\mathscr{O}'_2$ from $\mathscr{P}_2$ (see Step 5). Finally in Step 6 we apply Lemma 4.16 to show that $\mathscr{O}'_2$ can be slightly modified to obtain the desired $\mathscr{O}_2$.

Proof of Lemma 10.1. We start with several definitions. Choose a new constant ν such that $1/T, 1/a_1^{\mathscr{Q}} \ll \nu \ll \delta$. Let $\overline{\varepsilon} : \mathbb{N}^{k-1} \to (0,1]$ be a function such that for any $\mathbf{a} = (a_1, \ldots, a_{k-1}) \in \mathbb{N}^{k-1}$, we have $0 < \overline{\varepsilon}(\mathbf{a}) \ll \nu, \|\mathbf{a}\|_\infty^{-k}$. Now given $\overline{\varepsilon}$, we define

$$t' := t_{9.1}(k, T, T^{4^k}, 1/a_1^{\mathscr{Q}}, \nu, \overline{\varepsilon}).$$

Observe that $0 < \varepsilon \ll 1/k, 1/T, 1/a_1^{\mathscr{Q}}, \nu, 1/t'$. Thus we may assume that for any $\mathbf{a} \in [t']^{k-1}$, we have

$$0 < \varepsilon \ll \overline{\varepsilon}(\mathbf{a}), \mu_{9.1}(k, T, T^{4^k}, 1/a_1^{\mathscr{Q}}, \nu, \overline{\varepsilon}).$$

Step 1. *Constructing $\mathscr{P}_1$ as a refinement of $\mathscr{Q}_1$.*

Let

$$\mathscr{Q}_1^{(k)} := \{\mathcal{K}_k(\hat{Q}_1^{(k-1)}) : \hat{Q}_1^{(k-1)} \in \hat{\mathscr{Q}}_1^{(k-1)}\} \quad \text{and}$$
$$\mathscr{Q}_1'^{(k)} := \left(\mathscr{Q}_1^{(k)} \cup \left\{\binom{V_1}{k} \setminus \mathcal{K}_k(\mathscr{Q}_1^{(1)})\right\}\right) \setminus \{\emptyset\}.$$

Since $\mathscr{Q}_1$ is T-bounded, $|\mathscr{Q}_1^{(k)}| \leq T^{2^k}$ by Proposition 3.11(viii). Thus $|\mathscr{Q}_1'^{(k)}| \leq T^{4^k}$. Moreover, the fact that R is a regularity instance (and the definition of $\varepsilon_{3.14}$) implies that $\mathbf{a}^{\mathscr{O}} \in [T]^{k-1}$. We can apply Lemma 9.1 with the following objects and parameters.

object/parameter	V_1	$\mathscr{O}_1$	$\mathscr{Q}_1'^{(k)}$	$\{\mathscr{Q}_1^{(i)}\}_{i=1}^k$	T	T^{4^k}	$1/a_1^{\mathscr{Q}}$	ν	$\overline{\varepsilon}$
playing the role of	V	$\mathscr{O}$	$\mathscr{H}^{(k)}$	$\mathscr{Q}$	o	s	η	ν	ε

Observe that $\mathscr{O}_1$, $\{\mathscr{Q}_1^{(i)}\}_{i=1}^k$ and $\mathscr{Q}_1'^{(k)}$ playing the roles of $\mathscr{O}$, $\mathscr{Q}$ and $\mathscr{H}^{(k)}$, respectively, satisfy (O1)$_{9.1}$–(O4)$_{9.1}$ in Lemma 9.1. We obtain a family of partitions $\mathscr{P}_1 = \mathscr{P}_1(k-1, \mathbf{a}^{\mathscr{P}})$ such that the following hold.

(P11) $\mathscr{P}_1$ is $(1/a_1^{\mathscr{Q}}, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable and t'-bounded, and $a_j^{\mathscr{Q}}$ divides $a_j^{\mathscr{P}}$ for each $j \in [k-1]$.

(P12) $\mathscr{P}_1^{(j)} \prec \mathscr{Q}_1^{(j)}$ and $\mathscr{P}_1^{(j)} \prec_\nu \mathscr{O}_1^{(j)}$ for each $j \in [k-1]$.

Let

$$\varepsilon' := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \quad \text{and} \quad a_k^{\mathscr{P}} = a_k^{\mathscr{Q}} = a_k^{\mathscr{O}} := 1.$$

Hence $\varepsilon' \ll \nu, \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-k}$ by the definition of $\overline{\varepsilon}$.

By (P12), for each $j \in [k-1]$, $\hat{\mathbf{y}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}})$, and $b' \in [a_j^{\mathscr{P}}]$, either there exists $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}})$ and $b \in [a_j^{\mathscr{Q}}]$ such that $P_1^{(j)}(\hat{\mathbf{y}}, b') \subseteq Q_1^{(j)}(\hat{\mathbf{x}}, b)$ or $P_1^{(j)}(\hat{\mathbf{y}}, b') \cap Q_1^{(j)}(\hat{\mathbf{x}}, b) = \emptyset$ for all $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}})$ and $b \in [a_j^{\mathscr{Q}}]$. This allows us to describe $\mathscr{P}_1$ in terms of $\mathscr{Q}_1$ in the following way. For each $j \in [k-1]$, $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}})$, and $b \in [a_j^{\mathscr{Q}}]$, we define

$$A_j(\hat{\mathbf{x}}, b) := \{(\hat{\mathbf{y}}, b') \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}] : P_1^{(j)}(\hat{\mathbf{y}}, b') \subseteq Q_1^{(j)}(\hat{\mathbf{x}}, b)\} \text{ and}$$
$$A_j := \bigcup_{\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}}), b \in [a_j^{\mathscr{Q}}]} A_j(\hat{\mathbf{x}}, b). \tag{10.1}$$

For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})$, let

$$\hat{A}_k(\hat{\mathbf{x}}) := \{\hat{\mathbf{y}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}}) : \hat{P}_1^{(k-1)}(\hat{\mathbf{y}}) \subseteq \hat{Q}_1^{(k-1)}(\hat{\mathbf{x}})\}, \text{ and}$$
$$\hat{A}_k := \bigcup_{\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})} \hat{A}_k(\hat{\mathbf{x}}).$$

The density function $d_{\mathbf{a}^{\mathscr{Q}}, k}$ for $\mathscr{Q}_1$ naturally gives rise to a density function for $\mathscr{P}_1$. Indeed, for each $\hat{\mathbf{y}} \in A(k, k-1, \mathbf{a}^{\mathscr{P}})$, we define

$$d_{\mathbf{a}^{\mathscr{P}}, k}(\hat{\mathbf{y}}) := \begin{cases} d_{\mathbf{a}^{\mathscr{Q}}, k}(\hat{\mathbf{x}}) & \text{if } \hat{\mathbf{y}} \in \hat{A}_k(\hat{\mathbf{x}}) \text{ for some } \hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}}) \text{ and} \\ 0 & \text{if } \hat{\mathbf{y}} \notin \hat{A}_k. \end{cases}$$

Recall that $\mathscr{P}_1$ is a $(1/a_1^{\mathscr{Q}}, \varepsilon', \mathbf{a}^{\mathscr{P}})$-equitable family of partitions, $\mathscr{Q}_1$ is a $(1/a_1^{\mathscr{Q}}, \varepsilon, \mathbf{a}^{\mathscr{Q}})$-equitable family of partitions, and $\varepsilon \ll \varepsilon' \ll \nu, \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-k}$. Thus Lemma 4.5 implies that for each $j \in [k-1]$, $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$, and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$, we have

$$|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))| = (1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1} \text{ and } |\mathcal{K}_{j+1}(\hat{Q}_1^{(j)}(\hat{\mathbf{x}}))| = (1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{Q}})^{-\binom{j+1}{i}} n^{j+1}. \tag{10.2}$$

By (P11), for each $j \in [k-2]$, $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$ and $b \in [a_{j+1}^{\mathscr{P}}]$, we have

$$|P_1^{(j+1)}(\hat{\mathbf{y}}, b)| = (1/a_{j+1}^{\mathscr{P}} \pm \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}))|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))| = (1 \pm 2\nu) \prod_{i=1}^{j+1} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1}. \tag{10.3}$$

It will be convenient to restrict our attention to the k-graph $G_1^{(k)}$ which consists of the crossing k-sets of $H_1^{(k)}$ with respect to $\mathscr{Q}_1^{(1)}$ (rather than $H_1^{(k)}$ itself).

Claim 1. *Let* $G_1^{(k)} := H_1^{(k)} \cap \bigcup_{\hat{\mathbf{y}} \in \hat{A}_k} \mathcal{K}_k(\hat{P}_1^{(k-1)}(\hat{\mathbf{y}}))$. *Then*

(G11) $\mathscr{P}_1$ *is an* $(\varepsilon', \mathbf{a}^{\mathscr{P}}, d_{\mathbf{a}^{\mathscr{P}},k})$*-equitable partition of* $G_1^{(k)}$.
(G12) $|G_1^{(k)} \triangle H_1^{(k)}| \leq \nu \binom{n}{k}$.

Proof. Consider $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})$ and $\hat{\mathbf{y}} \in \hat{A}_k(\hat{\mathbf{x}})$. Note that (10.2) implies that $|\mathcal{K}_k(\hat{P}_1^{(k-1)}(\hat{\mathbf{y}}))| \geq \varepsilon' |\mathcal{K}_k(\hat{Q}_1^{(k-1)}(\hat{\mathbf{x}}))|$. Also by (P1)$_{10.1}$, $H_1^{(k)}$ is $(\varepsilon, d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{Q}_1^{(k-1)}(\hat{\mathbf{x}})$. Thus by Lemma 4.1(ii) $H_1^{(k)}$ is $(\varepsilon', d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}_1^{(k-1)}(\hat{\mathbf{y}})$. Together with the definition of $d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{y}})$ this in turn shows that for all $\hat{\mathbf{y}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}})$ we have that $G_1^{(k)}$ is $(\varepsilon', d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}_1^{(k-1)}(\hat{\mathbf{y}})$. Thus (G11) holds.

Note that (P12) and the definition of $\hat{A}_k$ imply that

$$G_1^{(k)} \triangle H_1^{(k)} \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}_1^{(1)}).$$

Since $\mathscr{Q}_1$ is $(1/a_1^{\mathscr{Q}}, \varepsilon, \mathbf{a}^{\mathscr{Q}})$-equitable and $1/a_1^{\mathscr{Q}} \ll \nu, 1/k$, we obtain

$$\left| \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}_1^{(1)}) \right| \overset{(3.5)}{\leq} \frac{k^2}{a_1^{\mathscr{Q}}} \binom{n}{k} \leq \nu \binom{n}{k}.$$

This proves (G12). □

Step 2. *Refining* $\mathscr{Q}_2$ *into* $\mathscr{P}_2$ *which mirrors* $\mathscr{P}_1$.

We have now set up the required definitions for the objects on V_1 and will now proceed with the objects on V_2. By using Lemma 4.13 with $\mathscr{Q}_2$, $\mathbf{a}^{\mathscr{P}}$, t' playing the roles of $\mathscr{P}, \mathbf{b}, t$, respectively, we can obtain a $(1/a_1^{\mathscr{Q}}, \varepsilon^{1/3}, \mathbf{a}^{\mathscr{P}})$-equitable family of partitions $\mathscr{P}_2 = \mathscr{P}_2(k-1, \mathbf{a}^{\mathscr{P}})$ such that $\mathscr{P}_2 \prec \mathscr{Q}_2$. By considering an appropriate $\mathbf{a}^{\mathscr{P}}$-labelling, we may assume that for each $j \in [k-1]$, $(\hat{\mathbf{y}}, b') \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}}) \times [a_j^{\mathscr{Q}}]$, we have

$$P_2^{(j)}(\hat{\mathbf{y}}, b') \subseteq Q_2^{(j)}(\hat{\mathbf{x}}, b) \text{ if and only if } (\hat{\mathbf{y}}, b') \in A_j(\hat{\mathbf{x}}, b).$$

Again Lemma 4.5 and the fact that $\varepsilon' \ll \nu, \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-k}$ imply that for each $j \in [k-1]$, $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$ and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$, we have

$$|\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))| = (1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} m^{j+1} \text{ and } |\mathcal{K}_{j+1}(\hat{Q}_2^{(j)}(\hat{\mathbf{x}}))| = (1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{Q}})^{-\binom{j+1}{i}} m^{j+1}. \tag{10.4}$$

Let

$$G_2^{(k)} := H_2^{(k)} \cap \bigcup_{\hat{\mathbf{y}} \in \hat{A}_k} \mathcal{K}_k(\hat{P}_2^{(k-1)}(\hat{\mathbf{y}})).$$

Similarly as in Claim 1, we conclude the following.

(G21) $\mathscr{P}_2$ is an $(\varepsilon', \mathbf{a}^{\mathscr{P}}, d_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of $G_2^{(k)}$.

(G22) $|G_2^{(k)} \triangle H_2^{(k)}| \leq \nu\binom{n}{k}$.

Step 3. *Modifying $\mathscr{O}_1$ into $\mathscr{O}_1'$ with $\mathscr{P} \prec \mathscr{O}_1'$.*

Recall that $\mathscr{P}_1 \prec_\nu \mathscr{O}_1$ by (P12). We next replace $\mathscr{O}_1$ by a very similar family of partitions $\mathscr{O}_1'$ such that $\mathscr{P}_1 \prec \mathscr{O}_1'$. To this end we apply Lemma 9.3 with $\mathscr{O}_1, \mathscr{P}_1$ playing the roles of $\mathscr{O}, \mathscr{P}$, respectively, and obtain $\mathscr{O}_1' = \mathscr{O}_1'(k-1, \mathbf{a}^{\mathscr{O}})$ such that

(O′11) $\mathscr{P}_1 \prec \mathscr{O}_1'$.

(O′12) $\mathscr{O}_1'$ is a $(1/a_1^{\mathscr{O}}, \varepsilon_0 + \nu^{1/20}, \mathbf{a}^{\mathscr{O}}, \nu^{1/20})$-equitable family of partitions which is an $(\varepsilon_0 + \nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $H_1^{(k)}$.

(O′13) for each $j \in [k-1]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have $|O'^{(j)}(\hat{\mathbf{x}}, b) \triangle O^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu^{1/2}\binom{n}{j}$.

Note that since $(\varepsilon_0/2, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance and $\nu \ll \varepsilon_0$, we have

$$\varepsilon_0 + \nu^{1/20} \leq \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-4^k} \cdot \varepsilon_{4.5}(\|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, k-1, k).$$

Thus Lemma 4.5 implies for any $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, we have

$$|\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))| \geq \varepsilon_0^{1/2} n^{j+1}. \tag{10.5}$$

Also, (O′12) implies that for all $j \in [k-2]$, $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_{j+1}^{\mathscr{O}}]$, we have

$$|O_1'^{(j+1)}(\hat{\mathbf{w}}, b)| \geq (1/a_{j+1}^{\mathscr{O}} - 2\varepsilon_0)|\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))| \geq \varepsilon_0^{2/3} n^{j+1}. \tag{10.6}$$

Moreover, by (O′12), (O′13) and (G12), we can apply Lemma 4.15 with $\mathscr{O}_1', \mathscr{O}_1', H_1^{(k)}$ and $G_1^{(k)}$ playing the roles of $\mathscr{P}, \mathscr{Q}, H^{(k)}$ and $G^{(k)}$ to obtain that

$$\mathscr{O}_1' \textit{ is an } (\varepsilon_0 + 2\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})\textit{-partition of } G_1^{(k)}. \tag{10.7}$$

Step 4. *Describing $\mathscr{O}_1'$ in terms of its refinement $\mathscr{P}_1$.*

We now describe how the partition classes and polyads of $\mathscr{O}_1'$ can be expressed in terms of $\mathscr{P}_1$. This description will be used to construct $\mathscr{O}_2'$ from $\mathscr{P}_2$ in Step 5.

For each $j \in [k-2]$, our next aim is to define $B_{j+1}(\hat{\mathbf{w}}, b)$ for $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_{j+1}^{\mathscr{O}}]$ in a similar way as we defined $A_{j+1}(\mathbf{x}, b)$ for $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$ and $b \in [a_{j+1}^{\mathscr{Q}}]$ in (10.1). To this end, for each $b \in [a_1^{\mathscr{O}}]$, let

$$B_1(b, b) := \{(b', b') \in \hat{A}(1, 0, \mathbf{a}^{\mathscr{P}}) \times [a_1^{\mathscr{P}}] : P_1^{(1)}(b', b') \subseteq O_1'^{(1)}(b, b)\}.$$

For each $j \in [k-1]$, let

$$\hat{B}_{j+1} := \left\{ \hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) : \left| \mathbf{u}_*^{(1)} \cap \{b' : (b', b') \in B_1(b, b)\} \right| \leq 1 \text{ for each } b \in [a_1^{\mathscr{O}}] \right\}.$$

Note that this easily implies that

$$\hat{\mathbf{u}} \in \hat{B}_{j+1} \text{ if and only if } \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\mathscr{O}_1'^{(1)}). \tag{10.8}$$

Consider any $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Let

$$\hat{B}_{j+1}(\hat{\mathbf{w}}) := \left\{ \hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) : \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}})) \right\}. \tag{10.9}$$

Together with (O′11) and Proposition 3.11(xi) this implies that

$$\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}})) = \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}})). \tag{10.10}$$

Moreover, if $j \in [k-2]$ and $b \in [a^{\mathscr{O}}_{j+1}]$, let

$$B_{j+1}(\hat{\mathbf{w}}, b) := \left\{ (\hat{\mathbf{u}}, b') \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) \times [a^{\mathscr{P}}_{j+1}] : P_1^{(j+1)}(\hat{\mathbf{u}}, b') \subseteq O_1'^{(j+1)}(\hat{\mathbf{w}}, b) \right\}. \tag{10.11}$$

Thus (O′11), (10.2) and (10.5) imply that for all $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$

$$|\hat{B}_{j+1}(\hat{\mathbf{w}})| \geq \frac{|\mathcal{K}_{j+1}(\hat{O'}_1^{(j)}(\hat{\mathbf{w}}))|}{(1+\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1}} \geq \frac{1}{2}\varepsilon_0^{1/2} \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{\binom{j+1}{i}}. \tag{10.12}$$

Similarly, (O′11), (10.3) and (10.6) imply that for all $j \in [k-1] \setminus \{1\}$, $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_j^{\mathscr{O}}]$,

$$|B_j(\hat{\mathbf{w}}, b)| \geq \frac{|O_1'^{(j-1)}(\hat{\mathbf{w}}, b)|}{(1+2\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j}{i}} n^{j}} \geq \frac{1}{2}\varepsilon_0^{2/3} \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{\binom{j}{i}} > 0. \tag{10.13}$$

Note that by Proposition 3.11(xi) and (O′11), for each $j \in [k-1]$, we have

$$\{\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}})) : \hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})\} \prec \{\mathcal{K}_{j+1}(\hat{O'}_1^{(j)}(\hat{\mathbf{w}})) : \hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})\}. \tag{10.14}$$

Together with (10.8) and Proposition 3.11(vi) applied to $\mathscr{O}_1'$, this implies that $\hat{\mathbf{u}} \in \hat{B}_{j+1}$ if and only if $\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})$ for some $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Thus for each $j \in [k-1]$,

$$\{\hat{B}_{j+1}(\hat{\mathbf{w}}) : \hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})\} \textit{ forms a partition of } \hat{B}_{j+1}. \tag{10.15}$$

The following observations relate polyads and partition classes of $\mathscr{O}_1'$ and $\mathscr{P}_1$. They will be used in the proof of Claim 3 to relate $\mathscr{O}_2'$ (which is constructed in Step 5) and $\mathscr{P}_2$.

Claim 2. (i) *For all $j \in [k-1] \setminus \{1\}$ and $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$, we have*

$$\bigcup_{b \in [a_j^{\mathscr{O}}]} B_j(\hat{\mathbf{w}}, b) = \hat{B}_j(\hat{\mathbf{w}}) \times [a_j^{\mathscr{P}}].$$

(ii) *For all $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, we have*

$$\left\{ (\hat{\mathbf{v}}, \mathbf{u}^{(j)}_{\mathbf{v}_*^{(1)}}) : \hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}), \hat{\mathbf{v}} \leq_{j,j-1} \hat{\mathbf{u}} \right\} \subseteq \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}}).$$

Proof. We first prove (i). Note that for all $j \in [k-1] \setminus \{1\}$, $(\hat{\mathbf{u}}, b') \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}]$ and $(\hat{\mathbf{w}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$ with $(\hat{\mathbf{u}}, b') \in B_j(\hat{\mathbf{w}}, b)$, we have

$$P_1^{(j)}(\hat{\mathbf{u}}, b') \subseteq O_1'^{(j)}(\hat{\mathbf{w}}, b) \subseteq \mathcal{K}_j(\hat{O}_1'^{(j-1)}(\hat{\mathbf{w}})).$$

Since $P_1^{(j)}(\hat{\mathbf{u}}, b') \subseteq \mathcal{K}_j(\hat{P}_1^{(j-1)}(\hat{\mathbf{u}}))$, this means $\mathcal{K}_j(\hat{P}_1^{(j-1)}(\hat{\mathbf{u}})) \cap \mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{w}})) \neq \emptyset$. By (10.14) this in turn implies that $\mathcal{K}_j(\hat{P}_1^{(j-1)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_j(\hat{O}_1'^{(j-1)}(\hat{\mathbf{w}}))$, and thus $\hat{\mathbf{u}} \in \hat{B}_j(\hat{\mathbf{w}})$. On the other hand, if $\hat{\mathbf{u}} \in \hat{B}_j(\hat{\mathbf{w}})$, then (O′11) implies that for each $b' \in [a_j^{\mathscr{P}}]$ there exists $b \in [a_j^{\mathscr{O}}]$ such that $P_1^{(j)}(\hat{\mathbf{u}}, b') \subseteq O_1'^{(j)}(\hat{\mathbf{w}}, b)$, and thus $(\hat{\mathbf{u}}, b') \in B_j(\hat{\mathbf{w}}, b)$.

We now prove (ii). Recall that for each $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, $\hat{O}_1'^{(j)}(\hat{\mathbf{w}})$ satisfies (3.12). Together with (O′11) this implies that

$$\hat{O}_1'^{(j)}(\hat{\mathbf{w}}) = \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} O_1'^{(j)}(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}}) \overset{(10.11)}{=} \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} \bigcup_{(\hat{\mathbf{v}}, b') \in B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}})} P_1^{(j)}(\hat{\mathbf{v}}, b'). \tag{10.16}$$

Then

$$\begin{aligned}
& (\hat{\mathbf{v}}, b') \in \{(\hat{\mathbf{v}}, \mathbf{u}^{(j)}_{\mathbf{v}_*^{(1)}}) : \hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}), \hat{\mathbf{v}} \leq_{j,j-1} \hat{\mathbf{u}}\} \\
\overset{(3.12),(10.9)}{\Longrightarrow} \quad & \exists \hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) : \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}})), P_1^{(j)}(\hat{\mathbf{v}}, b') \subseteq \hat{P}_1^{(j)}(\hat{\mathbf{u}}) \\
\overset{(3.2)}{\Longrightarrow} \quad & \exists (I, J) \in P_1^{(j)}(\hat{\mathbf{v}}, b') \times \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}})) : I \subseteq J \\
\overset{(3.2),(O'11)}{\Longrightarrow} \quad & P_1^{(j)}(\hat{\mathbf{v}}, b') \subseteq \hat{O}_1'^{(j)}(\hat{\mathbf{w}}) \\
\overset{(10.16)}{\Longrightarrow} \quad & (\hat{\mathbf{v}}, b') \in \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}}).
\end{aligned}$$

This proves the claim. □

Step 5. *Constructing $\mathscr{O}_2'$ from $\mathscr{P}_2$.*

Together $B_j(\mathbf{w}, b)$ and $\hat{B}_j(\mathbf{w})$ encode how $\mathscr{O}_1'$ can be refined into $\mathscr{P}_1$. We now use this information to construct $\mathscr{O}_2'$ from $\mathscr{P}_2$. Claim 3 then shows that this construction indeed yields a family of partitions (whose polyads can be expressed in terms of those of $\mathscr{P}_2$). Finally, Claim 4 shows that the partition classes are appropriately regular.

For each $b \in [a_1^{\mathscr{O}}]$, we let

$$O_2'^{(1)}(b, b) := \bigcup_{(b', b') \in B_1(b, b)} P_2^{(1)}(b', b'). \tag{10.17}$$

We also let $\mathscr{O}_2'^{(1)} := \{O_2'^{(1)}(b, b) : b \in [a_1^{\mathscr{O}}]\}$. Again, as in (10.8), this easily implies that for each $j \in [k-1]$

$$\hat{\mathbf{u}} \in \hat{B}_{j+1} \text{ if and only if } \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\mathscr{O}_2'^{(1)}). \tag{10.18}$$

Note that for each $b \in [a_1^{\mathscr{O}}]$, by (O′12), we have that

$$|O_2'^{(1)}(b, b)| = \sum_{(b', b') \in B_1(b, b)} |P_2^{(1)}(b', b')| = (1 \pm 2\nu^{1/20}) m / a_1^{\mathscr{O}}. \tag{10.19}$$

In analogy to (10.11), for each $j \in [k-1] \setminus \{1\}$, $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$, and $b \in [a_j^{\mathscr{O}}]$, we define

$$O_2'^{(j)}(\hat{\mathbf{w}}, b) := \bigcup_{(\hat{\mathbf{u}}, b') \in B_j(\hat{\mathbf{w}}, b)} P_2^{(j)}(\hat{\mathbf{u}}, b'), \tag{10.20}$$

and for each $j \in [k-1] \setminus \{1\}$, we let

$$\mathscr{O}_2'^{(j)} := \{O_2'^{(j)}(\hat{\mathbf{w}}, b) : \hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}), b \in [a_j^{\mathscr{O}}]\}.$$

Moreover, let $\mathscr{O}_2' := \{\mathscr{O}_2'^{(j)}\}_{j=1}^{k-1}$. Note that since $\mathscr{P}_2$ is a family of partitions, (10.13) and (10.20) imply that $O_2'^{(j)}(\hat{\mathbf{w}}, b)$ is nonempty for each $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{w}}, b) \in$

$\hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$. The construction of $\mathscr{O}'_2$ also gives rise to a natural description of all polyads. Indeed, as in (3.12), we define for all $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$

$$\begin{aligned} \hat{O}'^{(j)}_2(\hat{\mathbf{w}}) &:= \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} O'^{(j)}_2(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}^{(1)}_*}) && (10.21) \\ &\overset{(10.20)}{=} \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} \bigcup_{(\hat{\mathbf{v}}, b') \in B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}^{(1)}_*})} P^{(j)}_2(\hat{\mathbf{v}}, b'). && (10.22) \end{aligned}$$

Claim 3. *$\mathscr{O}'_2$ is a family of partitions on V_2. Moreover, for all $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, we have*

$$\mathcal{K}_{j+1}(\hat{O}'^{(j)}_2(\hat{\mathbf{w}})) = \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}^{(j)}_2(\hat{\mathbf{u}})). \tag{10.23}$$

Proof. We will prove Claim 3 by applying the criteria in Lemma 3.13. For each $j \in [k-1] \setminus \{1\}$, $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_j^{\mathscr{O}}]$, let $\phi^{(j)}(O'^{(j)}_2(\hat{\mathbf{w}}, b)) := b$. Let $\ell \in [k-1]$ be the largest number satisfying the following.

$(\text{OP1})_\ell$ $\{\mathscr{O}'^{(j)}_2\}_{j=1}^{\ell}$ is a family of partitions,

$(\text{OP2})_\ell$ Let $O^{(j)}_*(\cdot,\cdot)$ and $\hat{O}^{(j)}_*(\cdot)$ be the maps defined as in (3.8)–(3.11) for $\{\mathscr{O}'^{(j)}_2\}_{j=1}^{k}$ and $\{\phi^{(j)}\}_{j=2}^{k}$. Then $\{\phi^{(j)}\}_{j=2}^{\ell}$ is an $(a_1^{\mathscr{O}}, \dots, a_\ell^{\mathscr{O}})$-labelling of $\{\mathscr{O}'^{(j)}_2\}_{j=1}^{\ell}$ such that for each $j \in [\ell] \setminus \{1\}$ and $(\hat{\mathbf{w}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have

$$O^{(j)}_*(\hat{\mathbf{w}}, b) = O'^{(j)}_2(\hat{\mathbf{w}}, b),$$

and for each $j \in [\ell]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ we have

$$\hat{O}^{(j)}_*(\hat{\mathbf{w}}) = \hat{O}'^{(j)}_2(\hat{\mathbf{w}}).$$

It is easy to check that $(\text{OP1})_1$–$(\text{OP2})_1$ hold and thus $\ell \geq 1$. Since $\{\mathscr{O}'^{(j)}_2\}_{j=1}^{\ell}$ is a family of partitions, $\hat{\mathscr{O}}'^{(j)}_2$ is well-defined for each $j \in [\ell]$. Claim 2(ii) now allows us to express (the cliques spanned by) these polyads in terms of those of $\mathscr{P}^{(j)}_2$.

Subclaim 1. *For each $j \in [\ell]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, we have*

$$\mathcal{K}_{j+1}(\hat{O}'^{(j)}_2(\hat{\mathbf{w}})) = \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}^{(j)}_2(\hat{\mathbf{u}})).$$

Proof. Consider $j \in [\ell]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Note that

$$\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \hat{P}^{(j)}_2(\hat{\mathbf{u}}) \overset{(3.12)}{=} \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \bigcup_{\hat{\mathbf{v}} \leq_{j,j-1} \hat{\mathbf{u}}} P^{(j)}_2(\hat{\mathbf{v}}, \mathbf{u}^{(j)}_{\mathbf{v}^{(1)}_*}).$$

This together with (10.22) and Claim 2(ii) implies that $\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \hat{P}^{(j)}_2(\hat{\mathbf{u}}) \subseteq \hat{O}'^{(j)}_2(\hat{\mathbf{w}})$, thus we obtain

$$\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}^{(j)}_2(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\hat{O}'^{(j)}_2(\hat{\mathbf{w}})). \tag{10.24}$$

On the other hand, we have

$$\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}} \mathcal{K}_{j+1}(\hat{P}^{(j)}_2(\hat{\mathbf{u}})) \overset{(10.18)}{=} \mathcal{K}_{j+1}(\mathscr{O}'^{(1)}_2) = \bigcup_{\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})} \mathcal{K}_{j+1}(\hat{O}'^{(j)}_2(\hat{\mathbf{w}})). \tag{10.25}$$

(Here the final equality follows from $(\mathrm{OP1})_\ell$, $(\mathrm{OP2})_\ell$ and Proposition 3.11(vi) applied to $\{O_2'^{(j)}\}_{j=1}^{\ell}$.) Consider a $(j+1)$-set $J \in \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))$. Then by (10.25) there exists $\hat{\mathbf{u}}' \in \hat{B}_{j+1}$ such that $J \in \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}}'))$.

We claim that $\hat{\mathbf{u}}' \in \hat{B}_{j+1}(\hat{\mathbf{w}})$. Indeed if not, then by (10.15), there exists $\hat{\mathbf{w}}' \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}}) \setminus \{\hat{\mathbf{w}}\}$ such that $\hat{\mathbf{u}}' \in \hat{B}_{j+1}(\hat{\mathbf{w}}')$, thus we have

$$J \in \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}')} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})) \overset{(10.24)}{\subseteq} \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}')).$$

Hence $\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}})) \cap \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}'))$ is nonempty (as it contains J). However, since $\{\mathscr{O}_2'^{(i)}\}_{i=1}^{j}$ is a family of partitions, this contradicts Proposition 3.11(vi) and (ix). Hence, we have $\hat{\mathbf{u}}' \in \hat{B}_{j+1}(\hat{\mathbf{w}})$, thus $J \in \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}}))$. The fact that this holds for arbitrary $J \in \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))$ combined with (10.24) proves the subclaim. □

Now, if $\ell = k-1$, then $\mathscr{O}_2'$ is a family of partitions and Subclaim 1 implies the moreover part of Claim 3. Assume that $\ell < k-1$ for a contradiction. Now we show that $\{\mathscr{O}_2'^{(j)}\}_{j=1}^{\ell+1}$ and the maps $\{O_2'^{(j)}(\cdot,\cdot), \hat{O}_2'^{(j)}(\cdot)\}_{j=1}^{\ell+1}$ satisfy the conditions (FP1)–(FP3) in Lemma 3.13. Condition (FP1) follows from $(\mathrm{OP1})_\ell$, (10.13) and (10.20). Condition (FP3) also holds because of (10.21) and the assumption that $\ell < k-1$. Property $(\mathrm{OP1})_\ell$ implies that (FP2) holds when $j \in [\ell]$. To check that (FP2) also holds for $j = \ell+1$, consider $\hat{\mathbf{w}} \in \hat{A}(\ell+1, \ell, \mathbf{a}^{\mathscr{O}})$. By Claim 2(i) and (10.13), we have that $\{B_{\ell+1}(\hat{\mathbf{w}}, 1), \dots, B_{\ell+1}(\hat{\mathbf{w}}, a_{\ell+1}^{\mathscr{O}})\}$ forms a partition of $\hat{B}_{\ell+1}(\hat{\mathbf{w}}) \times [a_{\ell+1}^{\mathscr{P}}]$ into nonempty sets. Thus by (10.20) and Subclaim 1, $\{O_2'^{(\ell+1)}(\hat{\mathbf{w}}, 1), \dots, O_2'^{(\ell+1)}(\hat{\mathbf{w}}, a_{\ell+1}^{\mathscr{O}})\}$ forms a partition of $\mathcal{K}_{\ell+1}(\hat{O}_2'^{(\ell)}(\hat{\mathbf{w}}))$ into nonempty sets. Thus (FP2) holds for $j = \ell+1$.

Hence, by (10.17) we can apply Lemma 3.13 to see that $(\mathrm{OP1})_{\ell+1}$ and $(\mathrm{OP2})_{\ell+1}$ hold, a contradiction to the choice of ℓ. Thus $\ell = k-1$, which proves the claim. □

By Claim 3, $\mathscr{O}_2'$ is a family of partitions and (10.20) implies that $\mathscr{P}_2 \prec \mathscr{O}_2'$. Consider any $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Note that

$$\begin{aligned} |\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))| &\overset{(10.10)}{=} \sum_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} |\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}}))| \overset{(10.2)}{=} |\hat{B}_{j+1}(\hat{\mathbf{w}})|(1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1}, \\ |\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))| &\overset{(10.23)}{=} \sum_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} |\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}}))| \overset{(10.4)}{=} |\hat{B}_{j+1}(\hat{\mathbf{w}})|(1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} m^{j+1}. \end{aligned} \tag{10.26}$$

For notational convenience, for each $\hat{\mathbf{w}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{O}})$, let

$$O_1'^{(k)}(\hat{\mathbf{w}}, 1) := G_1^{(k)} \cap \mathcal{K}_k(\hat{O}_1'^{(k-1)}(\hat{\mathbf{w}})) \quad \text{and} \quad O_2'^{(k)}(\hat{\mathbf{w}}, 1) := G_2^{(k)} \cap \mathcal{K}_k(\hat{O}_2'^{(k-1)}(\hat{\mathbf{w}})).$$

Claim 4. *For all $j \in [k-1]$, $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_{j+1}^{\mathscr{O}}]$, we have that $O_2'^{(j+1)}(\hat{\mathbf{w}}, b)$ is*

- *$(\varepsilon_0 + 3\nu^{1/20}, 1/a_{j+1}^{\mathscr{O}})$-regular with respect to $\hat{O}_2'^{(j)}(\hat{\mathbf{w}})$ if $j \le k-2$, and*
- *$(\varepsilon_0 + 3\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k}(\hat{\mathbf{w}}))$-regular with respect to $\hat{O}_2'^{(j)}(\hat{\mathbf{w}})$ if $j = k-1$.*

Proof. To prove Claim 4, we will apply Lemma 9.2 (see (J1) and (J2) and the preceding discussion), which allows us to transfer information about $\mathscr{O}_1'$ and $\mathscr{P}_1$ to $\mathscr{O}_2'$ and $\mathscr{P}_2$.

Fix $j \in [k-1]$, $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, and $b \in [a^{\mathscr{O}}_{j+1}]$. Let

$$d := \begin{cases} 1/a^{\mathscr{O}}_{j+1} & \text{if } j \le k-2, \\ d_{\mathbf{a}^{\mathscr{O}},k}(\hat{\mathbf{w}}) & \text{if } j = k-1. \end{cases} \tag{10.27}$$

Consider an arbitrary j-graph $F^{(j)} \subseteq \hat{O}'^{(j)}_2(\hat{\mathbf{w}})$ with

$$|\mathcal{K}_{j+1}(F^{(j)})| \ge (\varepsilon_0 + 3\nu^{1/20})|\mathcal{K}_{j+1}(\hat{O}'^{(j)}_2(\hat{\mathbf{w}}))|. \tag{10.28}$$

To prove the claim, it suffices to show that $d(O'^{(j+1)}_2(\hat{\mathbf{w}}, b) \mid F^{(j)}) = d \pm (\varepsilon_0 + 3\nu^{1/20})$. For each $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$, let

$$d(\hat{\mathbf{y}}) := \begin{cases} \frac{1}{a^{\mathscr{P}}_{j+1}} |\{b' : (\hat{\mathbf{y}}, b') \in B_{j+1}(\hat{\mathbf{w}}, b)\}| & \text{if } j \le k-2, \\ d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) & \text{if } \hat{\mathbf{y}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}), j = k-1, \\ 0 & \text{if } \hat{\mathbf{y}} \notin \hat{B}_{j+1}(\hat{\mathbf{w}}), j = k-1. \end{cases} \tag{10.29}$$

Thus Claim 2(i) and the above definition implies that

$$\textit{if } \hat{\mathbf{y}} \notin \hat{B}_{j+1}(\hat{\mathbf{w}}), \textit{ then we have } d(\hat{\mathbf{y}}) = 0. \tag{10.30}$$

Subclaim 2. *For all $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$ and each $i \in [2]$, we have that $O'^{(j+1)}_i(\hat{\mathbf{w}}, b)$ is $(\varepsilon'^{1/2}, d(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}^{(j)}_i(\hat{\mathbf{y}})$.*

Proof. First, we note that if $j \le k-2$, then by (10.11) and (10.20), for $i \in [2]$, we have

$$O'^{(j+1)}_i(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(\hat{P}^{(j)}_i(\hat{\mathbf{y}})) = \bigcup_{b' :\, (\hat{\mathbf{y}}, b') \in B_{j+1}(\hat{\mathbf{w}}, b)} P^{(j+1)}_i(\hat{\mathbf{y}}, b').$$

Together with (G11), (G21) and (10.29) this shows that $O'^{(j+1)}_i(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(\hat{P}^{(j)}_i(\hat{\mathbf{y}}))$ is the disjoint union of $a^{\mathscr{P}}_{j+1} d(\hat{\mathbf{y}}) \le \|\mathbf{a}^{\mathscr{P}}\|_\infty$ hypergraphs, each of which is $(\varepsilon', 1/a^{\mathscr{P}}_{j+1})$-regular with respect to $\hat{P}^{(j)}_i(\hat{\mathbf{y}})$. Thus Lemma 4.3 together with the fact that $\varepsilon' \ll 1/\|\mathbf{a}^{\mathscr{P}}\|_\infty$ implies Subclaim 2 in this case.

If $j = k-1$, then we have $b = 1$. If $\hat{\mathbf{y}} \in \hat{B}_k(\hat{\mathbf{w}})$, then $\mathcal{K}_k(\hat{P}^{(k-1)}_i(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{O}'^{(k-1)}_i(\hat{\mathbf{w}}))$ by (10.9) and Claim 3. Thus

$$O'^{(k)}_i(\hat{\mathbf{w}}, 1) \cap \mathcal{K}_k(\hat{P}^{(k-1)}_i(\hat{\mathbf{y}})) = G^{(k)}_i \cap \mathcal{K}_k(\hat{P}^{(k-1)}_i(\hat{\mathbf{y}})).$$

Together with (G11), (G21) and (10.29) this implies Subclaim 2 in this case.

If $\hat{\mathbf{y}} \notin \hat{B}_k(\hat{\mathbf{w}})$, then by (10.9), (10.14) and Claim 3 we have

$$O'^{(k)}_i(\hat{\mathbf{w}}, 1) \cap \mathcal{K}_k(\hat{P}^{(k-1)}_i(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{O}'^{(k-1)}_i(\hat{\mathbf{w}})) \cap \mathcal{K}_k(\hat{P}^{(k-1)}_i(\hat{\mathbf{y}})) = \emptyset.$$

Since $d(\hat{\mathbf{y}}) = 0$ in this case, this proves Subclaim 2. $\square$

In order to show that $O'^{(j+1)}_2(\hat{\mathbf{w}}, b)$ is $(\varepsilon_0 + 3\nu^{1/20})$-regular with respect to $\hat{O}'^{(j)}_2(\hat{\mathbf{w}})$, we will transfer all the calculations about hypergraph densities from the hypergraphs on V_2 to the hypergraphs on V_1, because there we have much better control over their structure. To this end we first use Lemma 9.2 to show the existence of two hypergraphs $J^{(j)}_1, J^{(j)}_2$ on V_1, V_2, respectively, that exhibit a very similar structure in terms of their densities and where $J^{(j)}_2$ is very close to $F^{(j)}$. Consequently, $J^{(j)}_1$ also resembles $F^{(j)}$.

More precisely, we now apply Lemma 9.2 with $F^{(j)}, \{\mathscr{P}^{(i)}_2\}^j_{i=1}, \{\mathscr{P}^{(i)}_1\}^j_{i=1}, j$ playing the roles of $H^{(k-1)}, \mathscr{P}, \mathscr{Q}, k-1$ respectively (we can do this by (P11), (G21) and Claim 3). We obtain j-graphs $J^{(j)}_2 \subseteq \mathcal{K}_j(\mathscr{P}^{(1)}_2)$ on V_2 and $J^{(j)}_1 \subseteq \mathcal{K}_j(\mathscr{P}^{(1)}_1)$ on V_1 such that

(J1) $|J^{(j)}_2 \triangle F^{(j)}| \le \nu\binom{m}{j}$, and

(J2) $d(\mathcal{K}_{j+1}(J^{(j)}_2) \mid \hat{P}^{(j)}_2(\hat{\mathbf{y}})) = d(\mathcal{K}_{j+1}(J^{(j)}_1) \mid \hat{P}^{(j)}_1(\hat{\mathbf{y}})) \pm \nu$ for each $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$.

Our next aim is to estimate $|\mathcal{K}_{j+1}(J_2^{(j)})|$ in terms of $|\mathcal{K}_{j+1}(J_1^{(j)})|$.

$$\begin{aligned}
\left|\mathcal{K}_{j+1}(J_2^{(j)})\right| &= \sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} \left|\mathcal{K}_{j+1}(J_2^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right| \\
&\overset{(10.4)}{=} (1\pm\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} m^{j+1} \sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} d(\mathcal{K}_{j+1}(J_2^{(j)}) \mid \hat{P}_2^{(j)}(\hat{\mathbf{y}})) \\
&\overset{(J2),(10.2)}{=} (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right| \sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} (d(\mathcal{K}_{j+1}(J_1^{(j)}) \mid \hat{P}_1^{(j)}(\hat{\mathbf{y}}))\pm\nu) \\
&= (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left(\sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})}\left|\mathcal{K}_{j+1}(J_1^{(j)})\cap\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right| \pm\nu \sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})}\left|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right|\right) \\
&= \frac{m^{j+1}}{n^{j+1}}(|\mathcal{K}_{j+1}(J_1^{(j)})|\pm 5\nu n^{j+1}).
\end{aligned} \tag{10.31}$$

Similarly, we obtain

$$\begin{aligned}
\left|O_2'^{(j+1)}(\hat{\mathbf{w}},b)\cap\ \mathcal{K}_{j+1}(J_2^{(j)})\right| &= \sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})}\left|O_2'^{(j+1)}(\hat{\mathbf{w}},b)\cap\mathcal{K}_{j+1}(J_2^{(j)})\cap\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right| \\
&\overset{\text{Subcl. 2}}{=} \sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})}\left(d(\hat{\mathbf{y}})\left|\mathcal{K}_{j+1}(J_2^{(j)})\cap\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right| \pm \varepsilon'^{1/4}\left|\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right|\right) \\
&\overset{(10.4),(10.30)}{=} \left((1\pm\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} m^{j+1}\sum_{\hat{\mathbf{y}}\in\hat{B}_{j+1}(\hat{\mathbf{w}})} d(\hat{\mathbf{y}})d(\mathcal{K}_{j+1}(J_2^{(j)}) \mid \hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right) \pm\varepsilon'^{1/4}m^{j+1} \\
&\overset{(J2),(10.2)}{=} \left((1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right|\sum_{\hat{\mathbf{y}}\in\hat{B}_{j+1}(\hat{\mathbf{w}})} d(\hat{\mathbf{y}})d(\mathcal{K}_{j+1}(J_1^{(j)}) \mid \hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right)\pm 4\nu m^{j+1} \\
&= (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left(\sum_{\hat{\mathbf{y}}\in\hat{B}_{j+1}(\hat{\mathbf{w}})} d(\hat{\mathbf{y}})\left|\mathcal{K}_{j+1}(J_1^{(j)})\cap\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right|\right)\pm 4\nu m^{j+1} \\
&\overset{(10.30),\text{Subcl. 2}}{=} (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left(\sum_{\hat{\mathbf{y}}\in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})}\left|O_1'^{(j+1)}(\hat{\mathbf{w}},b)\cap\mathcal{K}_{j+1}(J_1^{(j)})\cap\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right|\right)\pm 5\nu m^{j+1} \\
&= \frac{m^{j+1}}{n^{j+1}}\left(\left|O_1'^{(j+1)}(\hat{\mathbf{w}},b)\cap\mathcal{K}_{j+1}(J_1^{(j)})\right|\pm 10\nu n^{j+1}\right).
\end{aligned} \tag{10.32}$$

Note that (J1) implies that

$$\left|\mathcal{K}_{j+1}(J_2^{(j)}))\triangle\mathcal{K}_{j+1}(F^{(j)})\right| \le \nu\binom{m}{j}\cdot m \le \nu m^{j+1} \tag{10.33}$$

Since $F^{(j)}\subseteq \hat{O}'^{(j)}_2(\hat{\mathbf{w}})$ by assumption, (10.33) implies that

$$\left|\mathcal{K}_{j+1}(J_2^{(j)})\setminus\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))\right|\le \nu m^{j+1}. \tag{10.34}$$

We can transfer (10.34) to the corresponding graphs on V_1 as follows:

$$
\begin{aligned}
\left|\mathcal{K}_{j+1}(J_1^{(j)}) \setminus \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))\right| &\overset{(10.9)}{=} \sum_{\hat{\mathbf{y}}\notin \hat{B}_{j+1}(\hat{\mathbf{w}})} \left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right| \\
&\overset{(10.2)}{\le} (1+\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1} \sum_{\hat{\mathbf{y}}\notin \hat{B}_{j+1}(\hat{\mathbf{w}})} d(\mathcal{K}_{j+1}(J_1^{(j)}) \mid \hat{P}_1^{(j)}(\hat{\mathbf{y}})) \\
&\overset{(J2)}{\le} \left((1+\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1} \sum_{\hat{\mathbf{y}}\notin \hat{B}_{j+1}(\hat{\mathbf{w}})} d(\mathcal{K}_{j+1}(J_2^{(j)}) \mid \hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right) + 2\nu n^{j+1} \\
&\overset{(10.4),(10.23)}{\le} \frac{n^{j+1}}{m^{j+1}} \left|\mathcal{K}_{j+1}(J_2^{(j)}) \setminus \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))\right| + 5\nu n^{j+1} \\
&\overset{(10.34)}{\le} 6\nu n^{j+1}. \qquad (10.35)
\end{aligned}
$$

Next we show that $|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))|$ is not too small:

$$
\begin{aligned}
\left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{O'}_1^{(j)}(\hat{\mathbf{w}}))\right| &\overset{(10.35)}{\ge} \left|\mathcal{K}_{j+1}(J_1^{(j)})\right| - 6\nu n^{j+1} \\
&\overset{(10.31)}{\ge} \frac{n^{j+1}}{m^{j+1}} \left|\mathcal{K}_{j+1}(J_2^{(j)})\right| - 11\nu n^{j+1} \\
&\overset{(10.33)}{\ge} \frac{n^{j+1}}{m^{j+1}} \left|\mathcal{K}_{j+1}(F^{(j)})\right| - 12\nu n^{j+1} \\
&\overset{(10.28)}{\ge} \frac{n^{j+1}}{m^{j+1}} (\varepsilon_0 + 3\nu^{1/20}) \left|\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))\right| - 12\nu n^{j+1} \\
&\overset{(10.12),(10.26)}{\ge} (\varepsilon_0 + 2\nu^{1/20}) \left|\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))\right|. \qquad (10.36)
\end{aligned}
$$

Recall that d was defined in (10.27). We now can combine our estimates to conclude that

$$
\begin{aligned}
\left|O_2'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(F^{(j)})\right| &\overset{(10.32),(10.33)}{=} \frac{m^{j+1}}{n^{j+1}} \left(\left|O_1'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(J_1^{(j)})\right| \pm 20\nu n^{j+1}\right) \\
&\overset{(10.7),(10.36),(O'12)}{=} \frac{m^{j+1}}{n^{j+1}} \left((d \pm (\varepsilon_0 + 2\nu^{1/20})) \left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))\right| \pm 20\nu n^{j+1}\right) \\
&= \frac{m^{j+1}}{n^{j+1}} (d \pm (\varepsilon_0 + 2\nu^{1/20})) \left(|\mathcal{K}_{j+1}(J_1^{(j)})| - |\mathcal{K}_{j+1}(J_1^{(j)}) \setminus \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))|\right) \pm 20\nu m^{j+1} \\
&\overset{(10.31),(10.35)}{=} (d \pm (\varepsilon_0 + 2\nu^{1/20})) \left|\mathcal{K}_{j+1}(J_2^{(j)})\right| \pm 40\nu m^{j+1} \\
&\overset{(10.33)}{=} (d \pm (\varepsilon_0 + 2\nu^{1/20})) \left|\mathcal{K}_{j+1}(F^{(j)})\right| \pm 50\nu m^{j+1} \\
&= (d \pm (\varepsilon_0 + 3\nu^{1/20})) \left|\mathcal{K}_{j+1}(F^{(j)})\right|. \qquad (10.37)
\end{aligned}
$$

Here, we obtain the final inequality since (10.12), (10.26) and (10.28) imply $|\mathcal{K}_{j+1}(F^{(j)})| \ge \varepsilon_0^2 m^{j+1}$ and $\nu \ll \varepsilon_0$. (10.37) holds for all $F^{(j)} \subseteq \hat{O}_2'^{(j)}(\hat{\mathbf{w}})$ satisfying (10.28), thus $O_2'^{(j+1)}(\hat{\mathbf{w}}, b)$ is $(\varepsilon_0 + 3\nu^{1/20}, d)$-regular with respect to $\hat{O}_2'^{(j)}(\hat{\mathbf{w}})$. This with the definition of d completes the proof of Claim 4. $\square$

Claim 4 and (10.19) show that $\mathscr{O}_2'$ is a $(1/a_1^{\mathscr{O}}, \varepsilon_0 + 3\nu^{1/20}, \mathbf{a}^{\mathscr{O}}, 2\nu^{1/20})$-equitable family of partitions which is also an $(\varepsilon_0 + 3\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $G_2^{(k)}$ (as defined in Section 3.5.3). Note that $(\varepsilon_0 + 3\nu^{1/20})/3 \le \varepsilon_0/2$, thus $((\varepsilon_0 + 3\nu^{1/20})/3, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance. Since $|G_2^{(k)} \triangle H_2^{(k)}| \leq \nu\binom{m}{k}$ by (G22), this means that we can apply Lemma 4.15 with the following objects and parameters.

object/parameter	$\mathscr{O}_2'$	$\mathscr{O}_2'$	ν	$\varepsilon_0 + 3\nu^{1/20}$	$d_{\mathbf{a}^{\mathscr{O}},k}$	$G_2^{(k)}$	$H_2^{(k)}$
playing the role of	$\mathscr{P}$	$\mathscr{Q}$	ν	ε	$d_{\mathbf{a},k}$	$H^{(k)}$	$G^{(k)}$

Hence $\mathscr{O}_2'$ is also an $(\varepsilon_0 + 4\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $H_2^{(k)}$.

Step 6. *Adjusting $\mathscr{O}_2'$ into an equipartition $\mathscr{O}_2$.*

Finally, we modify $\mathscr{O}_2'$ to turn it from an 'almost' equipartition into an equipartition $\mathscr{O}_2$. For this we apply Lemma 4.16 with $\mathscr{O}_2', H_2^{(k)}\varepsilon_0+4\nu^{1/20}, 2\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k}$ playing the roles of $\mathscr{P}, H^{(k)}, \varepsilon, \lambda, d_{\mathbf{a},k}$ respectively. This guarantees an $(\varepsilon_0 + 3\nu^{1/200}, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition $\mathscr{O}_2$ of $H_2^{(k)}$, which completes the proof. □

10.2. **Random samples.** To prove our results about random samples of hypergraphs, we will need the following lemma due to Czygrinow and Nagle. It states that ε-regularity of a random complex is inherited by a random sample (but with significantly worse parameters).

Lemma 10.2 (Czygrinow and Nagle [16])**.** *Suppose $0 < 1/m_0, 1/s, \varepsilon \ll \varepsilon', d_0, 1/\ell, 1/k \leq 1$ and $k, \ell \in \mathbb{N} \setminus \{1\}$ with $\ell \geq k$. Suppose $\mathscr{H} = \{H^{(j)}\}_{j=1}^k$ is an $(\varepsilon, (d_2, \dots, d_k))$-regular (ℓ, k)-complex with $H^{(1)} = \{V_1, \dots, V_\ell\}$ such that $d_i \in [d_0, 1]$, and $|V_i| > m_0$ for all $i \in [\ell]$. Let $s_1, \dots, s_\ell \geq s$ be integers such that $|V_i| \geq s_i$. Then for subsets $S_i \in \binom{V_i}{s_i}$ chosen uniformly at random, $\{H^{(j)}[S_1 \cup S_2 \cup \dots \cup S_\ell]\}_{j=1}^k$ is an $(\varepsilon', (d_2, \dots, d_k))$-regular (ℓ, k)-complex with probability at least $1 - e^{-\varepsilon s}$.*

Note that in [16], the lemma is only stated for the case $\ell = k$, but the case $\ell \geq k$ follows via a union bound. The next lemma generalizes Lemma 10.2 and shows how an equitable partition of a k-graph transfers with high probability to a random sample.

Lemma 10.3. *Suppose $0 < 1/n < 1/q \ll \varepsilon \ll \varepsilon' \ll 1/t, 1/k$, and $k \in \mathbb{N} \setminus \{1\}$. Suppose that $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of a k-graph H on vertex set V with $|V| = n$ and $\mathbf{a} \in [t]^{k-1}$. Then for a set $Q \in \binom{V}{q}$ chosen uniformly at random, with probability at least $1 - e^{-\varepsilon^3 q}$, there exists an $(\varepsilon', \mathbf{a}, d_{\mathbf{a},k})$-equitable family of partitions $\mathscr{Q}$ of $H[Q]$.*

The parameter ε' in Lemma 10.3 will be too large for our purposes. But we can combine Lemmas 10.1 and 10.3 to obtain the stronger assertion stated in $(\mathrm{Q1})_{6.1}$ of Lemma 6.1.

Proof of Lemma 10.3. We choose an additional constant ν such that

$$0 < \varepsilon \ll \nu \ll 1/t, 1/k, \varepsilon'.$$

Let Q be a set of q vertices selected uniformly at random in V. Write $\mathscr{P}^{(1)} = \{V_1, \dots, V_{a_1}\}$ and let $S_i := Q \cap V_i$. For $\mathbf{S} = (s_1, \dots, s_{a_1})$ with $\sum_{i=1}^{a_1} s_i = q$ and $s_i \in \mathbb{N} \cup \{0\}$, let $\mathcal{E}(\mathbf{S})$ be the event that $|S_i| = s_i$ for all $i \in [a_1]$, and let

$$I := \left\{ \mathbf{S} : s_i = (1 \pm \varepsilon)\frac{q}{a_1} \text{ for each } i \in [a_1] \right\}.$$

By Lemma 3.2, we conclude

$$\mathbb{P}\left[\bigvee_{\mathbf{S} \in I} \mathcal{E}(\mathbf{S})\right] \geq 1 - 2a_1 e^{-\varepsilon^2 q^2/(a_1^2 q)} \geq 1 - e^{-\varepsilon^{5/2} q}. \tag{10.38}$$

Recall that for $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ denotes the $(k, k-1)$-complex induced by $\hat{\mathbf{x}}$ in $\mathscr{P}$ as defined in (3.14) (as remarked at (3.14), for a family of partitions $\mathscr{P}$, $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ is indeed a $(k, k-1)$-complex). Let

$$A := \{\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) : d_{\mathbf{a},k}(\hat{\mathbf{x}}) \geq \nu\} \quad \text{and} \quad G := \bigcup_{\hat{\mathbf{x}} \in A} \Big(H \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))\Big) \cup \Big(H \setminus \mathcal{K}_k(\mathscr{P}^{(1)})\Big).$$

It is easy to see that $G \subseteq H$ and $|G \triangle H| \leq 2\nu\binom{n}{k}$.

For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, let

$$\hat{\mathcal{P}}'(\hat{\mathbf{x}}) := \hat{\mathcal{P}}(\hat{\mathbf{x}}) \cup \Big\{G \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))\Big\}.$$

Note that $\hat{\mathcal{P}}'(\hat{\mathbf{x}})$ is an $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}, d_{\mathbf{a},k}(\hat{\mathbf{x}})))$-regular (k, k)-complex for each $\hat{\mathbf{x}} \in A$ and $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ is a $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}))$-regular $(k, k-1)$-complex for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$.

For each $\hat{\mathbf{x}} \in A$, we define the following event:

($\hat{\mathcal{E}}(\hat{\mathbf{x}})$) $\hat{\mathcal{P}}'(\hat{\mathbf{x}})[Q]$ is an $(\varepsilon'/2, (1/a_2, \dots, 1/a_{k-1}, d_{\mathbf{a},k}(\hat{\mathbf{x}})))$-regular (k, k)-complex.

For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) \setminus A$, we also define the following event:

($\hat{\mathcal{E}}(\hat{\mathbf{x}})$) $\hat{\mathcal{P}}(\hat{\mathbf{x}})[Q]$ *is an* $(\varepsilon'/2, (1/a_2, \dots, 1/a_{k-1}))$*-regular* $(k, k-1)$*-complex.*

Note that for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) \setminus A$, the event $\hat{\mathcal{E}}(\hat{\mathbf{x}})$ implies that $\hat{\mathcal{P}}'(\hat{\mathbf{x}})[Q]$ is an $(\varepsilon'/2, (1/a_2, \dots, 1/a_{k-1}, d_{\mathbf{a},k}(\hat{\mathbf{x}}))$-regular complex as $d_{\mathbf{a},k}(\hat{\mathbf{x}}) \leq \nu \ll \varepsilon'$ and $(G \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})))[Q] = \emptyset$. Thus we have that

$$\bigwedge_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} \hat{\mathcal{E}}(\hat{\mathbf{x}}) \text{ implies that } \mathscr{P}[Q] \text{ is an } (\varepsilon'/2, d_{\mathbf{a},k})\text{-partition of } G[Q]. \tag{10.39}$$

Consider any $\hat{\mathbf{x}} \in A$. Since q is sufficiently large, we may apply Lemma 10.2 with the following objects and parameters.

object/parameter	$\hat{\mathcal{P}}'(\hat{\mathbf{x}})$	S_i	$1/a_1, \dots, 1/a_{k-1}$	$d_{\mathbf{a},k}(\hat{\mathbf{x}})$	$\varepsilon'/2$	$\nu/2$
playing the role of	$\mathcal{H}$	S_i	$d_1, \dots, d_{k-1}$	d_k	ε'	d_0

We obtain for any fixed $\mathbf{S} \in I$, that

$$\mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}}) \mid \mathcal{E}(\mathbf{S})] \geq 1 - e^{-\varepsilon^2 q}. \tag{10.40}$$

In a similar way, for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) \setminus A$, we can apply Lemma 10.2 to $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ to obtain that (10.40) holds, too. Thus for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, we obtain

$$\begin{aligned} \mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}})] &= \sum_{\mathbf{S} \in I} \mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}}) \mid \mathcal{E}(\mathbf{S})]\mathbb{P}[\mathcal{E}(\mathbf{S})] + \sum_{\mathbf{S} \notin I} \mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}}) \mid \mathcal{E}(\mathbf{S})]\mathbb{P}[\mathcal{E}(\mathbf{S})] \\ &\overset{(10.40)}{\geq} (1 - e^{-\varepsilon^2 q}) \sum_{\mathbf{S} \in I} \mathbb{P}[\mathcal{E}(\mathbf{S})] \overset{(10.38)}{\geq} (1 - e^{-\varepsilon^2 q})(1 - e^{-\varepsilon^{5/2} q}) \geq 1 - 2e^{-\varepsilon^{5/2} q}. \end{aligned}$$

Let $\mathcal{E}_0$ be the event that

$$|(H \setminus G)[Q]| \leq 3\nu\binom{q}{k}. \tag{10.41}$$

Since $|H \setminus G| \leq 2\nu\binom{n}{k}$, we may apply Lemma 3.3 with $n, H \setminus G, Q, \nu/2$ playing the roles of n, H, Q, ν to obtain

$$\mathbb{P}[\mathcal{E}_0] \geq 1 - e^{-\nu^3 q}.$$

As $|\hat{A}(k,k-1,\mathbf{a})| \leq t^{2^k}$ by Proposition 3.11(viii), we conclude

$$\mathbb{P}\left[\mathcal{E}_0 \wedge \bigwedge_{\hat{\mathbf{x}}\in \hat{A}(k,k-1,\mathbf{a})} \hat{\mathcal{E}}(\hat{\mathbf{x}})\right] \geq 1 - e^{-\nu^3 q} - 2t^{2^k} e^{-\varepsilon^{5/2}q} \geq 1 - e^{-\varepsilon^{8/3}q}. \tag{10.42}$$

Now suppose that $\mathcal{E}(\mathbf{S})$ holds for some $\mathbf{S} \in I$ and that $\mathcal{E}_0 \wedge \bigwedge_{\hat{\mathbf{x}}\in \hat{A}(k,k-1,\mathbf{a})} \hat{\mathcal{E}}(\hat{\mathbf{x}})$ holds. Then $\mathscr{P}$ induces a family of partitions $\mathscr{P}[Q]$ on Q which is $(1/a_1, \varepsilon'/2, \mathbf{a}, \varepsilon)$-equitable. Note $\varepsilon' \ll 1/t, 1/k$, thus $(\varepsilon'/6, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Since $\nu \ll \varepsilon' \ll 1/t$, by using (10.39), we can apply Lemma 4.15 with the following objects and parameters.

object/parameter	$\mathscr{P}[Q]$	$\mathscr{P}[Q]$	3ν	$\varepsilon'/2$	$d_{\mathbf{a},k}$	$G[Q]$	$H[Q]$	ε
playing the role of	$\mathscr{P}$	$\mathscr{Q}$	ν	ε	$d_{\mathbf{a},k}$	$H^{(k)}$	$G^{(k)}$	λ

This implies that $\mathscr{P}[Q]$ is an $(1/a_1, \varepsilon'/2 + \nu^{1/7}, \mathbf{a}, \nu^{1/7})$-equitable family of partitions on Q which is also an $(\varepsilon'/2 + \nu^{1/7}, d_{\mathbf{a},k})$-partition of $H[Q]$.

Finally, since $\nu \ll \varepsilon'$, Lemma 4.16 implies that there exists a family of partitions $\mathscr{Q}$ which is an $(\varepsilon', \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H[Q]$. By (10.38) and (10.42), this completes the proof. □

Next we proceed with the proof of Lemma 6.1. To prove (Q1)$_{6.1}$, we first apply the regular approximation lemma (Theorem 3.8) to obtain an ε-equitable partition $\mathscr{P}_1$ of a k-graph G that is very close to H. Lemma 10.3 implies that (with high probability) $G[Q]$ has a regularity partition $\mathscr{P}_2$ which has the same parameters as $\mathscr{P}_1$, except for a much worse regularity parameter ε'. However, we still have $\varepsilon' \ll \varepsilon_0$ and thus we can now apply Lemma 10.1 to G, $G[Q]$ and $\mathscr{P}_1, \mathscr{P}_2, \mathscr{O}_1$ to obtain an equitable partition $\mathscr{O}_2$ of $G[Q]$ which reflects $\mathscr{O}_1$. By Lemma 4.15, $\mathscr{O}_2$ is also an equitable partition of $H[Q]$. To prove (Q2)$_{6.1}$, we again apply Lemma 10.1 but with the roles of G and $G[Q]$ interchanged.

Proof of Lemma 6.1. Choose new constants η, ν so that $c \ll \eta \ll \nu \ll \delta$. Let $\overline{\varepsilon} : \mathbb{N}^{k-1} \to (0,1]$ be a function such that for all $\mathbf{b} \in \mathbb{N}^{k-1}$, we have

$$\overline{\varepsilon}(\mathbf{b}) \ll \|\mathbf{b}\|_\infty^{-k}.$$

Let $t_0 := t_{3.8}(\eta, \nu, \overline{\varepsilon})$.

By Theorem 3.8, there exists a t_0-bounded $(\eta, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable family of partitions $\mathscr{P}_1 = \mathscr{P}_1(k-1, \mathbf{a}^{\mathscr{P}})$, a k-graph G and a density function $d_{\mathbf{a}^{\mathscr{P}},k}$ such that the following hold.

(G1)$_{6.1}$ $\mathscr{P}_1$ is an $(\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), d_{\mathbf{a}^{\mathscr{P}},k})$-partition of G, and

(G2)$_{6.1}$ $|G \triangle H| \leq \nu\binom{n}{k}$.

(Here (G1)$_{6.1}$ follows from Theorem 3.8(ii), (3.16), and Lemma 4.6.)

Let $\varepsilon := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$ and $T := \|\mathbf{a}^{\mathscr{P}}\|_\infty$. As t_0 only depends on $\eta, \nu, \overline{\varepsilon}$, we may assume that $c \ll \varepsilon$. Together with the choice of $\overline{\varepsilon}$ and the fact that $1/T \leq 1/a_1^{\mathscr{P}} \leq \eta$, this implies

$$0 < 1/n < 1/q \ll c \ll \varepsilon \ll 1/T, 1/a_1^{\mathscr{P}} \ll \nu \ll \delta \ll \varepsilon_0 \leq 1.$$

Additionally, we choose ε' so that

$$0 < 1/n < 1/q \ll c \ll \varepsilon \ll \varepsilon' \ll 1/T, 1/a_1^{\mathscr{P}} \ll \nu \ll \delta \ll \varepsilon_0 \leq 1. \tag{10.43}$$

Let $\mathcal{E}_0$ be the event that

$$|G[Q] \triangle H[Q]| \leq 2\nu\binom{q}{k}.$$

Property $(G2)_{6.1}$ and Lemma 3.3 imply that

$$\mathbb{P}[\mathcal{E}_0] \geq 1 - e^{-\nu^3 q}. \tag{10.44}$$

Let $\mathcal{E}_1$ be the event that there exists a family of partitions $\mathscr{P}_2 = \mathscr{P}_2(k-1, \mathbf{a}^{\mathscr{P}})$ which is an $(\varepsilon', \mathbf{a}^{\mathscr{P}}, d_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of $G[Q]$. Since $\varepsilon \ll \varepsilon'$, Lemma 10.3 implies that

$$\mathbb{P}[\mathcal{E}_1] \geq 1 - e^{-\varepsilon^3 q}. \tag{10.45}$$

Thus (10.44) and (10.45) imply that

$$\mathbb{P}[\mathcal{E}_0 \wedge \mathcal{E}_1] \geq 1 - 2e^{-\varepsilon^3 q} \geq 1 - e^{-cq}. \tag{10.46}$$

Hence it suffices to show that the two statements $(Q1)_{6.1}$ and $(Q2)_{6.1}$ both hold if we condition on $\mathcal{E}_0 \wedge \mathcal{E}_1$.

First, assume $\mathcal{E}_0 \wedge \mathcal{E}_1$ holds and $\mathscr{O}_1$ exists as in $(Q1)_{6.1}$. As $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 4.15 with $\mathscr{O}_1$, $\mathscr{O}_1$, ν, ε_0, $d_{\mathbf{a},k}$, H and G playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $H^{(k)}$ and $G^{(k)}$, respectively, to conclude that $\mathscr{O}_1$ is also an $(\varepsilon_0+\delta/3, d_{\mathbf{a},k})$-partition of G.

Note that $(\varepsilon_0 + \delta/3)/2 \leq 2\varepsilon_0/3$, thus $((\varepsilon_0 + \delta/3)/2, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. By this and (10.43), we can apply Lemma 10.1 with the following objects and parameters.

object/parameter	n	q	$\mathscr{O}_1$	$\mathscr{P}_1$	$\mathscr{P}_2$	G	$G[Q]$	ε'	T	$\delta/3$	$\varepsilon_0+\delta/3$	$d_{\mathbf{a}^{\mathscr{P}},k}$	$d_{\mathbf{a},k}$
playing the role of	n	m	$\mathscr{O}_1$	$\mathscr{Q}_1$	$\mathscr{Q}_2$	$H_1^{(k)}$	$H_2^{(k)}$	ε	T	δ	ε_0	$d_{\mathbf{a}^{\mathscr{Q}},k}$	$d_{\mathbf{a}^{\mathscr{O}},k}$

Hence there exists an $(\varepsilon_0 + 2\delta/3, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_2$ of $G[Q]$. Since $\mathcal{E}_0$ holds and $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 4.15 with $\mathscr{O}_2$, $\mathscr{O}_2$, 2ν, $\varepsilon_0 + 2\delta/3$, $d_{\mathbf{a},k}$, $H[Q]$ and $G[Q]$ playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $G^{(k)}$ and $H^{(k)}$, respectively. Then we conclude that $\mathscr{O}_2$ is an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H[Q]$. Thus $\mathcal{E}_0 \wedge \mathcal{E}_1$ implies $(Q1)_{6.1}$.

Now assume $\mathcal{E}_0 \wedge \mathcal{E}_1$ holds and $\mathscr{O}_2$ exists as in $(Q2)_{6.1}$. As $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 4.15 with $\mathscr{O}_2$, $\mathscr{O}_2$, 2ν, ε_0, $d_{\mathbf{a},k}$, $H[Q]$ and $G[Q]$ playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $H^{(k)}$ and $G^{(k)}$, respectively. Thus $\mathscr{O}_2$ is an $(\varepsilon_0 + \delta/3, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $G[Q]$. By (10.43) and the fact that R is a regularity instance, we can apply Lemma 10.1 with the following objects and parameters.

object/parameter	q	n	$\mathscr{O}_2$	$\mathscr{P}_2$	$\mathscr{P}_1$	$G[Q]$	G	ε'	T	$\delta/3$	$\varepsilon_0+\delta/3$	$d_{\mathbf{a}^{\mathscr{P}},k}$	$d_{\mathbf{a},k}$
playing the role of	n	m	$\mathscr{O}_1$	$\mathscr{Q}_1$	$\mathscr{Q}_2$	$H_1^{(k)}$	$H_2^{(k)}$	ε	T	δ	ε_0	$d_{\mathbf{a}^{\mathscr{Q}},k}$	$d_{\mathbf{a}^{\mathscr{O}},k}$

Thus there exists a family of partitions $\mathscr{O}_1$ which is an $(\varepsilon_0 + 2\delta/3, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of G. By $(G2)_{6.1}$ and the fact that $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 4.15 with $\mathscr{O}_1$, $\mathscr{O}_1$, ν, $\varepsilon_0+2\delta/3$, $d_{\mathbf{a},k}$, H and G playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $G^{(k)}$ and $H^{(k)}$, respectively. We conclude that $\mathscr{O}_1$ is an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of H. Thus $\mathcal{E}_0 \wedge \mathcal{E}_1$ implies $(Q2)_{10.3}$. □

11. Applications

In this section we illustrate how Theorem 1.3 can be applied, first to counting subgraphs, then to the maximum cut problem.

11.1. **Testing the injective homomorphism density.** We first show how to test the (injective) homomorphism density, where a homomorphism of a k-graph F into a k-graph H is a function $f : V(F) \to V(H)$ that maps edges onto edges. Let $\mathrm{inj}(F, H)$ be the number of (vertex-)injective homomorphisms from F into H and let $t_{\mathrm{inj}}(F, H) := \mathrm{inj}(F, H)/(n)_{|V(F)|}$.

Corollary 11.1. *Suppose $p, \delta \in (0,1)$, $k \in \mathbb{N} \setminus \{1\}$, and F is a k-graph. Let $\mathbf{P}$ be the property that a k-graph H satisfies $t_{\rm inj}(F,H) = p \pm \delta$. Then $\mathbf{P}$ is testable.*

Before we continue with the proof of Corollary 11.1, we state a simple proposition.

Proposition 11.2. *Suppose $0 < 1/n \ll \nu, 1/k, 1/\ell$ and $\nu \ll \alpha, 1-\alpha$. Let F be an ℓ-vertex k-graph and H be an n-vertex k-graph. If $t_{\rm inj}(F,H) = \alpha \pm \nu$ for some $\alpha \in (0,1)$, then there exists an n-vertex k-graph G with $t_{\rm inj}(F,G) = \alpha \pm 1/n$ and $|G \triangle H| \leq (\frac{2\nu}{\min\{\alpha,1-\alpha\}})^{1/\ell}\binom{n}{k}$.*

Proof. Suppose first that $t_{\rm inj}(F,H) > \alpha + 1/n$. Let $\varepsilon := (\frac{2\nu}{\alpha})^{1/\ell}$. By an averaging argument, there exists a subgraph H' of H on εn vertices such that $t_{\rm inj}(F,H') > \alpha + 1/n$. Clearly, $|H'| \leq \varepsilon\binom{n}{k}$. Moreover, after removing all edges contained in H' from H, we reduce the number of injective homomorphisms from F to H by at least ${\rm inj}(F,H') \geq \alpha(\varepsilon n)_\ell \geq \nu(n)_\ell$. Thus if instead we remove a suitable number of these edges iteratively, we can reach a spanning subgraph G of H with $t_{\rm inj}(F,G) = \alpha \pm 1/n$ as any single edge removal decreases the number of homomorphisms by at most $2n^{\ell-2}$. The case $t_{\rm inj}(F,H) < \alpha - 1/n$ works similarly. □

Proof of Corollary 11.1. Let $\ell := |V(F)|$. We may assume that $|F| > 0$ as otherwise $t_{\rm inj}(F,H) = 1$ for every n-vertex graph H with $n \geq \ell$. By Theorem 1.3, it suffices to verify that $\mathbf{P}$ is regular reducible.

Suppose $\beta > 0$. We may assume that $\beta \ll p-\delta, 1/\ell$ if $p-\delta > 0$ and $\beta \ll 1-(p+\delta), 1/\ell$ if $p+\delta < 1$. We write $\beta' := \beta^{\ell+1}$ and $\beta'' := 2^{-\binom{\ell}{k}}\beta'$. We fix some function $\overline{\varepsilon} : \mathbb{N}^{k-1} \to (0,1)$ such that $\overline{\varepsilon}(\mathbf{a}) \ll \|\mathbf{a}\|_\infty^{-k}$ for all $(a_1,\dots,a_{k-1}) = \mathbf{a} \in \mathbb{N}^{k-1}$. We choose constants ε, η, and $n_0, T \in \mathbb{N}$ such that $1/n_0 \ll \varepsilon \ll 1/T \ll \eta \ll \beta, 1/k, 1/\ell$. In particular, we have $n_0 \geq n_{3.8}(\eta, \beta''\ell^{-k}/2, \overline{\varepsilon})$, $T \geq t_{3.8}(\eta, \beta''\ell^{-k}/2, \overline{\varepsilon})$ and $\varepsilon \ll \overline{\varepsilon}(\mathbf{a})$ for all $\mathbf{a} \in [T]^{k-1}$. For simplicity, we consider only n-vertex k-graphs H with $n \geq n_0$.

Let $\mathbf{I}$ be the collection of regularity instances $R = (\varepsilon'', \mathbf{a}, d_{\mathbf{a},k})$ such that

(R1)$_{11.1}$ $\varepsilon'' \in \{\varepsilon, 2\varepsilon, \dots, \lceil(\overline{\varepsilon}(\mathbf{a}))^{1/2}\varepsilon^{-1}\rceil\varepsilon\}$,
(R2)$_{11.1}$ $\mathbf{a} \in [T]^{k-1}$ and $a_1 > \eta^{-1}$, and
(R3)$_{11.1}$ $d_{\mathbf{a},k}(\hat{\mathbf{x}}) \in \{0, \varepsilon^2, 2\varepsilon^2, \dots, 1\}$ for every $\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})$.

Observe that by construction $|\mathbf{I}|$ is bounded by a function of β, k and ℓ. We define

$$\mathcal{R} := \left\{ (\varepsilon'', \mathbf{a}, d_{\mathbf{a},k}) \in \mathbf{I} : \sum_{J\colon |V(J)|=\ell} {\rm inj}(F,J) \cdot IC(J, d_{\mathbf{a},k})/\ell! = p \pm (\delta + \beta') \right\}.$$

First, suppose that an n-vertex k-graph H satisfies $\mathbf{P}$. Then

$$\frac{1}{\ell!} \sum_{J\colon |V(J)|=\ell} {\rm inj}(F,J) \cdot \mathbf{Pr}(J,H) = t_{\rm inj}(F,H) = p \pm \delta. \tag{11.1}$$

By applying the regular approximation lemma (Theorem 3.8) with $H, \eta, \beta''\ell^{-k}/2, \overline{\varepsilon}$ playing the roles of $H, \eta, \nu, \varepsilon$, we obtain a k-graph G and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ such that

(I) $\mathscr{P}$ is $(\eta, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable for some $\mathbf{a}^{\mathscr{P}} \in [T]^{k-1}$,
(II) G is perfectly $\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$, and
(III) $|G \triangle H| \leq \beta''\ell^{-k}\binom{n}{k}/2$.

Let $\varepsilon' := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$. By the choice of $\overline{\varepsilon}$ and η, we conclude that $0 < \varepsilon' \ll 1/\|\mathbf{a}^{\mathscr{P}}\|_\infty \leq 1/a_1^{\mathscr{P}} \ll \beta, 1/k, 1/\ell$ and by the choice of ε, we obtain $\varepsilon \ll \varepsilon'$. Note that if a k-graph J is (ε', d)-regular with respect to a $(k-1)$-graph J', then J is (ε'', d')-regular with respect to J' for some $d' \in \{0, \varepsilon^2, 2\varepsilon^2, \dots, 1\}$ and $\varepsilon'' \in \{\varepsilon, 2\varepsilon, \dots, \lceil \varepsilon'^{1/2}\varepsilon^{-1}\rceil\varepsilon\} \cap [2\varepsilon', 3\varepsilon']$. Thus there exists

$$R_G = (\varepsilon'', \mathbf{a}^{\mathscr{P}}, d^G_{\mathbf{a}^{\mathscr{P}},k}) \in \mathbf{I} \tag{11.2}$$

such that G satisfies R_G.

For every ℓ-vertex k-graph J, Proposition 3.1 with (III) and Corollary 4.10 imply that

$$IC(J, d^G_{\mathbf{a}^{\mathscr{P}},k}) = \mathbf{Pr}(J, G) \pm \beta''/2 = \mathbf{Pr}(J, H) \pm \beta''. \tag{11.3}$$

Hence

$$\begin{aligned} \frac{1}{\ell!} \sum_{J\colon |V(J)|=\ell} \mathrm{inj}(F, J) \cdot IC(J, d^G_{\mathbf{a}^{\mathscr{P}},k}) &\overset{(11.3)}{=} \frac{1}{\ell!} \sum_{J\colon |V(J)|=\ell} \mathrm{inj}(F, J) \cdot (\mathbf{Pr}(J, H) \pm \beta'') \\ &\overset{(11.1)}{=} p \pm (\delta + \beta'). \end{aligned} \tag{11.4}$$

By the definition of $\mathcal{R}$ and (11.2), this implies that $R_G \in \mathcal{R}$ and so H is indeed β-close to a graph G satisfying R_G, one of the regularity instances of $\mathcal{R}$.

Now we show that if $\alpha > \beta$ and H is α-far from $\mathbf{P}$, then H is $(\alpha - \beta)$-far from all $R \in \mathcal{R}$. We prove this by verifying the following statement: if H is $(\alpha - \beta)$-close to some $R \in \mathcal{R}$, then it is α-close to $\mathbf{P}$.

Suppose H is $(\alpha - \beta)$-close to some $R = (\varepsilon'', \mathbf{a}, d_{\mathbf{a},k}) \in \mathcal{R}$. Then there exists a k-graph G_R such that G_R satisfies R and $|H \triangle G_R| \le (\alpha - \beta)\binom{n}{k}$. By the definition of $\mathcal{R}$, we have $\sum_{J\colon |V(J)|=\ell} \mathrm{inj}(F, J) \cdot IC(J, d_{\mathbf{a},k})/\ell! = p \pm (\delta + \beta')$. Similarly to the calculations leading to (11.4), we obtain $t_{\mathrm{inj}}(F, G_R) = p \pm (\delta + 2\beta')$.

By Proposition 11.2, there exists a k-graph G such that $t_{\mathrm{inj}}(F, G) = p \pm \delta$ and $|G \triangle G_R| \le (\beta/2) \cdot \binom{n}{k}$. Therefore, G satisfies $\mathbf{P}$ and $|H \triangle G| \le |H \triangle G_R| + |G_R \triangle G| < \alpha\binom{n}{k}$ which implies that H is α-close to satisfying $\mathbf{P}$. Thus, $\mathbf{P}$ is indeed regular reducible. $\square$

11.2. **Testing the maximum cut size.** We proceed with another corollary of Theorem 1.3. For a given n-vertex k-graph H, we define the following parameter measuring the size of a largest ℓ-partite subgraph:

$$\mathrm{maxcut}_\ell(H) := \binom{n}{k}^{-1} \max_{\substack{\{V_1,\dots,V_\ell\} \text{ is a} \\ \text{partition of } V(H)}} \{|\mathcal{K}_k(V_1, \dots, V_\ell) \cap H|\}.$$

We let

$$c_{\ell,k}(n) := \binom{n}{k}^{-1} \sum_{\Lambda \in \binom{[\ell]}{k}} \prod_{\lambda \in \Lambda} \left\lfloor \frac{n + \lambda - 1}{\ell} \right\rfloor.$$

Thus $c_{\ell,k}(n)\binom{n}{k}$ is the number of edges of the complete ℓ-partite k-graph on n vertices whose vertex class sizes are as equal as possible. In particular, any n-vertex k-graph H satisfies $\mathrm{maxcut}_\ell(H) \le c_{\ell,k}(n)$.

Corollary 11.3. *Suppose $\ell, k \in \mathbb{N} \setminus \{1\}$ and $c = c(n)$ is such that $0 \le c \le c_{\ell,k}(n)$. Let $\mathbf{P}$ be the property that an n-vertex k-graph H satisfies $\mathrm{maxcut}_\ell(H) \ge c$. Then $\mathbf{P}$ is testable.*

Note that since the property of having a given edge density is trivially testable, it follows from Corollary 11.3 that the property of being strongly ℓ-colourable is also testable (in a strong colouring, we require all vertices of an edge to have distinct colours).

Before we prove Corollary 11.3, we need to introduce some notation and make a few observations. For a given vector $\mathbf{a} \in \mathbb{N}^{k-1}$, a density function $d_{\mathbf{a},k} : \hat{A}(k, k-1, \mathbf{a}) \to [0, 1]$, and a partition $\mathcal{L} = \{\Lambda_1, \dots, \Lambda_\ell\}$ of $[a_1]$, we define

$$\begin{aligned}\mathrm{cut}(d_{\mathbf{a},k}, \mathcal{L}) &:= k! \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} \sum_{\substack{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}):\\ \mathbf{x}_*^{(1)} \in \mathcal{K}_k(\mathcal{L})}} d_{\mathbf{a},k}(\hat{\mathbf{x}}) \quad \text{and}\\ \mathrm{maxcut}_\ell(d_{\mathbf{a},k}) &:= \max_{\substack{\mathcal{L} \text{ is a partition}\\ \text{of } [a_1] \text{ with } |\mathcal{L}|=\ell}} \mathrm{cut}(d_{\mathbf{a},k}, \mathcal{L}).\end{aligned}$$

Recall that if $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ is a family of partitions, then $P^{(1)}(1,1), \dots, P^{(1)}(a_1, a_1)$ denote the parts of $\mathscr{P}^{(1)}$.

Proposition 11.4. *Suppose $0 < 1/n \ll \varepsilon \ll \gamma, 1/T, 1/k, 1/\ell$. Suppose that $\mathbf{a} \in [T]^{k-1}$ and $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of an n-vertex k-graph H. Let $\mathcal{L} = \{\Lambda_1, \dots, \Lambda_\ell\}$ be a partition of $[a_1]$ and for each $i \in [\ell]$, let $U_i := \bigcup_{\lambda \in \Lambda_i} P^{(1)}(\lambda, \lambda)$. Then*

$$|\mathcal{K}_k(U_1, \dots, U_\ell) \cap H| = (\mathrm{cut}(d_{\mathbf{a},k}, \mathcal{L}) \pm \gamma)\binom{n}{k}.$$

Note that the $|\mathcal{K}_k(U_1, \dots, U_\ell) \cap H|\binom{n}{k}^{-1}$ is a lower bound for $\mathrm{maxcut}_\ell(H)$.

Proof. For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$ with $\mathbf{x}_*^{(1)} \in \mathcal{K}_k(\mathcal{L})$, we apply Lemma 4.5 to $\hat{\mathcal{P}}(\hat{\mathbf{x}})$. (Recall that $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ was defined in (3.14) and is an $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}))$-regular complex by Lemma 4.6(ii).) Since H is $(\varepsilon, d_{\mathbf{a},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$, we obtain

$$\begin{aligned}|\mathcal{K}_k(U_1, \dots, U_\ell) \cap H| &= \sum_{\substack{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}):\\ \mathbf{x}_*^{(1)} \in \mathcal{K}_k(\mathcal{L})}} |H \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))|\\ &= \sum_{\substack{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}):\\ \mathbf{x}_*^{(1)} \in \mathcal{K}_k(\mathcal{L})}} (d_{\mathbf{a},k}(\hat{\mathbf{x}}) \pm \varepsilon)|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))|\\ &= \sum_{\substack{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a}):\\ \mathbf{x}_*^{(1)} \in \mathcal{K}_k(\mathcal{L})}} (d_{\mathbf{a},k}(\hat{\mathbf{x}}) \pm \varepsilon)(1 \pm \gamma/2)\prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} n^k\\ &= (\mathrm{cut}(d_{\mathbf{a},k}, \mathcal{L}) \pm \gamma)\binom{n}{k}.\end{aligned}$$

We conclude the final equality since $\mathrm{cut}(d_{\mathbf{a},k}, \mathcal{L}) \le 1$. □

Proposition 11.5. *Suppose that $0 < 1/n \ll \nu \ll \beta, 1/k, 1/\ell$, and $c = c(n) \in [0, c_{\ell,k}(n)]$. If H is an n-vertex k-graph with $\mathrm{maxcut}_\ell(H) \ge c - \nu$, then there exists an n-vertex k-graph G with $\mathrm{maxcut}_\ell(G) \ge c$ and $|H \triangle G| \le \beta\binom{n}{k}$.*

Proof. Since $\mathrm{maxcut}_\ell(H) \ge c - \nu$, there is a partition $\{U_1, \dots, U_\ell\}$ of $V(H)$ such that

$$|\mathcal{K}_k(U_1, \dots, U_\ell) \cap H| \ge (c - \nu)\binom{n}{k}. \tag{11.5}$$

It is easy to see that there exists a partition $\{U_1', \dots, U_\ell'\}$ of $V(H)$ such that

(U′1) $\sum_{i=1}^{\ell} |U_i \triangle U_i'| \le \nu^{\frac{1}{5k}} n$, and
(U′2) $|\mathcal{K}_k(U_1', \dots, U_\ell')| \ge c\binom{n}{k}$.

Since $\nu \ll \beta$, we conclude

$$|\mathcal{K}_k(U'_1,\dots,U'_\ell)\cap H| \geq |\mathcal{K}_k(U_1,\dots,U_\ell)\cap H| - \sum_{i=1}^{\ell}|U_i\triangle U'_i|n^{k-1} \overset{(11.5),(\mathrm{U}'1)}{\geq} (c-\beta)\binom{n}{k}.$$

Together with (U′2), this shows that we can add at most $\beta\binom{n}{k}$ k-sets from $\mathcal{K}_k(U'_1,\dots,U'_\ell)\setminus H$ to H to obtain a k-graph G with $\mathrm{maxcut}_\ell(G)\geq c$ and $|H\triangle G|\leq \beta\binom{n}{k}$. □

Proof of Corollary 11.3. By Theorem 1.3, we only need to show that $\mathbf{P}$ is regular reducible. We assume that $\ell \geq k$, otherwise $\mathrm{maxcut}_\ell(H)=0$ for all k-graphs H. Suppose $\beta>0$.

Let $\bar\varepsilon : \mathbb{N}^{k-1}\to(0,1)$ be a function such that $\bar\varepsilon(\mathbf{a}) \ll \|\mathbf{a}\|_\infty^{-1}, 1/k, 1/\ell$. We choose constants ε,η,ν, and $n_0, T\in\mathbb{N}$ such that $0<1/n_0 \ll \varepsilon \ll 1/T \ll \eta,\nu \ll \beta, 1/k, 1/\ell$. For simplicity, we consider only n-vertex k-graphs H with $n\geq n_0$.

Let $\mathbf{I}$ be the collection of regularity instances $R=(\varepsilon'',\mathbf{a},d_{\mathbf{a},k})$ such that

(R1)$_{11.3}$ $\varepsilon''\in\{\varepsilon,2\varepsilon,\dots,\lceil(\bar\varepsilon(\mathbf{a}))^{1/2}\varepsilon^{-1}\rceil\varepsilon\}$,
(R2)$_{11.3}$ $\mathbf{a}\in[T]^{k-1}$, and
(R3)$_{11.3}$ $d_{\mathbf{a},k}(\hat{\mathbf{x}})\in\{0,\varepsilon^2,2\varepsilon^2,\dots,1\}$ for every $\hat{\mathbf{x}}\in\hat{A}(k,k-1,\mathbf{a})$.

Observe that by construction $|\mathbf{I}|$ is bounded by a function of β, k and ℓ. We define

$$\mathcal{R} := \left\{(\varepsilon'',\mathbf{a},d_{\mathbf{a},k})\in\mathbf{I} : \mathrm{maxcut}_\ell(d_{\mathbf{a},k})\geq c-\nu^{1/2}\right\}.$$

First, suppose that an n-vertex k-graph H satisfies $\mathrm{maxcut}_\ell(H)\geq c$. Then there exists a partition $\{V_1,\dots,V_\ell\}$ of $V(H)$ such that

$$|\mathcal{K}_k(V_1,\dots,V_\ell)\cap H| \geq c\binom{n}{k}. \tag{11.6}$$

Let

$$\mathscr{O}^{(1)} := \{V_1,\dots,V_\ell\}.$$

For each $j\in[k-2]$ and given $\mathscr{O}^{(j)}$, (3.4) naturally defines $\hat{\mathscr{O}}^{(j)}$, and we define

$$\mathscr{O}^{(j+1)} := \{\mathcal{K}_{j+1}(\hat{O}^{(j)}) : \hat{O}^{(j)}\in\hat{\mathscr{O}}^{(j)}\}.$$

By repeating this for each $j\in[k-2]$ in increasing order, we define a family of partitions $\mathscr{O} := \mathscr{O}(k-1,\mathbf{a}^{\mathscr{O}}) = \{\mathscr{O}^i\}_{i=1}^{k-1}$ with $\mathbf{a}^{\mathscr{O}} = (\ell,1,\dots,1)\in\mathbb{N}^{k-1}$. Let $\mathscr{Q}=\mathscr{Q}(k,\mathbf{a}^{\mathscr{Q}})$ be an arbitrary $(1/a_1^{\mathscr{Q}}, 1/n_0, \mathbf{a}^{\mathscr{Q}})$-equitable family of partitions on $V(H)$, where $\mathbf{a}^{\mathscr{Q}} = (\ell,1,\dots,1)\in\mathbb{N}^k$. It is easy to see that such $\mathscr{Q}$ indeed exists. Let

$$\begin{aligned}
\{H_1,\dots,H_s\} &:= \left(\left\{Q^{(k)}\cap H : Q^{(k)}\in\mathscr{Q}^{(k)}\right\}\cup\left\{H\setminus\mathcal{K}_k(\mathscr{Q}^{(1)})\right\}\right)\setminus\{\emptyset\},\\
\{H_{s+1},\dots,H_{s'}\} &:= \left(\left\{Q^{(k)}\setminus H : Q^{(k)}\in\mathscr{Q}^{(k)}\right\}\cup\left\{\binom{V(H)}{k}\setminus(\mathcal{K}_k(\mathscr{Q}^{(1)})\cup H)\right\}\right)\setminus\{\emptyset\},\\
\mathscr{H} &:= \{H_1,\dots,H_{s'}\}.
\end{aligned}$$

Note that $s'\leq 2\binom{\ell}{k}+2$. Since $|V(H)|\geq n_0$, we can apply Lemma 9.1 with the following objects and parameters.

object/parameter	$V(H)$	$\mathscr{O}$	$\mathscr{H}$	$\mathscr{Q}$	ℓ	s'	η	ν	$\bar\varepsilon$	T
playing the role of	V	$\mathscr{O}$	$\mathscr{H}^{(k)}$	$\mathscr{Q}$	o	s	η	ν	ε	t

Then we obtain k-graphs $G_1,\dots,G_{s'}$ partitioning $\binom{V(H)}{k}$ and a family of partitions $\mathscr{P}=\mathscr{P}(k-1,\mathbf{a}^{\mathscr{P}})$ such that

(I) $\mathscr{P}$ is $(\eta,\bar\varepsilon(\mathbf{a}^{\mathscr{P}}),\mathbf{a}^{\mathscr{P}})$-equitable for some $\mathbf{a}^{\mathscr{P}}\in[T]^{k-1}$,
(II) $\mathscr{P}^{(1)} \prec_\nu \mathscr{O}^{(1)}$,

(III) for each $i \in [s]$, G_i is perfectly $\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$, and
(IV) $\sum_{i=1}^{s} |G_i \triangle H_i| \leq \nu\binom{n}{k}$.

Let $\varepsilon' := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$ and $G := \bigcup_{i=1}^{s} G_i$. Lemma 4.3 together with (III) implies that G is perfectly $s\varepsilon'$-regular with respect to $\mathscr{P}$. Also (IV) implies that

$$|G \triangle H| \leq \nu\binom{n}{k}. \tag{11.7}$$

By the choice of ε, $\overline{\varepsilon}$ and η, we conclude that $0 < \varepsilon \ll \varepsilon' \ll 1/\|\mathbf{a}^{\mathscr{P}}\|_\infty \leq 1/a_1^{\mathscr{P}} \leq \eta \ll \beta, 1/k, 1/\ell$. Similarly as in the proof of Corollary 11.1, this implies that there exists

$$R_G = (\varepsilon'', \mathbf{a}^{\mathscr{P}}, d^G_{\mathbf{a}^{\mathscr{P}},k}) \in \mathbf{I} \tag{11.8}$$

such that G satisfies R_G.

Note that (II) implies that there exists a partition $\mathcal{L} := \{\Lambda_1, \dots, \Lambda_\ell\}$ of $[a_1^{\mathscr{P}}]$ such that

$$\sum_{i=1}^{\ell} \sum_{\lambda \in \Lambda_i} |P^{(1)}(\lambda,\lambda) \setminus V_i| \leq \nu n. \tag{11.9}$$

For each $i \in [\ell]$, let $U_i := \bigcup_{\lambda \in \Lambda_i} P^{(1)}(\lambda,\lambda)$. Then we obtain

$$\begin{aligned}
\mathrm{cut}(d^G_{\mathbf{a}^{\mathscr{P}},k}, \mathcal{L}) &\overset{\text{Prop. 11.4}}{=} \binom{n}{k}^{-1} |\mathcal{K}_k(U_1,\dots,U_\ell) \cap G| \pm \nu \\
&\overset{(11.7)}{=} \binom{n}{k}^{-1} \left(|\mathcal{K}_k(V_1,\dots,V_\ell) \cap H| \pm \sum_{i=1}^{\ell} \sum_{\lambda \in \Lambda_i} |P^{(1)}(\lambda,\lambda) \setminus V_i| n^{k-1} \right) \pm 2\nu \\
&\overset{(11.9)}{=} \binom{n}{k}^{-1} |\mathcal{K}_k(V_1,\dots,V_\ell) \cap H| \pm \nu^{1/2} \\
&\overset{(11.6)}{\geq} c - \nu^{1/2}.
\end{aligned}$$

By the definition of $\mathcal{R}$ and (11.8), this implies that $R_G \in \mathcal{R}$ and so H is indeed β-close to a graph G satisfying R_G, one of the regularity instances of $\mathcal{R}$.

Now we show that if H is α-far from satisfying $\mathbf{P}$, then H is $(\alpha-\beta)$-far from all $R \in \mathcal{R}$. Suppose H is $(\alpha-\beta)$-close to some $R = (\varepsilon'', \mathbf{a}, d_{\mathbf{a},k}) \in \mathcal{R}$. Then there exists a k-graph G_R such that G_R satisfies R and $|H \triangle G_R| \leq (\alpha-\beta)\binom{n}{k}$. Thus there is an $(\varepsilon'', \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{P}'$ of G_R. By the definition of $\mathcal{R}$, we have $\mathrm{maxcut}_\ell(d_{\mathbf{a},k}) \geq c - \nu^{1/2}$. By applying Proposition 11.4 with $G_R, c-\nu^{1/2}, \nu^{1/2}, d_{\mathbf{a},k}$ playing the roles of $H, c, \gamma, d_{\mathbf{a},k}$, we obtain that $\mathrm{maxcut}_\ell(G_R) \geq c - 2\nu^{1/2}$. Since $\nu \ll \beta, 1/k, 1/\ell$ and $c \in [0, c_{\ell,k}(n)]$, we can apply Proposition 11.5 with $G_R, 2\nu^{1/2}, \beta/2, c$ playing the roles of H, ν, β, c to obtain a k-graph G' such that $|G' \triangle G_R| \leq (\beta/2)\binom{n}{k}$ and $\mathrm{maxcut}_\ell(G) \geq c$. Then

$$|H \triangle G'| \leq |H \triangle G_R| + |G' \triangle G_R| \leq (\alpha - \beta + \beta/2)\binom{n}{k} < \alpha\binom{n}{k}.$$

Thus H is α-close to satisfying $\mathbf{P}$. Therefore, $\mathbf{P}$ is indeed regular reducible. □

References

1. N. Alon and O. Ben-Eliezer, *Removal lemmas for matrices*, arXiv:1609.04235 (2016).
2. N. Alon and E. Blais, *Testing Boolean function isomorphism*, Approximation, randomization, and combinatorial optimization, Lecture Notes in Comput. Sci., vol. 6302, Springer, Berlin, 2010, pp. 394–405.
3. N. Alon, S. Dar, M. Parnas, and D. Ron, *Testing of clustering*, SIAM J. Discrete Math. **16** (2003), 393–417.

4. N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy, *Efficient testing of large graphs*, Combinatorica **20** (2000), 451–476.
5. N. Alon, E. Fischer, I. Newman, and A. Shapira, *A combinatorial characterization of the testable graph properties: it's all about regularity*, SIAM J. Comput. **39** (2009), 143–167.
6. N. Alon and J. Fox, *Easily testable graph properties*, Combin. Probab. Comput. **24** (2015), 646–657.
7. N. Alon and A. Shapira, *Testing satisfiability*, J. Algorithms **47** (2003), 87–103.
8. ———, *Linear equations, arithmetic progressions, and hypergraph property testing*, Theory Comput. **1** (2005), 177–216.
9. ———, *A characterization of the (natural) graph properties testable with one-sided error*, SIAM J. Comput. **37** (2008), 1703–1727.
10. ———, *Every monotone graph property is testable*, SIAM J. Comput. **38** (2008), 505–522.
11. T. Austin and T. Tao, *Testability and repair of hereditary hypergraph properties*, Random Structures Algorithms **36** (2010), 373–463.
12. C. Avart, V. Rödl, and M. Schacht, *Every monotone 3-graph property is testable*, SIAM J. Discrete Math. **21** (2007), 73–92.
13. M. Ben Or, D. Coppersmith, M. Luby, and R. Rubinfeld, *Non-abelian homomorphism testing, and distributions close to their self-convolutions*, Random Structures Algorithms **32** (2008), 49–70.
14. I. Benjamini, O. Schramm, and A. Shapira, *Every minor-closed property of sparse graphs is testable*, Adv. Math. **223** (2010), 2200–2218.
15. C. Borgs, J. Chayes, L. Lovász, V. T. Sós, B. Szegedy, and K. Vesztergombi, *Graph limits and parameter testing*, STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, ACM, New York, 2006, pp. 261–270.
16. A. Czygrinow and B. Nagle, *On random sampling in uniform hypergraphs*, Random Structures Algorithms **38** (2011), 422–440.
17. G. Elek and B. Szegedy, *A measure-theoretic approach to the theory of dense hypergraphs*, Adv. Math. **231** (2012), 1731–1772.
18. E. Fischer, *The difficulty of testing for isomorphism against a graph that is given in advance*, SIAM J. Comput. **34** (2005), 1147–1158.
19. E. Fischer and I. Newman, *Testing versus estimation of graph properties*, SIAM J. Comput. **37** (2007), 482–501.
20. J. Fox, J. Pach, and A. Suk, *A polynomial regularity lemma for semialgebraic hypergraphs and its applications in geometry and property testing*, SIAM J. Comput. **45** (2016), 2199–2223.
21. P. Frankl and V. Rödl, *The uniformity lemma for hypergraphs*, Graphs Combin. **8** (1992), 309–312.
22. K. Friedl, G. Ivanyos, and M. Santha, *Efficient testing of groups*, STOC'05: Proceedings of the 37th Annual ACM Symposium on Theory of Computing, ACM, New York, 2005, pp. 157–166.
23. O. Goldreich, S. Goldwasser, and D. Ron, *Property testing and its connection to learning and approximation*, J. ACM **45** (1998), 653–750.
24. O. Goldreich and L. Trevisan, *Three theorems regarding testing graph properties*, Random Structures Algorithms **23** (2003), 23–57.
25. W. T. Gowers, *Lower bounds of tower type for Szemerédi's uniformity lemma*, Geom. Funct. Anal. **7** (1997), 322–337.
26. ———, *Hypergraph regularity and the multidimensional Szemerédi theorem*, Ann. of Math. **166** (2007), 897–946.
27. S. Janson, T. Łuczak, and A. Rucinski, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.
28. M. Karpinski and R. Markó, *Explicit bounds for nondeterministically testable hypergraph parameters*, arXiv:1509.03046 (2015).
29. ———, *On the complexity of nondeterministically testable hypergraph parameters*, arXiv:1503.07093 (2015).
30. Y. Kohayakawa, B. Nagle, and V. Rödl, *Efficient testing of hypergraphs (extended abstract)*, Automata, languages and programming, Lecture Notes in Comput. Sci., vol. 2380, Springer, Berlin, 2002, pp. 1017–1028.
31. Y. Kohayakawa, V. Rödl, and J. Skokan, *Hypergraphs, quasi-randomness, and conditions for regularity*, J. Combin. Theory Ser. A **97** (2002), 307–352.
32. G. Lior and A. Shapira, *Deterministic vs non-deterministic graph property testing*, Israel J. Math. **204** (2014), 397–416.
33. L. Lovász, *Large networks and graph limits*, American Mathematical Society Colloquium Publications, vol. 60, American Mathematical Society, Providence, RI, 2012.
34. L. Lovász and B. Szegedy, *Testing properties of graphs and functions*, Israel J. Math. **178** (2010), 113–156.

35. L. Lovász and K. Vesztergombi, *Non-deterministic graph property testing*, Combin. Probab. Comput. **22** (2013), 749–762.
36. I. Newman and C. Sohler, *Every property of hyperfinite graphs is testable*, SIAM J. Comput. **42** (2013), 1095–1112.
37. V. Rödl and M. Schacht, *Property testing in hypergraphs and the removal lemma*, STOC'07—Proceedings of the 39th Annual ACM Symposium on Theory of Computing, ACM, New York, 2007, pp. 488–495.
38. ———, *Regular partitions of hypergraphs: counting lemmas*, Combin. Probab. Comput. **16** (2007), 887–901.
39. ———, *Regular partitions of hypergraphs: regularity lemmas*, Combin. Probab. Comput. **16** (2007), 833–885.
40. ———, *Generalizations of the removal lemma*, Combinatorica **29** (2009), 467–501.
41. V. Rödl and J. Skokan, *Regularity lemma for k-uniform hypergraphs*, Random Structures Algorithms **25** (2004), 1–42.
42. R. Rubinfeld and M. Sudan, *Robust characterizations of polynomials with applications to program testing*, SIAM J. Comput. **25** (1996), 252–271.
43. I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, Colloq. Math. Soc. János Bolyai, vol. 18, North-Holland, Amsterdam-New York, 1978, pp. 939–945.
44. T. Tao, *A variant of the hypergraph removal lemma*, J. Combin. Theory Ser. A **113** (2006), 1257–1280.

School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom

E-mail address: f.joos@bham.ac.uk, j.kim.3@bham.ac.uk, d.kuhn@bham.ac.uk, d.osthus@bham.ac.uk