Large planar subgraphs in dense graphs

Daniela Kühn, Deryk Osthus and Anusch Taraz

Abstract

We prove sufficient and essentially necessary conditions in terms of the minimum degree for a graph to contain planar subgraphs with many edges. For example, for all positive γ every sufficiently large graph G with minimum degree at least $(2/3+\gamma)|G|$ contains a triangulation as a spanning subgraph, whereas this need not be the case when the minimum degree is less than $2|G|/3$.

1 Introduction

1.1 Results

In this paper we study the following extremal question: Given a function $m = m(n)$, how large does the minimum degree of a graph G of order n have to be in order to guarantee a planar subgraph with at least $m(n)$ edges?

If $m \leq n$, the answer is easy. Indeed, suppose that the minimum degree of G is at least one. Then every component C of G has a spanning tree with $|C|-1 \geq |C|/2$ edges. So G has a (planar) spanning forest with at least $n/2$ edges, which is best possible if G consists of independent edges. Similarly, it is easy to see that if G has minimum degree at least two, then G contains a planar subgraph with n edges, which is best possible if G is a cycle.

On the other hand, if G is bipartite, then the facial cycles of any planar subgraph have length at least four and so Euler's formula implies that no planar subgraph of G has more than $2n-4$ edges. So as long as the minimum degree is at most $n/2$, we cannot hope for a planar subgraph with more than $2n-4$ edges. Our first theorem shows that a much smaller minimum degree already guarantees a planar subgraph with roughly $2n$ edges.

Theorem 1 *For every $0<\varepsilon<1$ there exists $n_0 = n_0(\varepsilon)$ such that every graph G of order $n \geq n_0$ and minimum degree $\delta \geq 1500\sqrt{n}/\varepsilon^2$ contains a planar subgraph with at least $(2-\varepsilon)n$ edges.*

This is essentially best possible in two ways. Firstly, there are graphs with minimum degree $\sqrt{n}/2$ and girth at least 6 ([6], see also [3]). Hence Euler's formula shows that any planar subgraph of such a graph can have at most $\frac{3}{2}(n-2)$ edges (as all of its facial cycles have length at least 6). Secondly, for $\delta \leq n/2$ consider the graph consisting of $n/2\delta$ disjoint copies of the complete bipartite graph $K_{\delta,\delta}$. It obviously has minimum degree δ, but again by Euler's formula it cannot contain a planar subgraph with more than $(2\cdot 2\delta-4)n/2\delta =$

$2n - 2n/\delta$ edges. This shows that as long as the minimum degree δ of G is $o(n)$, we cannot ask for a planar subgraph of G with $2n - C$ edges, where C does not depend on n. So if we want at least $2n - C$ edges in a planar subgraph, then a necessary condition is that $\delta \geq 2n/C$, i.e. δ must be linear in n. Our second theorem shows that the linearity of δ is also sufficient.

Theorem 2 *For every $\gamma > 0$ there exists $C = C(\gamma)$ such that every graph G of order n and minimum degree at least γn contains a planar subgraph with at least $2n - C$ edges.*

As we have already seen, this is best possible up to the value of the constant C as long as the minimum degree is at most $n/2$. If however the minimum degree is a little larger than this, we can already guarantee a planar subgraph which is a triangulation apart from a constant number of missing edges:

Theorem 3 *For every $\gamma > 0$ there exists $C = C(\gamma)$ such that every graph G of order n and minimum degree at least $(1/2 + \gamma)n$ contains a planar subgraph with at least $3n - C$ edges.*

Again, this is best possible in the sense that the constant C has to depend on γ and the additional term γn in the bound on the minimum degree cannot be replaced by a sublinear one (see Proposition 14).

Finally, we seek a spanning triangulation, i.e. a planar subgraph with $3n - 6$ edges. As pointed out to us by Bollobás, the following 3-partite graph G shows that a minimum degree of $2n/3$ is necessary for this. G is obtained from two disjoint cliques C_1 and C_2 of order $n/3$ by adding an independent set X of $n/3$ new vertices and joining each of them to all the vertices in the two cliques. So G has minimum degree $2n/3 - 1$. Observe that any spanning triangulation in G would have two facial triangles T_1 and T_2 which share an edge and are such that T_i contains a vertex of C_i $(i = 1, 2)$. But this is impossible since every triangle of G containing a vertex of C_i can have at most one vertex outside C_i, namely in X. However, to guarantee a triangulation, it suffices to increase the minimum degree by a small amount:

Theorem 4 *For every $\gamma > 0$ there exists an integer $n_0 = n_0(\gamma)$ such that every graph G of order $n \geq n_0$ and minimum degree at least $(2/3 + \gamma)n$ contains a triangulation as a spanning subgraph.*

In [19] the first two authors show that for sufficiently large graphs a minimum degree of $2n/3$ suffices. However, the proof of this is rather more involved than that of Theorem 4.

We also obtain an analogue of Theorem 4 for quadrangulations, i.e. plane subgraphs with $2n - 4$ edges in which every face is bounded by a 4-cycle.

Theorem 5 *For every $\gamma > 0$ there exists an integer $n_0 = n_0(\gamma)$ such that every graph G of order $n \geq n_0$ and minimum degree at least $(1/2 + \gamma)n$ contains a quadrangulation as a spanning subgraph.*

The disjoint union of two cliques of order $n/2$ shows that apart from the error term γn, the minimum degree in Theorem 5 cannot be reduced.

1.2 Open questions and related results

There is a conjecture of Bollobás and Komlós [12] which would immediately imply Theorems 4 and 5. It asserts that for every $\gamma > 0$ and all $r, \Delta \in \mathbb{N}$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that every graph G of order $n \geq n_0$ and minimum degree at least $(1 - \frac{1}{r} + \gamma)n$ contains a copy of every graph H of order n whose chromatic number is at most r, whose maximum degree is at most Δ and whose band-width is at most αn. (The *band-width* of a graph H is the smallest integer k for which there exists an enumeration $v_1, \ldots, v_{|H|}$ of the vertices of H such that every edge $v_i v_j \in H$ satisfies $|i - j| \leq k$.) Indeed, to derive e.g. Theorem 4 from this conjecture it suffices to find for all $n \in \mathbb{N}$ a 3-partite triangulation of order n which has both bounded maximum degree and bounded band-width. It is easy to see that such triangulations exist (e.g. modify the graph H_1 in Fig. 3 below).

Theorems 1–5 give a fairly accurate picture of the maximum size of a planar subgraph when we consider graphs whose minimum degree δ is much larger than $\sqrt{n}$. However, we are not aware of any nontrivial lower bounds when δ lies between 2 and $\sqrt{n}$. An easy upper bound is obtained as follows. For $\ell \geq 3$ let $\delta_{2\ell} = \delta_{2\ell}(n)$ be the largest integer such that there are graphs G of order n, minimum degree at least $\delta_{2\ell}$ and girth at least 2ℓ. (The order of magnitude of $\delta_{2\ell}$ is only known for $\ell = 3, 4$ and 6, see e.g. [2, 6].) So all facial cycles in a planar subgraph of such a graph G have length at least 2ℓ and thus Euler's formula gives us an upper bound on the size of a planar subgraph of G. We believe that in general this upper bound is close to the truth (except maybe when the minimum degree is only a little larger than $\delta_{2\ell+2}$).

The problem of finding a large planar subgraph in a random graph was investigated by Schlatter [20], the case of triangulations was already considered earlier by Bollobás and Frieze [5].

1.3 Algorithmic aspects

Our proofs immediately show that the planar subgraphs guaranteed by Theorems 1–5 can be found in polynomial time. For graphs with high minimum degree we therefore obtain improved approximation algorithms for the maximum planar subgraph problem which in a given graph G asks for a planar subgraph with the maximum number of edges. Călinescu et al. [7] showed that this problem is Max SNP-hard: there is a constant ε such that there cannot exist a polynomial time approximation algorithm with approximation ratio better than $1 - \varepsilon$, unless $P = NP$. Recently, Faria, Figueiredo and Mendonça [11] proved that it is Max SNP-hard even for cubic graphs. The best known approximation algorithm for arbibrary input graphs has an approximation ratio of 4/9 [7]. (Note that a ratio of 1/3 is already achieved by producing spanning trees for all connected components.) On the other hand, our proof of Theorem 4 implies that for any $\gamma > 0$ the maximum planar subgraph problem can be solved in polynomial time for graphs with minimum degree at least $(2/3 + \gamma)n$. Our remaining results give improved approximation algorithms for graphs whose minimum degree is sufficiently large for the respective results to apply.

The paper is organized as follows. In Section 2 we give a brief sketch of the proofs of Theorems 1–5. In Section 3 we collect some notation and all the information about the Regularity lemma and the Blow-up lemma we need for the proofs of Theorems 2–5. The proofs themselves are then given in the final section.

2 Sketch of proofs

The proof of Theorem 1 is rather different from those of the other results. In particular, it relies neither on the Regularity lemma nor on the Blow-up lemma. The strategy is to repeatedly find a suitable greedy covering of part of the vertices of the original graph G with disjoint complete bipartite graphs $K_{2,s}$, where s is large. (Note that if s is large then the planar graph $H := K_{2,s}$ has roughly $2|H|$ edges.) These partial coverings (which will overlap a little) are then combined into a single planar graph of the required size.

We now give a sketch of the proofs of Theorems 2–5. The structure of these proofs is similar: we first apply the Regularity lemma (Lemma 7) to obtain a partition of the vertices of G into a large but constant number of clusters. Since G has large minimum degree, this is also true for the 'reduced graph' R (whose vertices are the clusters and whose edges correspond to the pairs of clusters which are regular and have sufficient density). We will use this to cover almost all vertices of R by suitable disjoint graphs H of bounded size. Then we apply the Blow-up lemma (Lemma 10) to find spanning planar graphs P of the required density within the subgraphs H' of G corresponding to these graphs H. However, we also have to ensure that the exceptional vertices of G (i.e. the small proportion of those vertices of G which do not belong to some such H') can be incorporated into these planar graphs P without reducing their density. This also follows from the Blow-up lemma provided that we can assign each exceptional vertex v to some H which contains enough clusters with many neighbours of v in such a way that to each H we assign only a small number of exceptional vertices.

In the proof of Theorem 2 the graphs H will be stars of bounded size and the planar graphs P we seek within the graphs H' will be quadrangulations. For Theorem 3 we want the planar graphs P to be triangulations, which means that the graphs H can no longer be bipartite. Thus an obvious choice for H would be a triangle, but we cannot hope to cover almost all vertices of the reduced graph R by disjoint triangles since its minimum degree may be only a little larger than $|R|/2$. However, a recent result of Komlós (Theorem 13) implies that we can take H to be the complete 3-partite graph $K_{a,a,1}$ (where a is large) as it is in some sense close to being bipartite.

In the proof of Theorem 4 the minimum degree of the reduced graph R exceeds $2|R|/3$ and hence the Theorem of Corrádi and Hajnal [8] implies that R can be covered by disjoint triangles. However, this is not sufficient for our purposes as this time we seek a single triangulation containing all vertices of G (instead of a disjoint union of boundedly many triangulations as in the proof of Theorem 3). So we have to 'glue together' the different triangulations corresponding to the triangles covering R. For this we use suitable edges of R joining these triangles (as well as some additional vertices of G). Thus instead of merely covering R by disjoint triangles, we will start with the second power of a Hamilton path of R. The latter is guaranteed by a result of Fan and Kierstead [10].

The proof of Theorem 5 is similar to that of Theorem 4 but the gluing process is somewhat simpler. Instead of the second power, this time it suffices to work with an 'ordinary' Hamilton path.

3 Notation and tools

Throughout this paper we omit floors and ceilings whenever this does not affect the argument. We write $e(G)$ for the number of edges of a graph G, $|G|$ for its order, $\delta(G)$ for its minimum degree, $\Delta(G)$ for its maximum degree and $\chi(G)$ for its chromatic number. If this is not ambiguous, we also write n for the order of a graph G. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and the set of its neighbours by $N_G(x)$. Given disjoint $A, B \subseteq V(G)$, an A–B *edge* is an edge of G with one endvertex in A and the other in B, the number of these edges is denoted by $e_G(A, B)$ or $e(A, B)$ if this is unambiguous. We write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all A–B edges in G. More generally, we write (A, B) for a bipartite graph with vertex classes A and B. Given a plane graph G, a *facial cycle* of G is a cycle in G which is the boundary of a face. G is a *triangulation* if all its faces are bounded by triangles and a *quadrangulation* if all faces are bounded by 4-cycles. So by Euler's formula a triangulation has $3n - 6$ edges whereas a quadrangulation has $2n - 4$ edges.

In the remainder of this section we collect all the information we need about the Regularity lemma and the Blow-up lemma. See [18] and [12] for surveys about these. Let us start with some more notation. The *density* of a bipartite graph $G = (A, B)$ is defined to be

$$d(A, B) := \frac{e(A, B)}{|A||B|}.$$

Given $\varepsilon > 0$, we say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A, B) - d(X, Y)| < \varepsilon$. Given $d \in [0, 1]$, we say that G is *(ε, d)-super-regular* if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) > d$ and, furthermore, if $d_G(a) > d|B|$ for all $a \in A$ and $d_G(b) > d|A|$ for all $b \in B$.

We will often use the following simple fact.

Proposition 6 *Given an ε-regular bipartite graph (A, B) of density $> d$ and a set $X \subseteq A$ with $|X| \geq \varepsilon|A|$, there are less than $\varepsilon|B|$ vertices in B which have at most $(d - \varepsilon)|X|$ neighbours in X.*

We will use the following degree form of Szemerédi's Regularity lemma which can be easily derived from the classical version. Proofs of the latter are for example included in [4] and [9].

Lemma 7 (Regularity lemma) *For all $\varepsilon > 0$ and all integers k_0 there is an $N = N(\varepsilon, k_0)$ such that for every number $d \in [0,1]$ and for every graph G there exist a partition of $V(G)$ into $V_0, V_1, \dots, V_k$ and a spanning subgraph G' of G such that the following holds:*

- $k_0 \leq k \leq N$,
- $|V_0| \leq \varepsilon |G|$,
- $|V_1| = \cdots = |V_k| =: L$,
- $d_{G'}(x) > d_G(x) - (d+\varepsilon)|G|$ *for all vertices* $x \in G$,
- *for all* $i \geq 1$ *the graph* $G'[V_i]$ *is empty,*
- *for all* $1 \leq i < j \leq k$ *the graph* $(V_i, V_j)_{G'}$ *is* ε*-regular and has density either* 0 *or* $> d$.

The sets V_i $(i \geq 1)$ are called *clusters*, V_0 is called the *exceptional set.* Given clusters and G' as in Lemma 7, the *reduced graph R* is the graph whose vertices are $V_1, \dots, V_k$ and in which V_i is joined to V_j whenever $(V_i, V_j)_{G'}$ is ε-regular and has density $> d$. Thus V_iV_j is an edge of R if and only if G' has an edge between V_i and V_j.

Proposition 8 *Let H be a subgraph of the reduced graph R with $\Delta(H) \leq \Delta$. Then each vertex V_i of H contains a subset V_i' of size $(1-\varepsilon\Delta)L$ such that for every edge V_iV_j of H the graph $(V_i', V_j')_{G'}$ is $(\varepsilon/(1-\varepsilon\Delta), d-(1+\Delta)\varepsilon)$-super-regular.*

Proof. Consider an edge V_iV_j of H. By Proposition 6, there are less than εL vertices in V_i which have at most $(d-\varepsilon)L$ neighbours in V_j (in the graph G'). So for every vertex V_i of H we can choose a set $V_i' \subseteq V_i$ of size $(1-\varepsilon\Delta)L$ such that for each neighbour V_j of V_i in H all vertices $x \in V_i'$ have more than $(d-\varepsilon)L$ neighbours in V_j. It can be easily checked that for every edge V_iV_j of H the graph $(V_i', V_j')_{G'}$ is $(\varepsilon/(1-\varepsilon\Delta), d-(1+\Delta)\varepsilon)$-super-regular. □

We will often use the following well-known and simple fact. Its proof is the only place in this paper where the degree form of the Regularity lemma is more convenient than the classical form.

Proposition 9 *For every $\gamma > 0$ there exist $\varepsilon_0 = \varepsilon_0(\gamma)$ and $d_0 = d_0(\gamma)$ such that for all $\varepsilon \leq \varepsilon_0$, $d \leq d_0$ and every $c \geq 0$ an application of Lemma 7 to a graph G of minimum degree at least $(c+\gamma)|G|$ yields a reduced graph R of minimum degree at least $(c+\gamma/2)|R|$.*

Proof. Suppose that there is a vertex $V_i \in R$ whose degree in R is less than $(c+\gamma/2)k$. Let W denote the union of all those clusters V_j $(j \neq i)$ for which $(V_i, V_j)_{G'}$ has density 0. Let u be any vertex in V_i. Then

$$\begin{aligned} d_{G'}(u) &\leq |N_{G'}(u) \cap W| + d_R(V_i) \cdot L + |N_{G'}(u) \cap V_0| < 0 + (c+\gamma/2)kL + \varepsilon n \\ &\leq (c+\gamma/2+\varepsilon)n. \end{aligned}$$

But on the other hand, Lemma 7 states that $d_{G'}(u) > d_G(u) - (d+\varepsilon)n \geq (c+\gamma-d-\varepsilon)n$, a contradiction, provided that $\gamma \geq 2d + 4\varepsilon$. □

We will also use the Blow-up lemma of Komlós, Sárközy and Szemerédi [14]. It implies that dense regular pairs behave like complete bipartite graphs with respect to containing bounded degree graphs as subgraphs.

Lemma 10 (Blow-up lemma) *Given a graph R on $\{1,\dots,r\}$ and numbers $d,c,\Delta>0$, there are positive numbers $\varepsilon_0=\varepsilon_0(d,\Delta,r,c)$ and $\alpha=\alpha(d,\Delta,r,c)\le 1/2$ such that the following holds. Given $L\in\mathbb{N}$ and $\varepsilon\le\varepsilon_0$, let $R(L)$ be the graph obtained from R by replacing each vertex $i\in R$ with a set V_i of L new vertices and joining all vertices in V_i to all vertices in V_j whenever ij is an edge of R. Let G be a spanning subgraph of $R(L)$ such that for every edge $ij\in R$ the graph $(V_i,V_j)_G$ is (ε,d)-super-regular. Then G contains a copy of every subgraph H of $R(L)$ with $\Delta(H)\le\Delta$. Furthermore, we can additionally require that for vertices $x\in H\subseteq R(L)$ lying in V_i their images in the copy of H in G are contained in (arbitrary) given sets $C_x\subseteq V_i$ provided that $|C_x|\ge cL$ for each such x and provided that in each V_i there are at most αL such vertices x.*

We say that the vertices x in Lemma 10 are *image restricted to C_x*.

4 Proofs

4.1 Planar subgraphs of size $2n-\varepsilon n$

In our proof of Theorem 1 we will use the following well-known upper bound on the number of edges of $K_{2,s}$-free graphs (see e.g. [3, Ch. VI, Thm. 2.2 and 2.3]).

Theorem 11 *Let $s\ge 2$ be an integer. Then every graph G with $e(G)\ge\sqrt{s}n^{3/2}$ contains a copy of $K_{2,s}$. Moreover, every bipartite graph $G=(A,B)$ with $e(G)\ge\sqrt{s}|A||B|^{1/2}+|B|$ contains a copy of $K_{2,s}$ with 2 vertices in A and s vertices in B.*

Proof of Theorem 1. Throughout the proof we assume that n is sufficiently large for our estimates to hold. For all $k\ge 1$ set $s_k:=2^{k^2+2}/\varepsilon^k$. We first greedily choose as many disjoint copies of K_{2,s_1} in G as possible. Let P_1 be the union of all these K_{2,s_1}'s, $X_1:=V(P_1)$ and let $Y_1:=V(G)\setminus X_1$. Thus $G[Y_1]$ is K_{2,s_1}-free and so Theorem 11 implies that $e(G[Y_1])\le\sqrt{s_1}|Y_1|^{3/2}$. Let Y_1' be the set of all those vertices in Y_1 which have at most $\delta/2$ neighbours in X_1. Then

$$\delta|Y_1'|/2\le 2e(G[Y_1])\le 2\sqrt{s_1}n^{3/2},$$

and thus

$$|Y_1'|\le\frac{4\sqrt{s_1}n^{3/2}}{\delta}. \tag{1}$$

Let $Y_1^*:=Y_1\setminus Y_1'$. Next we greedily choose (as often as possible) disjoint copies of K_{2,s_2} in $(X_1,Y_1^*)_G$ having 2 vertices in X_1 and s_2 vertices in Y_1^*. Let P_2 be the union of all these K_{2,s_2}'s, $X_2:=V(P_2)\cap X_1$ and $Y_2:=V(P_2)\cap Y_1^*$. Let Y_2' be the set of all those vertices in $Y_1^*\setminus Y_2$ which have at most $\delta/2^2$ neighbours in X_2. Thus each vertex in Y_2' has at least $\delta/2^2$ neighbours in $X_1\setminus X_2$ and so

$$e(X_1\setminus X_2,Y_2')\ge\delta|Y_2'|/2^2.$$

On the other hand, $(X_1 \setminus X_2, Y_2')_G$ does not contain a K_{2,s_2} with 2 vertices in $X_1 \setminus X_2$ and s_2 vertices in Y_2'. Thus Theorem 11 implies

$$e(X_1 \setminus X_2, Y_2') \leq \sqrt{s_2} \cdot |X_1 \setminus X_2| \cdot |Y_2'|^{1/2} + |Y_2'| \leq \sqrt{s_2} n^{3/2} + |Y_2'|$$

and therefore

$$|Y_2'| \leq \frac{\sqrt{s_2} n^{3/2}}{\delta/2^2 - 1} \leq \frac{5\sqrt{s_2} n^{3/2}}{\delta}. \tag{2}$$

Let $Y_2^* := Y_1^* \setminus (Y_2 \cup Y_2')$ and greedily choose (again as often as possible) disjoint copies of K_{2,s_3} in $(X_2, Y_2^*)_G$ having 2 vertices in X_2 and s_3 vertices in Y_2^*. Let P_3 be the union of all these K_{2,s_3}'s, $X_3 := V(P_3) \cap X_2$ and $Y_3 := V(P_3) \cap Y_2^*$. Let Y_3' be the set of all those vertices in $Y_2^* \setminus Y_3$ which have at most $\delta/2^3$ neighbours in X_3. Let $Y_3^* := Y_2^* \setminus (Y_3 \cup Y_3')$ and continue in this fashion until $P_i = \emptyset$ (and thus $X_i = Y_i = \emptyset$ and $Y_i' = Y_{i-1}^*$). Let i be the smallest index such that $P_i = \emptyset$. Thus $i \leq \sqrt{\log n}$ since $s_{\sqrt{\log n}} > n$.

Using that $|X_{k-1} \setminus X_k| \leq |X_{k-1}| \leq 2n/s_{k-1}$ for all $3 \leq k \leq i$, a calculation similar to the case $k = 2$ shows that

$$|Y_k'| \leq \frac{5 \cdot 2^{k-1} \sqrt{s_k} n^{3/2}}{\delta s_{k-1}}. \tag{3}$$

Moreover, since $Y_i' = Y_{i-1}^*$,

$$|X_1| + \sum_{k=2}^{i-1} |Y_k| + \sum_{k=1}^{i} |Y_k'| = n. \tag{4}$$

Set

$$P := (P_1 - X_2) \cup (P_2 - X_3) \cup \cdots \cup (P_{i-2} - X_{i-1}) \cup P_{i-1}.$$

Clearly P is a planar subgraph of G. Notice that when removing X_k from P_{k-1}, we destroy at most $s_{k-1}|X_k|$ of its edges, but this is negligible compared to $e(P_k) = 2|Y_k|$, as s_k grows rather rapidly with k. Also, recall that $|X_k| \leq 2n/s_k$ for $k \geq 2$. Hence

$$\begin{aligned}
e(P) &\geq \sum_{k=1}^{i-1} e(P_k) - \sum_{k=2}^{i-1} s_{k-1}|X_k| \\
&\geq \frac{2s_1|X_1|}{s_1+2} + \sum_{k=2}^{i-1} 2|Y_k| - \sum_{k=2}^{i-1} \frac{2s_{k-1}n}{s_k} \\
&\overset{(4)}{=} 2(n - \sum_{k=1}^{i} |Y_k'|) - \frac{4|X_1|}{s_1+2} - \sum_{k=2}^{i-1} \frac{\varepsilon n}{2^{2k-2}} \\
&\overset{(1,2,3)}{\geq} 2n - \frac{32n^{3/2}}{\delta\sqrt{\varepsilon}} - \frac{80n^{3/2}}{\delta\varepsilon} - \sum_{k=3}^{i} \frac{5 \cdot 2^k \sqrt{s_k} n^{3/2}}{\delta s_{k-1}} - \frac{4n}{s_1} - \frac{\varepsilon n}{3} \\
&\geq 2n - \frac{\varepsilon n}{9} - \sum_{k=3}^{i} \frac{5\varepsilon^{k/2-1} n^{3/2}}{2^{k^2/2-3k+2}\delta} - \frac{\varepsilon n}{2} - \frac{\varepsilon n}{3} \\
&\geq 2n - \frac{80n^{3/2}}{\delta} - \frac{17\varepsilon n}{18} \\
&\geq (2-\varepsilon)n,
\end{aligned}$$

as required. $\square$

We remark that the proof of Theorem 1 shows that we can let ε be any function of n with $\varepsilon(n) \leq 1$. Note that it does not make sense to take $\varepsilon(n) \leq n^{-1/4}$.

4.2 Planar subgraphs of size $2n - C$

For the proof of Theorem 2 we need the following simple proposition.

Proposition 12 *Given $0 < \gamma \leq 1/2$ and a graph G of minimum degree at least γn, there exists a set $\mathcal{S}$ of disjoint substars of G such that every vertex of G lies in some $S \in \mathcal{S}$ and such that each such S satisfies $1 \leq \Delta(S) \leq 1/\gamma$.*

Proof. Construct the stars in $\mathcal{S}$ greedily as follows. Suppose that we have already covered a set $X \subseteq V(G)$ with a set $\mathcal{S}'$ of disjoint substars of G such that $1 \leq \Delta(S) \leq 1/\gamma$ for every $S \in \mathcal{S}'$. Choose $x \in V(G) \setminus X$. If x has a neighbour y outside X, we may add the star consisting of the edge xy to $\mathcal{S}'$. So suppose that all neighbours of x lie in X. If x is joined to a leaf y of some star $S \in \mathcal{S}'$ then, if $|S| \geq 3$, we can replace S by $S - y$ and add the new star xy to $\mathcal{S}'$ or, if $|S| = 2$, we can replace S by $S \cup xy$. If x is only joined to midpoints of stars in $\mathcal{S}'$, then one such star must have at most $1/\gamma - 1$ leaves and so we can add x to this star. $\square$

Proof of Theorem 2. Clearly, we may assume that $\gamma \leq 1/2$. Let $\varepsilon_0(\gamma)$ and $d_0(\gamma) =: d$ be as given in Proposition 9. Let $\varepsilon_0(d/2, 8/\gamma, 1+2/\gamma, \gamma/4) =: \varepsilon^*$ and $\alpha(d/2, 8/\gamma, 1+2/\gamma, \gamma/4) =: \alpha$ be as defined in the Blow-up lemma (Lemma 10). Put

$$\varepsilon := \min\left\{\varepsilon_0(\gamma), \frac{\varepsilon^*}{2}, \frac{\gamma^3\alpha}{72}, \frac{\gamma d}{6}\right\}.$$

Clearly, it suffices to show that every graph G whose order n is sufficiently large compared with γ contains a planar subgraph with at least $2n - 4N(\varepsilon, 2)$ vertices, where $N(\varepsilon, 2)$ is given by the Regularity lemma (Lemma 7). So throughout the proof we assume that n is sufficiently large.

We first apply the Regularity lemma to G to obtain an exceptional set V_0 and clusters $V_1, \dots, V_k$ where $2 \leq k \leq N(\varepsilon, 2)$. Let L and G' be as defined in the Regularity lemma and let R denote the reduced graph. Thus Proposition 9 implies that $\delta(R) \geq \gamma k/2$. So by Proposition 12 there exists a set $\mathcal{S}$ of disjoint substars of R such that every vertex of R lies in some star from $\mathcal{S}$ and such that $1 \leq \Delta(S) \leq 2/\gamma$ for each $S \in \mathcal{S}$.

Next we apply Proposition 8 to obtain sets $V_i' \subseteq V_i$ of size $(1 - 2\varepsilon/\gamma)L =: L'$ such that for all the edges V_iV_j of R lying in some star from $\mathcal{S}$ the graph $(V_i', V_j')_{G'}$ is $(2\varepsilon, d - (1 + 2/\gamma)\varepsilon)$-super-regular. Henceforth we will think of R and of the stars in $\mathcal{S}$ as graphs whose vertices are the new sets V_i'. Add all vertices of G which do not lie in some V_i' to the exceptional set V_0. By adding further vertices to V_0 if necessary, we may assume that L' is even. We still denote the enlarged exceptional set by V_0. Thus $|V_0| \leq \varepsilon n + 2\varepsilon kL/\gamma + k \leq 3\varepsilon n/\gamma$.

Given a vertex $v \in V_0$ and a star $S \in \mathcal{S}$, we say that S is *v-friendly* if there is a vertex $V_i' \in S$ such that v has at least $\gamma L'/4$ neighbours in V_i'. Let N_v denote the number of v-friendly stars $S \in \mathcal{S}$. Then

$$\gamma n/2 < (\gamma - 3\varepsilon/\gamma)n \leq d_G(v) - |V_0| \leq N_v \cdot (1 + 2/\gamma)L' + \sum_{S \in \mathcal{S}} |S| \cdot \gamma L'/4,$$

and therefore, since $\sum_{S \in \mathcal{S}} |S| = k$,

$$N_v > \frac{\gamma}{3L'} \cdot \left(\frac{\gamma n}{2} - \frac{\gamma k L'}{4} \right) \geq \frac{\gamma^2 n}{12 L'}.$$

So

$$\frac{2|V_0|}{\alpha L'} \leq \frac{6\varepsilon n}{\gamma \alpha L'} < N_v$$

for every vertex $v \in V_0$. But this implies that we can greedily assign each vertex $v \in V_0$ to a v-friendly star $S \in \mathcal{S}$ in such a way that to every $S \in \mathcal{S}$ we assign at most $\alpha L'/2$ vertices from V_0.

Consider a fixed $S \in \mathcal{S}$ and let $X \subseteq V_0$ be the set of all vertices assigned to S. Let U_1 be the centre of S and let $U_2, \dots, U_{|S|}$ be its other vertices. So each U_ℓ is a set of the form V_i'. Fix any bipartite quadrangulation P_S of maximum degree $4\Delta(S) \leq 8/\gamma$ whose vertex classes are U_1 and $U_2 \cup \dots \cup U_{|S|}$ such that for each $\ell \leq |S|$ there is a set $\mathcal{C}_\ell$ of at least $L'/4 \geq |X|$ facial 4-cycles of P_S with the property that, firstly, each $C \in \mathcal{C}_\ell$ has two of its vertices in U_ℓ, secondly, these vertices are distinct for different $C \in \mathcal{C}_\ell$ and thirdly, each facial 4-cycle of P_S lies in at most one such $\mathcal{C}_\ell$. Recalling that L' is even, it is not difficult to see that such quadrangulations exist (see Fig. 1).

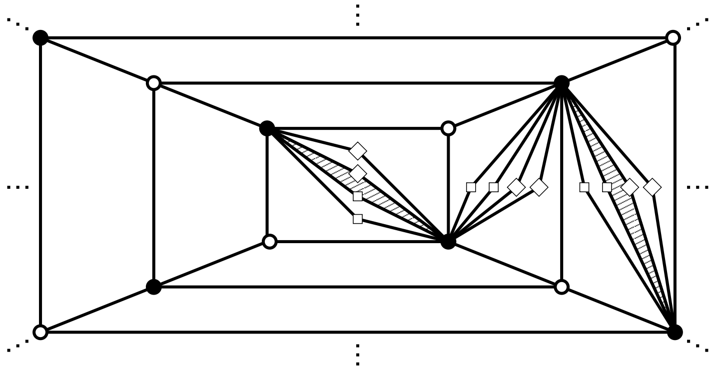

Figure 1: A quadrangulation P_S which corresponds to a star S with three leaves. The black vertices belong to U_1. The shaded faces indicate a possible choice for $\mathcal{C}_1$.

As each edge of S corresponds to a $(2\varepsilon, d/2)$-super-regular subgraph of G', the Blow-up lemma (Lemma 10) implies that the subgraph of G' corresponding to S (that is $G'[U_1 \cup \dots \cup U_{|S|}]$) contains a spanning copy of P_S such that every vertex $v \in X$ is joined to two opposite vertices on some facial 4-cycle of P_S and such that these 4-cycles differ for distinct vertices $v \in X$. Indeed, this can be achieved as follows. By definition, each $v \in X$ has at least $\gamma L'/4$ neighbours in some U_ℓ ($1 \leq \ell \leq |S|$). Assign v to a cycle $C_v \in \mathcal{C}_\ell$ such that these cycles C_v differ for distinct such v. When applying the Blow-up lemma, for each $v \in X$ the two vertices in $V(C_v) \cap U_\ell$ are image restricted to the neighbourhood of v in U_ℓ. (This can be done since the vertices in $V(C_v) \cap U_\ell$ are distinct for different v.)

The graph obtained from P_S by inserting all the vertices $v \in X$ in their facial 4-cycles C_v is still a quadrangulation. Hence G contains a planar subgraph which is a disjoint union of $|\mathcal{S}|$ quadrangulations and thus has $2n - 4|\mathcal{S}| \geq 2n - 4N(\varepsilon, 2)$ edges. □

4.3 Planar subgraphs of size $3n - C$

The *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as $(\chi(H) - 1)|H|/(|H| - \sigma)$, where σ denotes the minimum size of the smallest colour class in an optimal colouring of H. For the proof of Theorem 3 we need the following result of Komlós [13, Thm. 8].

Theorem 13 *For every $\varepsilon > 0$ and every graph H there exists an integer $k_0 = k_0(H, \varepsilon)$ such that all but at most εk vertices of every graph R of order $k \geq k_0$ and minimum degree $\delta(R) \geq (1 - 1/\chi_{cr}(H))k$ can be covered by disjoint copies of H.*

Note that Theorem 13 immediately implies that for all $\varepsilon, \gamma > 0$ there exists an integer $n_0 = n_0(\varepsilon, \gamma)$ such that every graph R of order $n \geq n_0$ and minimum degree at least γn contains a planar graph with at least $2n - \varepsilon n$ edges. Indeed, let $H := K_{2,s}$ in Theorem 13, where s is sufficiently large compared to ε and γ. Then the critical chromatic number of H is close to one and the disjoint union of all copies of H given by Theorem 13 is a planar subgraph of R of the required size. Similarly, as there exist large triangulations whose critical chromatic number is close to 2 (e.g. modify the graph in Fig. 2 below), Theorem 13 implies that Theorem 3 is true for large n if we only ask for a planar subgraph with $3n - \varepsilon n$ edges.

Proof of Theorem 3. By making γ smaller, we may assume that $1/\gamma$ is an integer divisible by 4. Let $\varepsilon_0(\gamma)$ and $d_0(\gamma) =: d$ be as given in Proposition 9. Set $a := 2/\gamma$ and $H := K_{a,a,1}$, the complete 3-partite graph with vertex classes of size a, a and 1. Let $\varepsilon_0(d/2, 8a, 2a+1, \gamma/4) =: \varepsilon^*$ and $\alpha(d/2, 8a, 2a+1, \gamma/4) =: \alpha$ be as defined in the Blow-up lemma (Lemma 10). Put

$$\varepsilon := \min\left\{\varepsilon_0(\gamma), \frac{\varepsilon^*}{2}, \frac{\gamma^3\alpha}{640}, \frac{\gamma d}{12}\right\}$$

and let $k_0 := k_0(H, \varepsilon)$ be defined as in Theorem 13. Clearly, it suffices to show that every graph G whose order n is sufficiently large compared with γ contains a planar subgraph with at least $3n - 6N(\varepsilon, k_0)$ vertices, where $N(\varepsilon, k_0)$ is given by the Regularity lemma (Lemma 7).

We first apply the Regularity lemma to G to obtain an exceptional set V_0 and clusters $V_1, \dots, V_k$ where $k_0 \le k \le N(\varepsilon, k_0)$. Let L and G' be as defined in the Regularity lemma and let R denote the reduced graph. Thus Proposition 9 implies that $\delta(R) \ge (1/2 + \gamma/2)k$. As $\chi_{cr}(H) = 2(2a+1)/2a = 2 + 1/a$ and therefore $\delta(R) \ge (1 - 1/\chi_{cr}(H))k$, we can apply Theorem 13 to obtain a set $\mathcal{H}$ of disjoint copies of H in R such that all but at most εk vertices of R lie in the union H' of all these copies. As $\Delta(H') = 2a$, we may apply Proposition 8 to find for every $V_i \in V(H')$ a set $V_i' \subseteq V_i$ of size $(1 - 2a\varepsilon)L =: L'$ such that for every edge $V_iV_j \in H'$ the graph $(V_i', V_j')_{G'}$ is $(2\varepsilon, d - (1+2a)\varepsilon)$-super-regular. We add all vertices of G which do not lie in some V_i' to the exceptional set V_0 and still denote this enlarged set by V_0. Thus

$$|V_0| \le \varepsilon n + \varepsilon k L + 2a\varepsilon k L \le 4a\varepsilon n.$$

Put $R' := R[V(H')]$. We will think of R' and of the graphs in $\mathcal{H}$ as graphs whose vertices are the new sets V_i'.

Given a vertex $v \in V_0$ and $S \in \mathcal{H}$, we say that S is *v-friendly* if there are vertices V_i' and V_j' lying in different classes of the $K_{a,a} \subseteq S$ such that v has at least $\gamma L'/4$ neighbours in both V_i' and V_j'. Let N_v denote the number of v-friendly $S \in \mathcal{H}$. Then

$$(1/2+\gamma/2)n < (1/2+\gamma-4a\varepsilon)n \le d_G(v) - |V_0| \le N_v(2a+1)L' + |\mathcal{H}|(a+1+\gamma a/4)L'$$

and therefore

$$\begin{aligned} N_v &> \frac{(1/2+\gamma/2)n}{(2a+1)L'} - \frac{k(a+1+\gamma a/4)L'}{(2a+1)^2 L'} \\ &\ge \frac{n}{(2a+1)L'} \cdot \left(\frac{1}{2} + \frac{\gamma}{2} - \frac{a(1+1/a+\gamma/4)}{2a}\right) \ge \frac{n\gamma}{5L'} \cdot \frac{\gamma}{8} = \frac{\gamma^2 n}{40L'}. \end{aligned}$$

So

$$\frac{2|V_0|}{\alpha L'} \le \frac{8a\varepsilon n}{\alpha L'} < N_v$$

for every vertex $v \in V_0$. But this implies that we can successively assign each vertex $v \in V_0$ to a v-friendly $S \in \mathcal{H}$ in such a way that to every $S \in \mathcal{H}$ we assign at most $\alpha L'/2$ vertices from V_0.

Consider a fixed $S \in \mathcal{H}$ and the set $X \subseteq V_0$ of all vertices assigned to S. Let P_S be any 3-partite plane graph which satisfies the following three properties. Firstly, the classes of P_S have sizes aL', aL' and L' respectively. Secondly, $\Delta(P_S) \le 8a$ and, thirdly, P_S is a triangulation apart from $|X|$ disjoint facial 4-cycles and the vertices of each of these 4-cycles lie in the two larger vertex classes of P_S. Such plane graphs exist, see e.g. Fig. 2.

Since each edge of S corresponds to a $(2\varepsilon, d/2)$-super-regular subgraph of G', the Blow-up lemma (Lemma 10) implies that the subgraph of G' corresponding to S contains a spanning copy of P_S where every vertex $v \in X$ is joined to all vertices on one of the facial 4-cycles in P_S and these 4-cycles differ for distinct vertices from X. (The latter can be achieved in a similar way as in the proof of Theorem 2.) Thus by inserting the vertices from X into these facial 4-cycles of P_S we obtain a triangulation. Proceeding similarly for every element of $\mathcal{H}$, we obtain a spanning planar subgraph of G which is the disjoint union of $|\mathcal{H}|$ triangulations and thus has $3n - 6|\mathcal{H}| \ge 3n - 6N(\varepsilon, k_0)$ edges. $\square$

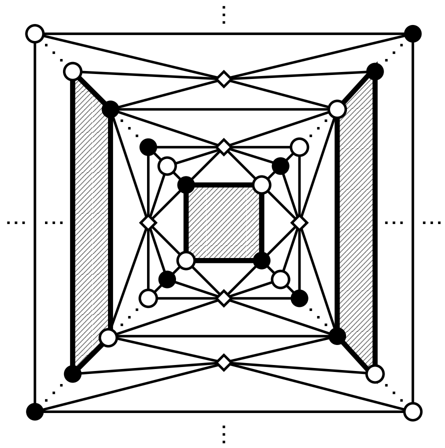

Figure 2: A triangulation apart from the shaded faces (into which the exceptional vertices will be inserted)

As a special case, the following proposition implies that the constant C in Theorem 3 must depend on γ and that the extra γn in the condition on the minimum degree cannot be replaced by a sublinear term.

Proposition 14 *For all positive integers k and n which satisfy $n/2+k = r(2k+1)$ for some integer $r \geq 2$ there is a graph G of order n and minimum degree $n/2+k$ which does not contain a planar subgraph with more than $3n-6-n/12k$ edges.*

Proof. Let G be the graph obtained from a disjoint union of r cliques $G_1, \dots, G_r$ of order $2k+1$ by adding a set Y of $n/2-k$ new vertices and joining every vertex in Y to every vertex in $V(G_1) \cup \dots \cup V(G_r) =: X$. So G has order n and minimum degree $n/2+k$. Consider a planar subgraph P of G with a maximum number of edges. Put $C := 3n-6-e(P)$. We will show that $C \geq n/12k$. Let E be a set of C edges such that $P+E$ is a triangulation, T say. Thus $E \cap E(G) = \emptyset$. Call an edge $e \in E$ *useful for G_i* if either

- e has an endvertex in G_i (and thus both endvertices of e lie in X) or
- e has both endvertices in Y and is an edge of a facial triangle of T which contains a vertex of G_i.

We claim that for every i there is an edge in E which is useful for G_i. Since a given edge from E lies in two faces of T and hence is useful for at most two cliques G_i, this would imply that

$$C = |E| \geq \frac{r}{2} = \frac{n/2+k}{4k+2} \geq \frac{n}{8k+4} \geq \frac{n}{12k},$$

as desired. So fix $i \leq r$ and let us now show that there is an edge in E which is useful for G_i. Suppose not. Then every vertex of G_i lies in a facial triangle of T which is contained in G. So each such triangle contains at least one edge of

G_i. We say that all these facial triangles of T are of *type I* and all other facial triangles (i.e. those which do not contain an edge of G_i) are of *type II*. So no vertex of $X - V(G_i)$ lies in a facial triangle of type I and thus there are facial triangles of type II. Since T is a triangulation, there is a path in the dual graph from a triangle of type I to a triangle of type II. Hence there is a triangle of type I which shares an edge with some triangle D of type II. But D cannot be contained in G, and so it contains an edge e from E. It is now easy to check that e is useful for G_i, a contradiction. □

4.4 Triangulations and Quadrangulations

The *square* G^2 of a graph G is the graph obtained from G by adding an edge between every two vertices of distance two in G. For the proof of Theorem 4 we will use the following result of Fan and Kierstead [10]. (It was extended to arbitrary powers of Hamilton cycles by Komlós, Sárközy and Szemerédi [17], see also [16].)

Theorem 15 *Every graph of minimum degree at least* $2|G|/3$ *contains the square of a Hamilton path.*

Proof of Theorem 4. Clearly, we may assume that $\gamma < 1/3$. Apply Proposition 9 to obtain $\varepsilon_0(\gamma)$ and $d_0(\gamma)$. Put $d := \min\{\gamma, d_0(\gamma)\}$. Let $\varepsilon_0(d/2, 8, 3, (d/2)^4) =: \varepsilon^*$ and $\alpha(d/2, 8, 3, (d/2)^4) =: \alpha$ be as given in the Blow-up lemma (Lemma 10). Set

$$\varepsilon := \min\left\{\varepsilon_0(\gamma), \frac{\varepsilon^*}{3}, \frac{\gamma\alpha}{252}, \frac{d^3}{16}\right\}.$$

and $k_0 := \max\{2/\varepsilon, 20/\gamma\}$. Throughout the proof we assume that n is sufficiently large for our estimates to hold.

Apply the Regularity lemma (Lemma 7) to G to obtain an exceptional set V_0 and clusters $V_1, \dots, V_k$ where $k_0 \le k \le N(\varepsilon, k_0)$. Let L and G' be as defined in the Regularity lemma. By adding at most 2 of the V_i to the exceptional set V_0 if necessary, we may assume that 3 divides k. We still denote the enlarged exceptional set by V_0. Thus $|V_0| \le \varepsilon n + 2L \le \varepsilon n + 2n/k_0 \le 2\varepsilon n$. Let R denote the reduced graph. By Proposition 9 we have $\delta(R) \ge (2/3 + \gamma/2)k - 2$. So Theorem 15 implies that R contains the square of a Hamilton path P. As $\Delta(P^2) = 4$, we may apply Proposition 8 to obtain adjusted clusters $V_i' \subseteq V_i$ ($i \ge 1$) of size $(1 - 4\varepsilon)L =: L'$ such that every edge of P^2 corresponds to a $(2\varepsilon, d - 5\varepsilon)$-super-regular subgraph of G'. We add all vertices that do not lie in some V_i' to the exceptional set V_0. Thus $|V_0| \le 2\varepsilon n + 4\varepsilon kL \le 6\varepsilon n$. Given a vertex $x \in R$, we will write $V'(x)$ for the adjusted cluster corresponding to x. Since $|V'(x)|, |V'(y)| \ge L/2$ for every edge $xy \in R$, it follows from the ε-regularity of the original pair that the graph $(V'(x), V'(y))_{G'}$ corresponding to xy is 2ε-regular and has density $> d - \varepsilon$.

Partition the vertices of P^2 into $k' := k/3$ disjoint sets $D_1, \dots, D_{k'}$, each containing 3 consecutive vertices of P. So the vertices in each D_i induce a triangle of P^2. For all $1 \le i < k'$ let N_i be the number of vertices of R which are joined to at least five of the six vertices in $D_i \cup D_{i+1}$. Then

$$6\delta(R) - 2e(R[D_i \cup D_{i+1}]) \le e_R(D_i \cup D_{i+1}, V(R) \setminus (D_i \cup D_{i+1})) \le 6N_i + 4|R| \quad (5)$$

and thus

$$N_i \ge \delta(R) - 2|R|/3 - e(R[D_i \cup D_{i+1}])/3 \ge \gamma k/2 - 2 - 5 > 0.$$

So for each $1 \le i < k'$ we can find a vertex $a_i \in R$ as well as vertices $s_i, t_i \in D_i$ and $u_{i+1}, w_{i+1} \in D_{i+1}$ with $s_i u_{i+1} \in P^2$ and such that in R each of $s_i, t_i, u_{i+1}, w_{i+1}$ is joined to a_i. (Here the vertices a_i need not be distinct for different i.) As each edge of R corresponds to a 2ε-regular subgraph of G' of density $> d - \varepsilon$, it easily follows from repeated applications of Proposition 6 that there are vertices $x_i \ne y_i$ in $V'(a_i)$ such that in the graph G' their common neighbourhood in each of $V'(s_i), V'(t_i), V'(u_{i+1}), V'(w_{i+1})$ has size at least $(d - 3\varepsilon)^2 L'$. Moreover, all these vertices x_i and y_i can be chosen to be distinct. Roughly speaking, the proof now proceeds as follows. We apply the Blow-up lemma to obtain for all i an (almost-) triangulation which is a spanning subgraph of the subgraph of G' corresponding to D_i. (Each exceptional vertex will also be added to one of these triangulations.) The vertices x_i and y_i will be used to 'glue together' all these triangulations into a single triangulation containing all vertices of G. In this gluing process we will also use two edges between $V'(s_i)$ and $V'(u_{i+1})$.

So let $S_i \subseteq V'(s_i)$ be any set consisting of $(d - 3\varepsilon)^3 L'$ vertices which lie in the common neighbourhood of x_i and y_i but are not of the form x_j or y_j $(1 \le j < k')$. Note that this is possible since $(d - 3\varepsilon)^3 L' \le (d - 3\varepsilon)^2 L' - 2k'$. Define T_i, U_{i+1} and W_{i+1} similarly. Since we still have $|U_{i+1}| \ge 2\varepsilon L'$, we can apply Proposition 6 again to find a set $S'_i \subseteq S_i$ of size $(d - 3\varepsilon)^4 L' \le |S_i| - 2\varepsilon L'$ such that in G' each vertex from S'_i has at least $(d - 3\varepsilon)|U_{i+1}| \ge (d - 3\varepsilon)^4 L'$ neighbours in U_{i+1}.

Remove all x_i and y_i from the adjusted clusters to which they belong (but do not add them to V_0). Then the sizes of the clusters thus obtained lie between $L' - 2k'$ and L'. Set $\ell := \lfloor (L' - 2k')/4 \rfloor$. By moving a constant number of vertices into V_0 if necessary, we may assume that for all $1 \le i \le k'$ every cluster belonging to D_i has size $4\ell =: L''$. We still denote by $V'(x)$ the (re)-adjusted cluster corresponding to a vertex $x \in R$ and by V_0 the enlarged exceptional set. Thus $|V_0| \le 7\varepsilon n$ and each pair of clusters in D_i still corresponds to a $(3\varepsilon, d/2)$-super-regular subgraph of G'. Furthermore, we can easily ensure that each newly adjusted cluster of the form $V'(s_i)$, $V'(t_i)$, $V'(u_i)$ or $V'(w_i)$ still contains S'_i, T_i, U_i or W_i respectively.

Let H_1, H_2 and H_3 be the 3-partite plane graphs of order $3L''$ given in Fig 3. So each H_i has maximum degree 8 and all of its vertex classes have size $L'' = 4\ell$. Moreover, both H_1 and H_2 are triangulations apart from two disjoint facial 4-cycles. In H_1 the vertices on these 4-cycles lie in the same two vertex classes while in H_2 one of the 4-cycles has its vertices in the first and second vertex class and the other one in the second and third vertex class. H_3 is a triangulation apart from one facial 4-cycle.

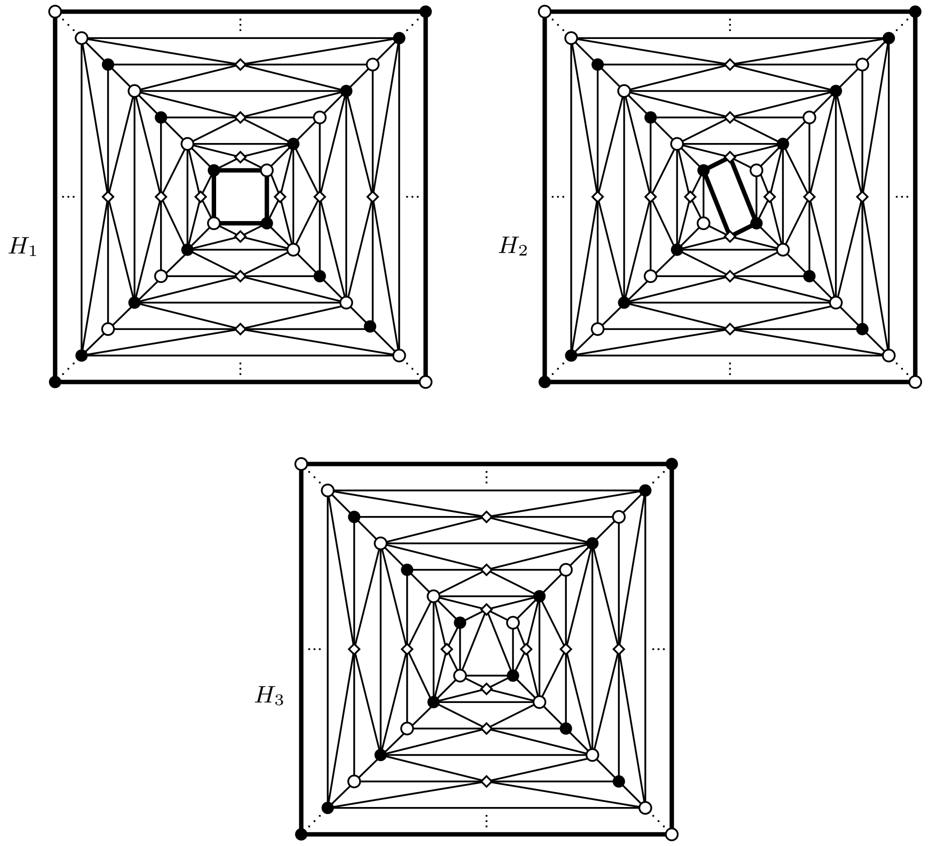

Figure 3: The graphs H_1, H_2 and H_3, the only non-triangular facial cycles are indicated with thick lines

The Blow-up lemma implies that for all $1 \leq i \leq k'$ the subgraph of G' corresponding to $R[D_i]$ contains a spanning copy of each of H_1, H_2 and H_3. However, before we apply the Blow-up lemma we also have to take care of the exceptional vertices. So given a vertex $v \in V_0$ and $1 \leq i \leq k'$, we say that D_i is *v-friendly* if each of the three newly adjusted clusters in D_i contains at least $\gamma L''$ neighbours of v. Let N_v denote the number of v-friendly D_i's. Then

$$(2/3 + \gamma/2)n < d_G(v) - |V_0| \leq N_v \cdot 3L'' + k' \cdot (2+\gamma)L''.$$

Thus

$$N_v > \frac{n}{3L''}\left(\frac{2}{3} + \frac{\gamma}{2} - \frac{k(2+\gamma)L''}{3n}\right) \geq \frac{n}{3L''}\left(\frac{\gamma}{2} - \frac{\gamma}{3}\right) = \frac{\gamma n}{18L''}$$

and hence

$$\frac{2|V_0|}{\alpha L''} \leq \frac{14\varepsilon n}{\alpha L''} < N_v$$

for every $v \in V_0$. This shows that we can successively assign each exceptional vertex $v \in V_0$ to some v-friendly D_i in such a way that to each D_i we assign at most $\alpha L''/2$ vertices.

We are now ready to construct our spanning triangulation of G. We first apply the Blow-up lemma to find a spanning copy P_1 of H_3 in the subgraph of G' corresponding to $R[D_1]$ so that the vertices of the unique facial 4-cycle in P_1 lie alternately in S_1' and T_1 and so that every exceptional vertex v assigned to D_1 is joined to all vertices on some facial triangle of P_1 where these facial triangles are disjoint for distinct such vertices $v \in V_0$. (This can be done in a similar way as in the proof of Theorem 2 since H_3 contains at least $\alpha L''/2$ disjoint facial triangles which are also disjoint from the unique facial 4-cycle of H_3.) Let $x_S^1, y_S^1 \in S_1'$ and $x_T^1, y_T^1 \in T_1$ be the vertices of the facial 4-cycle of P_1 and call this cycle C_{ST}^1.

For $1 < i < k'$, we now say that D_i is of *type I* if the unordered pairs s_i, t_i and u_i, w_i coincide and of *type II* if they differ. The pair s_i, t_i will be used to 'glue' the (almost-) triangulation P_i corresponding to D_i to that corresponding to D_{i+1}, whereas the pair u_i, w_i will be used to 'glue' P_i to the (almost-) triangulation corresponding to D_{i-1}. As the next step, we apply the Blow-up lemma to find a spanning copy P_2 of H_1 if D_2 is of type I, or of H_2 if it is of type II, in the subgraph of G' corresponding to $R[D_2]$ such that the vertices of one facial 4-cycle lie alternately in S_2' and T_2, the vertices of the other facial 4-cycle lie alternately in U_2 and W_2 and such that every exceptional vertex v assigned to D_2 is joined to all vertices on some facial triangle of P_2. (Again, these facial triangles are disjoint for distinct such vertices v.) Let $x_S^2, y_S^2 \in S_2'$ and $x_T^2, y_T^2 \in T_2$ be the vertices of the first facial 4-cycle C_{ST}^2 and let $x_U^2, y_U^2 \in U_2$ and $x_W^2, y_W^2 \in W_2$ be the vertices of the other facial 4-cycle C_{UW}^2. As, by definition of S_1', each of x_S^1, y_S^1 has at least $(d-3\varepsilon)^4 L'$ neighbours in U_2, we may also require that x_U^2 is joined to x_S^1 and y_U^2 is joined to y_S^1. (To achieve this, we restrict the image of x_U^2 to the neighbourhood of x_S^1 in U_2 and the image of y_U^2 to the neighbourhood of y_S^1 in U_2.) Furthermore, by definition of S_1', T_1, U_2 and W_2, both x_1 and y_1 are joined to all vertices of C_{ST}^1 and C_{UW}^2. Thus x_1 and y_1 may be used to 'glue' P_1 and P_2 together in order to obtain a planar graph which is a triangulation apart from one facial 4-cycle, namely C_{ST}^2 (Fig. 4).

We may continue in this fashion to obtain a spanning triangulation. Indeed, for $P_{k'}$ we again choose a copy of H_3 such that the vertices on the unique facial 4-cycle $C_{UW}^{k'}$ of $P_{k'}$ lie alternately in $U_{k'}$ and $W_{k'}$ and such that one of the two vertices from $U_{k'}$ on $C_{UW}^{k'}$ is joined to $x_S^{k'-1}$ while the other one is joined to $y_S^{k'-1}$. Thus if we glue $P_{k'}$ into the planar graph constructed in the previous step, we obtain a triangulation T. As each exceptional vertex v is joined to all vertices on some facial triangle of T and all these are distinct, we can add the exceptional vertices to T to obtain a triangulation containing all vertices of G. $\square$

Proof of Theorem 5 (sketch). The proof proceeds in a similar way as that of Theorem 4 except for a few modifications (and simplifications) which we describe below. We may now assume that the reduced graph R has even order and contains a Hamilton path P (instead of the square of a Hamilton path). We partition P into $|P|/2 := k'$ independent edges $D_1, \dots D_{k'}$. We then adjust the clusters such that each edge D_i corresponds to a $(2\varepsilon, d-2\varepsilon)$-super-regular

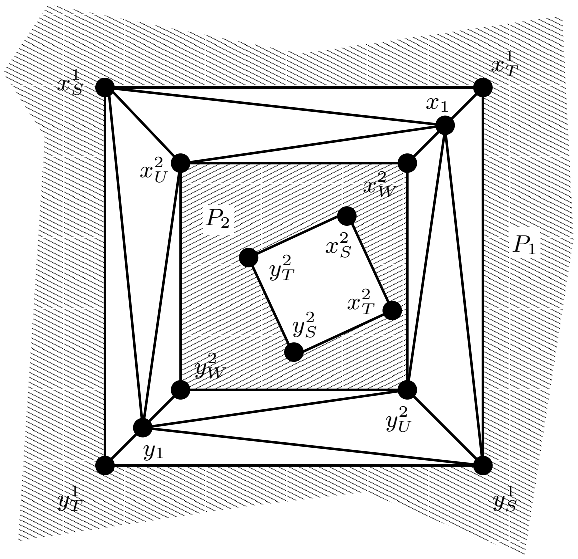

Figure 4: Gluing two almost-triangulations P_1 and P_2

subgraph of G'. A calculation similar to (5) shows that for every pair D_i, D_{i+1} there is a vertex $a_i \in R$ which is joined to both a vertex $s_i \in D_i$ and a vertex $u_{i+1} \in D_{i+1}$. We choose two vertices $x_i, y_i \in V'(a_i)$ which have many common neighbours in both $V'(s_i)$ and $V'(u_{i+1})$. Finally, we apply the Blow-up lemma to obtain spanning quadrangulations P_i of the subgraphs of G' corresponding to the D_i which are 'glued together' into a single quadrangulation P using the vertices x_i and y_i (Fig. 5). These quadrangulations are chosen so that every

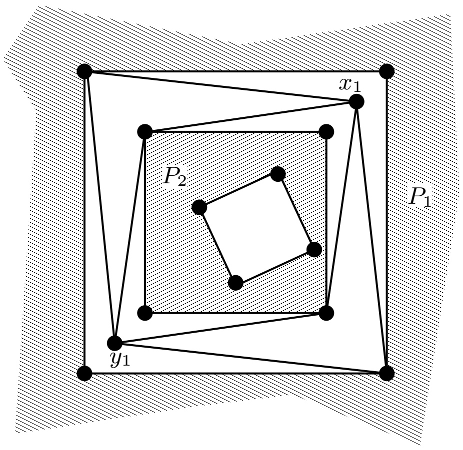

Figure 5: Gluing two quadrangulations P_1 and P_2

exceptional vertex v is joined to two opposite vertices on some facial 4-cycle where these 4-cycles are disjoint for distinct exceptional vertices v. So all the exceptional vertices can be added to P to obtain a spanning quadrangulation of G. □

As remarked towards the end of Section 1, the planar graphs guaranteed by Theorems 2–5 can be constructed in polynomial time: both the Regularity lemma and the Blow-up lemma can be implemented in polynomial time (see [1] and [15]). As the order of the reduced graph is constant, the remaining steps can also be carried out in polynomial time.

Acknowledgements

We would like to thank Béla Bollobás, Matthias Jüngel and Carsten Thomassen for useful discussions.

References

[1] N. Alon, R.A. Duke, H. Lefmann, V. Rödl and R. Yuster, The algorithmic aspects of the Regularity lemma, *J. Algorithms* **16** (1994), 80–109.

[2] C. T. Benson, Minimal regular graphs of girths eight and twelve, *Canad. J. Math.* **26** (1966), 1091–1094.

[3] B. Bollobás, *Extremal Graph Theory*, Academic Press 1978.

[4] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics 184, Springer-Verlag 1998.

[5] B. Bollobás and A. Frieze, Spanning maximal subgraphs of random graphs, *Random Struct. Alg.* **2** (1991), 225-231.

[6] W.G. Brown, On hamiltonian regular graphs of girth six, *J. London Math. Soc.* **42** (1967), 514–520.

[7] G. Călinescu, C.G. Fernandes, U. Finkler and H. Karloff, A better approximation algorithm for finding planar subgraphs, *J. Algorithms* **28** (1998), 105-124.

[8] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423–439.

[9] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics 173, Springer-Verlag 1997.

[10] G. Fan and H.A. Kierstead, Hamilton square paths, *J. Combin. Theory B* **67** (1996), 167–182.

[11] L. Faria, C.M.H. de Figueiredo and C.F.X. Mendonça, On the complexity of the approximation of nonplanarity parameters for cubic graphs, *Discrete Applied Math.* **141** (2004), 119-134.

[12] J. Komlós, The Blow-up Lemma, *Comb. Probab. Comput.* **8** (1999), 161–176.

[13] J. Komlós, Tiling Turán theorems, *Combinatorica* **20** (2000), 203–218.

[14] J. Komlós, G. N. Sárközy and E. Szemerédi, Blow-up Lemma, *Combinatorica* **17** (1997), 109–123.

[15] J. Komlós, G. N. Sárközy and E. Szemerédi, An algorithmic version of the blow-up lemma, *Random Struct. Alg.* **12** (1998), 297–312.

[16] J. Komlós, G. N. Sárközy and E. Szemerédi, On the Pósa-Seymour conjecture, *J. Graph Theory* **29** (1998), 167–176.

[17] J. Komlós, G. N. Sárközy and E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Ann. Combin.* **2** (1998), 43–60.

[18] J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, *Bolyai Society Mathematical Studies 2, Combinatorics, Paul Erdős is Eighty (Vol. 2)* (D. Miklós, V. T. Sós and T. Szőnyi eds.), Budapest (1996), 295–352.

[19] D. Kühn and D. Osthus, Spanning triangulations in graphs, *J. Graph Theory*, to appear.

[20] D. Schlatter, *On the Evolution of Random Discrete Structures: Poset Dimension and Maximum Planar Subgraphs*, Diploma thesis, Humboldt-Universität zu Berlin, 2002.

Daniela Kühn
Mathematisches Seminar
Universität Hamburg
Bundesstraße 55
D - 20146 Hamburg
Germany

kuehn@math.uni-hamburg.de

Deryk Osthus and Anusch Taraz
Institut für Informatik
Humboldt-Universität zu Berlin
Unter den Linden 6
D - 10099 Berlin
Germany

osthus@informatik.hu-berlin.de
taraz@informatik.hu-berlin.de