# Partitions of graphs with high minimum degree or connectivity

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**Abstract**

We prove that there exists a function $f(\ell)$ such that the vertex set of every $f(\ell)$-connected graph $G$ can be partitioned into sets $S$ and $T$ such that each vertex in $S$ has at least $\ell$ neighbours in $T$ and both $G[S]$ and $G[T]$ are $\ell$-connected. This implies that there exists a function $g(\ell, H)$ such that every $g(\ell, H)$-connected graph contains a subdivision $TH$ of $H$ so that $G - V(TH)$ is $\ell$-connected. We also prove an analogue with connectivity replaced by minimum degree. Furthermore, we show that there exists a function $h(\ell)$ such that the vertex set of every graph $G$ of minimum degree at least $h(\ell)$ can be partitioned into sets $S$ and $T$ such that both $G[S]$ and $G[T]$ have minimum degree at least $\ell$ and the bipartite subgraph between $S$ and $T$ has average degree at least $\ell$.

## 1 Introduction

It is well-known that the vertex set of every graph $G$ of minimum degree at least $2\ell - 1$ can be partitioned into sets $S$ and $T$ such that the bipartite subgraph $(S, T)_G$ of $G$ between $S$ and $T$ has minimum degree at least $\ell$. Moreover, Hajnal [4] and Thomassen [11] showed that for every $\ell$ there exists $k = k(\ell)$ such that the vertex set of every graph of minimum degree at least $k$ can be partitioned into sets $S$ and $T$ such that the graphs $G[S]$ and $G[T]$ induced by these sets both have minimum degree at least $\ell$. Later, Stiebitz [9] proved that $k = 2\ell + 1$ suffices. The complete graph on $2\ell + 1$ vertices shows that this is best possible. Hajnal [4] and Thomassen [11] also proved an analogue of their result where minimum degree is replaced by connectivity.

On the other hand, it is easily seen that we cannot simultaneously require that $G[S]$, $G[T]$ and $(S, T)_G$ have large minimum degree or connectivity if $G$ has large minimum degree or connectivity (see Proposition 6). In Section 2 we show that we can nevertheless strengthen the results of Hajnal and Thomassen by requiring that each vertex in $S$ has many neighbours in $T$:

**Theorem 1** *For every $\ell \in \mathbb{N}$ there exists $k = k(\ell) \in \mathbb{N}$ such that the vertex set of every graph $G$ of minimum degree at least $k$ can be partitioned into non-empty sets $S$ and $T$ such that both $G[S]$ and $G[T]$ have minimum degree at least $\ell$ and every vertex in $S$ has at least $\ell$ neighbours in $T$.*

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**Theorem 2** *For every $\ell \in \mathbb{N}$ there exists $k = k(\ell) \in \mathbb{N}$ such that the vertex set of every $k$-connected graph $G$ can be partitioned into non-empty sets $S$ and $T$ such that both $G[S]$ and $G[T]$ are $\ell$-connected and every vertex in $S$ has at least $\ell$ neighbours in $T$.*

Theorem 2 can be applied to show the existence of non-separating structures in highly connected graphs. A well-known result in this area is the theorem of Thomassen [10] which states that every $(\ell+3)$-connected graph $G$ contains an induced cycle $C$ such that $G - V(C)$ is $\ell$-connected. In our case, if $H$ is a structure whose existence is guaranteed by high connectivity, then Theorem 2 implies that in every highly connected graph $G$ there exists a copy of $H$ such that $G - V(H)$ is still highly connected. Indeed, if $S, T$ is a partition as in Theorem 2, then any copy of $H$ in $G[S]$ will do. For example, an application of the result of Mader that every graph of sufficiently large average degree contains a subdivision of a given graph $H$ (see e.g. [3, Thm. 3.6.1]) immediately yields the following.

**Corollary 3** *For every $\ell \in \mathbb{N}$ and every graph $H$ there exists $k = k(\ell, H) \in \mathbb{N}$ such that every $k$-connected graph $G$ contains a subdivision $TH$ of $H$ such that $G - V(TH)$ is $\ell$-connected.*

Similarly, an 'induced analogue' of Corollary 3 follows from the result of [6] that for every $s \in \mathbb{N}$ and every graph $H$ there exists $d = d(s, H)$ such that every $K_{s,s}$-free graph $G$ with average degree at least $d$ contains an induced subdivision of $H$:

**Corollary 4** *For all $\ell, s \in \mathbb{N}$ and every graph $H$ there exists $k = k(\ell, s, H) \in \mathbb{N}$ such that every $k$-connected $K_{s,s}$-free graph $G$ contains an induced subdivision $TH$ of $H$ such that $G - V(TH)$ is $\ell$-connected.*

Analogous results for graphs of large minimum degree can be deduced from Theorem 1.

Strengthening the theorem of Mader, Larman and Mani [7] showed that every sufficiently highly connected graph $G$ contains a subdivision of a given graph $H$ with prescribed branch vertices. Thus the following conjecture of Thomassen, which greatly strengthens Theorem 2, would even imply the existence of a non-separating such subdivision.

**Conjecture** (Thomassen [12]) *For every $\ell \in \mathbb{N}$ there exists $k = k(\ell) \in \mathbb{N}$ such that if $G$ is a $k$-connected graph and $X \subseteq V(G)$ consists of at most $\ell$ vertices then the vertex set of $G$ can be partitioned into non-empty sets $S$ and $T$ such that $X \subseteq S$, each vertex in $S$ has at least $\ell$ neighbours in $T$ and both $G[S]$ and $G[T]$ are $\ell$-connected.*

If true, this would also imply a conjecture of Lovász (see [12]) that there exists a function $f(\ell)$ such that, for any two vertices $x$ and $y$ in an $f(\ell)$-connected graph $G$, there exists an induced $x$–$y$ path $P$ such that $G - V(P)$ is $\ell$-connected. Kriesell [5] observed that Tutte's theorem on 3-connected graphs can be used to show that $f(1) = 3$ and proved that $f(2) = 5$.

Finally, for graphs of high minimum degree we can strengthen the result of Hajnal and Thomassen in a different direction. It turns out that instead of asking for high minimum degree on one side of the bipartite graph $(S,T)_G$ as in Theorem 1, one can require $(S,T)_G$ to have high average degree:

**Theorem 5** *The vertex set of every graph $G$ of minimum degree at least $2^{32}\ell$ can be partitioned into non-empty sets $S$ and $T$ such that both $G[S]$ and $G[T]$ have minimum degree at least $\ell$, $(S,T)_G$ has average degree at least $\ell$ and $|S|,|T| \geq |G|/2^{18}$.*

In Section 4 we present some results about partitions of graphs of high average degree or chromatic number.

Let us now introduce some notation. Given a graph $G$, we write $e(G)$ for the number of its edges, $d(G) := 2e(G)/|G|$ for its average degree, $\delta(G)$ for its minimum degree and $\chi(G)$ for its chromatic number. If $S$ and $T$ are disjoint sets of vertices of $G$, then we denote by $G[S]$ the subgraph of $G$ induced by $S$ and write $(S,T)_G$ for the bipartite subgraph of $G$ whose vertex classes are $S$ and $T$ and whose edges are all the $S$–$T$ edges in $G$. We denote by $e_G(S,T)$ or, if this is unambiguous, by $e(S,T)$ the number of these edges and call $(S,T)_G$ the *bipartite subgraph of $G$ between $S$ and $T$*. We write $N(x)$ for the neighbourhood of a vertex $x \in G$ and $d_G(x)$ or $d(x)$ for the degree of $x$ in $G$.

## 2 Proof of Theorems 1 and 2

Before proving Theorems 1 and 2, let us first observe that these results are best possible in the sense that one cannot additionally require that each vertex in $T$ has many neighbours in $S$, i.e. that $(S,T)_G$ has large minimum degree.

**Proposition 6** *For every $k \in \mathbb{N}$ there is a $k$-connected graph $G$ whose vertex set cannot be partitioned into non-empty sets $S$ and $T$ such that each vertex of $G$ has a neighbour in both $S$ and $T$.*

**Proof.** Let $G_k^n$ be the bipartite graph whose vertex classes are $X := \{1,\dots,n\}$ and the set $X^{(k)}$ consisting of all $k$-element subsets of $X$, and in which $x \in X$ is joined to $Y \in X^{(k)}$ if $x \in Y$. It is easy to see that $G_k^n$ is $k$-connected if $n$ is sufficiently large compared to $k$. Consider any partition of $V(G_k^n)$ into non-empty sets $S$ and $T$. As we may assume that $n \geq 2k-1$, one of $S$, $T$ contains at least $k$ vertices from $X$. Suppose that this is true for $S$ and let $Y \in X^{(k)}$ be a vertex of $G_k^n$ corresponding to a set of $k$ vertices in $X \cap S$. Then $Y$ has no neighbour in $T$, which proves the proposition. $\square$

In the proof of Theorems 1 and 2 we will apply the following quantitative versions (see Hajnal [4, Remark 3 and Thm. 4.3] or Stiebitz [9]) of the results mentioned in Section 1.

**Theorem 7** *The vertex set of every graph $G$ of minimum degree at least $3\ell$ can be partitioned into non-empty sets $S$ and $T$ such that both $G[S]$ and $G[T]$ have minimum degree at least $\ell$.*

**Theorem 8** *The vertex set of every $8\ell$-connected graph $G$ can be partitioned into non-empty sets $S$ and $T$ such that both $G[S]$ and $G[T]$ are $\ell$-connected.*

In the proof of Theorem 1 we will also use the following simple proposition (see e.g. [3, Prop. 1.2.2]) and a lemma which is an easy consequence of Theorem 7.

**Proposition 9** *Every graph $G$ contains a subgraph of minimum degree at least $d(G)/2$.*

**Lemma 10** *The vertex set of every graph $G$ of minimum degree at least $3\ell$ can be partitioned into non-empty sets $S$ and $T$ such that both $G[S]$ and $G[T]$ have minimum degree at least $\ell$ and every subgraph of $G[S]$ has average degree less than $6\ell$.*

**Proof.** Theorem 7 implies that $V(G)$ can be partitioned into non-empty sets $S$ and $T$ such that both $\delta(G[S]) \geq \ell$ and $\delta(G[T]) \geq \ell$. Choose $S$ and $T$ satisfying these properties such that $S$ is minimal. Then $S, T$ are as desired. Indeed, if $G[S]$ had a subgraph of average degree at least $6\ell$, then by Proposition 9 it would also contain a subgraph $H$ of minimum degree at least $3\ell$. Apply Theorem 7 to obtain a partition $S', T'$ of $V(H)$. Let $X$ be a maximal subset of $S \setminus T'$ such that $S' \subseteq X$ and $\delta(G[X]) \geq \ell$. Then each vertex in $S \setminus (T' \cup X)$ has at least $3\ell - \ell = 2\ell$ neighbours in $G$ outside $X$. Hence $X$, $V(G) \setminus X$ is a partition contradicting the minimality of $S$. $\square$

The following lemma shows that if $S, T$ is a partition as provided by Lemma 10, then we can alter it by successively moving vertices from $S$ to $T$ to obtain a partition $S'$, $T'$ satisfying Theorem 1 except that $G[S']$ is only required to have large average degree. Theorem 1 will then immediately follow since $G[S']$, having large average degree, contains a subgraph of large minimum degree. The 'moreover' part of Lemma 11 will only be used in the proof of Theorem 2.

**Lemma 11** *Let $c, k, \ell, r \in \mathbb{N}$ be such that $\ell \geq r$ and $k \geq 2^6 c\ell r$. Let $G$ be a graph of minimum degree at least $k$ and let $S, T$ be a partition of $V(G)$ such that $d(G[S]) \geq \ell$, $\delta(G[T]) \geq \ell$ and every subgraph of $G[S]$ has average degree less than $c\ell$. Then there exists $S' \subseteq S$ such that, writing $T' := V(G) \setminus S'$, every vertex in $S'$ has at least $r$ neighbours in $T'$, $d(G[S']) \geq \ell/8$ and $\delta(G[T']) \geq r$. Moreover, $T'$ can be obtained from $T$ by successively adding vertices having at least $r$ neighbours in the superset of $T$ already constructed.*

**Proof.** Let $A \subseteq S$ be the set of all vertices in $S$ having at least $4c\ell$ neighbours in $S$, and let $C := S \setminus A$. Thus $4c\ell|A| \leq 2e(G[S])$, and so, as every subgraph of $G[S]$ has average degree less than $c\ell$,

$$e(G[A]) \leq \frac{c\ell|A|}{2} \leq \frac{e(G[S])}{4}. \tag{1}$$

Define $B$ to be the set of all those vertices in $S$ which have less than $r$ neighbours in $T$. (So if $B = \emptyset$, then $S' := S$ and $T' := T$ would be a partition as required in the lemma.) Since $\delta(G) \geq k \geq 4c\ell + r$, we have $B \subseteq A$. Also

$$\frac{k|B|}{2} \leq (k - r)|B| \leq 2e(G[S]). \tag{2}$$

Let $B'$ be the set of all those vertices in $B$ which have at least $r$ neighbours in $C$. For every vertex $x \in B'$ choose a set $N_x$ of $r$ neighbours in $C$. Let $e_S(\bigcup_{x\in B'} N_x)$ denote the number of all edges in $G[S]$ which are incident to some vertex in $\bigcup_{x\in B'} N_x$. (So here we also count those edges with both endvertices in $\bigcup_{x\in B'} N_x$.) Then

$$e_S(\bigcup_{x\in B'} N_x) \le |B'|4c\ell r \overset{(2)}{\le} \frac{16c\ell r \cdot e(G[S])}{k} \le \frac{e(G[S])}{4}. \tag{3}$$

Thus if we add $\bigcup_{x\in B'} N_x$ to $T$ then we can ensure that every vertex in $B'$ has at least $r$ neighbours in the resulting superset of $T$ while maintaining high average degree in the resulting subgraph of $G[S]$. We now have to deal with the vertices in $B \setminus B'$. As some of these vertices may have (nearly) all their neighbours in $B \setminus B'$, but we nevertheless want to ensure that every vertex in $S'$ has many neighbours in $T'$, we will move all of $B \setminus B'$ into $T'$. For each vertex of $B \setminus B'$ which has most of its neighbours outside $B \setminus B'$ (and thus in $A \setminus (B \setminus B')$), we will also move some of these neighbours into $T'$ in order to ensure that $G[T']$ has large minimum degree. The main difficulty is that the removal of these neighbours should not decrease the average degree of the resulting subgraph of $G[S]$ too much.

So let $A_1 := B \setminus B'$, $A_2 := A \setminus A_1$ and let $A'_1 \subseteq A_1$ be the set of all those vertices which have at least $k/2$ neighbours in $A_2$. Put $C' := C \setminus \bigcup_{x\in B'} N_x$ (Fig. 1). We will show that we can add $A_1 \cup \bigcup_{x\in B'} N_x$ to $T$ together with a set $A'_2 \subseteq A_2$ such that

(i) each vertex in $A'_1$ has at least $r$ neighbours in $A'_2$,

(ii) $e(A'_2, C') \le e(A_2, C')/2$.

As we shall see, the partition thus obtained is as required in the lemma. However, let us first show that there exists a set $A'_2$ satisfying (i) and (ii).

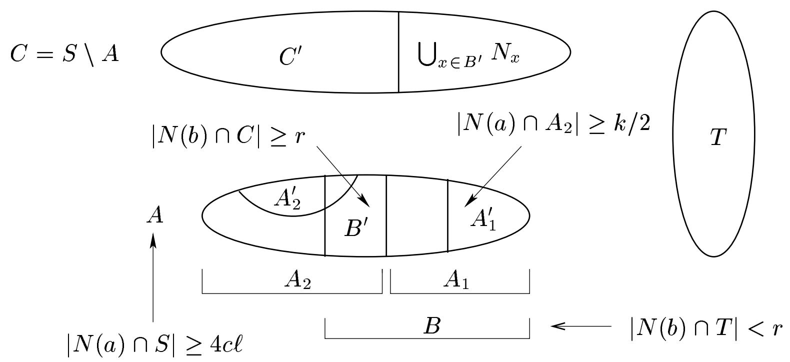


Figure 1: The set-up of the proof of Lemma 11.

Let $a_1, a_2, \dots$ be any enumeration of the vertices in $A_2$ such that their degrees in $(A_2, C')_G$ form a non-increasing sequence. For every $x \in A'_1$ let $R_x$ be the set consisting of the $r$ rightmost neighbours of $x$ in $a_1, a_2, \dots$. Note that, writing $[a_i]$ for the set $\{a_1, \dots, a_i\}$, *If $a \in R_x$ then $x$ has at least $k/2 - r$ neighbours in $[a]$.* $\quad (*)$

Let $A_2' := \bigcup_{x \in A_1'} R_x$. Then clearly $A_2'$ satisfies condition (i). To show that $A_2'$ also satisfies (ii), we first prove the following

**Claim.** *For all $a \in A_2$ at most half of the vertices in $[a]$ are contained in $A_2'$.*

Suppose not and let $a$ be a vertex in $A_2$ such that more than half of the vertices in $[a]$ are contained in $A_2'$. Consider the bipartite subgraph $H$ of $G[S]$ between $[a]$ and the set $A_1^*$ consisting of all those vertices $x \in A_1'$ for which $[a] \cap R_x \neq \emptyset$. Since $[a] \cap A_2'$ contains at most $r$ vertices lying in a common set $R_x$, it follows that $|A_1^*| \geq |[a]|/(2r)$. Moreover, $(*)$ implies that each vertex in $A_1^*$ has degree at least $k/2 - r \geq k/4$ in $H$. So if $|A_1^*| \geq |[a]|$, then

$$e(H) \geq \frac{k|A_1^*|}{4} \geq \frac{k|H|}{8} \geq \frac{c\ell|H|}{2};$$

while if $|A_1^*| \leq |[a]|$, we have

$$e(H) \geq \frac{k|A_1^*|}{4} \geq \frac{k|[a]|}{8r} \geq \frac{k|H|}{16r} \geq \frac{c\ell|H|}{2}.$$

Thus in each case $H$ is a subgraph of $G[S]$ of average degree at least $c\ell$, contradicting our assumption. This proves the claim.

By the claim there exists an injection which sends every vertex $a \in A_2'$ to a vertex in $A_2 \setminus A_2'$ preceding $a$ in the sequence $a_1, a_2, \ldots$. The choice of the enumeration $a_1, a_2, \ldots$ now implies (ii).

Put $T' := T \cup A_1 \cup A_2' \cup \bigcup_{x \in B'} N_x$ and $S' := V(G) \setminus T'$. Let us first show that $G[S']$ has large average degree. Clearly,

$$e(G[S']) \geq e(A_2 \setminus A_2', C') \overset{\text{(ii)}}{\geq} e(A_2, C')/2.$$

Thus if $e(A_2, C') \geq e(G[S])/4$ then $e(G[S']) \geq e(G[S])/8$, as required. To verify the other case, note that

$$\begin{aligned} e(G[S']) &\geq e(G[S]) - e(G[A]) - e_S(\bigcup_{x \in B'} N_x) - e(A_1, C') - e(A_2', C') \\ &\overset{(1),(3)}{\geq} e(G[S]) - \frac{e(G[S])}{4} - \frac{e(G[S])}{4} - |B \setminus B'|r - e(A_2', C') \\ &\overset{(2),\text{(ii)}}{\geq} \frac{e(G[S])}{4} - \frac{e(A_2, C')}{2}. \end{aligned}$$

Thus $e(G[S']) \geq e(G[S])/8$ also holds in the case when $e(A_2, C') \leq e(G[S])/4$. This proves that $d(G[S']) \geq \ell/8$. Furthermore, every vertex in $S' \subseteq S \setminus A_1 = S \setminus (B \setminus B')$ has at least $r$ neighbours in $(T \cup \bigcup_{x \in B'} N_x) \subseteq T'$.

Let us now verify that $T'$ can be obtained from $T$ by successively adding vertices having at least $r$ neighbours in the superset of $T$ already constructed. Clearly, as $B \cap \bigcup_{x \in B'} N_x = \emptyset$, $T_1 := T \cup \bigcup_{x \in B'} N_x$ can be constructed in this way. As $A_2' \subseteq (A \setminus B) \cup B'$, every vertex in $A_2'$ has at least $r$ neighbours in

$T_1$, and thus also $T_1 \cup A_2'$ can be constructed. Since by (i) every vertex in $A_1'$ has at least $r$ neighbours in $A_2'$, we can now construct $T_1 \cup A_2' \cup A_1'$. Finally, to construct $T$, we now successively move vertices from $A_1 \setminus A_1'$ to the other side of the partition as long as they have at least $r$ neighbours there. Suppose that we are not able to exhaust all of $A_1 \setminus A_1'$ in this way, but are left with some non-empty set $X$. As every vertex in $X \subseteq A_1 \setminus A_1' \subseteq B \setminus B'$ has less than $r$ neighbours in $C$, less than $k/2$ neighbours in $A_2$, less than $r$ neighbours in $T' \setminus X$ and as $X \cup C \cup A_2 \cup (T' \setminus X) = V(G)$, it follows that the graph $G[X]$ has minimum degree at least $k - k/2 - 2r \geq c\ell$, contradicting our assumption on $G[S]$. Thus $T'$ can be constructed in the required way. In particular as $\delta(G[T]) \geq \ell \geq r$ this implies that $\delta(G[T']) \geq r$, so $S'$ and $T'$ are as desired.

□

**Proof of Theorem 1.** We will prove the theorem for $k \geq 2^{10} \cdot 6\ell^2 = 2^{11} \cdot 3\ell^2$. Apply Lemma 10 to obtain a partition $S'', T''$ with $\delta(G[S'']) \geq 16\ell$, $\delta(G[T'']) \geq 16\ell$ and such that every subgraph of $G[S'']$ has average degree less than $6 \cdot 16\ell$. Then apply Lemma 11 to find a partition $S', T'$ such that $d(G[S']) \geq 2\ell$, $\delta(G[T']) \geq \ell$ and each vertex in $S'$ has at least $\ell$ neighbours in $T'$. By Proposition 9, the graph $G[S']$ contains a subgraph $H$ of minimum degree at least $\ell$. Then $S := V(H)$, $T := V(G) \setminus S$ is a partition as required.

□

To prove Theorem 2, we need the following weak version of a theorem of Mader (see e.g. [3, Thm. 1.4.2]) and an analogue to Lemma 10.

**Theorem 12** *Every graph $G$ has a $\lfloor d(G)/4 \rfloor$-connected subgraph.*

**Lemma 13** *The vertex set of every $9\ell$-connected graph $G$ can be partitioned into non-empty sets $S$ and $T$ such that $G[T]$ is $\ell$-connected, the average degree of $G[S]$ is at least $\ell$ and every subgraph of $G[S]$ has average degree at most $32\ell$.*

**Proof.** Theorem 8 implies that $V(G)$ can be partitioned into non-empty sets $S$ and $T$ such that $d(G[S]) \geq \ell$, $G[T]$ is $\ell$-connected and $|T| \geq 2\ell$. Choose $S$ and $T$ satisfying these properties such that $S$ is minimal. We will show that $S$ and $T$ are as desired. First note that $G[S]$ cannot be $8\ell$-connected. For otherwise, by Theorem 8, $S$ can be partitioned into two non-empty sets $S_1$ and $S_2$ such that each $G[S_i]$ is $\ell$-connected. As $G$ was $2\ell$-connected and $|S|, |T| \geq 2\ell$, there are at least $2\ell$ independent $S$–$T$ edges in $G$. So some $S_i$, $S_2$ say, must contain endvertices of at least $\ell$ of these edges. But then $G[T \cup S_2]$ is $\ell$-connected, and so $S_1$, $T \cup S_2$ is a partition contradicting the minimality of $S$.

Let us now show that $G[S]$ does not even contain an $8\ell$-connected subgraph (and hence, by Theorem 12, no subgraph of average degree at least $32\ell$). Suppose that $H$ is an $8\ell$-connected subgraph of $G[S]$. As $G[S]$ is not $8\ell$-connected, it has a cut-set $X$ with $|X| < 8\ell$. Let $C$ be the unique component of $G[S] - X$ such that $V(H) \subseteq V(C) \cup X$. Let $S' := V(C) \cup X$ and $T' := V(G) \setminus S'$. Then $G[T']$ is $\ell$-connected. Indeed, if it has a cut-set $Y$ with $|Y| < \ell$, then, as $G[T]$ is $\ell$-connected, there is a component $D$ of $G[T'] - Y$ such that $T \subseteq V(D) \cup Y$. Since there is no path from $F := T' \setminus (V(D) \cup Y)$ to $C$ within $G[S]$ which avoids

$X$ and no path in $G$ from $F$ to $T$ which avoids $S' \cup Y = V(C) \cup X \cup Y$, it follows that $X \cup Y$ separates $C$ from $F$ in $G$, contradicting the fact that $G$ is $9\ell$-connected (Fig. 2). We may now successively move vertices from $S'$ to $T'$

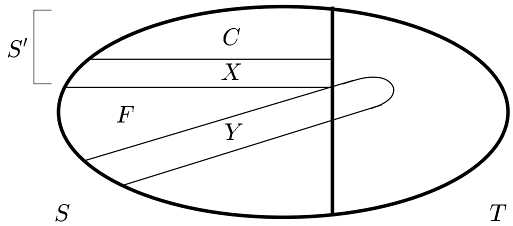


Figure 2: $X \cup Y$ separates $C$ from $F$ in $G$.

if they have less than $8\ell$ neighbours in the subset of $S'$ already constructed. (Thus each of these vertices has at least $\ell$ neighbours in the superset of $T'$ and therefore adding it preserves $\ell$-connectedness.) This process must terminate as $H$ is a subgraph of $G[S']$ with minimum degree at least $8\ell$. The partition obtained in this way contradicts the minimality of $S$. $\square$

**Proof of Theorem 2.** We will prove the theorem for $k \geq 2^{16}\ell^2$. First apply Lemma 13 to obtain a partition $S'', T''$ such that $d(G[S'']) \geq 2^5\ell$, $G[T'']$ is $2^5\ell$-connected and such that every subgraph of $G[S'']$ has average degree less than $32 \cdot 2^5\ell$. Now apply Lemma 11 to find a partition $S'$ and $T'$ such that each vertex in $S'$ has at least $\ell$ neighbours in $T'$, $d(G[S']) \geq 4\ell$, $\delta(G[T']) \geq \ell$ and $T'$ can be obtained from $T''$ by successively adding vertices having at least $\ell$ neighbours in the superset of $T''$ already constructed. Thus $G[T']$ is $\ell$-connected, since $G[T'']$ is. By Theorem 12, $G[S']$ contains an $\ell$-connected subgraph $H$. As each vertex in $S' \setminus V(H)$ has $\ell$ neighbours in $T'$, $G[T' \cup (S' \setminus V(H))] = G - V(H)$ is $\ell$-connected. Thus $S := V(H)$, $T := V(G) \setminus S$ is a partition as required. $\square$

It would be interesting to know whether the quadratic bounds on $k$ in Theorems 1 and 2 can be replaced by linear ones.

# 3 Proof of Theorem 5

Similarly to Section 2, before proving the theorem, let us first show that it is best possible in the sense that we cannot always partition the vertex set of a graph of large minimum degree into non-empty sets $S$ and $T$ satisfying Theorems 1 and 5 simultaneously, i.e. we cannot always find $S, T$ such that both $G[S]$ and $G[T]$ have large minimum degree, $(S, T)_G$ has large average degree and every vertex in $S$ has many neighbours in $T$.

**Proposition 14** *For every $k \in \mathbb{N}$ there is a $k$-connected graph $G$ whose vertex set cannot be partitioned into non-empty sets $S$ and $T$ such that $\delta(G[S]) \geq 1$, $\delta(G[T]) \geq 1$, $d((S, T)_G)) \geq 1$ and each vertex in $S$ has at least one neighbour in $T$.*

Note that Proposition 6 is a special case of Proposition 14, but has the advantage that its proof is less technical. Moreover, we will refer to the proof of Proposition 6 again in Section 4 to deduce a result about partitions of graphs of large chromatic number and large minimum degree.

**Proof of Proposition 14.** Let $G^n_k$, $X$ and $X^{(k)}$ be as defined in the proof of Proposition 6 and suppose that $S, T$ is a partition of $V(G^n_k)$ contradicting Proposition 14. Then $0 < |X \cap S| < k$. (Indeed, if $|X \cap S| \geq k$ and $Y$ is a set consisting of any $k$ vertices in $X \cap S$, then all neighbours of the vertex $Y \in X^{(k)}$ are contained in $S$, contradicting our assumptions on $S$ and $T$.) Moreover, as any vertex $Y$ in $X^{(k)} \cap S$ must contain at least one of the at most $k-1$ points in $X \cap S$ (when $Y$ is viewed as a $k$-element subset of $X$), we have

$$|X^{(k)} \cap S| \leq \sum_{i=1}^{k-1} \binom{k-1}{i} \binom{n}{k-i} \leq 2^k n^{k-1}.$$

If $n$ is sufficiently large, this implies

$$\begin{aligned} e(S,T) &\leq |X \cap S| \cdot \binom{n}{k-1} + |X^{(k)} \cap S| \cdot k \\ &\leq k(1+2^k)n^{k-1} < \frac{1}{2} \cdot \binom{n}{k} \leq \frac{|G^n_k|}{2}. \end{aligned}$$

Thus $d((S,T)_{G^n_k}) < 1$, a contradiction. □

By Proposition 9, every graph $G$ of large average degree contains a subgraph $H$ of large minimum degree. The following lemma implies that if $G$ has many vertices of large degree, then $H$ can be chosen to contain a constant fraction of the vertices in $G$.

**Lemma 15** *Let $\ell \in \mathbb{N}$ and $0 < \alpha \leq 1$. Let $G$ be a graph and let $X \subseteq V(G)$ be a set of at least $\alpha|G|$ vertices having degree at least $\ell$ in $G$. Then $G$ has a subgraph of minimum degree at least $\alpha\ell/4$ which contains at least $|X|/4$ of the vertices in $X$.*

**Proof.** By Proposition 9, $G$ contains a subgraph of minimum degree at least $\alpha\ell/2$. Let $H \subseteq G$ be maximal with minimum degree at least $\alpha\ell/4$ and suppose that it contains less than $|X|/4$ of the vertices in $X$. By the choice of $H$, every vertex of $G - V(H)$ has less than $\alpha\ell/4$ neighbours in $H$. Thus $G - V(H)$ still has at least $3|X|/4 \geq 3\alpha|G|/4$ vertices having degree at least $\ell - \alpha\ell/4 = (1-\alpha/4)\ell$ in $G - V(H)$. Thus

$$d(G - V(H)) \geq \frac{(3\alpha|G|/4) \cdot (1-\alpha/4)\ell}{|G|} \geq \frac{3\alpha}{4} \cdot \frac{3\ell}{4} \geq \frac{\alpha\ell}{2}.$$

By Proposition 9, $G - V(H)$ contains a subgraph $H'$ of minimum degree at least $\alpha\ell/4$. But $H \cup H'$ contradicts the maximality of $H$. □

We will need the following special case of Chernoff's inequality (see [1, Thm. A.11 and A.13]).

**Lemma 16** *Let $X_1, \dots, X_n$ be independent 0-1 random variables with $\mathbb{P}(X_i = 1) = p$ for all $i \le n$, and let $X := \sum_{i=1}^n X_i$. Then $\mathbb{P}(|X - \mathbb{E}X| \ge \mathbb{E}X/2) \le 2\mathrm{e}^{-\mathbb{E}X/16}$ and $\mathbb{P}(|X - \mathbb{E}X| \ge \mathbb{E}X/4) \le 2\mathrm{e}^{-\mathbb{E}X/64}$.*

**Proof of Theorem 5.** Let $k := 2^{32}\ell$ and let $(A, B)_G$ be a spanning bipartite subgraph of $G$ of minimum degree at least $k/2$. Suppose that $|A| \ge |B|$. Delete edges if necessary to obtain a bipartite graph $H$ in which each vertex in $A$ has precisely $k/2$ neighbours in $B$. (Then $B$ may contain some vertices of degree less than $k/2$.) Consider a random partition of $B$ into sets $B_1, B_2$ where each vertex of $B$ is included in $B_1$ with probability $1/2$ independently of all other vertices in $B$. Call a vertex $a \in A$ *good* if it has at least $k/8$ neighbours in each $B_i$, and *bad* otherwise. Given $a \in A$, write $N_a$ for the number of those neighbours of $a$ in $H$ that lie in $B_1$. Lemma 16 together with the fact that $\mathbb{E}(N_a) = k/4$ implies that

$$\mathbb{P}(a \text{ is bad}) \le \mathbb{P}(|N_a - \mathbb{E}(N_a)| \ge \mathbb{E}(N_a)/2) \le 2\mathrm{e}^{-\mathbb{E}(N_a)/16} \le 1/4.$$

Hence the expected number of bad vertices in $A$ is at most $|A|/4$. Thus there exists a partition $B_1, B_2$ of $B$ such that the set $A'$ of good vertices in $A$ satisfies

$$|A'| \ge \frac{3|A|}{4}. \tag{4}$$

Let $H' := H - (A \setminus A')$. Now consider a random partition of $A'$ into $A'_1$ and $A'_2$ where again each vertex of $A'$ is included in $A'_1$ with probability $1/2$ independently of all other vertices in $A'$. Then, by Lemma 16,

$$\begin{aligned}\mathbb{P}\left(\big||A'_1| - |A'|/2\big| \ge |A'|/8\right) &= \mathbb{P}\left(\big||A'_1| - \mathbb{E}(|A'_1|)\big| \ge \mathbb{E}(|A'_1|)/4\right) \\ &\le 2\mathrm{e}^{-\mathbb{E}(|A'_1|)/64} \le 1/3. \end{aligned} \tag{5}$$

Let us say that a vertex $b \in B$ is *good* if $d_{H'}(b) \ge 3k/2^{17}$ and if each $A'_i$ contains at least one quarter of the neighbours of $b$ in $H'$, otherwise call $b$ *bad*. Given a vertex $b \in B$, let $N'_b$ denote the number of those neighbours of $b$ in $H'$ that lie in $A'_1$. Using Lemma 16 again, if $d_{H'}(b) \ge 3k/2^{17}$ we have

$$\mathbb{P}(b \text{ is bad}) \le \mathbb{P}(|N'_b - \mathbb{E}(N'_b)| \ge \mathbb{E}(N'_b)/2) \le 2\mathrm{e}^{-\mathbb{E}(N'_b)/16} \le 1/2^{14}.$$

Call an edge of $H'$ *good* if it is incident to a good vertex of $B$, and *bad* otherwise. Let $Y$ denote the number of bad edges of $H'$. Let $E'$ be the set of all those edges in $H'$ whose endvertex in $B$ has degree less than $3k/2^{17}$ in $H'$. As

$$|E'| \le \frac{3k|B|}{2^{17}} \le \frac{3k|A|}{2^{17}} \overset{(4)}{\le} \frac{k|A'|}{2^{15}} = \frac{e(H')}{2^{14}},$$

it follows that

$$\begin{aligned}\mathbb{E}(Y) &= \sum_{e \in E(H')} \mathbb{P}(\text{the endvertex of } e \text{ in } B \text{ is bad }) \\ &\le |E'| + \sum_{e \in E(H') \setminus E'} \mathbb{P}(\text{the endvertex of } e \text{ in } B \text{ is bad}) \\ &\le |E'| + e(H')/2^{14} \le e(H')/2^{13}.\end{aligned}$$

Now Markov's inequality implies that

$$\mathbb{P}(Y \geq e(H')/2^{12}) \leq \mathbb{P}(Y \geq 2\mathbb{E}Y) \leq 1/2.$$

Together with (5) this implies that there exists a partition $A'_1, A'_2$ of $A'$ such that at most $e(H')/2^{12}$ of the edges of $H'$ are bad and such that $\big|\,|A'_1| - |A'|/2\,\big| \leq |A'|/8$. Thus

$$|A'_i| \geq \frac{3|A'|}{8} \tag{6}$$

for $i = 1, 2$. We will now show that there exist sets $A^*_i \subseteq A'_i$ and $B^*_i \subseteq B_i$ such that each $(A^*_i, B^*_i)_{H'}$ has large minimum degree and $(A^*_2, B^*_1)_{H'}$ contains a constant fraction of the edges of $H'$. Once we have obtained these sets, we will suitably extend $A^*_1 \cup B^*_1$ to $S$ and $A^*_2 \cup B^*_2$ to $T$. To find $A^*_1$ and $B^*_1$, consider the graph $H_1 := (A'_1, B_1)_{H'}$. As every vertex in $A'_1$ has at least $k/8$ neighbours in $B_1$ and

$$|A'_1| \overset{(6)}{\geq} \frac{3|A'|}{8} \overset{(4)}{\geq} \frac{|A|}{4} \geq \frac{|G|}{8} \geq \frac{|H_1|}{8}, \tag{7}$$

we may apply Lemma 15 with $X := A'_1$ and $\alpha := 1/8$ to obtain sets $A^*_1 \subseteq A'_1$ and $B^*_1 \subseteq B_1$ such that the minimum degree of $(A^*_1, B^*_1)_{H'}$ is at least $k/2^8$ and

$$|A^*_1| \geq \frac{|A'_1|}{4} \overset{(7)}{\geq} \frac{|G|}{2^5}. \tag{8}$$

Thus

$$e_{H'}(A^*_1, B^*_1) \geq \frac{k|A^*_1|}{2^8} \overset{(8)}{\geq} \frac{k|A'_1|}{2^{10}} \overset{(6)}{\geq} \frac{3k|A'|}{2^{13}}.$$

But as at most $e(H')/2^{12} = k|A'|/2^{13}$ edges in $H' \supseteq (A^*_1, B^*_1)_{H'}$ are bad, it follows that at least $k|A'|/2^{12}$ of the edges from $(A^*_1, B^*_1)_{H'}$ are good in $H'$. In other words,

$$\sum_{b \in B^*_1,\, b \text{ is good}} d_{H'}(b) \geq k|A'|/2^{12},$$

and therefore there are also at least $k|A'|/2^{14}$ edges in $H'$ joining (the good vertices in) $B^*_1$ to $A'_2$.

Now consider the graph $(A'_2, B^*_1)_{H'}$ and let $A^\diamond_2$ be the set of all vertices in $A'_2$ which have at least $k/2^{15}$ neighbours in $B^*_1$. Then

$$\frac{k|A^\diamond_2|}{2} + \frac{k|A'|}{2^{15}} \geq e_{H'}(A'_2, B^*_1) \geq \frac{k|A'|}{2^{14}},$$

and therefore

$$|A^\diamond_2| \geq \frac{|A'|}{2^{14}} \overset{(4)}{\geq} \frac{|A|}{2^{15}} \geq \frac{|G|}{2^{16}}. \tag{9}$$

Now consider the graph $(A^\diamond_2, B_2)_{H'}$. As in this graph the degree of every vertex in $A^\diamond_2$ is at least $k/8$, we may apply Lemma 15 with $X := A^\diamond_2$ and $\alpha := 1/2^{16}$ to obtain sets $A^*_2 \subseteq A^\diamond_2$ and $B^*_2 \subseteq B_2$ such that $(A^*_2, B^*_2)_{H'}$ has minimum degree at least $k/2^{21}$ and

$$|A^*_2| \geq \frac{|A^\diamond_2|}{4} \overset{(9)}{\geq} \frac{|G|}{2^{18}}.$$

Since every vertex in $A_2^\diamond \supseteq A_2^*$ has at least $k/2^{15}$ neighbours in $B_1^*$, it follows that
$$e_{H'}(A_2^*, B_1^*) \geq \frac{k|A_2^*|}{2^{15}} \geq \frac{k|G|}{2^{33}}.$$

Let $S' := A_1^* \cup B_1^*$ and $T' := A_2^* \cup B_2^*$. Then $|S'| \overset{(8)}{\geq} |G|/2^5$, $|T'| \geq |G|/2^{18}$, $\delta(G[S']) \geq k/2^8 \geq \ell$, $\delta(G[T']) \geq k/2^{21} \geq \ell$ and $(S',T')_G$ has at least $k|G|/2^{33}$ edges. Choose $S \supseteq S'$ and $T \supseteq T'$ such that $S$ and $T$ are disjoint, both $G[S]$ and $G[T]$ have minimum degree at least $k/2^{21}$ and so that $S \cup T$ is maximal. Then $S \cup T = V(G)$. Indeed, otherwise every vertex in $G - (S \cup T)$ has at least $k - 2k/2^{21}$ neighbours outside $S \cup T$, i.e. $\delta(G - (S \cup T)) \geq k/2$. But then $S, V(G) \setminus S$ is a partition contradicting the choice of $S, T$. Note that $e_G(S,T) \geq e_G(S',T') \geq e_{H'}(A_2^*, B_1^*)$ and so $d((S,T)_G) \geq k/2^{32} = \ell$, as desired. □

It would be interesting to know whether a result analogous to Theorem 5 holds for graphs of high connectivity.

# 4 Partitions with constraints on the average degree or the chromatic number

We now derive a partition result for graphs of large average degree: the vertex set of every such graph can be partitioned into $S$ and $T$ such that each of $G[S]$, $G[T]$ and $(S,T)_G$ still has large average degree. This follows straightforwardly from the next theorem, which is a consequence of a result of Porter [8]. (The bounds in Porter's result were improved to best possible ones by Bollobás and Scott [2].)

**Theorem 17** *The vertex set of every graph $G$ can be partitioned into non-empty sets $S$ and $T$ such that $(S,T)_G$ has at least $e(G)/2$ edges while both $G[S]$ and $G[T]$ have at most $e(G)/4 + |G|$ edges.*

**Corollary 18** *The vertex set of every graph $G$ can be partitioned into non-empty sets $S$ and $T$ such that $(S,T)_G$ has at least $e(G)/2$ edges while both $G[S]$ and $G[T]$ have at least $e(G)/4 - 3|G|$ edges.*

**Proof.** Put $n := |G|$ and $m := e(G)$. Let $S$, $T$ be a partition satisfying the conditions in Theorem 17 such that $e(S,T)$ is minimal. Let us show that $S, T$ is as required in the corollary. Suppose not. Then $e(G[S]) < m/4 - 3n$ (say). Thus
$$\sum_{t \in T} d_{(S,T)_G}(t) = e(S,T) > m/2 + 2n \geq 2e(G[T]) = \sum_{t \in T} d_{G[T]}(t),$$
and hence there exists a vertex $t \in T$ which has more neighbours in $S$ than in $T$. But then $S' := S \cup \{t\}$ and $T' := T \setminus \{t\}$ contradicts the choice of $S$, $T$ since $e(G[S']) \leq e(G[S]) + n < m/4 - 2n$, $e(G[T']) \leq e(G[T])$ and $e(S,T) > e(S',T') \geq e(S,T) - n > m/2 + n$. □

Note that Corollary 18 implies that for every $\varepsilon > 0$ the vertex set of every graph $G$ with $m$ edges and average degree at least $6/\varepsilon$ can be partitioned into $S$ and $T$ such that both $G[S]$ and $G[T]$ contain at least $m(1/4 - \varepsilon)$ edges and $(S,T)_G$ contains at least $m/2$ edges.

For a bipartite graph $G = (A,B)$ this (with $m/2$ replaced by $m(1/2 - \varepsilon)$) can also be proved using a straightforward probabilistic argument which we outline below. Consider a random partition of $A$ into $A_1$ and $A_2$. Then with non-zero probability most of the edges of $G$ will have the property that their endvertex $b$ in $B$ is *good* in the sense that about half of the neighbours of $b$ lies in $A_1$ and the other half lies in $A_2$. Now consider any enumeration of the good vertices in $B$ such that their degrees in $G$ form a non-increasing sequence, and let $B_1$ be the set of all good vertices with odd indices. It is easily seen that $A_1 \cup B_1$, $A_2 \cup (B \setminus B_1)$ is a partition as desired. Since every graph $G$ contains a spanning bipartite subgraph $G'$ with $e(G') \geq e(G)/2$, this argument also gives Corollary 18 with much weaker bounds.

To conclude this section, let us prove a simple result about partitions of graphs of large chromatic number and large minimum degree: the vertex set of every such graph can be partitioned into sets $S$ and $T$ such that both $G[S]$ and $G[T]$ have large chromatic number and $(S,T)_G$ has large minimum degree. Note that we cannot additionally require $G[S]$ and $G[T]$ to have large minimum degree. In fact, there need not even exist a partition $S,T$ such that each of $G[S]$, $G[T]$ and $(S,T)_G$ has minimum degree at least one. (Let $H_k^n$ be the graph obtained from the graph $G_k^n$ defined in the proof of Proposition 6 by making $G_k^n[X]$ complete. Then $\chi(H_k^n) \geq n$ and $\delta(H_k^n) = k$, but the proof of Proposition 6 shows that there is no partition $S,T$ having the properties in question.)

**Proposition 19** *The vertex set of every graph $G$ can be partitioned into non-empty sets $S$ and $T$ such that $\chi(G[S]) = \lceil \chi(G)/2 \rceil$, $\chi(G[T]) = \lfloor \chi(G)/2 \rfloor$ and every vertex $x \in G$ has at least $d^*(x) := \min\{\lfloor \chi(G)/2 \rfloor, d(x)/2\}$ neighbours in the partition set not containing $x$.*

**Proof.** Let $S$, $T$ be a partition with $\chi(G[S]) = \lceil \chi(G)/2 \rceil$ and $\chi(G[T]) = \lfloor \chi(G)/2 \rfloor$ and such that $e(S,T)$ is maximal under these conditions. (By partitioning $V(G)$ into the vertices in the first $\lceil \chi(G)/2 \rceil$ colour classes and the remaining $\lfloor \chi(G)/2 \rfloor$ colour classes, it is clear that such a partition exists.) Let $x$ be a vertex in $S$ and suppose that $x$ has less than $d^*(x) \leq \chi(G[T])$ neighbours in $T$. Then the chromatic number of $G[T \cup \{x\}]$ is still $\chi(G[T])$. As $\chi(G) \leq \chi(G[S \setminus \{x\}]) + \chi(G[T \cup \{x\}])$, this implies that $\chi(G[S \setminus \{x\}]) = \chi(G[S])$. But as $d^*(x) \leq d(x)/2$, we have $e(S \setminus \{x\}, T \cup \{x\}) > e(S,T)$, contradicting the choice of $S,T$. Similarly it can be shown that each vertex $x \in T$ has at least $d^*(x)$ neighbours in $S$. Thus $S,T$ is a partition as required. $\square$

It is not hard to show that the vertex set of every graph $G$ of large chromatic number and large average degree can be partitioned into $S,T$ such that both $G[S]$ and $G[T]$ have large chromatic number and each of $G[S]$, $G[T]$ and $(S,T)_G$ has large average degree. The idea is to use an averaging argument to find a set $X \subseteq V(G)$ (consisting of some of the colour classes of $G$) such that $G[X]$

still has large chromatic number but only a small fraction of the edges of $G$ is incident to vertices in $X$. Then one can apply Corollary 18 to $G - X$ to obtain a partition $S'$, $T'$. Adding one half of the colour classes of $G[X]$ to each of $S'$ and $T'$ gives a partition of $V(G)$ as desired.

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