Minors in graphs of large girth

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Abstract

We show that for every odd integer $g \geq 5$ there exists a constant c such that every graph of minimum degree r and girth at least g contains a minor of minimum degree at least $cr^{(g+1)/4}$. This is best possible up to the value of the constant c for $g = 5, 7$ and 11. More generally, a well-known conjecture about the minimal order of graphs of given minimum degree and large girth would imply that our result gives the correct order of magnitude for all odd values of g. The case $g = 5$ of our result implies Hadwiger's conjecture for C_4-free graphs of sufficiently large chromatic number.

1 Introduction

Kostochka [9] and Thomason [17] independently showed that there exists a constant c such that every graph of average degree at least $cr\sqrt{\log r}$ contains the complete graph K_r of order r as minor. Random graphs show that this gives the correct order of magnitude. Recently, Thomason [18] determined the asymptotic value of the above constant c.

On the other hand, Thomassen [19] observed that if the girth of a graph G is large, then G contains (complete) minors whose minimum degree is much larger than that of G itself. In this paper we use probabilistic arguments to give more precise asymptotic bounds: for example, we show that every graph G of girth at least five and minimum degree r contains a minor of minimum degree $c_1 r^{3/2}$ and that if the girth of G is at least seven then G contains a minor of minimum degree at least $c_2 r^2$. In both cases the bound on the minimum degree is best possible up to the value of the constant. More generally, the main results of this paper are as follows:

Theorem 1 *Let $k \geq 1$ and $r \geq 3$ be integers and put $g := 4k + 3$. Then every graph G of minimum degree r and girth at least g contains a minor of minimum degree at least $(r-1)^{k+1}/48 = (r-1)^{(g+1)/4}/48$.*

Theorem 2 *Let $k \geq 1$ and $r \geq \max\{5k, 2\cdot 10^6\}$ be integers and put $g := 4k+1$. Then every graph G of minimum degree at least $4r$ and girth at least g contains a minor of minimum degree at least $r^{k+1/2}/288 = r^{(g+1)/4}/288$.*

In addition to the two cases mentioned above, this is also best possible up to the value of the constant for graphs G of girth 11. In fact, we will see in Section 4 that the above results would give the correct order of magnitude (as a function of r) for arbitrary girth g if there exist graphs of minimum degree r

and odd girth g whose order is at most $c(r-1)^{(g-1)/2}$. The minimum order of such graphs is known to lie between $(r-1)^{(g-1)/2}$ and $4(r-1)^{g-2}$, and it has been conjectured (see e.g. Bollobás [4, p. 164]) that the lower bound gives the proper order of magnitude.

An application of the result of Kostochka and Thomason mentioned in the beginning to the minors obtained in Theorems 1 and 2 immediately yields the following.

Corollary 3 *For all odd integers $g \geq 3$ there exists $c = c(g) > 0$ such that every graph of minimum degree r and girth at least g contains a K_t minor for some*

$$t \geq \frac{cr^{\frac{g+1}{4}}}{\sqrt{\log r}}.$$

□

As every graph of chromatic number at least r contains a subgraph of minimum degree at least $r-1$ and every such graph contains a bipartite subgraph of minimum degree at least $(r-1)/2$, Corollary 3 in turn implies that Hadwiger's conjecture (that every graph of chromatic number r contains K_r as minor) is true for C_4-free graphs of sufficiently large chromatic number:

Corollary 4 *There exists an integer r_0 such that every C_4-free graph of chromatic number $r \geq r_0$ contains a K_r minor.* □

In fact, in [11] we show that similar results (with weaker bounds) even hold for $K_{s,s}$-free graphs whose minimum degree (respectively chromatic number) is sufficiently large compared with s. In Section 2 we also give a simple argument which implies that Hadwiger's conjecture holds for all graphs of girth at least 19 (Corollary 8).

At the other extreme, given an integer t, Theorem 1 with $r = 3$ shows that every graph G of minimum degree at least three contains a minor of minimum degree at least t if its girth is sufficiently large. This fact was first observed by Thomassen [19], who obtained a bound on the girth linear in t. Diestel and Rempel [7] reduced it to $6\log_2 t + 4$. Theorem 1 applied with $r = 3$ and $k = \lceil \log_2 t + 5 \rceil$ shows that the constant 6 can be reduced to 4:

Corollary 5 *Let $t \geq 3$ be an integer. Then every graph of minimum degree at least 3 and girth at least $4\log_2 t + 27$ contains a minor of minimum degree at least t. Hence there exists a constant c such that every graph of minimum degree at least 3 and girth at least $4\log_2 t + 2\log_2\log_2 t + c$ contains a K_t minor.* □

(The second part of Corollary 5 immediately follows from the first by an application of the result of Kostochka and Thomason mentioned above.) As already observed in [7], the existence of 3-regular graphs of girth at least g and order at most $c2^{g/2}$ (which is a special case of the conjecture mentioned earlier) would show that Corollary 5 is asymptotically best possible in the sense that the constant 4 in the leading terms cannot be reduced any further (see Section 4).

The minimal order of such 3-regular graphs is known to lie between $c_1 2^{g/2}$ and $c_2 2^{3g/4}$.

Mader [16] proved that for every $\varepsilon > 0$ and every graph H with $\Delta(H) \geq 3$ there exists an integer g such that every graph G of average degree at least $\Delta(H) - 1 + \varepsilon$ and girth at least g contains H as a topological minor. (His bound on g is at least linear in $|H|$ and also depends on ε.) This implies that for every $\varepsilon > 0$ and every integer t there exists an integer g such that every graph of average degree at least $2 + \varepsilon$ and girth at least g contains a minor of minimum degree t. Indeed, first apply the special (and much easier) case $\Delta(H) = 3$ of Mader's result to obtain a 3-regular graph of large girth as a minor and then the observation of Thomassen mentioned before Corollary 5 to this minor. In [12] we strengthen Mader's result for the case when H is a large clique: for all $\varepsilon > 0$ every graph of average degree at least $t - 2 + \varepsilon$ and girth at least 1000 contains a topological K_t minor if t is sufficiently large compared with ε. Also, based on techniques of Mader [15], in [10] we proved that for large t every graph of minimum degree at least $t - 1$ and girth at least 15 contains a topological K_t minor. This implies the conjecture of Hajós for all graphs of girth at least 15 and sufficiently large chromatic number. See also [13] for related results.

The paper is organized as follows. In Section 2 we introduce necessary definitions and collect some tools which we will need later on. We will also apply an idea of Mader to prove Corollary 8. In Section 3 we then prove Theorems 1 and 2. In the final section we show that Theorem 1 and 2 are best possible up to the value of the constant provided that the conjecture mentioned above is true (which is known to be the case for girth $5, 7$ and 11).

2 Notation, tools and preliminary observations

The *length* of a cycle C or a path P is the number of its edges. The *girth* of a graph G is the length of its shortest cycle and denoted by $g(G)$. The *distance* between two vertices x, y of a graph G is the length of the shortest path joining x to y and denoted by $d_G(x, y)$. Given $\ell \in \mathbb{N}$, the *ℓ-ball $B^\ell_G(x)$ in G around a vertex x* is the subgraph of G induced by all its vertices of distance at most ℓ from x. If $P = x_1 \dots x_\ell$ is a path and $1 \leq i \leq j \leq \ell$, we write $x_i P x_j$ for its subpath $x_i \dots x_j$.

We write $e(G)$ for the number of edges of a graph G and $|G|$ for its order. We denote the degree of a vertex $x \in G$ by $d_G(x)$. The *average degree* of a graph G is defined to be $2e(G)/|G|$ and denoted by $d(G)$. Given $A, B \subseteq V(G)$, an *A–B edge* is an edge of G with one endvertex in A and the other in B, the number of these edges is denoted by $e_G(A, B)$. If A and B are disjoint, we write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all A–B edges in G.

A graph H is a *minor* of G if for every vertex $h \in H$ there is a connected subgraph G_h of G such that all the G_h are disjoint and G contains a G_h–$G_{h'}$ edge whenever hh' is an edge in H. We say that H is the *minor of G obtained by contracting the G_h*. (The vertex set of) G_h is called the *branch set* corresponding to h.

Before we collect some tools which we will need in the proofs of Theorems 1 and 2, let us present a simple proposition which shows that if G is a graph of large girth, then G contains minors whose minimum degree is much larger than that of G itself. Its proof is the same as the beginning of Mader's proof of his main result of [15]. We include it here as it implies a counterpart (Corollary 8) to Corollary 4 for graphs of small chromatic number. Moreover, it should help to illustrate the basic ideas underlying the proofs of Theorems 1 and 2, which use a probabilistic version of Mader's argument.

Proposition 6 *Let $k \geq 1$ and $r \geq 3$ be integers. Then every graph of girth at least $8k+3$ and minimum degree r contains a minor of minimum degree at least $r(r-1)^k$.*

Proof. Let X be a maximal set of vertices of G that have pairwise distance at least $2k+1$ from each other. Thus for distinct $x, y \in X$ the balls $B^k_G(x)$ and $B^k_G(y)$ are disjoint. Extend the $B^k_G(x)$ $(x \in X)$ to disjoint connected subgraphs of G by first adding each vertex of distance $k+1$ from X to one of the $B^k_G(x)$ to which it is adjacent. Then add each vertex of distance $k+2$ from X to one of the subgraphs constructed in the previous step to which it is adjacent. Continue in this fashion until each vertex of G lies in one of the constructed subgraphs and denote the subgraph obtained from $B^k_G(x)$ in this way by T_x. The choice of X implies that each vertex of G has distance at most $2k$ from X. So each vertex of T_x has distance at most $2k$ from x in T_x. Therefore, as $g(G) \geq 4k+2$, each T_x is an induced subtree of G. In particular $B^k_G(x)$ is a tree in which every vertex that is not a leaf has degree at least r and in which every leaf has distance k from x. So $B^k_G(x)$ (and thus also T_x) has at least $r(r-1)^{k-1}$ leaves. Hence T_x sends at least $r(r-1)^k$ edges to vertices outside T_x. As $g(G) \geq 8k+3$, at most one edge of G joins T_x to a given other tree T_y $(y \in X \setminus \{x\})$. Thus the graph obtained from G by contracting the trees T_x $(x \in X)$ has minimum degree at least $r(r-1)^k$, as required. □

An application of the result of Kostochka and Thomason mentioned at the beginning of Section 1 to the minor obtained in Proposition 6 for $k = 1$ shows that for sufficiently large r every graph G of minimum degree r and girth at least 11 contains a K_{r+1} minor. For small r, we will apply the following result of Mader (see [14] or [4, Ch. VII.1]).

Theorem 7 *For all integers $t \geq 4$ every graph of average degree $> 16(t-2)\log_2(t-2)$ contains a K_t minor. Moreover, every graph of average degree > 10 contains a K_7 minor.*

As above, combining this with Proposition 6 leads to the observation that Hadwiger's conjecture is true for all graphs of girth at least 19:

Corollary 8 *Every graph of girth at least 19 and minimum degree r contains a K_{r+1} minor. In particular, every graph of girth at least 19 and chromatic number r contains a K_r minor.* □

In the proof of Theorem 2 we shall use the following two simple facts.

Proposition 9 *Every graph G with at least one edge contains a subgraph of average degree at least $d(G)$ and minimum degree greater than $d(G)/2$.*

Proposition 10 *The vertex set of every graph G can be partitioned into disjoint sets A, B such that the minimum degree of $(A, B)_G$ is at least $\delta(G)/2$.*

Moreover, we will need the following Chernoff type bound (see [1, Thm. A.13]).

Lemma 11 *Let $X_1, \dots, X_n$ be independent 0-1 random variables with $\mathbb{P}(X_i = 1) = p$ for all $i \leq n$, and let $X := \sum_{i=1}^{n} X_i$. Then for all $0 < \varepsilon < 1$ we have*

$$\mathbb{P}(X \leq (1-\varepsilon)\mathbb{E}X) \leq \mathrm{e}^{-\varepsilon^2 \mathbb{E}X/2}.$$

3 Proof of Theorems 1 and 2

In the proof of Proposition 6 we covered the entire vertex set of our graph G with suitable disjoint rooted trees T_x and considered the minor M obtained by contracting these trees. Amongst other properties, these trees had radius between k and $2k$. If we could choose them all of radius at most k while still maintaining sufficiently many edges between the trees, this would reduce the bound on the girth from $8k+3$ to $4k+3$. We will achieve this in the proof of Theorem 1 by choosing the roots of the trees at random, albeit at the expense that there will be a small number of vertices which do not lie in any of the trees. The case when $k = 1$ (i.e. when the trees are stars) of the first part of the proof is similar to an argument of Alon which shows the existence of small dominating sets (i.e. sets of vertices to which every vertex has distance at most one) in graphs of large minimum degree (see [1, Thm. 2.2]).

Proof of Theorem 1. We may assume that $(r-1)^k \geq 48$. Consider a random subset X of $V(G)$ which is obtained by including each vertex in X with probability $p := 4/(r-1)^k$ independently of all other vertices. The branch sets of our minor will be trees of radius at most k whose roots are the elements of X. As $g(G) \geq 2k+2$ and $\delta(G) = r$, for each vertex $x \in G$ the graph $B^k_G(x)$ is a tree with at least $(r-1)^k$ leaves. Call an edge $e = xy$ of G *bad* if $d(x, X) > k$ or $d(y, X) > k$. Then

$$\begin{aligned} \mathbb{P}(xy \text{ is bad}) &\leq \mathbb{P}(B^k_G(x) \cap X = \emptyset) + \mathbb{P}(B^k_G(y) \cap X = \emptyset) \\ &= (1-p)^{|B^k_G(x)|} + (1-p)^{|B^k_G(y)|} \leq 2(1-p)^{(r-1)^k} \\ &\leq 2\mathrm{e}^{-p(r-1)^k} = 2/\mathrm{e}^4, \end{aligned}$$

and so

$$\mathbb{E}(\text{number of bad edges}) \leq 2e(G)/\mathrm{e}^4.$$

Markov's inequality now implies

$$\mathbb{P}(> e(G)/9 \text{ edges are bad}) \leq 18/\mathrm{e}^4 \leq 1/3.$$

Moreover, the expected size of X is $p|G|$, and so again, by Markov's inequality,

$$\mathbb{P}(|X| > 2p|G|) \leq 1/2.$$

Thus with probability at least $1 - 1/2 - 1/3 > 0$ there is an outcome X with $|X| \leq 2p|G|$ and so that at most $e(G)/9$ edges of G are bad.

Extend the vertices in X to disjoint connected subgraphs G_x $(x \in X)$ of G with $x \in G_x$ by first adding each vertex of distance one from X to a vertex in X to which it is adjacent, then adding each vertex of distance two from X to one of the subgraphs constructed in the previous step to which it is now adjacent etc. Continue in this fashion until each vertex of G of distance at most k from X is contained in one of the graphs G_x thus obtained. Then each vertex of G_x has distance at most k from x. As $g(G) \geq 2k+2$, every G_x is an induced subtree of G. So each edge of G that is not bad and does not lie in $\bigcup_{x \in X} E(G_x)$ joins distinct G_x. Moreover, since $g(G) \geq 4k+3$, there is at most one edge of G joining a given pair of graphs G_x. Thus for the minor M of G whose branch sets are the G_x we have

$$\begin{aligned} d(M) &\geq \frac{2(e(G) - |\bigcup_{x \in X} E(G_x)| - e(G)/9)}{|X|} \geq \frac{16e(G) - 18|G|}{9 \cdot 2p|G|} \\ &\geq \frac{4r-9}{9p} \geq \frac{4(r-1)}{9p} \cdot \frac{3}{8} \geq \frac{(r-1)^{k+1}}{24}. \end{aligned}$$

(The fourth inequality holds since $r \geq 3$.) By Proposition 9 the graph M contains a subgraph of minimum degree at least $(r-1)^{k+1}/48$, as desired.

□

A result of Györi [8] states that every C_6-free bipartite graph can be made into a graph of girth at least 7 by deleting at most half of its edges. This implies that the assertion of the $g = 7$ case of Theorem 1 remains true for C_6-free graphs (with a modified constant).

To prove Theorem 2, we will again cover a large part of our graph G with disjoint trees of radius k whose roots are chosen at random (similarly as in the proof of Theorem 1). However, this time the girth is not large enough to ensure that between every pair of these trees there is at most one edge. To deal with such multiple edges we choose the trees more carefully and prove that firstly there are still many (good) edges joining leaves of distinct trees and secondly that only a small fraction of these edges is redundant in the sense that there are many additional (good) edges joining the same pair of trees.

Proof of Theorem 2. First apply Proposition 10 to obtain a bipartite subgraph $G_1 = (A, B)_G$ of G of minimum degree at least $2r$. We may assume that $|A| \geq |B|$. Delete edges if necessary to obtain a bipartite subgraph G_2 of G_1 in which the degree of every vertex in A is precisely $2r$. Thus $d(G_2) \geq 2r$. Now apply Proposition 9 to obtain a subgraph $H = (C, D)_{G_2}$ of G_2 of minimum degree at least $r+1$ and average degree at least $2r$ and where every vertex in C has degree at most $2r$.

We now assign orientations to the edges of H as follows. For every vertex $x \in H$ choose any $r+1$ of its neighbours in H and orient the edges between x

and these neighbours from x towards these. We thus obtain a graph $\vec{H}$ in which every edge has either none, one or two orientations and in which the outdegree of every vertex is precisely $r+1$. We say that a path $x_0 \dots x_\ell$ in $\vec{H}$ is *directed* from x_0 to x_ℓ if each edge $x_i x_{i+1}$ is oriented from x_i towards x_{i+1}. So $x_i x_{i+1}$ may additionally be oriented from x_{i+1} to x_i. Given two vertices x and y of $\vec{H}$ we write $\vec{d}(x,y)$ for the length of the shortest directed path from x to y (and set $\vec{d}(x,y) := \infty$ if such a path does not exist). Given $\ell \in \mathbb{N}$, we write $\vec{S}^\ell(x)$ for the set of all those vertices $y \in \vec{H}$ with $\vec{d}(x,y) = \ell$. We define $\vec{B}^\ell(x)$ to be the subgraph of $\vec{H}$ which consists of all directed paths of length at most ℓ starting at x. Note that if $\ell < 2k$, then, as $g(H) \geq 4k$, the graph $\vec{B}^\ell(x)$ is an induced subtree of $\vec{H}$ with root x in which every edge is oriented away from the root (and possibly also towards it). As the outdegree of every vertex of $\vec{H}$ is $r+1$, every vertex of $\vec{B}^\ell(x)$ which is not a leaf has either $r+1$ or $r+2$ neighbours in $\vec{B}^\ell(x)$ and every leaf has distance precisely ℓ from x. In particular,

$$r^\ell \leq |\vec{S}^\ell(x)| \leq (r+1)^\ell.$$

Consider a random subset X of $V(H)$ which is obtained by including each vertex of H in X with probability

$$p := \frac{1}{4(r+1)^{k-1/2}}$$

independently of all other vertices. For some suitable outcome X, the branch sets of the desired minor in G will be subtrees of H of radius k and with roots in X. Call a vertex $v \in H$ *good* if it satisfies the following three conditions.

(i) $|\vec{S}^k(v) \cap X| \geq \sqrt{r}/6$.

(ii) $\vec{B}^{k-1}(v) \cap X = \emptyset$.

(iii) Each component of $\vec{B}^k(v) - v$ contains at most one vertex of $\vec{S}^k(v) \cap X$.

We will now show that the probability that a given vertex v is good is quite large. First note that, as $r \geq 5k$,

$$\begin{aligned}\mathbb{E}(|\vec{S}^k(v) \cap X|) = p|\vec{S}^k(v)| \geq pr^k &= \frac{\sqrt{r}}{4} \cdot \left(\frac{r}{r+1}\right)^{k-1/2} \\ &\geq \frac{\sqrt{r}}{4}\left(1-\frac{1}{r}\right)^k \geq \frac{\sqrt{r}}{4}\left(1-\frac{k}{r}\right) \geq \frac{\sqrt{r}}{5}.\end{aligned}$$

Since $r \geq 2 \cdot 10^6$, Lemma 11 with $\varepsilon := 1/6$ implies that

$$\mathbb{P}(|\vec{S}^k(v) \cap X| \leq \sqrt{r}/6) \leq 1/25. \tag{1}$$

Secondly,

$$\mathbb{E}(|\vec{B}^{k-1}(v) \cap X|) = p|\vec{B}^{k-1}(v)| \leq 2p|\vec{S}^{k-1}(v)| \leq \frac{1}{2\sqrt{r+1}},$$

and hence, as $r \geq 625$, Markov's inequality implies

$$\mathbb{P}(|\vec{B}^{k-1}(v) \cap X| \geq 1) \leq 1/50. \tag{2}$$

Finally, given a component L of $\vec{B}^k(v) - v$, let $S(v, L) := L \cap \vec{S}^k(v)$. Writing $\sum_{x,y}$ for the sum over all unordered pairs $x \neq y$ of vertices in $S(v, L)$, we have

$$\begin{aligned} \mathbb{P}(|S(v,L) \cap X| \geq 2) &\leq \sum_{x,y} \mathbb{P}(x, y \in X) \leq \binom{|S(v,L)|}{2} p^2 \\ &\leq \frac{((r+1)^{k-1}p)^2}{2} = \frac{1}{32(r+1)}. \end{aligned}$$

As the outdegree of v is $r+1$ and so $\vec{B}^k(v) - v$ has precisely $r+1$ components, it follows that

$$\mathbb{E}(\text{number of components } L \text{ of } \vec{B}^k(v) - v \text{ for which } |S(v,L) \cap X| \geq 2) \leq 1/32.$$

Hence Markov's inequality implies that

$$\mathbb{P}(\text{there is a component } L \text{ of } \vec{B}^k(v) - v \text{ for which } |S(v,L) \cap X| \geq 2) \leq 1/32. \tag{3}$$

From (1), (2) and (3) together it now follows that

$$\mathbb{P}(v \text{ is not good}) \leq 1/10.$$

Call an edge e of H *good* if both of its endvertices are good. Thus

$$\mathbb{P}(e \text{ is not good}) \leq 1/5,$$

and therefore

$$\mathbb{E}(\text{number of edges of } H \text{ which are not good}) \leq e(H)/5.$$

Hence Markov's inequality shows that

$$\mathbb{P}(\geq e(H)/2 \text{ edges of } H \text{ are not good}) \leq 2/5. \tag{4}$$

Moreover, $\mathbb{E}(|X|) = p|H|$, and so Markov's inequality implies that

$$\mathbb{P}(|X| \geq 2p|H|) \leq 1/2. \tag{5}$$

Now (4) and (5) show that with probability at least $1 - 2/5 - 1/2 > 0$ there is an outcome X with $|X| \leq 2p|H|$ and such that at least half of the edges of H are good. Let U be the set of all good vertices of H. We say that a vertex $x \in X$ *belongs to* a vertex $u \in U$ if $\vec{d}(u, x) = k$. So condition (i) in the definition of a good vertex implies that at least $\sqrt{r}/6$ vertices in X belong to u. As $g(H) > 2k$, there exists precisely one directed path P_{ux} of length k from u to a vertex x belonging to u. Given $x \in X$, let U_x denote the set of all the good vertices to which x belongs, and let H_x be the union of all paths P_{ux} over all $u \in U_x$. If $U_x = \emptyset$, we put $H_x := x$. As $g(H) \geq 2k+2$, each H_x is an induced subtree of H and U_x is the set of its leaves.

Let us now prove the following claim.

If $x, y \in X$ are distinct, x belongs to $u \in U$, y belongs to $u' \in U$ and $u \neq u'$ then P_{ux} and $P_{u'y}$ are disjoint. $(*)$

Suppose not and let z be the first vertex on P_{ux} that is contained in $P_{u'y}$. Note that $|uP_{ux}z| \neq |u'P_{u'y}z|$ would imply the existence of either a directed u'–x path or a directed u–y path of length $< k$. Hence $|uP_{ux}z| = |u'P_{u'y}z|$ (and thus in particular $z \neq u$), as both u and u' are good vertices (cf. condition (ii)). So if L is the component of $\vec{B}^k(u) - u$ containing z, then both x and y lie in $L \cap \vec{S}^k(u) \cap X$, contradicting condition (iii) for u.

For every $u \in U$ choose a vertex $x_u \in X$ which belongs to u uniformly at random independently of the other elements of U. For every $x \in X$ we then define T_x to be the subtree of H_x consisting of the paths P_{ux} for all those $u \in U_x$ with $x_u = x$. If there are no such paths we set $T_x := x$. So every choice of the x_u ($u \in U$) yields a family T_x ($x \in X$) of trees. Note that $(*)$ implies that T_x and T_y are disjoint whenever $x \neq y$. We will show that with non-zero probability the x_u will have the property that the minor M of $H \subseteq G$ whose branch sets are the T_x ($x \in X$) thus defined has large average degree. To do this, we will show that with non-zero probability there are only a few pairs T_x, T_y such that H contains many good T_x–T_y edges. Then a large fraction of the good edges of H will join different pairs T_x, T_y and thus will correspond injectively to edges of M. As X is relatively small, this will imply that M has large average degree.

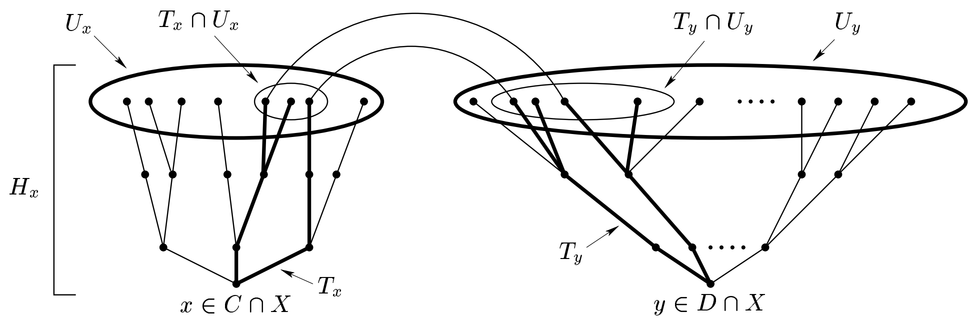

Figure 1: Illustrating a cycle of length 14 in the final part of the proof of Theorem 2 for $k = 3$ and $g = 13$.

Suppose that $x, y \in X$ are given, and let us first estimate the expected number of good edges of H joining T_x to T_y. Recall that by definition, every good T_x–T_y edge joins $T_x \cap U_x$ to $T_y \cap U_y$. As $g(G) \geq 4k+1$, for every component L of $H_x - x$ there is at most one edge in H joining L to H_y. Similarly, for every component L of $H_y - y$ there is at most one edge in H joining L to H_x. So in particular the U_x–U_y edges in H are independent and their number is at most $\min\{d_{H_x}(x), d_{H_y}(y)\} \leq \min\{d_H(x), d_H(y)\}$. But as $g(H) \geq 2k$, every vertex in U_x has distance precisely k from x in $H = (C, D)_{G_2}$. Thus either $U_x \subseteq C$ or $U_x \subseteq D$, and the same is true for U_y. So if H contains a U_x–U_y edge, then one of U_x, U_y must be contained in C while the other one is contained in D. Hence one of x, y lies in C. As every vertex in C has degree at most $2r$ in H, it follows that H contains at most $2r$ edges joining U_x to U_y.

Consider a U_x–U_y edge u_1u_2 with $u_1 \in U_x$ and $u_2 \in U_y$. Then $u_1 \notin U_y$ and $u_2 \notin U_x$, since $g(H) \geq 2k+2$. So the probability that u_1u_2 is a T_x–T_y edge equals the probability that $x_{u_1} = x$ and $x_{u_2} = y$ which in turn is the inverse of the product of the number of vertices in X belonging to u_1 with the number of vertices in X belonging to u_2; so by (i) this probability is at most $(6/\sqrt{r})^2$. Hence

$$\begin{aligned}\mathbb{E}(\text{number of good } T_x\text{–}T_y \text{ edges in } H) &\leq e_H(U_x, U_y) \cdot \left(\frac{6}{\sqrt{r}}\right)^2 \\ &\leq 2r \cdot \frac{36}{r} = 72.\end{aligned}$$

So Markov's inequality implies that

$$\mathbb{P}(H \text{ contains at least 144 good } T_x\text{–}T_y \text{ edges}) \leq 1/2. \tag{6}$$

Given a good edge u_1u_2, call it *overloaded* if there are at least 144 good edges of H which are distinct from u_1u_2 and join $T_{x_{u_1}}$ to $T_{x_{u_2}}$. For $i = 1, 2$ let X_i be the set of all vertices in X belonging to u_i. (So $X_1 \cap X_2 = \emptyset$.) For all $x \in X_1$, $y \in X_2$ let $\mathcal{A}_{xy}$ be the event that there are at least 144 good T_x–T_y edges which are distinct from u_1u_2. As the U_x–U_y edges of H are independent, and thus the event that $x_{u_1} = x$ and $x_{u_2} = y$ is independent from $\mathcal{A}_{xy}$, we have

$$\begin{aligned}\mathbb{P}(u_1u_2 \text{ is overloaded}) &\leq \sum_{x \in X_1,\ y \in X_2} \mathbb{P}(x_{u_1} = x,\ x_{u_2} = y \text{ and } \mathcal{A}_{xy} \text{ is true}) \\ &\overset{(6)}{\leq} \sum_{x \in X_1,\ y \in X_2} \frac{1}{|X_1|} \cdot \frac{1}{|X_2|} \cdot \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}(\text{number of overloaded edges}) &= \sum_{u_1u_2 \in E(H[U])} \mathbb{P}(u_1u_2 \text{ is overloaded}) \\ &\leq e_H(U,U)/2.\end{aligned}$$

But this means that for all $u \in U$ the vertices x_u can be chosen in such a way that for the trees T_x $(x \in X)$ thus defined at most half of the good edges of H are overloaded. Let F be the subgraph of H which consists of all those good edges that are not overloaded. Thus

$$e(F) \geq e_H(U,U)/2 \geq e(H)/4 \geq r|H|/4.$$

Consider the minor M of H whose branch sets are the T_x and let $e = u_1u_2$ be an edge of F. Recall that as $g(H) \geq 2k+2$, the endpoints of e must lie in distinct T_x, i.e. $x_{u_1} \neq x_{u_2}$. As e is not overloaded, there are less than 144 other edges of F joining $T_{x_{u_1}}$ to $T_{x_{u_2}}$. Thus to each edge of M there correspond at most 144 edges of F, i.e. $e(M) \geq e(F)/144$. Hence

$$d(M) \geq \frac{2e(F)}{144|X|} \geq \frac{r|H|}{144 \cdot 2 \cdot 2p|H|} \geq \frac{r^{k+1/2}}{144}.$$

Proposition 9 implies that M contains a subgraph of minimum degree at least $r^{k+1/2}/288$, as desired. □

We remark that the constants in Theorems 1 and 2 could be improved a little by more careful calculations. Furthermore, the proof of the case $k = 1$ (i.e. $g = 5$) of Theorem 2 can easily be modified to give the following.

Theorem 12 *There exists a constant $c > 0$ such that for all integers $t \geq 2$ every $K_{2,t}$-free graph G of minimum degree d contains a minor of minimum degree at least $cd^{3/2}/t$.*

Proof. By choosing c sufficiently small, we may assume that $r := \lfloor d/4 \rfloor \geq 2 \cdot 10^6$. It then suffices to make the following minor changes in the proof of the case $k = 1$ of Theorem 2. Define H, p, X, U, H_x and T_x as before. For every vertex $x \in X$ there are now less than t edges (instead of at most one) joining a given leaf of the star H_x to leaves of a given other star H_y ($y \in X$). So H contains at most $2rt$ edges joining U_x to U_y. Similarly as before, this shows that with probability at most $1/2$ the graph H contains at least $144t$ good T_x–T_y edges. This time we call a good edge u_1u_2 overloaded if there are at least $144t$ good edges of H which are *disjoint* from u_1u_2 and join $T_{x_{u_1}}$ to $T_{x_{u_2}}$. Again, it follows that for all $u \in U$ the vertices x_u can be chosen so that at most half of the good edges are overloaded. But for each good edge u_1u_2 which is not overloaded there are at most $144t + 2t$ other good edges joining $T_{x_{u_1}}$ to $T_{x_{u_2}}$ (as there are at most t edges joining u_1 to leaves of $T_{x_{u_2}}$ and vice versa). Thus the minor M of H whose branch sets are the T_x has average degree at least $r^{3/2}/146t$. By Proposition 9, M contains a subgraph of minimum degree at least $r^{3/2}/292t$, as desired. □

More generally, in [11] we prove that for all $t \geq s \geq 2$ every $K_{s,t}$-free graph of minimum degree at least r contains a graph of minimum degree $r^{1+\frac{1}{2(s-1)}-o(1)}$ as minor. This implies that for sufficiently large r every $2r$-connected $K_{s,t}$-free graph is r-linked (see [11]).

4 Upper bounds

The following simple proposition (which generalizes [7, Prop. 2.2]) shows that the existence of small graphs of large girth can be used to prove upper bounds on the minimum degree of minors in graphs of large girth.

Proposition 13 *Let $c, \ell > 0$ and let d, r be integers such that $r \geq 2$. Suppose that G is a graph of maximum degree at most cr and order at most $c(r-1)^\ell$ which contains a minor of minimum degree d. Then $d < 2c(r-1)^{(\ell+1)/2}$.*

Proof. Suppose that H is a minor of G of minimum degree d. Let $W \subseteq V(G)$ be a branch set corresponding to a vertex of H. As each vertex of W sends at most cr edges to other branch sets, $|W| \geq d/cr$. Hence

$$c(r-1)^\ell \geq |G| \geq \frac{d|H|}{cr} > \frac{d^2}{cr} \geq \frac{d^2}{2c(r-1)}.$$

This shows that $d < 2c(r-1)^{(\ell+1)/2}$, as required. □

We will now use Proposition 13 to observe that the truth of the following well-known conjecture (see e.g. Bollobás [4, p. 164]) would show that for fixed girth Theorems 1 and 2 are best possible up to the value of the constant and also that the constant 4 in Corollary 5 cannot be replaced by a smaller one.

Conjecture 14 *There exists a constant c such that for all integers $r, g \geq 3$ there is a graph of minimum degree at least r and girth at least g whose order is at most $c(r-1)^{\lfloor \frac{g-1}{2} \rfloor}$.*

An observation of Tutte (see [4, Ch. III, Thm. 1.2]) shows that this would be close to best possible: Consider any vertex x in a graph G of minimum degree at least r and girth at least g. Then the graph obtained from the $\lfloor \frac{g-1}{2} \rfloor$-ball around x by deleting any edges between vertices of distance $\lfloor \frac{g-1}{2} \rfloor$ from x is a tree. Since $\delta(G) \geq r$, this tree (and so also G) has at least $(r-1)^{\lfloor \frac{g-1}{2} \rfloor}$ vertices. This argument also shows that any graph G demonstrating the truth of Conjecture 14 must have maximum degree at most cr. (Indeed, take for x a vertex of maximum degree in G.) Thus by Proposition 13 with $\ell := \lfloor \frac{g-1}{2} \rfloor$, such a graph G has no minor of minimum degree at least $2c(r-1)^{\frac{1}{2}\lfloor \frac{g+1}{2} \rfloor}$. In other words, the truth of Conjecture 14 would imply that Theorem 1 is best possible up to the value of the constant, and so is Theorem 2 if the girth g is fixed. Furthermore, it would also imply that Theorems 1 and 2 give the correct order of magnitude even for graphs of fixed even girth.

There are several constructions which show that for infinitely many values of r there are graphs of girth at least 5 and minimum degree r whose order is at most $3(r-1)^2$ (see e.g. Brown [6, Thm. 3.4(b)] or the proof of [5, Thm. 1.3.3]). For $g = 7, 11$ Benson [2] showed that for infinitely many integers r there are graphs of minimum degree r and girth at least g whose order is at most $3(r-1)^{\frac{g-1}{2}}$. Together with the above this implies the following

Proposition 15 *For $g = 5, 7$ and 11 there are infinitely many integers r for which there exists a graph of minimum degree r and girth at least g that does not contain a minor of minimum degree at least $6(r-1)^{\frac{g+1}{4}}$.* □

The best known general upper bound for the minimal order of graph of minimum degree at least r and girth at least g was proved by Sauer. It implies that for $r \geq 3$ and odd $g \geq 3$ the minimal order of such graphs is at most $4(r-1)^{g-2}$ (see [4, Ch. III, Thm. 1.4]).

Turning to the case $r = 3$, Weiss [20] proved that a construction of Biggs and Hoare [3] yields infinitely many integers g for which there are 3-regular graphs of girth g and order at most $c2^{3g/4}$. Together with Proposition 13 this implies that for infinitely many integers t there are 3-regular graphs of girth at least $\frac{8}{3}\log t - c'$ that have no minor of minimum degree t (as was already observed by Diestel and Rempel [7]). In particular, the constant 4 in Corollary 5 cannot be replaced by a number smaller than 8/3. Again, the constant 4 in Corollary 5 would be best possible if Conjecture 14 holds for $r = 3$.

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