Complete minors in $K_{s,s}$-free graphs

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Abstract

We prove that for a fixed integer $s \geq 2$ every $K_{s,s}$-free graph of average degree at least r contains a K_p minor where $p = r^{1+\frac{1}{2(s-1)}+o(1)}$. A well-known conjecture on the existence of dense $K_{s,s}$-free graphs would imply that the value of the exponent is best possible. Our result implies Hadwiger's conjecture for $K_{s,s}$-free graphs whose chromatic number is sufficiently large compared with s.

1 Introduction

For every $r > 0$ define $p = p(r)$ to be the largest integer such that all graphs G of average degree at least r contain the complete graph K_p on p vertices as a minor. Kostochka [7] and Thomason [15] independently proved that there exists a positive constant c such that

$$p(r) \geq c\frac{r}{\sqrt{\log r}}, \tag{1}$$

which improved a bound of Mader. Random graphs show that (1) gives the correct order of magnitude. Recently, Thomason [16] showed that $p(r) = (1 + o(1))\gamma r/\sqrt{\log r}$ for an explicit constant γ.

In this paper we prove that if the graph G is locally sparse in the sense that it does not contain a fixed complete bipartite graph $K_{s,s}$ as a subgraph, then G has a K_p minor where p is asymptotically much larger than the average degree of G:

Theorem 1 *For every integer $s \geq 2$ there exists an r_s such that every $K_{s,s}$-free graph of average degree at least $r \geq r_s$ contains a K_p minor for all*

$$p \leq \frac{r^{1+\frac{1}{2(s-1)}}}{(\log r)^3}.$$

As every graph of chromatic number k contains a subgraph of minimum degree at least $k-1$, this implies Hadwiger's conjecture for $K_{s,s}$-free graphs of sufficiently large chromatic number:

Corollary 2 *For every integer $s \geq 2$ there exists an integer k_s such that every $K_{s,s}$-free graph of chromatic number $k \geq k_s$ contains a K_k minor.* □

In Section 3 we will see that there exists an absolute constant α so that we can take $k_s := s^{\alpha s}$ in Corollary 2.

A simple observation (Proposition 14) shows that the bound on p in Theorem 1 is best possible up to the logarithmic term, provided that there exist $K_{s,s}$-free graphs G with at least $c_s|G|^{2-1/s}$ edges. These are known to exist for $s = 2, 3$ and have been conjectured to exist also in general (see e.g. Bollobás [2, p. 362]).

In [11, Cor. 3] we showed that if G is not only $K_{s,s}$-free but has large girth, then it contains even larger complete minors than those guaranteed by Theorem 1: for every odd integer g there exists a positive constant c such that every graph of average degree at least r and girth at least g contains a K_p minor for all $p \leq cr^{\frac{g+1}{4}}/\sqrt{\log r}$. Note that the case $g = 5$ immediately implies the case $s = 2$ of Theorem 1 because every $K_{2,2}$-free graph G can be made into a bipartite graph (which has girth at least 5) by deleting at most half of the edges of G. Related results concerning topological minors in graphs of large girth can be found in [9, 10, 12, 13, 14]. For example, a result in [9] implies the conjecture of Hajós for all graphs of girth at least 15 and sufficiently large chromatic number.

We now turn to an application of Theorem 1 to highly connected graphs. Bollobás and Thomason [3] proved that every $22k$-connected graph is k-linked. As is well known and easy to see, the graph obtained from K_{3k-1} by deleting k independent edges shows that the function $22k$ cannot be replaced by anything smaller than $3k - 2$. On the other hand, a result in [3] states that if a graph G is $2k$-connected and contains a minor H with $2\delta(H) \geq |H| + 4k - 2$ then G is k-linked. Together with Theorem 1 this immediately implies the following.

Corollary 3 *For every integer $s \geq 2$ there exists an integer k_s such that for all $k \geq k_s$ every $2k$-connected $K_{s,s}$-free graph is k-linked.* □

Mader [13, Cor. 1] showed that for $k \geq 2$ one cannot replace $2k$ by $2k - 1$.

Note that Theorem 1 is far from being true if we forbid a non-bipartite graph H instead of a $K_{s,s}$. Indeed, recall that there are graphs of average degree r containing no complete graph of order at least $c'r/\sqrt{\log r}$ as minor. These graphs can be made bipartite (and thus H-free) by deleting at most half of their edges. In particular, the resulting graphs G contain no complete graph as minor whose order exceeds the average degree of G. However, replacing average degree with chromatic number might help:

Problem 4 *Given an integer $s \geq 3$, does there exist a function $\omega_s(k)$ tending to infinity such that every K_s-free graph of chromatic number k contains a K_p minor for all $p \leq k \cdot \omega_s(k)$?*

In other words, the question is whether for K_s-free graphs of sufficiently large chromatic number Hadwiger's conjecture is true with room to spare. For a survey on Hadwiger's conjecture and related questions see e.g. [5].

This paper is organized as follows. In Section 2 we introduce some notation and state several results which we will need later on. Theorem 1 is then proved in Section 3. The methods are related to those in [11]. The final section is concerned with upper bounds for the size of the complete minor in Theorem 1.

2 Notation and tools

All logarithms in this paper are base e, where e denotes the Euler number. We write $e(G)$ for the number of edges of a graph G and $|G|$ for its order. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and the set of its neighbours by $N_G(x)$. We denote by $\delta(G)$ the minimum degree of G, by $\Delta(G)$ its maximum degree and by $d(G) := 2e(G)/|G|$ the average degree of G. Given $A, B \subseteq V(G)$, an *A–B edge* is an edge of G joining a vertex in A and to a vertex in B, the number of these edges is denoted by $e_G(A, B)$. If A and B are disjoint, we write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all A–B edges in G. More generally, we write (A, B) for a bipartite graph with vertex classes A and B.

A graph H is a *minor* of G if for every vertex $h \in H$ there is a connected subgraph G_h of G such that all the G_h are disjoint and G contains a G_h–$G_{h'}$ edge whenever hh' is an edge in H. The vertex set of G_h is called the *branch set corresponding to h*. A *subdivision* of a graph G is a graph TG obtained from G by replacing the edges of G with internally disjoint paths. So if G contains a subdivision of a graph H then H is a minor of G.

Let us now collect some results which will be used in the proof of Theorem 1. We will need the following two easy propositions.

Proposition 5 *Every graph G contains a subgraph of average degree at least $d(G)$ and minimum degree at least $d(G)/2$.*

Proposition 6 *The vertex set of every graph G can be partitioned into disjoint sets A, B such that the minimum degree of $(A, B)_G$ is at least $\delta(G)/2$.*

Moreover, we will use the following Chernoff type bound (see e.g. [4, Cor. 2.3]).

Lemma 7 *Let $X_1, \dots, X_n$ be independent 0-1 random variables with $\mathbb{P}(X_i = 1) = p$ for all $i \leq n$, and let $X := \sum_{i=1}^n X_i$. Then*

$$\mathbb{P}(X \leq \mathbb{E}X/2 \text{ or } X \geq 2\mathbb{E}X) \leq 2\mathrm{e}^{-\mathbb{E}X/12}.$$

A proof of the next lemma can be found in [2, Ch. VI, Lemma 2.1].

Lemma 8 *Let (A, B) be a bipartite $K_{s,t}$-free graph and suppose that on average each vertex in A has d neighbours in B. Then*

$$|A|\binom{d}{s} \leq t\binom{|B|}{s}.$$

Lemma 8 can be used to prove the following upper bound on the number of edges of a $K_{s,t}$-free graph (see e.g. [2, Ch. VI, Thm. 2.3]).

Theorem 9 *Let $t \geq s \geq 1$ be integers. Then every $K_{s,t}$-free graph G has at most $t|G|^{2-1/s}$ edges.*

Finally, we will need the following special case of [8, Cor. 19].

Lemma 10 *Let ℓ, t be integers with $\ell \geq 8t$. Let $G = (A, B)$ be a $K_{t,t}$-free bipartite graph such that $|A| \geq \ell^{12t}|B|$ and $d_G(a) = \ell$ for every vertex $a \in A$. Then G contains a subdivision of some graph of average degree at least $\ell^9/2^{14}$.*

3 Dense Minors in $K_{s,t}$-free graphs

Instead of proving Theorem 1, we will prove the following slightly more general result on the existence of dense minors in $K_{s,t}$-free graphs.

Theorem 11 *For all integers $t \geq s \geq 2$ and all $r \geq (100t)^{16s}$ every $K_{s,t}$-free graph G of average degree r contains a minor of average degree at least*

$$d := \frac{r^{1+\frac{1}{2(s-1)}}}{10^9 t^4 (\log r)^{2+\frac{1}{s+1}}}. \tag{2}$$

Note that asymptotically the restriction on the range of r is not too severe: if $r \leq t^s$, then (2) is already smaller than the trivial lower bound of r on the average degree of the densest minor of G.

Proof of Theorem 1. Theorem 1 immediately follows by an application of (1) to the minor obtained from the $s = t$ case of Theorem 11. □

Furthermore, Theorem 11 shows that there exists an absolute constant α so that we can take $k_s := s^{\alpha s}$ in Corollary 2. (Indeed, given a $K_{s,s}$-free graph G of chromatic number $r+1$, apply Theorem 11 to a subgraph H of G of minimum degree at least r. If $r \geq s^{\alpha s}$ where α is sufficiently large compared with the constant c appearing in (1), then this shows that H contains a K_{r+1} minor, since then the value d in (2) satisfies $cd/\sqrt{\log d} \geq r + 1$.)

Our aim in the proof of Theorem 11 is to find disjoint stars in G such that a large fraction of the edges of G joins two distinct stars. If the number of these stars is not too large and if only a few edges join the same pair of stars, then the minor of G obtained by contracting the stars (and deleting all other vertices) has large average degree, as desired. We will find such stars by first choosing the set X of their centres at random and then assigning vertices $v \in G$ with distance one to X to one of the centres adjacent to v in a suitable way. For this to work we need that G is 'almost regular'. The following lemma allows us to assume this at the expense of only a small loss of the average degree.

Lemma 12 *For all integers $t \geq 2$ and all $r \geq 10^9 t^4$ every $K_{t,t}$-free graph G of average degree at least r either contains a subdivision of some graph of average degree at least r^3 or a bipartite subgraph H such that $\delta(H) \geq \frac{r}{400t\log r}$ and $\Delta(H) \leq r$.*

Proof. Apply Propositions 5 and 6 to obtain a bipartite subgraph G' of G of minimum degree at least $d := \lceil r/4 \rceil$. Let A be the larger vertex class of G' and delete edges if necessary to obtain a (bipartite) subgraph G'' with $d_{G''}(a) = d$ for all $a \in A$. Let B be the set of all vertices in $G'' - A$ that are not isolated and put $G^* := (A, B)_{G''}$. So $d_{G^*}(a) = d$ for all vertices $a \in A$ and thus $d(G^*) \geq d$ (since $|A| \geq |B|$). Put $N := \lceil 1 + (6t+1)\log d \rceil$ and note that

$$\frac{d}{N} \geq \frac{d}{8t\log d} \geq 10^5 t^2. \tag{3}$$

Partition B into N disjoint sets $B_1, \dots, B_N$ such that

$$\begin{aligned} \mathrm{e}^{i-1} &\leq d_{G^*}(x) < \mathrm{e}^i \quad && \forall x \in B_i,\ i = 1, \dots, N-1 \\ \mathrm{e}^{N-1} &\leq d_{G^*}(x) && \forall x \in B_N. \end{aligned}$$

Then there exists an index i such that $e_{G^*}(A, B_i) \geq e(G^*)/N$. First assume that $i \leq \log d$. Then Proposition 5 implies that $(A, B_i)_{G^*}$ contains a subgraph H with $\delta(H) \geq d((A, B_i)_{G^*})/2 \geq d/2N$. As $\Delta(H) \leq \Delta((A, B_i)_{G^*}) \leq d$, H is as required in the lemma.

Next assume that $i = N$. Let A^* be the set of all those vertices in A which send at least $\lfloor \sqrt{d}/(2N)^{1/9} \rfloor =: \ell$ edges in G^* to B_N. Then

$$d|A^*| + \ell|A| \geq e_{G^*}(A, B_N) \geq \frac{e(G^*)}{N} = \frac{d|A|}{N},$$

and therefore

$$|A^*| \geq \left(\frac{d}{N} - \ell \right) \frac{|A|}{d} \geq \frac{|A|}{2N}. \tag{4}$$

Moreover, $d|A| = e(G^*) \geq \mathrm{e}^{N-1}|B_N| \geq d^{6t+1}|B_N|$. Together with (4) this implies that

$$|A^*| \geq \frac{d^{6t}|B_N|}{2N} \geq \ell^{12t}|B_N|.$$

Let H^* be the graph obtained from $(A^*, B_N)_{G^*}$ by deleting edges if necessary such that $d_{H^*}(a) = \ell$ for all $a \in A^*$. Since $\ell \geq 8t \geq 2$ by (3), Lemma 10 implies that H^* (and hence G) contains a subdivision of some graph of average degree at least

$$\frac{\ell^9}{2^{14}} \geq r^3 \cdot \frac{d^{3/2}}{4^3 \cdot 2^{15+9}N} \overset{(3)}{\geq} r^3.$$

So we may assume that $\log d < i < N$. Set $k := \lfloor d/2N \rfloor$ and let A_p be a random subset of A which is obtained by including every vertex into A_p with probability $p := 2k/\mathrm{e}^{i-1}$ independently of all other vertices. Then for every vertex $b \in B_i$ we have

$$2k \leq d_{G^*}(b)p = \mathbb{E}(|N_{G^*}(b) \cap A_p|) \leq 2\mathrm{e}k \leq d/2. \tag{5}$$

Let us call a vertex $b \in B_i$ *bad* if $|N_{G^*}(b) \cap A_p| \leq k$ or $|N_{G^*}(b) \cap A_p| \geq d$. So (3), (5) and Lemma 7 together imply that the probability that a given vertex $b \in B_i$ is bad is at most $2\mathrm{e}^{-k/6} \leq 1/24$. So the expected number of bad vertices in B_i is at most $|B_i|/24$. Hence Markov's inequality implies that

$$\mathbb{P}(\geq |B_i|/6 \text{ vertices of } |B_i| \text{ are bad}) \leq 1/4. \tag{6}$$

Moreover

$$2k|A| \leq \frac{d|A|}{N} \leq e_{G^*}(A, B_i) \leq \mathrm{e}^i|B_i|,$$

and so $|A| \leq \mathrm{e}^i|B_i|/2k$. Hence

$$\mathbb{E}(|A_p|) = p|A| \leq \frac{p\mathrm{e}^i|B_i|}{2k} = \mathrm{e}|B_i|.$$

Thus Markov's inequality shows that

$$\mathbb{P}(|A_p| \geq 4|B_i|) \leq \mathrm{e}/4.$$

Together with (6) this implies that with probability at least $1 - 1/4 - \mathrm{e}/4 > 0$ there exists an outcome A_p such that $|A_p| \leq 4|B_i|$ and at most $|B_i|/6$ vertices of B_i are bad. Let H' be the subgraph of G^* induced by A_p and those vertices in B_i that are not bad. Then $\Delta(H') \leq d$ and $e(H') \geq 5k|B_i|/6$. Moreover, $|H'| \leq |A_p| + |B_i| \leq 5|B_i|$, and so the average degree of H' is at least $k/3$. By Proposition 5, H' has a subgraph H with

$$\delta(H) \geq \frac{k}{6} \overset{(3)}{\geq} \frac{d}{100t\log d} \geq \frac{r}{400t\log r}.$$

So H is as required in the lemma. □

Proof of Theorem 11. Apply Lemma 12 to G to obtain (without loss of generality) a bipartite subgraph H with $\Delta(H) \leq r$ and

$$\delta := \delta(H) \geq \frac{r}{400t\log r}. \tag{7}$$

Define ε by

$$r^{\varepsilon} = \frac{r^{\frac{1}{2(s-1)}}}{32t(r/\delta)^{\frac{1}{s+1}}}. \tag{8}$$

Put $\ell := r^{1-\varepsilon}$ and let X be a random subset of $V(H)$ which is obtained by including each vertex into X with probability $p := 2\ell/\delta$ independently of all other vertices. The branch sets of our minor of large average degree will consist of stars whose centres are precisely the vertices in X. Since $r \geq (100t)^{16(s-1)}$, for every vertex $v \in H$ we have

$$\begin{aligned}\mathbb{P}(v \in X) = p = \frac{2r}{r^{\varepsilon}\delta} &\leq \frac{2 \cdot 400t\log r}{r^{\varepsilon}} \\ &\leq \frac{2 \cdot 32t \cdot (400t)^2}{r^{\frac{1}{4(s-1)}}} \cdot \frac{(\log r)^2}{r^{\frac{1}{4(s-1)}}} \leq 1/20.\end{aligned} \tag{9}$$

Call a vertex $v \in H$ *good* if it satisfies the following two conditions.

(i) $v \notin X$.

(ii) $|N_H(v) \cap X| \geq \ell$.

We will now show that with large probability a given vertex $v \in H$ is good. First note that

$$\mathbb{E}(|N_H(v) \cap X|) = d_H(v) \cdot p \geq 2\ell.$$

As $\ell \geq \sqrt{r}$, Lemma 7 implies that

$$\mathbb{P}(|N_H(v) \cap X| < \ell) \leq 2\mathrm{e}^{-\sqrt{r}/6} \leq 1/20.$$

Together with (9) this implies that the probability that a given vertex $v \in H$ is not good is at most $1/10$. Call an edge $uv \in H$ *good* if both u and v are good. So the probability that a given edge $uv \in H$ is not good is at most $1/5$ and therefore

$$\mathbb{E}(\text{number of edges which are not good}) \le e(H)/5.$$

So Markov's inequality implies that

$$\mathbb{P}(\ge e(H)/2 \text{ edges are not good}) \le 2/5.$$

Using Markov's inequality once more, we see that

$$\mathbb{P}(|X| \ge 2p|H|) \le 1/2.$$

Thus with probability at least $1 - 2/5 - 1/2 > 0$ there is an outcome X with $|X| \le 2p|H|$ and for which at least half of the edges of H are good. Let U be the set of good vertices of H. So $e_H(U, U)$ is precisely the number of good edges of H. For every $x \in X$ put $U_x := U \cap N_H(x)$. Note that, since H is bipartite, $H[U_x]$ consists of isolated vertices. Given a vertex $u \in U$, let $X_u := X \cap N_H(u)$. So condition (ii) implies that $|X_u| \ge \ell$.

For every vertex $u \in U$ choose a vertex $x_u \in X_u$ uniformly at random, independently of all other vertices in U. For all $x \in X$, let S_x be the set of all those $u \in U_x$ with $x_u = x$. Note that the S_x are disjoint and their union is U. Moreover, every good edge of H joins vertices in distinct S_x. We will now show that with positive probability the minor M of H whose branch sets are the $S_x \cup \{x\}$ ($x \in X$) has large average degree. For this, we will show that with positive probability a large fraction of good edges joins different pairs S_x, S_y and thus corresponds to different edges of M. As $|X|$ (i.e. the number of vertices of M) is relatively small, this will imply that M has large average degree. Thus, given a good edge $uv \in H$, we say that

- *uv is of type I* if there exists a good edge $ab \ne uv$ joining S_{x_u} to S_{x_v} such that ab and uv are disjoint,
- *uv is of type II* if there exists a good edge $ab \ne uv$ joining S_{x_u} to S_{x_v} such that a is an endvertex of uv and $|N_H(a) \cap U_{x_w}| \le \ell/30$, where w is the endvertex of uv distinct from a (Fig. 1),
- *uv is of type III* if there exists a good edge $ab \ne uv$ joining S_{x_u} to S_{x_v} such that a is an endvertex of uv and $|N_H(a) \cap U_{x_w}| > \ell/30$, where w is the endvertex of uv distinct from a.

Note that for all distinct $x, y \in X$ the graph $H[U_x \cup U_y]$ does not contain a $K_{s-1,t}$ (since this would form a $K_{s,t}$ together with either x or y). So Theorem 9 implies that

$$e_H(U_x, U_y) \le 4tr^{2-\frac{1}{s-1}}. \tag{10}$$

Recall that the S_x–S_y edges are precisely those U_x–U_y edges uv (with $u \in U_x$ and $v \in U_y$) for which u has chosen x and v has chosen y, i.e. for which $x = x_u$ and

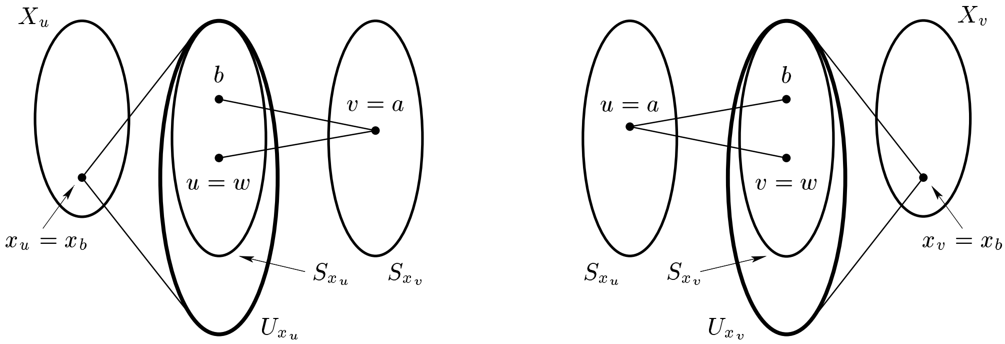

Figure 1: The two possibilities for an edge uv to be of type II

$y = x_v$. Since the probability that $x = x_u$ and $y = x_v$ is $|X_u|^{-1}|X_v|^{-1} \le \ell^{-2}$, it follows that

$$\begin{aligned}\mathbb{P}(\text{there is a good } S_x\text{–}S_y \text{ edge}) &\le e_H(U_x, U_y) \cdot \left(\frac{1}{\ell}\right)^2 \\ &\overset{(10)}{\le} 4tr^{2\varepsilon - \frac{1}{s-1}} \overset{(8)}{\le} \frac{1}{60}. \end{aligned} \tag{11}$$

So given a good edge uv we have

$$\begin{aligned}\mathbb{P}(uv \text{ is of type I}) &= \sum_{x \in X_u,\, y \in X_v} \mathbb{P}(uv \text{ is of type I and } x = x_u \text{ and } y = x_v) \\ &= \sum_{x \in X_u,\, y \in X_v} \mathbb{P}(\text{there is a good } S_x\text{–}S_y \text{ edge disjoint from } uv) \cdot \frac{1}{|X_u|} \cdot \frac{1}{|X_v|} \\ &\overset{(11)}{\le} \frac{1}{60}.\end{aligned}$$

Moreover, given x_u and x_v, in the definition of a type II edge uv there are at most two possibilities for a and at most $\ell/30$ candidates for b and $\mathbb{P}(x_b = x_w) \le 1/\ell$. Thus

$$\begin{aligned}\mathbb{P}(uv \text{ is of type II}) &= \sum_{x \in X_u,\, y \in X_v} \mathbb{P}(uv \text{ is of type II and } x = x_u \text{ and } y = x_v) \\ &\le \sum_{x \in X_u,\, y \in X_v} 2 \cdot \frac{\ell}{30} \cdot \frac{1}{\ell} \cdot \frac{1}{|X_u|} \cdot \frac{1}{|X_v|} = \frac{4}{60}.\end{aligned}$$

Hence

$$\mathbb{E}(\text{number of good edges which are type I or II}) \le e_H(U, U)/12,$$

and so Markov's inequality implies that

$$\mathbb{P}(\ge e_H(U, U)/4 \text{ good edges are of type I or II}) \le 1/3. \tag{12}$$

It remains to show that also with only small probability a large fraction of the good edges is of type III. This trivially holds for $s = 2$. Indeed, as $\ell/30 \ge t$, the vertices a and x_w in the definition of a type III edge form a $K_{2,t}$ together with any t vertices in $N_H(a) \cap U_{x_w}$. Thus there are no good edges of type III in this case. So suppose that $s \geq 3$. Given a vertex $y \in X$, let V_y be the set of all those vertices in U which have at least $\ell/30$ neighbours in U_y. So $V_y \subseteq U \setminus U_y$. As H is $K_{s,t}$-free, Lemma 8 implies that

$$|V_y| \binom{\ell/30}{s} \leq t \binom{|U_y|}{s}.$$

Thus

$$|V_y| \leq \left(\frac{32}{\ell}\right)^s t \cdot |U_y|^s \leq (32r^{\varepsilon})^s t. \tag{13}$$

Given distinct good edges uv and ub and vertices $x, y \in X$, we say that the ordered quadruple uv, ub, x, y *forms a configuration of type III* if $u \in U_x$, $v, b \in U_y$ and if u has at least $\ell/30$ neighbours in U_y. So each configuration of type III can be obtained by first selecting a vertex $v \in U$, then selecting a vertex $y \in X_v$, then selecting a neighbour u of v which lies in V_y (i.e. which lies in U and sends at least $\ell/30$ edges to U_y), then we select a vertex $x \in X_u$ and finally we select a neighbour b of u in $U_y \setminus v$. We say that a configuration of type III *survives* if u has chosen x and both v and b have chosen y, i.e. if $x = x_u$ and $y = x_v = x_b$. Thus the probability that it survives is precisely $|X_u|^{-1}|X_v|^{-1}|X_b|^{-1} \leq |X_u|^{-1}|X_v|^{-1}/\ell$. Hence

$$\begin{aligned}
& \mathbb{E}(\text{number of good edges which are of type III}) \\
\leq\ & \mathbb{E}(\text{number of surviving configurations of type III}) \\
\leq\ & \sum_{v \in U} \sum_{y \in X_v} \sum_{u \in N_H(v) \cap V_y} \sum_{x \in X_u} \sum_{b \in N_H(u) \cap U_y \setminus v} \frac{1}{|X_u||X_v|\ell} \\
\overset{(13)}{\leq}\ & \frac{|H|(32r^{\varepsilon})^s tr}{r^{1-\varepsilon}} \\
\overset{(8)}{=}\ & |H| 32^s t \left(\frac{r^{\frac{1}{2(s-1)}}}{32t}\right)^{s+1} \cdot \frac{\delta}{r} \\
\overset{(s \geq 3)}{\leq}\ & \frac{\delta |H|}{32} \leq \frac{e(H)}{16} \leq \frac{e_H(U,U)}{8}.
\end{aligned}$$

Hence Markov's inequality implies that also for $s \geq 3$

$$\mathbb{P}(\geq e_H(U,U)/4 \text{ good edges are of type III}) \leq 1/2.$$

Together with (12) this shows that for every $u \in U$ there exists a choice of x_u such that at most $e_H(U,U)/2$ good edges are of type I, II or III. Let F be the set of all good edges which are not of type I, II or III.

Consider the minor M of H whose branch sets are the sets $S_x \cup \{x\}$ (for all $x \in X$). As H is bipartite, every edge in F joins distinct branch sets and, by definition of F, no two edges in F join the same pair of branch sets. Thus

$e(M) \geq |F|$ and so

$$\begin{aligned} d(M) &\geq \frac{2|F|}{|X|} \geq \frac{e_H(U,U)}{|X|} \geq \frac{e(H)}{2 \cdot 2p|H|} \geq \frac{\delta}{8p} = \frac{\delta^2}{16r^{1-\varepsilon}} \\ &\overset{(8)}{=} \frac{r^{1+\frac{1}{2(s-1)}}}{16 \cdot 32t} \cdot \left(\frac{\delta}{r}\right)^{2+\frac{1}{s+1}} \\ &\overset{(7)}{\geq} \frac{r^{1+\frac{1}{2(s-1)}}}{16 \cdot 32t \cdot (400t \log r)^{2+\frac{1}{s+1}}} \geq \frac{r^{1+\frac{1}{2(s-1)}}}{10^9 t^4 (\log r)^{2+\frac{1}{s+1}}}, \end{aligned} \tag{14}$$

as required. □

Note that for regular graphs G the logarithmic term in (2) is not necessary. Indeed, we only have to replace the graph H in the proof of Theorem 11 with a bipartite subgraph obtained from G by an application of Proposition 6, and then (14) shows that this subgraph contains a minor of the required average degree. Moreover, for non-regular graphs the exponent $2+\frac{1}{s+1}$ of the logarithmic term can be reduced to $1+\frac{1}{2(s-1)}$. However, we do not give the details as we conjecture that (as in the case $s=2$, see [11, Thm. 12]) the logarithmic term in (2) can be removed altogether.

4 Upper bounds

In this section we observe that the truth of the following well-known conjecture about the existence of dense $K_{s,t}$-free graphs would imply that for fixed s and t Theorems 1 and 11 are best possible up to the logarithmic term.

Conjecture 13 *For all integers $t \geq s \geq 2$ there exists a positive constant $c = c(s,t)$ such that for all integers n there is a $K_{s,t}$-free graph G of order n with at least $cn^{2-1/s}$ edges.*

(See e.g. Bollobás [2, p. 362] for the case $s = t$ which of course would already imply the general case.) In other words, the conjecture states that the upper bound on the number of edges of a $K_{s,t}$-free graph in Theorem 9 gives the correct order of magnitude. Conjecture 13 is known to be true for all $t \geq s$ with $s = 2, 3$ (see [2, Ch. VI]). Furthermore, Alon, Rónyai and Szabó [1] proved the conjecture for all $t \geq s \geq 2$ with $t > (s-1)!$ by modifying a construction of [6]. The following proposition immediately implies that Theorems 1 and 11 are best possible up to the logarithmic term, provided that Conjecture 13 holds.

Proposition 14 *For every $c > 0$ and every $s \geq 2$ there exists a constant $C = C(c,s)$ such that whenever G is a graph with $e(G) \geq c|G|^{2-1/s}$ then every minor H of G satisfies*

$$d(H) \leq C \cdot d(G)^{1+\frac{1}{2(s-1)}}.$$

Proof. Put $n := |G|$, $r := d(G)$ and $d := d(H)$. For every vertex $h \in H$ let $V_h \subseteq V(G)$ be the branch set corresponding to h. Then

$$nr = 2e(G) \geq \sum_{h \in H} \sum_{v \in V_h} d_G(v) \geq \sum_{h \in H} d_H(h) = 2e(H) \geq d^2,$$

and so $d \leq \sqrt{nr}$. But $r \geq 2cn^{1-1/s}$, i.e. $n \leq (r/2c)^{\frac{s}{s-1}}$. Therefore

$$d \leq \frac{r^{\frac{1}{2}(1+\frac{s}{s-1})}}{(2c)^{\frac{s}{2(s-1)}}},$$

as required. □

In general, for $s \geq 4$ the best known lower bound on the maximum number of edges of a $K_{s,s}$-free graph G is $c|G|^{2-2/(s+1)}$ (see e.g. [2, Ch. VI, Thm. 2.10]). Together with Proposition 14, this still yields an upper bound of $c'r^{1+\frac{1}{s-1}}$ for the size of the complete minor in Theorem 1 and the average degree of the minor in Theorem 11.

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