THE ROBUST COMPONENT STRUCTURE OF DENSE REGULAR GRAPHS AND APPLICATIONS

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ABSTRACT. In this paper, we study the large-scale structure of dense regular graphs. This involves the notion of robust expansion, a recent concept which has already been used successfully to settle several longstanding problems. Roughly speaking, a graph is robustly expanding if it still expands after the deletion of a small fraction of its vertices and edges. Our main result allows us to harness the useful consequences of robust expansion even if the graph itself is not a robust expander. It states that every dense regular graph can be partitioned into 'robust components', each of which is a robust expander or a bipartite robust expander. We apply our result to obtain (amongst others) the following.

(i) We prove that whenever $\varepsilon > 0$, every sufficiently large 3-connected D-regular graph on n vertices with $D \geq (1/4+\varepsilon)n$ is Hamiltonian. This asymptotically confirms the only remaining case of a conjecture raised independently by Bollobás and Häggkvist in the 1970s.

(ii) We prove an asymptotically best possible result on the circumference of dense regular graphs of given connectivity. The 2-connected case of this was conjectured by Bondy and proved by Wei.

1. INTRODUCTION

1.1. **The robust component structure of dense regular graphs.** Our main result states that any dense regular graph G is the vertex-disjoint union of a bounded number of 'robust components'. Each such component has a strong expansion property that is highly 'resilient' and almost all edges of G lie inside these robust components. In other words, the result implies that the large scale structure of dense regular graphs is remarkably simple. This can be applied e.g. to Hamiltonicity problems in dense regular graphs. Note that the structural information obtained in this way is quite different from that given by Szemerédi's regularity lemma.

The crucial notion in our partition is that of robust expansion. This is a structural property which has close connections to Hamiltonicity. Given a graph G on n vertices, $S \subseteq V(G)$ and $0 < \nu \leq \tau < 1$, we define the *ν-robust neighbourhood* $RN_{\nu,G}(S)$ of S to be the set of all those vertices of G with at least νn neighbours in S. We say G is a *robust (ν,τ)-expander* if, for every $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1-\tau)n$, we have that $|RN_{\nu,G}(S)| \geq |S| + \nu n$.

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There is an analogous notion of robust outexpansion for digraphs. This was first introduced in [25] and has been instrumental in proving several longstanding conjectures. For example, Kühn and Osthus [23] recently settled a conjecture of Kelly from 1968 (for large tournaments) by showing that every sufficiently large dense regular robust outexpander has a Hamilton decomposition. Another example is the recent proof [20, 21] of Sumner's universal tournament conjecture from 1971.

Our main result allows us to harness the useful consequences of robust expansion even if the graph itself is not a robust expander. For this, we introduce the additional notion of 'bipartite robust expanders'. Let G be a bipartite graph with vertex classes A and B. Then clearly G is not a robust expander. However, we can obtain a bipartite analogue of robust expansion by only considering sets $S \subseteq A$ with $\tau|A| \leq |S| \leq (1-\tau)|A|$. This notion extends in a natural way to graphs which are 'close to bipartite'.

Roughly speaking, our main result (Theorem 3.1) implies the following.

(†) For all $r \in \mathbb{N}$ and all $\varepsilon > 0$, any sufficiently large D-regular graph on n vertices with $D \geq (\frac{1}{r+1} + \varepsilon)n$ has a vertex partition into at most r robust expander components and bipartite robust expander components, so that the number of edges between these is $o(n^2)$.

We give a formal statement of this in Section 3. In Section 5 we obtain a generalisation to almost regular graphs. (Here, G is 'almost regular' if $\Delta(G) - \delta(G) = o(n)$.)

In the special case of dense vertex-transitive graphs (which are always regular), Christofides, Hladký and Máthé [7] introduced a partition into 'iron connected components'. (Iron connectivity is closely related to robust expansion.) They applied this to resolve the dense case of a question of Lovász [28] on Hamilton paths (and cycles) in vertex-transitive graphs. It would be very interesting to obtain a similar partition result for further classes of graphs. In particular, it might be possible to generalise Theorem 3.1 to sparser graphs.

In the current paper, we apply (†) to give an approximate solution to a longstanding conjecture on Hamilton cycles in regular graphs (Theorem 1.2) as well as an asymptotically optimal result on the circumference of dense regular graphs of given connectivity (Theorem 1.4). We are also confident that our robust partition result will have applications to other problems.

1.2. **An application to Hamilton cycles in regular graphs.** Consider the classical result of Dirac that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. Suppose we wish to strengthen this by reducing the degree threshold at the expense of introducing some other condition(s). The two extremal examples for Dirac's theorem (i.e. the disjoint union of two cliques and the almost balanced complete bipartite graph) make it natural to consider regular graphs with some connectivity property, see e.g. the recent survey of Li [26] and handbook article of Bondy [5].

In particular, Szekeres (see [14]) asked for which D every 2-connected D-regular graph G on n vertices is Hamiltonian. Jackson [14] showed that $D \geq n/3$ suffices. This improved earlier results of Nash-Williams [29], Erdős and Hobbs [10] and Bollobás and Hobbs [3]. Hilbig [13] improved the degree condition to $n/3 - 1$, unless G is the Petersen graph or another exceptional graph. As discussed later on in this section, this bound is best possible.

Bollobás [2] as well as Häggkvist (see [14]) independently made the natural and far more general conjecture that any t-connected regular graph on n vertices with degree at least $n/(t+1)$ is Hamiltonian. However, the following counterexample (see Figure 1(i)),

due to Jung [18] and independently Jackson, Li and Zhu [17], disproves this conjecture for $t > 3$.

For m divisible by four, construct G as follows. Let C_1, C_2 be two disjoint copies of K_{m+1} and let A, B be two disjoint independent sets of orders $m, m-1$ respectively. Add every edge between A and B. Add a set of $m/2$ independent edges from each of C_1 and C_2 to A so that together these edges form a matching of size m. Delete $m/4$ independent edges in each of C_1, C_2 so that G is m-regular. Then G has $4m+1$ vertices and is $m/2$-connected. However G is not Hamiltonian since $G \setminus A$ has $|A|+1$ components (in other words, G is not 1-tough).

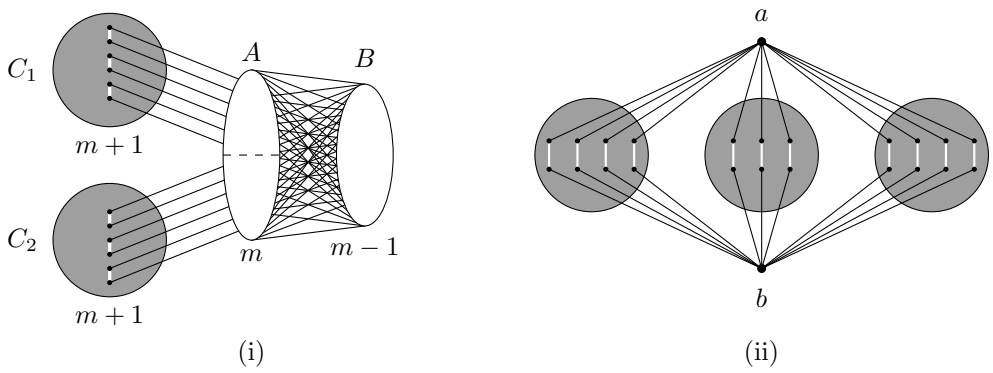

Figure 1: Extremal examples for Conjecture 1.1.

Jackson, Li and Zhu [17] believe that the conjecture of Bollobás and Häggkvist is true in the remaining open case when $t = 3$.

Conjecture 1.1. *Let G be a 3-connected D-regular graph on $n \geq 13$ vertices such that $D \geq n/4$. Then G contains a Hamilton cycle.*

The 3-regular graph obtained from the Petersen graph by replacing one vertex with a triangle shows that the conjecture does not hold for $n = 12$. The graph in Figure 1(i) is extremal and the bound on D is tight.

As mentioned earlier, there exist non-Hamiltonian 2-connected regular graphs on n vertices with degree close to $n/3$ (see Figure 1(ii)). Indeed, we can construct such a graph G as follows. Start with three disjoint cliques on $3m$ vertices each. In the ith clique choose disjoint sets A_i and B_i with $|A_i| = |B_i|$ and $|A_1| = |A_3| = m$ and $|A_2| = m-1$. Remove a perfect matching between A_i and B_i for each i. Add two new vertices a and b, where a is connected to all vertices in the sets A_i and b is connected to all vertices in all the sets B_i. Then G is a $(3m-1)$-regular 2-connected graph on $n = 9m+2$ vertices. However, G is not Hamiltonian because $G \setminus \{a, b\}$ has three components. Therefore none of the conditions – degree, order or connectivity – of Conjecture 1.1 can be relaxed.

There have been several partial results in the direction of Conjecture 1.1. Fan [11] and Jung [18] independently showed that every 3-connected D-regular graph contains a cycle of length at least $3D$, or a Hamilton cycle. Li and Zhu [27] proved Conjecture 1.1 in the case when $D \geq 7n/22$ and Broersma, van den Heuvel, Jackson and Veldman [6] proved it for $D \geq 2(n+7)/7$. In [17] it is proved that, if G satisfies the conditions of the conjecture,

any longest cycle in G is dominating provided that n is not too small. (Here, a subgraph H of a graph G is *dominating* if $G \setminus V(H)$ is an independent set.) By considering robust partitions, we are able to prove an approximate version of the conjecture.

Theorem 1.2. *For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every 3-connected D-regular graph on $n \geq n_0$ vertices with $D \geq (1/4 + \varepsilon)n$ is Hamiltonian.*

In fact, if D is at least a little larger than $n/5$ but G is not Hamiltonian we also determine the asymptotic structure of G (see Theorem 7.11). In a recent paper [19] we use this to prove the exact version of Conjecture 1.1 for large n. Note that [19] does not supersede this paper but rather uses it as an essential tool.

There are also natural analogues of the above results and questions for directed graphs. Here, a D-regular directed graph is such that every vertex has both in- and outdegree equal to D. An *oriented graph* is a digraph without 2-cycles.

Conjecture 1.3.

(a) *For each $D > 2$, every D-regular oriented graph G on n vertices with $D \geq (n-1)/4$ is Hamiltonian.*
(b) *Every strongly 2-connected D-regular digraph on n vertices with $D \geq n/3$ is Hamiltonian.*
(c) *For each $D > 2$, every D-regular strongly 2-connected oriented graph G on n vertices with $D \geq n/6$ is Hamiltonian.*

(a) was conjectured by Jackson [15], (b) and (c) were raised in [22], which also contains a more detailed discussion of these conjectures.

1.3. **An application to the circumference of regular graphs.** More generally, we also consider the circumference of dense regular graphs of given connectivity. Bondy [4] conjectured that, for $r \geq 3$, every sufficiently large 2-connected D-regular graph G on n vertices with $D \geq n/r$ has circumference $c(G) \geq 2n/(r-1)$. (Here the circumference $c(G)$ of G is the length of the longest cycle in G.) This was confirmed by Wei [31], who proved the conjecture for all n and in fact showed that $c(G) \geq 2n/(r-1) + 2(r-3)/(r-1)$, which is best possible. We are able to extend this (asymptotically) to t-connected dense regular graphs.

Theorem 1.4. *Let $t, r \in \mathbb{N}$. For all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds. Whenever G is a t-connected D-regular graph on $n \geq n_0$ vertices where $D \geq (1/r + \varepsilon)n$, the circumference of G is at least* $\min\{t/(r-1), 1-\varepsilon\}n$.

This is asymptotically best possible. Indeed, in Proposition 8.1 we show that, for every $t, r \in \mathbb{N}$, there are infinitely many n such that there exists a graph G on n vertices which is $((n-t)/(r-1)-1)$-regular and t-connected with $c(G) \leq tn/(r-1) + t$. Moreover, as discussed above, the first extremal example in Figure 1 shows that in general $\min\{t/(r-1), 1-\varepsilon\}n$ cannot be replaced by $\min\{t/(r-1), 1\}n$.

Theorem 1.4 shows that the conjecture of Bollobás and Häggkvist is in fact close to being true after all – any t-connected regular graph with degree slightly higher than $n/(t+1)$ contains an almost spanning cycle.

1.4. **An application to bipartite regular graphs.** One can consider similar questions about dense regular bipartite graphs. Häggkvist [12] conjectured that every 2-connected D-regular bipartite graph on n vertices with $D \geq n/6$ is Hamiltonian. If true, this result

would be best possible. Indeed, it was essentially verified by Jackson and Li [16] who proved it in the case when $D \geq (n+38)/6$. Recently, Li [26] conjectured a bipartite analogue of Conjecture 1.1, i.e. that every 3-connected D-regular bipartite graph on n vertices with $D \geq n/8$ is Hamiltonian.

Restricting to bipartite graphs strengthens the structural information implied by our main result (†) considerably. So it seems likely that one can use our partition result to make progress towards these and other related conjectures.

One might ask if a bipartite analogue of the conjecture of Bollobás and Häggkvist holds, i.e. whether every t-connected D-regular bipartite graph on n vertices with $D \geq n/2(t+1)$ contains a Hamilton cycle. However, as in the general case, it turns out that this is false for $t > 3$. Indeed, for each $t \geq 2$ and infinitely many $D \in \mathbb{N}$, Proposition 8.2 guarantees a D-regular bipartite graph G on $8D+2$ vertices that is t-connected and contains no Hamilton cycle. (This observation generalises one from [26], who considered the case when $t = 3$.)

As in the general case, one may also consider the circumference of dense regular bipartite graphs. Indeed, the argument for Theorem 1.4 yields the following bipartite analogue. Again, it is asymptotically best possible (see Proposition 8.2(i)).

Theorem 1.5. *Let $t, r \in \mathbb{N}$, where r is even. For all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds. Whenever G is a t-connected D-regular bipartite graph on $n \geq n_0$ vertices where $D \geq (1/r+\varepsilon)n$, the circumference of G is at least $\min\{2tn/(r-2), n\} - \varepsilon n$.*

1.5. **Organisation of the paper and sketch proof of Theorem 1.2.** This paper is organised as follows. In the remainder of this section we sketch the proof of Theorem 1.2. Section 2 lists some notation which will be used throughout. In Section 3 we state our robust partition result (Theorem 3.1) which formalises (†). We prove it in Section 4, which also contains a sketch of the argument. In Section 5 we derive a version of Theorem 3.1 for almost regular graphs. In Section 6 we show how to find suitable path systems covering the robust components obtained from Theorem 3.1. These tools are then used in Section 7 to prove Theorem 1.2 and used in Section 8 to prove Theorems 1.4 and 1.5.

In order to see how our partition result (†) may be applied, we now briefly outline the argument used to prove Theorem 1.2.

Let $\varepsilon > 0$ and suppose that G is a 3-connected D-regular graph on n vertices, where $D \geq (1/4+\varepsilon)n$. Now (†) gives us a robust partition $\mathcal{V}$ of G containing exactly k robust expander components and ℓ bipartite robust expander components, where $k + \ell \leq 3$.

However, Theorem 3.1 actually gives the stronger bound that $k + 2\ell \leq 3$, so there are only five possible choices of (k, ℓ) (see Proposition 3.2). Assume for simplicity that $\mathcal{V}$ consists of three robust expander components G_1, G_2, G_3. So $(k, \ell) = (3, 0)$. The result of [25] mentioned in Section 1.1 implies that G_i contains a Hamilton cycle for $i = 1, 2, 3$. In fact, it can be used to show (see Corollary 6.9) that G_i is Hamilton p-linked for each bounded p. (Here a graph G is *Hamilton p-linked* if, whenever $x_1, y_1, \dots, x_p, y_p$ are distinct vertices, there exist vertex-disjoint paths $P_1, \dots, P_p$ such that P_j connects x_j to y_j, and such that together the paths $P_1, \dots, P_p$ cover all vertices of G.) This means that the problem of finding a Hamilton cycle in G can be reduced to finding only a suitable set of *external edges*, where an edge is external if it has endpoints in different G_i. We use the assumption of 3-connectivity to find these external edges (in Section 7).

The cases where $\ell > 0$ are more difficult since a bipartite graph does not contain a Hamilton cycle if it is not balanced. So as well as suitable external edges, we need to find some 'balancing edges' incident to the bipartite robust expander component. (Note that

if $\ell > 0$ we must have $\ell = 1$ and $k \leq 1$.) Suppose for example that we have $k = \ell = 1$ and that we have a bipartite robust expander component with vertex classes A, B where $|A| = |B| + 1$, as well as a robust expander component X and an edge e joining A to X and an edge f joining B to X, where e and f are disjoint (so e and f are external edges). Then one possible set of balancing edges consists e.g. of two further external edges incident to A. Another example would be one edge inside A. These balancing edges are guaranteed by our assumption that G is regular. We construct them in Section 7.

2. Notation

For $A \subseteq V(G)$, complements are always taken within the entire graph G, so that $\overline{A} := V(G) \setminus A$. Given $A \subseteq V(G)$, we write $N(A) := \bigcup_{a \in A} N(a)$. For $x \in V(G)$ and $A \subseteq V(G)$ we write $d_A(x)$ for the number of edges xy with $y \in A$. For $A, B \subseteq V(G)$, we write $e(A, B)$ for the number of edges with exactly one endpoint in A and one endpoint in B. (Note that A, B are not necessarily disjoint.) Define $e'(A, B) := e(A, B) + e(A \cap B)$. So $e'(A, B) = \sum_{a \in A} d_B(a) = \sum_{b \in B} d_A(b)$ and if A, B are disjoint then $e'(A, B) = e(A, B)$. For a digraph G, we write $\delta^0(G)$ for the minimum of its minimum indegree and minimum outdegree.

For distinct $x, y \in V(G)$ and a path P with endpoints x and y, we sometimes write $P = xPy$ to emphasise this. Given disjoint subsets A, B of $V(G)$, we say that P is an *AB-path* if P has one endpoint in A and one endpoint in B. We call a vertex-disjoint collection of paths a *path system*. We will often think of a path system $\mathcal{P}$ as a graph with edge set $\bigcup_{P \in \mathcal{P}} E(P)$, so that e.g. $V(\mathcal{P})$ is the union of the vertex sets of each path in $\mathcal{P}$.

Throughout we will omit floors and ceilings where the argument is unaffected. The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph or digraph), then there is a non-decreasing function $f : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq f(b)$ and $1/n \leq f(a)$. Hierarchies with more constants are defined in a similar way.

3. Robust partitions of regular graphs

In this section we list the definitions which are required to state our main result. For a graph G on n vertices, $0 < \nu < 1$ and $S \subseteq V(G)$, recall that the *ν-robust neighbourhood* $RN_{\nu,G}(S)$ of S to be the set of all those vertices with at least νn neighbours in S. Also, recall that for $0 < \nu \leq \tau < 1$ we say that G is a *robust (ν, τ)-expander* if, for all sets S of vertices satisfying $\tau n \leq |S| \leq (1 - \tau)n$, we have that $|RN_{\nu,G}(S)| \geq |S| + \nu n$. For $S \subseteq X \subseteq V(G)$ we write $RN_{\nu,X}(S) := RN_{\nu,G[X]}(S)$.

We now introduce the concept of 'bipartite robust expansion'. Let $0 < \nu \leq \tau < 1$. Suppose that H is a (not necessarily bipartite) graph on n vertices and that A, B is a partition of $V(H)$. We say that H is a *bipartite robust (ν, τ)-expander with bipartition A, B* if every $S \subseteq A$ with $\tau|A| \leq |S| \leq (1 - \tau)|A|$ satisfies $|RN_{\nu,H}(S) \cap B| \geq |S| + \nu n$. Note that the order of A and B matters here. We do not mention the bipartition if it is clear from the context.

Note that for $0 < \nu' \leq \nu \leq \tau \leq \tau' < 1$, any robust (ν, τ)-expander is also a robust (ν', τ')-expander (and the analogue holds in the bipartite case).

Given $0 < \rho < 1$, we say that $U \subseteq V(G)$ is a *ρ-component* of a graph G on n vertices if $|U| \geq \sqrt{\rho}n$ and $e_G(U, \overline{U}) \leq \rho n^2$. Let $0 < \rho \leq \nu \leq 1$. Let G be a graph containing a ρ-component U and let $S \subseteq U$. We say that S is *ν-expanding in U* if $|RN_{\nu,U}(S)| \geq |S| + \nu|U|$, and *non-$\nu$-expanding* otherwise. So if $G[U]$ is a robust (ν, τ)-expander for some τ, then all $S \subseteq U$ satisfying $\tau|U| \leq |S| \leq (1-\tau)|U|$ are ν-expanding in U.

Recall that $\overline{U_1} = V(G) \setminus U_1$ and similarly for U_2. Suppose that G is a graph on n vertices and that $U \subseteq V(G)$. We say that $G[U]$ is *ρ-close to bipartite (with bipartition U_1, U_2)* if

(C1) U is the union of two disjoint sets U_1 and U_2 with $|U_1|, |U_2| \geq \sqrt{\rho}n$;
(C2) $||U_1| - |U_2|| \leq \rho n$;
(C3) $e(U_1, \overline{U_2}) + e(U_2, \overline{U_1}) \leq \rho n^2$.

Note that (C1) and (C3) together imply that U is a ρ-component. Suppose that G is a graph on n vertices and that $U \subseteq V(G)$. Let $0 < \rho \leq \nu \leq \tau < 1$. We say that $G[U]$ is a *(ρ, ν, τ)-robust expander component of G* if

(E1) U is a ρ-component;
(E2) $G[U]$ is a robust (ν, τ)-expander.

We say that $G[U]$ is a *bipartite (ρ, ν, τ)-robust expander component (with bipartition A, B) of G* if

(B1) $G[U]$ is ρ-close to bipartite with bipartition A, B;
(B2) $G[U]$ is a bipartite robust (ν, τ)-expander with bipartition A, B.

We say that U is a *(ρ, ν, τ)-robust component* if it is either a (ρ, ν, τ)-robust expander component or a bipartite (ρ, ν, τ)-robust expander component.

Our main result states that any sufficiently dense regular graph has a partition into (bipartite) robust expander components. Let $k, \ell, D \in \mathbb{N}$ and $0 < \rho \leq \nu \leq \tau < 1$. Given a D-regular graph G on n vertices, we say that $\mathcal{V}$ is a *robust partition of G with parameters ρ, ν, τ, k, ℓ* if the following conditions hold.

(D1) $\mathcal{V} = \{V_1, \dots, V_k, W_1, \dots, W_\ell\}$ is a partition of $V(G)$;
(D2) for all $1 \leq i \leq k$, $G[V_i]$ is a (ρ, ν, τ)-robust expander component of G;
(D3) for all $1 \leq j \leq \ell$, there exists a partition A_j, B_j of W_j such that $G[W_j]$ is a bipartite (ρ, ν, τ)-robust expander component with respect to A_j, B_j;
(D4) for all $X, X' \in \mathcal{V}$ and all $x \in X$, we have $d_X(x) \geq d_{X'}(x)$. In particular, $d_X(x) \geq D/m$, where $m := k + \ell$;
(D5) for all $1 \leq j \leq \ell$ we have $d_{B_j}(u) \geq d_{A_j}(u)$ for all $u \in A_j$ and $d_{A_j}(v) \geq d_{B_j}(v)$ for all $v \in B_j$; in particular, $\delta(G[A_j, B_j]) \geq D/2m$;
(D6) $k + 2\ell \leq \lfloor (1 + \rho^{1/3})n/D \rfloor$;
(D7) for all $X \in \mathcal{V}$, all but at most ρn vertices $x \in X$ satisfy $d_X(x) \geq D - \rho n$.

As we shall see, (D6) can be derived from (D1)–(D5) but it is useful to state it explicitly. Our main result is the following theorem, which we prove in the next section. As mentioned in the introduction, we can use Theorem 3.1 to derive a version for almost regular graphs (see Section 5).

Theorem 3.1. *For all $\alpha, \tau > 0$ and every non-decreasing function $f : (0,1) \to (0,1)$, there exists $n_0 \in \mathbb{N}$ such that the following holds. For all D-regular graphs G on $n \geq n_0$ vertices where $D \geq \alpha n$, there exist ρ, ν with $1/n_0 \leq \rho \leq \nu \leq \tau$; $\rho \leq f(\nu)$ and $1/n_0 \leq f(\rho)$, and $k, \ell \in \mathbb{N}$ such that G has a robust partition $\mathcal{V}$ with parameters ρ, ν, τ, k, ℓ.*

When the degree of G is large, (D6) implies that there are only a small number of possible choices for k and ℓ.

Proposition 3.2. *Let $n, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho \ll \nu \ll \tau \ll 1/r < 1$ and $\rho^{1/3} \leq \varepsilon/2$. Let G be a D-regular graph on n vertices where $D \geq (1/r + \varepsilon)n$ and let $\mathcal{V}$ be a robust partition of G with parameters ρ, ν, τ, k, ℓ. Then $k + 2\ell \leq r - 1$ and so $\ell \leq \lfloor (r-1)/2 \rfloor$ and $k \leq r - 1 - 2\ell$. In particular,*

(i) *if $r = 4$ then $(k, \ell) \in \mathcal{S}$, where $\mathcal{S} := \{(1,0), (2,0), (3,0), (0,1), (1,1)\}$;*
(ii) *if $r = 5$ then $(k, \ell) \in \mathcal{S} \cup \{(4,0), (2,1), (0,2)\}$.*

Proof. It suffices to show that $k + 2\ell \leq r - 1$. By (D6) and our assumption that $\rho^{1/3} \leq \varepsilon/2$ we have

$$k + 2\ell \leq \left\lfloor \frac{1 + \varepsilon/2}{1/r + \varepsilon} \right\rfloor = \left\lfloor \frac{r + r\varepsilon/2}{1 + r\varepsilon} \right\rfloor = r - 1,$$

as required. □

We will prove Theorem 1.2 separately for each of these cases in Proposition 3.2(i). Proposition 3.2 is the only point of the proof where we need the full strength of the degree condition $D \geq (1/4 + \varepsilon)n$. Furthermore, Proposition 3.2(ii) implies that a $\lceil n/4 \rceil$-regular graph could have any of the structures specified by (i), as well as $(k, \ell) = (4,0), (2,1), (0,2)$. Note also that Figure 1(i) has $(k, \ell) = (2,1)$ and Figure 1(ii) has $(k, \ell) = (3,0)$.

4. The proof of Theorem 3.1

We begin by giving a brief sketch of the argument.

4.1. **Sketch proof of Theorem 3.1.** The basic proof strategy is to successively refine an appropriate partition of G. So let G be a D-regular graph on n vertices, where D is linear in n. Suppose that G is not a (bipartite) robust expander. Then $V(G)$ contains a set S such that N is not much larger than S, where $N := RN_{\nu,G}(S)$ for appropriate ν. Consider a minimal S with this property. Since G is regular, N cannot be significantly smaller than S. One can use this to show that there are very few edges between $S \cup N$ and $X := V(G) \setminus (S \cup N)$. Moreover, one can show that S and N are either almost identical or almost disjoint. In the former case, $G[S \cup N]$ is a robust expander and in the latter $G[S \cup N]$ is close to a bipartite robust expander. So in both cases, $S \cup N$ is a (bipartite) robust expander component. Similarly, if X is non-empty, it is either a (bipartite) robust expander component or we can partition it further along the above lines. In this way, we eventually arrive at the desired partition.

4.2. **Preliminary observations.** We will often use the following simple observation about ρ-components.

Lemma 4.1. *Let $n, D \in \mathbb{N}$ and $\rho, \rho', \gamma > 0$ such that $\rho \leq \rho'$ and $\gamma \geq \rho + \rho'$. Let G be a D-regular graph on n vertices and let U be a ρ-component of G. Then*

(i) *$|U| \geq D - \sqrt{\rho}n$;*
(ii) *if W, W' is a partition of U and W is a ρ'-component of G, then $e(W', \overline{W'}) \leq \gamma n^2$;*
(iii) *if $D \geq 2\sqrt{\rho'}n$, then U is a ρ'-component of G.*

Let $X \subseteq V(G)$ have bipartition X_1, X_2 such that $G[X]$ is ρ-close to bipartite with bipartition X_1, X_2. Then

(iv) *$|X_1|, |X_2| \geq D - 2\sqrt{\rho}n$;*

(v) *if $D \geq 3\sqrt{\rho'}n$, then X is ρ'-close to bipartite.*

Proof. To prove (i), note that

$$|U|D = \sum_{x\in U} d_G(x) = 2e_G(U) + e_G(U,\overline{U}) \leq |U|^2 + \rho n^2.$$

So $|U| \geq D - \rho n^2/|U| \geq D - \sqrt{\rho}n$, as required. To see (ii), note that

$$e(W', \overline{W'}) = e(W', W) + e(W', \overline{U}) \leq e(\overline{W}, W) + e(U, \overline{U}) \leq (\rho + \rho')n^2 \leq \gamma n^2.$$

To see (iii), note first that $e(U,\overline{U}) \leq \rho n^2 \leq \rho' n^2$. Furthermore, (i) implies that $|U| \geq D - \sqrt{\rho}n \geq \sqrt{\rho'}n$.

We now prove (iv). Since $e'(X_1, \overline{X_2}) \leq 2e(X_1, \overline{X_2}) \leq 2\rho n^2$ and since G is D-regular, we have that

$$|X_1|D - 2\rho n^2 \leq e'(X_1, V(G)) - e'(X_1, \overline{X_2}) = e'(X_1, X_2) \leq |X_1||X_2|. \tag{4.1}$$

So $|X_2| \geq D - 2\rho n^2/|X_1| \geq D - 2\sqrt{\rho}n$. A similar argument shows that $|X_1| \geq D - 2\sqrt{\rho}n$. Finally, (v) holds since (C2) and (C3) are immediate, and (C1) follows from (iv). □

The following lemma implies that, for any regular graph G and any $S \subseteq V(G)$ that is not too small, the robust neighbourhood of S cannot be significantly smaller than S itself.

Lemma 4.2. *Let $n, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \alpha < 1$. Let U be a ρ-component of a D-regular graph G on n vertices, where $D \geq \alpha n$. Let $S \subseteq U$ satisfy $|S| \geq \tau|U|$. Write $N := RN_{\nu,U}(S)$ and let $Y := S \setminus N$ and $W := V(G) \setminus (S \cup N)$. Then*

(i) *$e(S,Y) \leq \nu n^2$ and $e(S,W) \leq 2\nu n^2$;*
(ii) *$|N| \geq |S| - \sqrt{\nu}n$;*
(iii) *$|N| \geq D - \sqrt{\nu}n$.*

Proof. To prove (i), note that $e(S,Y) = e_{G[U]}(S,Y) \leq |Y|\nu|U| \leq \nu n^2$. Moreover, $e'(S, \overline{N} \cap U) = \sum_{x\in\overline{N}\cap U} d_S(x) \leq \nu|U|^2 \leq \nu n^2$. Since U is a ρ-component of G, we have that $e(U,\overline{U}) \leq \rho n^2$. Hence

$$e'(S,\overline{N}) = e'(S, \overline{N} \cap U) + e(S,\overline{U}) \leq (\nu + \rho)n^2 \leq 2\nu n^2. \tag{4.2}$$

This proves (i) as $e(S,W) \leq e'(S,\overline{N})$. We now prove (ii). Certainly $e'(S,N) \leq \sum_{x\in N} d(x) = D|N|$. Similarly

$$e'(S,N) = D|S| - e'(S,\overline{N}) \overset{(4.2)}{\geq} D|S| - 2\nu n^2. \tag{4.3}$$

Then $|N| \geq |S| - 2\nu n^2/D \geq |S| - \sqrt{\nu}n$, which proves (ii). Finally, we prove (iii). Lemma 4.1(i) implies that $|U| \geq D - \sqrt{\rho}n$, so

$$|S| \geq \tau|U| \geq \tau D/2. \tag{4.4}$$

Moreover, (4.3) implies that $|S||N| \geq e'(S,N) \geq D|S| - 2\nu n^2$ and hence

$$|N| \geq D - \frac{2\nu}{|S|}n^2 \overset{(4.4)}{\geq} D - \frac{4\nu}{\tau D}n^2 \geq D - \sqrt{\nu}n,$$

as required. □

The next lemma gives some sufficient conditions for $G[U]$ to be close to bipartite when G is a regular graph and $U \subseteq V(G)$.

Lemma 4.3. *Let $n, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \gamma' \leq \gamma \ll \alpha < 1$ where $\gamma' \leq \gamma^{7/6}$. Suppose that G is a D-regular graph on n vertices where $D \geq \alpha n$. Let Y, Z be disjoint subsets of $V(G)$ such that*

(i) $|Y| \geq \gamma n$;
(ii) $||Y| - |Z|| \leq \gamma n$;
(iii) $e(Y, \overline{Z}) \leq \gamma' n^2$.

Then $G[Y \cup Z]$ is $\gamma^{1/3}$-close to bipartite with bipartition Y, Z.

Proof. First note that (C2) certainly holds with $\gamma^{1/3}$ playing the role of ρ. Since $e'(Y, \overline{Z}) \leq 2e(Y, \overline{Z}) \leq 2\gamma' n^2$ and G is D-regular, we have that

$$|Y|D - 2\gamma' n^2 \leq e'(Y, V(G)) - e'(Y, \overline{Z}) = e'(Y, Z) \leq |Y||Z|. \tag{4.5}$$

So $|Z| \geq D - 2\gamma' n^2/|Y| \geq 2\gamma^{1/6} n$ and $|Y| \geq |Z| - \gamma n \geq \gamma^{1/6} n$. Thus (C1) holds. We also have that

$$\begin{aligned} e(Z, \overline{Y}) &\leq e'(Z, \overline{Y}) = |Z|D - e'(Y, Z) \overset{(4.5)}{\leq} (|Z| - |Y|)D + 2\gamma' n^2 \\ &\overset{\text{(ii)}}{\leq} D\gamma n + 2\gamma' n^2 \leq 3\gamma n^2. \end{aligned}$$

So $e(Y, \overline{Z}) + e(Z, \overline{Y}) \leq 4\gamma n^2 \leq \gamma^{1/3} n^2$ and therefore (C3) holds. □

We now show that if a regular graph G contains a non-expanding set S whose intersection with its robust neighbourhood is small, then G contains an induced subgraph which is close to bipartite.

Lemma 4.4. *Let $n, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \alpha < 1$. Suppose that G is a D-regular graph on n vertices where $D \geq \alpha n$. Let $U \subseteq V(G)$ be a ρ-component of G. Suppose that $S \subseteq U$ is non-ν-expanding in U and $|S| \geq \tau |U|$. Let $N := RN_{\nu,U}(S)$, $Y := S \setminus N$ and $Z := N \setminus S$. Then*

(i) $||Y| - |Z|| \leq \sqrt{\nu} n$;
(ii) *if also $|Y| > \sqrt{\nu} n$, then $G[Y \cup Z]$ is $\nu^{1/6}$-close to bipartite with bipartition Y, Z.*

Proof. Let $X := S \cap N$. So $S = X \cup Y$ and $N = X \cup Z$. Since S is non-ν-expanding in U, we have that $|N| < |S| + \nu |U|$. By Lemma 4.2(ii) we have that

$$|S| - \sqrt{\nu} n \leq |N| < |S| + \nu |U| \leq |S| + \sqrt{\nu} n,$$

which proves (i). To prove (ii), let $W := \overline{S \cup N} = \overline{X \cup Y \cup Z}$. Note that Lemma 4.2(i) implies that

$$e(Y, \overline{Z}) = e(Y, S \cup W) \leq e(S, Y) + e(S, W) \leq 3\nu n^2. \tag{4.6}$$

Set $\gamma' := 3\nu$ and $\gamma := \sqrt{\nu}$. Then $\gamma' \leq \gamma^{5/6}$. So we can apply Lemma 4.3 to see that $G[Y \cup Z]$ is $\nu^{1/6}$-close to bipartite with bipartition Y, Z. □

The next proposition formalises the fact that, if a graph G contains a subset U that is close to bipartite; we may add or remove any small set of vertices so that it is still close to bipartite (with slightly weaker parameters). We omit the proof as it is straightforward to check that (C1)–(C3) are satisfied.

Proposition 4.5. *Let $n, D \in \mathbb{N}$, $0 < 1/n \ll \rho_1, \rho_2 \ll \alpha < 1$ and let $\rho \geq \rho_1 + 2\rho_2$. Suppose that G is a D-regular graph on n vertices where $D \geq \alpha n$ and let $U \subseteq V(G)$ be such that $G[U]$ is ρ_1-close to bipartite, with bipartition A, B. Suppose that $A', B' \subseteq V(G)$ are disjoint and $|A \triangle A'| + |B \triangle B'| \leq \rho_2 n$. Let $U' := A' \cup B'$. Then $G[U']$ is ρ-close to bipartite with bipartition A', B'.*

4.3. **Properties of non-expanding subsets.** In this subsection we prove that a ρ-component is either a robust expander component, a bipartite robust expander component, or the union of two ρ'-components (where $\rho \ll \rho'$). This forms the core of the proof of Theorem 3.1.

For this, we first show that if U is a ρ-component in a regular graph G such that $G[U]$ is not a robust expander, then either $G[U]$ is close to bipartite, or U can be decomposed into two ρ'-components. To prove this, we consider a non-expanding set S and its robust neighbourhood N. We use our previous results to show that either $S \cup N$ and its complement in U are both ρ'-components or $G[U]$ is ρ'-close to bipartite.

Lemma 4.6. *Let $n, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho \ll \nu \ll \rho' \ll \tau \ll \alpha < 1$. Let U be a ρ-component of a D-regular graph G on n vertices where $D \geq \alpha n$. Suppose that $G[U]$ is not a robust (ν, τ)-expander. Then at least one of the following hold:*

(i) *U has a partition U_1, U_2 such that each of U_1, U_2 is a ρ'-component of G;*
(ii) *$G[U]$ is ρ'-close to bipartite.*

Proof. Since $G[U]$ is not a robust (ν, τ)-expander, there exists $S \subseteq U$ with

$$\tau|U| \leq |S| \leq (1-\tau)|U| \tag{4.7}$$

and $|RN_{\nu,U}(S)| < |S| + \nu|U|$. Let $N := RN_{\nu,U}(S)$, $X := S \cap N$, $Y := S \setminus N$, $Z := N \setminus S$ and $W := V(G) \setminus (S \cup N)$. We consider two cases, depending on the size of Y.

Case 1. $|Y| \leq \sqrt{\nu} n$.

In this case, we will show that (i) holds. Let $U_1 := S \cup N = S \cup Z$ so that $\overline{U_1} = W$. Then Lemma 4.2(iii) implies that $|U_1| \geq |N| \geq D - \sqrt{\nu} n \geq \sqrt{\rho'} n$. By Lemma 4.4(i), we have

$$|Z| \leq |Y| + \sqrt{\nu} n \leq 2\sqrt{\nu} n \tag{4.8}$$

$$\leq \tau D/4 \leq \tau |U|/2, \tag{4.9}$$

where the last inequality holds since $|U| \geq D - \sqrt{\rho} n$ by Lemma 4.1(i). Now Lemma 4.2(i) implies that

$$e(U_1, \overline{U_1}) = e(S, W) + e(Z, W) \leq 2\nu n^2 + |Z| n \overset{(4.8)}{\leq} 3\sqrt{\nu} n^2.$$

So U_1 is a $3\sqrt{\nu}$-component of G. Moreover, (4.9) and (4.7) together imply that

$$|U_1| = |S| + |Z| \leq (1 - \tau/2)\,|U|.$$

Let $U_2 := U \setminus U_1$. Then $|U_2| \geq \tau|U|/2 \geq \sqrt{\rho'} n$. Since U is a ρ-component, U_1 is a $3\sqrt{\nu}$-component, and $\rho + 3\sqrt{\nu} \leq \rho'$, we can apply Lemma 4.1(ii) with $U_1, U_2, \rho, 3\sqrt{\nu}, \rho'$ playing the roles of $W, W', \rho, \rho', \gamma$ respectively to see that $e(U_2, \overline{U_2}) \leq \rho' n^2$. Thus U_2 is a ρ'-component of G and so (i) holds.

Case 2. $|Y| > \sqrt{\nu} n$.

Let $U_1 := Y \cup Z = S \triangle N$. Lemma 4.4(ii) implies that $G[U_1]$ is $\nu^{1/6}$-close to bipartite with bipartition Y, Z. Therefore (C1) and (C3) imply that U_1 is a $\nu^{1/6}$-component. Moreover,

$|U_1| \geq 2(D - 2\nu^{1/12}n) \geq 2(D - 2\sqrt{\rho'}n)$ by Lemma 4.1(iv). Let $U_2 := U \setminus U_1$. Now Lemma 4.1(ii) with $U_1, U_2, \rho, \nu^{1/6}, (\rho'/3)^2$ playing the roles of $W, W', \rho, \rho', \gamma$ implies that $e(U_2, \overline{U_2}) \leq (\rho'/3)^2 n^2$. If $|U_2| \geq \rho' n/3$ then U_2 is a $(\rho'/3)^2$-component. So Lemma 4.1(i) implies that $|U_2| \geq D - \rho' n/3$ and thus U_2 is actually a ρ'-component of G. So (i) holds in this case.

Thus we may assume that $|U_2| < \rho' n/3$. Let $Y' := Y \cup U_2$ and $Z' := Z$. Then Y', Z' are disjoint subsets whose union is U. Note that $|Y' \triangle Y| + |Z' \triangle Z| = |U_2| < \rho' n/3$. Now Proposition 4.5 with $U_1, U, Y, Z, Y', Z', \nu^{1/6}, \rho'/3, \rho'$ playing the roles of $U, U', A, B, A', B', \rho_1, \rho_2, \rho$ implies that $G[U]$ is ρ'-close to bipartite with bipartition Y', Z'. So (ii) holds. □

The following lemma is a bipartite analogue of Lemma 4.6. It states that if G is a regular graph and $U \subseteq V(G)$ such that $G[U]$ is close to bipartite and $G[U]$ is not a bipartite robust expander, then U can be decomposed into two components. The proof is similar to that of Lemma 4.6 – we find the partition by considering a non-expanding set and its robust neighbourhood – and can be found in [30].

Lemma 4.7. *Let $n, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho \ll \nu \ll \rho' \ll \tau \ll \alpha < 1$. Let G be a D-regular graph on n vertices where $D \geq \alpha n$. Suppose that $U \subseteq V(G)$ is such that $G[U]$ is ρ-close to bipartite with bipartition A, B and $G[U]$ is not a bipartite robust (ν, τ)-expander with bipartition A, B. Then there is a partition U_1, U_2 of U such that U_1, U_2 are ρ'-components.*

4.4. **Adjusting partitions.** The results of this subsection will be needed to ensure (D4), (D5) and (D7) in the proof of Theorem 3.1.

The next two lemmas state that (bipartite) robust expanders are indeed robust, in the sense that the expansion property cannot be destroyed by adding or removing a small number of vertices. We omit the proofs as they follow from the definitions in a straightforward way.

Lemma 4.8. *Let $0 < \nu \ll \tau \ll 1$. Suppose that G is a graph and $U, U' \subseteq V(G)$ are such that $G[U]$ is a robust (ν, τ)-expander and $|U \triangle U'| \leq \nu|U|/2$. Then $G[U']$ is a robust $(\nu/2, 2\tau)$-expander.*

Lemma 4.9. *Let $0 < \nu \ll \tau \ll 1$. Suppose that $U \subseteq V(G)$ and that $G[U]$ is a bipartite robust (ν, τ)-expander with bipartition A, B. Let $W, A', B' \subseteq V(G)$ be such that $|W| \leq \nu|A|/2$; A' and B' are disjoint; and $|A \triangle A'| + |B \triangle B'| \leq \nu|A|/2$. Then*

(i) *$G[U \setminus W]$ is a bipartite robust $(\nu/2, 2\tau)$-expander with bipartition $A \setminus W, B \setminus W$;*
(ii) *$G[A' \cup B']$ is a bipartite robust $(\nu/2, 2\tau)$-expander with bipartition A', B'.*

We now extend Lemma 4.9 by showing that, after adding and removing a small number of vertices, a bipartite robust component is still a bipartite robust component, with slightly weaker parameters. The proof may be found in [30].

Lemma 4.10. *Let $0 < 1/n \ll \rho \leq \gamma \ll \nu \ll \tau \ll \alpha < 1$ and suppose that G is a D-regular graph on n vertices where $D \geq \alpha n$.*

(i) *Suppose that $G[A \cup B]$ is a bipartite (ρ, ν, τ)-robust expander component of G with bipartition A, B. Let $A', B' \subseteq V(G)$ be such that $|A \triangle A'| + |B \triangle B'| \leq \gamma n$. Then $G[A' \cup B']$ is a bipartite $(3\gamma, \nu/2, 2\tau)$-robust expander component of G with bipartition A', B'.*

(ii) *Suppose that $G[U]$ is a bipartite (ρ,ν,τ)-robust expander component of G. Let $U' \subseteq V(G)$ be such that $|U \triangle U'| \leq \gamma n$. Then $G[U']$ is a bipartite $(3\gamma, \nu/2, 2\tau)$-robust expander component of G.*

In any ρ-component, almost all vertices have very few neighbours outside the component. In particular, most vertices have more neighbours within their own component than in any other. The following lemma allows us to move a small number of vertices in a partition into ρ-components so that this property holds for all vertices.

Lemma 4.11. *Let $m, n, D \in \mathbb{N}$ and $0 < 1/n \ll \rho \ll \alpha, 1/m \leq 1$. Let G be a D-regular graph on n vertices where $D \geq \alpha n$. Suppose that $\mathcal{U} := \{U_1, \dots, U_m\}$ is a partition of $V(G)$ such that U_i is a ρ-component for each $1 \leq i \leq m$. Then G has a vertex partition $\mathcal{V} := \{V_1, \dots, V_m\}$ such that*

(i) *$|U_i \triangle V_i| \leq \rho^{1/3} n$;*
(ii) *V_i is a $\rho^{1/3}$-component for each $1 \leq i \leq m$;*
(iii) *if $x \in V_i$ then $d_{V_i}(x) \geq d_{V_j}(x)$ for all $1 \leq i, j \leq m$. In particular, $d_V(x) \geq D/m$ for all $x \in V$ and all $V \in \mathcal{V}$;*
(iv) *for all but at most $\rho^{1/3} n$ vertices $x \in V_i$ we have $d_{V_i}(x) \geq D - 2\sqrt{\rho} n$.*

Proof. First note that the second part of (iii) follows from the first. For each $1 \leq i \leq m$, let X_i be the collection of vertices $y \in U_i$ with $d_{\overline{U_i}}(y) \geq \sqrt{\rho} n$. Since U_i is a ρ-component, we have $|X_i| \leq \sqrt{\rho} n$. Let $W_i := U_i \setminus X_i$. Then each $x \in W_i$ satisfies

$$d_{W_i}(x) = d(x) - d_{\overline{U_i} \cup X_i}(x) \geq d(x) - \sqrt{\rho} n - |X_i| \geq d(x) - 2\sqrt{\rho} n. \tag{4.10}$$

Let $X := \bigcup_{1 \leq i \leq m} X_i$. Among all partitions $X_1', \dots, X_m'$ of X, choose one such that $\sum_{1 \leq i \leq m} e(V_i, \overline{V_i})$ is minimal, where $V_i := W_i \cup X_i'$. It is easy to see that $d_{V_i}(x) \geq d_{V_j}(x)$ for all $x \in X_i'$ and all $1 \leq i, j \leq m$. So (iii) holds for all $x \in X_i'$ and $i \leq m$. Moreover, if $x \in W_i$, then (4.10) implies that $d_{V_i}(x) \geq d_{W_i}(x) \geq d(x) - 2\sqrt{\rho} n \geq d(x)/2$. So (iii) also holds for each vertex in W_i. Furthermore,

$$\sum_{1 \leq i \leq m} e(V_i, \overline{V_i}) \leq \sum_{1 \leq i \leq m} e(U_i, \overline{U_i}) \leq \rho m n^2 \leq \rho^{1/3} n^2$$

and hence each V_i is a $\rho^{1/3}$-component, so (ii) holds.

Note that $U_i \cap V_i \supseteq W_i$, so

$$|U_i \triangle V_i| \leq \sum_{1 \leq i \leq m} |X_i'| = |X| \leq m\sqrt{\rho} n \leq \rho^{1/3} n, \tag{4.11}$$

which proves (i). Finally, (4.10) and the fact that $|V_i \setminus W_i| \leq |X| \leq \rho^{1/3} n$ by (4.11) together imply (iv). □

The next lemma shows that, in a bipartite robust expander component, we can adjust the bipartition slightly so that any vertex has at least as many neighbours in the opposite class as within its own class. The resulting graph will still be a bipartite robust expander component. The proof is very similar to that of Lemma 4.11 and may be found in [30].

Lemma 4.12. *Let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \alpha < 1$ and let G be a D-regular graph on n vertices where $D \geq \alpha n$. Suppose that U is a bipartite (ρ,ν,τ)-robust component of G. Then there exists a bipartition A, B of U such that*

(i) *U is a bipartite $(3\sqrt{\rho}, \nu/2, 2\tau)$-robust component with partition A, B;*

(ii) $d_B(u) \geq d_A(u)$ *for all* $u \in A$, *and* $d_A(v) \geq d_B(v)$ *for all* $v \in B$;

4.5. **Proof of the main result.** We are now ready to prove Theorem 3.1 – that every sufficiently dense regular graph has a robust partition. The first part of the proof is an iteration of Lemmas 4.6 and 4.7 – we begin with the trivial partition of $V(G)$ and successively refine it by applying Lemma 4.7 to those components which are close to bipartite and Lemma 4.6 to the others, until we obtain a partition into robust components. We then use Lemma 4.11 to adjust the partition slightly and Lemma 4.12 to achieve an appropriate bipartition of the bipartite robust expander components.

Proof of Theorem 3.1. Let $t := 3\lceil 2/\alpha \rceil$. Define further constants satisfying

$$0 < 1/n_0 \ll \rho_1 \ll \nu_1 \ll \rho_2 \ll \nu_2 \ll \ldots \ll \rho_t \ll \nu_t \ll \tau' \ll \alpha, \tau$$

so that $1/n_0 \leq f(\rho_1)$ and $3^{3/2}\rho_i^{1/6} \leq f(\nu_i/4)$ for all $1 \leq i \leq t$. We first prove the following claim.

Claim. *There is some* $1 \leq i < t$ *and a partition* $\mathcal{U}$ *of* $V(G)$ *such that* U *is a* (ρ_i, ν_i, τ')*-robust component for each* $U \in \mathcal{U}$.

To see this, let $\mathcal{U}_1 := \{V(G)\}$. Note that $V(G)$ is certainly a ρ_1-component of G, and $|\mathcal{U}_1| = 1$. Suppose, for some i with $1 \leq i < t$, we have inductively defined a partition $\mathcal{U}_i$ of $V(G)$ such that U is a ρ_i-component for each $U \in \mathcal{U}_i$ and $2|\mathcal{U}_i| + |\mathcal{W}_i| \geq i + 1$, where $\mathcal{W}_i$ is the collection of all those $U \in \mathcal{U}_i$ which are ρ_i-close to bipartite. If each $U \in \mathcal{U}_i$ is a (ρ_i, ν_i, τ')-robust component, then we are done by setting $\mathcal{U} := \mathcal{U}_i$. Otherwise, we obtain $\mathcal{U}_{i+1}$ from $\mathcal{U}_i$ as follows.

There is some $U \in \mathcal{U}_i$ which is not a (ρ_i, ν_i, τ')-robust component. If $U \in \mathcal{W}_i$, then apply Lemma 4.7 with $\rho_i, \nu_i, \rho_{i+1}, \tau'$ playing the roles of ρ, ν, ρ', τ to obtain a partition U_1, U_2 of U such that U_1, U_2 are ρ_{i+1}-components. Let $\mathcal{U}_{i+1} := (\mathcal{U}_i \setminus \{U\}) \cup \{U_1, U_2\}$. Lemma 4.1(v) implies that $\mathcal{W}_i \setminus \{U\} \subseteq \mathcal{W}_{i+1}$, where $\mathcal{W}_{i+1}$ is the collection of all those $U \in \mathcal{U}_{i+1}$ which are ρ_{i+1}-close to bipartite. Thus $|\mathcal{U}_{i+1}| = |\mathcal{U}_i| + 1$ and $|\mathcal{W}_{i+1}| \geq |\mathcal{W}_i| - 1$.

So suppose next that $U \in \mathcal{U}_i \setminus \mathcal{W}_i$. Apply Lemma 4.6 with $\rho_i, \nu_i, \rho_{i+1}, \tau'$ playing the roles of ρ, ν, ρ', τ. If (i) holds, then U has a partition U_1, U_2 such that U_1, U_2 are ρ_{i+1}-components. As before, we let $\mathcal{U}_{i+1} := (\mathcal{U}_i \setminus \{U\}) \cup \{U_1, U_2\}$. So $|\mathcal{U}_{i+1}| = |\mathcal{U}_i| + 1$ and $|\mathcal{W}_{i+1}| \geq |\mathcal{W}_i|$. Otherwise, Lemma 4.6(ii) holds. Then U is ρ_{i+1}-close to bipartite. We let $\mathcal{U}_{i+1} := \mathcal{U}_i$. Then $|\mathcal{U}_{i+1}| = |\mathcal{U}_i|$ and $|\mathcal{W}_{i+1}| \geq |\mathcal{W}_i| + 1$.

Note that in each case we have $2|\mathcal{U}_{i+1}| + |\mathcal{W}_{i+1}| \geq i + 2$. Moreover, Lemma 4.1(iii) implies that each $W \in \mathcal{U}_i \setminus \{U\}$ is a ρ_{i+1}-component. Therefore each $W \in \mathcal{U}_{i+1}$ is a ρ_{i+1}-component.

It remains to show that this process must stop before we define $\mathcal{U}_t$. Suppose not, i.e. suppose we have defined $\mathcal{U}_t$. Since each $W \in \mathcal{U}_t$ is a ρ_t-component, Lemma 4.1(i) implies that $|W| \geq (\alpha - \sqrt{\rho_t})n$ for all $W \in \mathcal{U}_t$. Moreover, $|\mathcal{U}_t| > t/3$ since $3|\mathcal{U}_t| \geq 2|\mathcal{U}_t| + |\mathcal{W}_t| \geq t + 1$. Altogether, this implies that

$$|V(G)| \geq \frac{t}{3}(\alpha - \sqrt{\rho_t})n \geq \frac{2}{\alpha}(\alpha - \sqrt{\rho_t})n > n,$$

a contradiction. This completes the proof of the claim.

Set $\rho' := \rho_i$, $\nu' := \nu_i$, $\rho := 3^{3/2}\rho'^{1/6}$ and $\nu := \nu'/4$. So

$$\rho = 3^{3/2}\rho'^{1/6} \leq f(\nu'/4) = f(\nu) \quad \text{and} \quad 1/n_0 \leq f(\rho_1) \leq f(\rho) \tag{4.12}$$

and every $U \in \mathcal{U}$ is a (ρ', ν', τ')-robust component of G. So there exist $k, \ell \in \mathbb{N}$ such that $\mathcal{U} = \{U_1, \dots, U_k, Z_1, \dots, Z_\ell\}$, where U_i is a (ρ', ν', τ')-robust expander component for all $1 \le i \le k$, and Z_j is a bipartite (ρ', ν', τ')-robust expander component for all $1 \le j \le \ell$. Let $m := k + \ell$. Note that for each $1 \le i \le k$, we have $|U_i| \ge D - \sqrt{\rho'}n$ (by Lemma 4.1(i) and since U_i is a ρ'-component). Moreover, for each $1 \le j \le \ell$, $|Z_j| \ge 2(D - 2\sqrt{\rho'}n)$ by Lemma 4.1(iv). Thus

$$n = \sum_{1 \le i \le k} |U_i| + \sum_{1 \le j \le \ell} |Z_j| \ge (D - 2\sqrt{\rho'}n)(k + 2\ell)$$

and so

$$k + 2\ell \le \left\lfloor \frac{n}{D - 2\sqrt{\rho'}n} \right\rfloor \le \left\lfloor (1 + \rho'^{1/3}) \frac{n}{D} \right\rfloor. \tag{4.13}$$

In particular, $1/m \ge \alpha/2$.

To achieve (D4), we apply Lemma 4.11 with ρ' playing the role of ρ to $\mathcal{U}$ to obtain a new partition $\mathcal{V} = \{V_1, \dots, V_k, W_1, \dots, W_\ell\}$ of $V(G)$ satisfying (i)–(iv), so in particular

$$|U_i \triangle V_i|, |Z_j \triangle W_j| \le \rho'^{1/3} n \tag{4.14}$$

for all $1 \le i \le k$ and all $1 \le j \le \ell$. We claim that $\mathcal{V}$ satisfies (D1)–(D7).

Now (D1) certainly holds, (D4) follows from Lemma 4.11(iii) and (D7) follows from Lemma 4.11(iv). To prove (D2), note that V_i is a $\rho'^{1/3}$-component by Lemma 4.11(ii) and $|V_i| \ge D/2 \ge \sqrt{\rho}n$. Thus V_i is a ρ-component, i.e. (E1) holds. Now, by (4.14) and Lemma 4.8 with ν', τ', U_i, V_i playing the roles of ν, τ, U, U', we have that $G[V_i]$ is a robust $(\nu'/2, 2\tau')$-expander and thus also a robust (ν, τ)-expander. So (E2) holds, proving (D2).

To check (D3), recall that $G[Z_j]$ is a bipartite (ρ', ν', τ')-robust expander component. Then (4.14) and Lemma 4.10(ii) applied with $\rho', \rho'^{1/3}, \nu', \tau', Z_j, W_j$ playing the roles of $\rho, \gamma, \nu, \tau, U, U'$ imply that $G[W_j]$ is a bipartite $(3\rho'^{1/3}, \nu'/2, 2\tau')$-robust expander component. Now for each $1 \le j \le \ell$, apply Lemma 4.12 to W_j with $3\rho'^{1/3}, \nu'/2, 2\tau'$ playing the roles of ρ, ν, τ to obtain a bipartition A_j, B_j of W_j satisfying (i) and (ii). Lemma 4.12(i) implies that $G[W_j]$ is a bipartite (ρ, ν, τ)-robust expander component with bipartition A_j, B_j. So (D3) holds. Lemma 4.12(ii) implies that (D5) holds. Finally, (D6) follows from (4.13). □

5. Extending Theorem 3.1 to almost regular graphs

In this section, we prove an extension of Theorem 3.1 which states that every dense almost regular graph has a robust partition. We first extend the definition of a robust partition to graphs which may not be regular. Let $k, \ell, D \in \mathbb{N}$ and $0 < \rho \le \nu \le \tau < 1$. Given a graph G on n vertices, we say that $\mathcal{V}$ is a *robust partition of G with parameters ρ, ν, τ, k, ℓ* if (D1)–(D7) hold with $\delta(G)$ playing the role of D. Note that, for D-regular graphs, this coincides with the definition given in Section 3.

Theorem 5.1. *For all $\alpha, \tau > 0$ and every non-decreasing function $f : (0,1) \to (0,1)$, there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds. For all graphs G on $n \ge n_0$ vertices with $\alpha n \le \delta(G) \le \Delta(G) \le \delta(G) + \gamma n$, there exist ρ, ν with $1/n_0, \gamma \le \rho \le \nu \le \tau$; $\rho \le f(\nu)$ and $1/n_0 \le f(\rho)$, and $k, \ell \in \mathbb{N}$ such that G has a robust partition $\mathcal{V}$ with parameters ρ, ν, τ, k, ℓ.*

The proof proceeds by taking two copies of G and adding a small number of edges between them to obtain a regular graph G', whose degree is only slightly higher than $\Delta(G)$. We apply Theorem 3.1 to obtain a robust partition $\mathcal{V}$ of G'. The construction of G' implies that every robust component in $\mathcal{V}$ lies entirely in one copy of G. So there is a partition of $\mathcal{V}$ into two parts, one of which must be a robust partition of G.

In order to construct G' from G, we need some preliminaries. We say that a non-decreasing sequence $(d_i)_{1\le i\le n}$ of positive integers is *bipartite graphic* if there exists a bipartite graph G with vertex classes A and B with $|A| = |B| = n$ such that the ith vertex of each of A and B has degree d_i. The following theorem of Alon, Ben-Shimon and Krivelevich [1] gives a sufficient condition for a sequence to be bipartite graphic. Their result is stated differently to the statement below, but the two forms are equivalent, as observed in [8].

Theorem 5.2. *Suppose that $(d_i)_{1\le i\le n}$ is a non-decreasing sequence of positive integers. Then $(d_i)_{1\le i\le n}$ is bipartite graphic if $nd_1 \ge (d_1+d_n)^2/4$.*

We also need the following result (Lemma 3.8 from [24]).

Lemma 5.3. *Suppose that $0 < \nu \le \tau \le \varepsilon < 1$ are such that $\varepsilon \ge 2\nu/\tau$. Let G be a graph on n vertices with minimum degree $\delta(G) \ge (1/2+\varepsilon)n$. Then G is a robust (ν,τ)-expander.*

We are now able to deduce Theorem 5.1 from Theorem 3.1.

Proof that Theorem 3.1 implies Theorem 5.1. Define $f' : (0,1) \to (0,1)$ by $f'(x) := \min\{f(x)/4, \alpha x/2\}$ and let $\tau' := \min\{\tau, \alpha^2/20\}$. Apply Theorem 3.1 with α, τ', f' playing the roles of α, τ, f to obtain $n_0 \in \mathbb{N}$. Let $\gamma := 1/4n_0$. Let G be a graph on $n \ge n_0$ vertices with $\alpha n \le \delta(G) \le \Delta(G) \le \delta(G) + \gamma n$. Let $D := \delta(G)$. Order the vertices $v_1, \dots, v_n$ of G in order of increasing degree.

Obtain a graph G'' from G as follows. We let $W_1 := \{w_1, \dots, w_n\}$ and $W_2 := \{x_1, \dots, x_n\}$ be disjoint sets of vertices and let G'' have vertex set $W_1 \cup W_2$. We add the edges w_iw_j and x_ix_j whenever $v_iv_j \in E(G)$. Choose a constant β such that $\gamma = \beta(1-\beta)$ and $\gamma \le \beta \le 2\gamma$.

Let $d_i := D + \beta n - d_G(v_{n+1-i})$. Then $(d_i)_{1\le i\le n}$ is a non-decreasing sequence and $(\beta-\gamma)n \le d_1 \le d_n \le \beta n$. Observe that if $(d_i)_{1\le i\le n}$ is bipartite graphic, then we can add edges to G'' between W_1 and W_2 to obtain a $(D+\beta n)$-regular graph G'. Since $(d_1+d_n)^2/4 \le \beta^2 n^2 = (\beta-\gamma)n^2 \le nd_1$, Theorem 5.2 implies that such a G' exists. Note that

$$\Delta(G'[W_1, W_2]) = d_n \le \beta n. \tag{5.1}$$

Theorem 3.1 applied to G' implies that there exist ρ', ν with $1/n_0 \le \rho' \le \nu \le \tau'$; $\rho' \le f'(\nu)$ and $1/n_0 \le f'(\rho')$, and $k', \ell' \in \mathbb{N}$ such that G' has a robust partition $\mathcal{V}$ with parameters $\rho', \nu, \tau', k', \ell'$. Note that $\beta \le 2\gamma = 1/2n_0 \le \nu/2$.

Claim. *Let $U \in \mathcal{V}$ be arbitrary. Then U is contained entirely within one of W_1, W_2.*

To see this, let $U_i := U \cap W_i$ for $i = 1, 2$. Assume, for a contradiction, that $U_1, U_2 \ne \emptyset$. Then

$$|U_i| \ge \delta(G'[U_i]) \overset{\text{(D4),(5.1)}}{\ge} \frac{D}{k'+\ell'} - \beta n \overset{\text{(D6)}}{\ge} \frac{D}{2(1+\rho^{1/3})n/D} - \beta n \ge (\alpha^2/4 - \beta)n \ge \alpha^2 n/5. \tag{5.2}$$

In particular this implies that $\tau'|U| \le |U_i| \le (1-\tau')|U|$. The fact that $\beta \le \nu/2$ and (5.1) imply that $RN_{\nu,U}(U_i) \subseteq U_i$. Then U cannot be a robust expander component. So U is a bipartite robust expander component, with bipartition A, B, say. Let $A_i := A \cap U_i$ for $i = 1, 2$ and define B_i analogously. Similarly as in (5.2), using (D5) instead of (D4), one can show that $|A_i|, |B_i| \ge (\alpha^2/8 - \beta)n \ge \alpha^2 n/10$. In particular, $\tau'|A| \le |A_i| \le (1-\tau')|A|$. Without loss of generality, suppose that $|A_1| - |B_1| \ge |A_2| - |B_2|$. Then (C2) implies that $|A_1| - |B_1| \ge -\rho' n$ and so $|RN_{\nu,U}(A_1) \cap B| \le |B_1| \le |A_1| + \rho' n < |A_1| + \nu|U|$, a contradiction. (Here we used the fact that $|U| > \alpha n/2$ and $\rho' \le f'(\nu)$.) This completes the proof of the claim.

So there is a partition $\mathcal{V}_1, \mathcal{V}_2$ of $\mathcal{V}$ such that $U \subseteq W_i$ for all $U \in \mathcal{V}_i$. For $i = 1, 2$, let k_i be the number of robust expander components and ℓ_i the number of bipartite robust expander components in $\mathcal{V}_i$. Let $\rho := 4\rho'$. We claim that, for at least one of $i = 1, 2$, we have that $\mathcal{V}_i$ is a robust partition of G with parameters $\rho, \nu, \tau', k_i, \ell_i$. Suppose that, for both $i = 1, 2$, we have $k_i + 2\ell_i > \lfloor (1+\rho^{1/3})n/D \rfloor$. Then

$$k' + 2\ell' \ge 2\left\lfloor \frac{(1+\rho^{1/3})n}{D} \right\rfloor + 2 > \left\lfloor \frac{2(1+\rho^{1/3})n}{D} \right\rfloor \ge \left\lfloor \frac{2(1+\rho'^{1/3})n}{D+\beta n} \right\rfloor,$$

contradicting (D6) for $\mathcal{V}$. So without loss of generality, we have that $\mathcal{V}_1$ satisfies (D6). It is easy to check that the remaining properties (D1)–(D5) and (D7) are also satisfied by $\mathcal{V}_1$. Therefore $\mathcal{V}_1$ is a robust partition of G with parameters $\rho, \nu, \tau', k_1, \ell_1$ and hence also with parameters $\rho, \nu, \tau, k_1, \ell_1$. □

6. How to obtain a long cycle given a robust partition

The main result of this section is Lemma 6.2 which implies that, given a suitable set $\mathcal{P}$ of paths joining up the robust components of a robust partition, one can extend $\mathcal{P}$ into a Hamilton cycle. Actually, in the proof of Theorem 1.4 we will need to consider the more general notion of a weak robust subpartition, defined below.

6.1. **Definitions and the main statement.** Let $k, \ell \in \mathbb{N}$ and $0 < \rho \le \nu \le \tau \le \eta < 1$. Given a graph G on n vertices, we say that $\mathcal{U}$ is a *weak robust subpartition in G with parameters $\rho, \nu, \tau, \eta, k, \ell$* if the following conditions hold.

- (D1′) $\mathcal{U} = \{U_1, \dots, U_k, Z_1, \dots, Z_\ell\}$ is a collection of disjoint subsets of $V(G)$;
- (D2′) for all $1 \le i \le k$, $G[U_i]$ is a (ρ, ν, τ)-robust expander component of G;
- (D3′) for all $1 \le j \le \ell$, there exists a partition A_j, B_j of Z_j such that $G[Z_j]$ is a bipartite (ρ, ν, τ)-robust expander component with respect to A_j, B_j;
- (D4′) $\delta(G[X]) \ge \eta n$ for all $X \in \mathcal{U}$;
- (D5′) for all $1 \le j \le \ell$, we have $\delta(G[A_j, B_j]) \ge \eta n/2$.

A weak robust subpartition $\mathcal{U}$ is weaker than a robust partition in the sense that the graph is not necessarily regular and $\mathcal{U}$ need not involve the entire graph, and we can make small adjustments to the partition while still maintaining (D1′)–(D5′) with slightly worse parameters. This is formalised by the following statement.

Proposition 6.1. *Let $k, \ell, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho \le \nu \le \tau \le \eta \le \alpha^2/2 < 1$.*

(i) *Suppose that G is a D-regular graph on n vertices where $D \ge \alpha n$. Let $\mathcal{V}$ be a robust partition of G with parameters ρ, ν, τ, k, ℓ. Then $\mathcal{V}$ is a weak robust subpartition in G with parameters $\rho, \nu, \tau, \eta, k, \ell$.*

(ii) *Suppose that H is a graph and $\mathcal{U}$ is a weak robust subpartition in H with parameters $\rho, \nu, \tau, \eta, k, \ell$. Let $\mathcal{U}' \subseteq \mathcal{U}$ be non-empty. Then $\mathcal{U}'$ is a weak robust subpartition in H with parameters $\rho, \nu, \tau, \eta, k', \ell'$ for some $k' \leq k$ and $\ell' \leq \ell$.*

Proof. We only prove (i) since (ii) is clear. Note that properties (D1′)–(D3′) are immediate. Note (D6) implies that $k + 2\ell \leq \lfloor (1 + \rho^{1/3})/\alpha \rfloor \leq 2/\alpha$. So $D/(k + \ell) \geq \alpha^2 n/2$. Together with (D4) and (D5) this shows that (D4′) and (D5′) hold. This completes the proof. □

For a path system $\mathcal{P}$, we say that a vertex x is an *endpoint* of $\mathcal{P}$ if x is an endpoint of some path in $\mathcal{P}$. Define the *internal vertices* of $\mathcal{P}$ similarly. If every endpoint of a path system $\mathcal{P}$ lies in some $U \subseteq V(G)$, we say that $\mathcal{P}$ is *U-anchored*. When $\mathcal{U}$ is a collection of disjoint subsets of $V(G)$, we say that $\mathcal{P}$ is *$\mathcal{U}$-anchored* if it is $\bigcup_{U \in \mathcal{U}} U$-anchored. Given a path P in G, we say that P' is an *extension* of P if P' is a path which contains P as a subpath. An *Euler tour* in a (multi)graph is a closed walk that visits every vertex and uses each edge exactly once.

Given a graph G with $U \subseteq V(G)$ and a path system $\mathcal{P}$ in G, we write $\mathrm{End}_{\mathcal{P}}(U)$ and $\mathrm{Int}_{\mathcal{P}}(U)$ for, respectively, the number of endpoints/internal vertices of $\mathcal{P}$ which lie in U. Given disjoint sets $A, B \subseteq V(G)$, we say that $\mathcal{P}$ is *(A, B)-balanced* if

- $\mathrm{End}_{\mathcal{P}}(A) = \mathrm{End}_{\mathcal{P}}(B) > 0$; and
- $|A| - \mathrm{Int}_{\mathcal{P}}(A) = |B| - \mathrm{Int}_{\mathcal{P}}(B)$.

Suppose that G is a graph and $\mathcal{U}$ is a collection of disjoint subsets of $V(G)$. Let $\mathcal{P}$ be a $\mathcal{U}$-anchored path system in G (so all endpoints of the paths in $\mathcal{P}$ lie in $\bigcup_{U \in \mathcal{U}} U$). We define the *reduced multigraph $R_{\mathcal{U}}(\mathcal{P})$ of $\mathcal{P}$ with respect to $\mathcal{U}$* to be the multigraph with vertex set $\mathcal{U}$ in which we add a distinct edge between $U, U' \in \mathcal{U}$ whenever $\mathcal{P}$ contains a path with one endpoint in U and one endpoint in U'. So $R_{\mathcal{U}}(\mathcal{P})$ might contain loops.

Let $k, \ell \in \mathbb{N}$, let $0 < \rho \leq \nu \leq \tau \leq \eta < 1$ and let $0 < \gamma < 1$. Suppose that G is a graph on n vertices with a weak robust subpartition $\mathcal{U} = \{U_1, \dots, U_k, Z_1, \dots, Z_\ell\}$ with parameters $\rho, \nu, \tau, \eta, k, \ell$, so that the bipartition of Z_j specified by (D3′) is A_j, B_j. We say that $\mathcal{P}$ is a *$\mathcal{U}$-tour with parameter γ* if

(T1) $\mathcal{P}$ is a $\mathcal{U}$-anchored path system;
(T2) $R_{\mathcal{U}}(\mathcal{P})$ has an Euler tour;
(T3) for all $U \in \mathcal{U}$ we have $|V(\mathcal{P}) \cap U| \leq \gamma n$;
(T4) for all $1 \leq j \leq \ell$, $\mathcal{P}$ is (A_j, B_j)-balanced.

We will often think of $R_{\mathcal{U}}(\mathcal{P})$ as a walk rather than a multigraph. So in particular, we will often say that '$R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour'. The aim of this section is to prove the following lemma, stating that every graph with a weak robust subpartition $\mathcal{U}$ and a $\mathcal{U}$-tour contains a cycle which covers every vertex within the components of $\mathcal{U}$.

Lemma 6.2. *Let $k, \ell, n \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho, \gamma \ll \nu \leq \tau \ll \eta < 1$. Suppose that G is a graph on n vertices and that $\mathcal{U}$ is a weak robust subpartition in G with parameters $\rho, \nu, \tau, \eta, k, \ell$. Suppose further that G contains a $\mathcal{U}$-tour $\mathcal{P}$ with parameter γ. Then there is a cycle in G which contains $\mathcal{P}$ and every vertex in $\bigcup_{U \in \mathcal{U}} U$.*

Since by Proposition 6.1(i) every robust partition is also a weak robust subpartition, Lemma 6.2 immediately implies the following result which will be used in the proof of Theorem 1.2 while for the proof of Theorem 1.4 we will need Lemma 6.2 itself.

Corollary 6.3. *Let $k, \ell, n, D \in \mathbb{N}$ and suppose that $0 < 1/n \ll \rho, \gamma \ll \nu \leq \tau \ll \alpha < 1$. Suppose that G is a D-regular graph on n vertices where $D \geq \alpha n$, with a robust partition*

$\mathcal{V}$ with parameters ρ, ν, τ, k, ℓ. Suppose further that G contains an $\mathcal{V}$-tour with parameter γ. Then G contains a Hamilton cycle.

The following corollary of Lemma 6.2 will be used in [9].

Corollary 6.4. *Let $n \in \mathbb{N}$ and suppose that $0 < 1/n \ll \gamma \ll \eta < 1/2$. Suppose that G is a graph and let A, B, V_0 be a partition of $V(G)$ with $|A| = |B| = n$. Let H be a spanning subgraph of $G[A, B]$ such that $\delta(H) \geq (1/2 + \eta)n$. Suppose further that G contains an (A, B)-balanced path system $\mathcal{P}$ with $|V(\mathcal{P}) \cap (A \cup B)| \leq \gamma n$ and such that every vertex in V_0 lies in the interior of some path in $\mathcal{P}$. Then G contains a Hamilton cycle C with $E(\mathcal{P}) \subseteq E(C)$ and $E(C) \setminus E(\mathcal{P}) \subseteq E(H)$.*

Proof. Let ν, τ be new constants such that $\gamma \ll \nu \ll \tau \ll \eta$. Let $\mathcal{P}'$ be the path system obtained from $\mathcal{P}$ by iterating the following process: if uvw is a subpath in $\mathcal{P}$ with $v \in V_0$, then we replace uvw with an edge uw. So $V(\mathcal{P}') \subseteq A \cup B$. Let $H' := H \cup \bigcup_{P \in \mathcal{P}'} P$. Let G'' be the graph obtained from $G \setminus V_0$ by deleting all edges in $G[A, B] - H$ and let $G' := G'' \cup \bigcup_{P \in \mathcal{P}'} P$. Note that H' is γ-close to bipartite (when viewed as a subgraph of G') and $\delta(H') \geq (1/2 + \eta)n$. We claim that H' is a bipartite robust (ν, τ)-expander with bipartition A, B. To see this, it suffices to show that H has this property. Consider any set $S \subseteq A$ with $\tau n \leq |S| \leq (1 - \tau)n$. Suppose first that $|S| \geq n/2$. Then every vertex in B has at least $\eta n \geq \nu n$ neighbours in S. So $RN_{\nu,H}(S) = B$. Thus we may assume that $|S| \leq n/2$. Let $N := RN_{\nu,H}(S) \cap B$. Then

$$(1/2 + \eta)n|S| \leq e_H(S, N) + e_H(S, B \setminus N) \leq |S||N| + \nu n^2 \leq |S||N| + \nu n|S|/\tau$$

and so $|N| \geq (1/2 + \eta - \nu/\tau)n \geq (1 + \eta)n/2 \geq |S| + \nu n$, as required. Since $V(H') = V(G')$ it follows that $\mathcal{U} = \{V(G')\}$ is a weak robust subpartition in G' with parameters $\gamma, \nu, \tau, 1/4, 0, 1$. Moreover, $\mathcal{P}'$ is a $\mathcal{U}$-tour with parameter γ. Lemma 6.2 with $\gamma, 1/4$ playing the roles of ρ, η implies that there is a Hamilton cycle C' in G' which contains $\mathcal{P}'$. C' corresponds to the required Hamilton cycle C in G. □

The remainder of this section is devoted to the proof of Lemma 6.2.

6.2. **Spanning path systems in robust expanders.** In this subsection, we prove Corollary 6.9, which states that when p is not too large, every robust expander G is Hamilton p-linked, i.e. given distinct vertices $y_1, y_1', \dots, y_p, y_p'$, there exist p vertex-disjoint paths joining y_i to y_i' for all $i \leq p$ such that together these paths cover all the vertices of G. This, combined with a bipartite analogue in the next subsection, will be the main tool in proving Lemma 6.2: the y_i and y_i' will be suitable endpoints of the paths in the $\mathcal{U}$-tour $\mathcal{P}$.

We now define an analogue of robust expansion for digraphs. Let $0 < \nu \leq \tau < 1$. Given any digraph G on n vertices and $S \subseteq V(G)$, the *ν-robust outneighbourhood* $RN^+_{\nu,G}(S)$ of S is the set of all those vertices of G which have at least νn inneighbours in S. G is called a *robust (ν, τ)-outexpander* if $|RN^+_{\nu,G}(S)| \geq |S| + \nu n$ for all $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1 - \tau)n$.

The next lemma is a directed analogue of Lemma 4.8. Its proof follows immediately from the definition.

Lemma 6.5. *Let $0 < \nu \ll \tau \ll 1$. Suppose that G is a digraph and $U \subseteq W \subseteq V(G)$ are such that $G[U]$ is a robust (ν, τ)-outexpander and $|U \setminus W| \leq \nu|U|/2$. Then $G[W]$ is a robust $(\nu/2, 2\tau)$-outexpander.*

The next lemma shows that the diameter of a robust outexpander is small. Again, it follows immediately from the definition of robust outexpansion.

Lemma 6.6. *Let $n \in \mathbb{N}$ and $0 < 1/n \ll \nu \ll \tau \ll \eta \leq 1$. Suppose that G is a robust (ν, τ)-outexpander on n vertices with $\delta^0(G) \geq \eta n$. Then, given any distinct vertices $x, y \in V(G)$, there exists a path P in G from x to y such that $|V(P)| \leq 1/\nu$.*

We will need the following result of Kühn, Osthus and Treglown [25], which states that a robust outexpander whose minimum degree is not too small contains a (directed) Hamilton cycle.

Theorem 6.7 ([25])**.** *Let $n \in \mathbb{N}$ and suppose that $0 < 1/n \ll \nu \leq \tau \ll \eta < 1$. Let G be a robust (ν, τ)-outexpander on n vertices with $\delta^0(G) \geq \eta n$. Then G contains a Hamilton cycle.*

We say that a digraph G is *p-ordered Hamilton* if, given $x_1, \dots, x_p \in V(G)$, G contains a Hamilton cycle which traverses $x_1, \dots, x_p$ in this order.

Corollary 6.8. *Let $n, p \in \mathbb{N}$ and suppose that $0 < 1/n \ll \nu \ll \tau \ll \eta < 1$ and $p \leq \nu^3 n$. Let G be a robust (ν, τ)-outexpander on n vertices with $\delta^0(G) \geq \eta n$. Then G is p-ordered Hamilton.*

Proof. Let $x_1, \dots, x_p \in V(G)$. We claim that we can find a path P in G joining x_1, x_p which traverses $x_1, \dots, x_p$ in this order and such that $|V(P)| \leq \nu n/2$. To see this, suppose for some $i \leq p-1$ we have found a path P_i joining x_1, x_i with $|V(P_i)| \leq 2i/\nu$ which traverses $x_1, \dots, x_i$ in this order and such that $x_{i+1}, \dots, x_p$ do not lie in P_i. Let $G_i := G \setminus ((V(P_i) \setminus \{x_i\}) \cup \{x_{i+2}, \dots, x_p\})$. Note that $n - |V(G_i)| \leq 2p/\nu \leq \nu n/2$. So Lemma 6.5 implies that G_i is a robust $(\nu/2, 2\tau)$-outexpander. Apply Lemma 6.6 with G_i, x_i, x_{i+1} playing the roles of G, x, y to obtain a path, which, when appended to P_i, gives a path P_{i+1} joining x_1, x_{i+1} which traverses $x_1, \dots, x_{i+1}$ in this order such that $x_{i+2}, \dots, x_p$ do not lie in P_{i+1} and $|V(P_{i+1})| \leq |V(P_i)| + 2/\nu \leq 2(i+1)/\nu$. Set $P := P_p$. This proves the claim.

Let G' be the graph obtained from $G \setminus V(P)$ by adding a new vertex z such that $N^-_{G'}(z) := N^-_{G \setminus V(P)}(x_1)$ and $N^+_{G'}(z) := N^+_{G \setminus V(P)}(x_p)$. Then $\delta^0(G') \geq \delta^0(G) - \nu n/2 \geq \eta |G'|/2$ and G' is a robust $(\nu/2, 2\tau)$-outexpander. Therefore we can apply Theorem 6.7 to find a directed Hamilton cycle in G'. This corresponds to a Hamilton cycle in G which traverses $x_1, \dots, x_p$ in this order. □

The following corollary states that robust (out)expanders are Hamilton p-linked provided that p is not too large.

Corollary 6.9. *Let $n, p \in \mathbb{N}$ and suppose that $0 < 1/n \ll \nu \ll \tau \ll \eta < 1$ and let $p \leq \nu^4 n$.*

(i) *Let G be a robust (ν, τ)-outexpander on n vertices with $\delta^0(G) \geq \eta n$. Then G is Hamilton p-linked.*
(ii) *Let H be a robust (ν, τ)-expander on n vertices with $\delta(H) \geq \eta n$. Then H is Hamilton p-linked.*

Proof. To prove (i), let $y_1, \dots, y_p, y'_1, \dots, y'_p \in V(G)$. Obtain G^* from G as follows. For each $1 \leq i \leq p$ (where indices are considered modulo p), replace the pair y_{i+1}, y'_i with a new vertex z_i such that $N^+_{G^*}(z_i) := N^+_G(y_{i+1})$ and $N^-_{G^*}(z_i) := N^-_G(y'_i)$. Then it is easy to see that G^* is a robust $(\nu/2, 2\tau)$-outexpander. Corollary 6.8 implies that G^* contains a

Hamilton cycle which traverses $z_1, \dots, z_p$ in this order. This corresponds to a collection $P_1, \dots, P_p$ of vertex-disjoint paths such that P_i joins y_i to y_i' and all the P_i together cover $V(G)$, proving (i).

To prove (ii), let G be the digraph obtained from H by replacing each edge xy with directed edges $\overrightarrow{xy}$ and $\overrightarrow{yx}$. Then G is a robust (ν, τ)-outexpander with $\delta^0(G) \geq \eta n$. Now (i) implies that G is Hamilton p-linked. For each $xy \in E(H)$, any path system in G uses at most one of $\overrightarrow{xy}, \overrightarrow{yx}$. So H is Hamilton p-linked. □

6.3. **Spanning path systems in bipartite robust expanders.** Given $p \in \mathbb{N}$ and a bipartite graph G with vertex classes A, B, we say that G is (A, B)-*Hamilton p-linked* if, given any $Y := \{y_1, y_1', y_2, y_2', \dots, y_p, y_p'\} \subseteq V(G)$ with $|Y \cap A| = |Y \cap B| = p$, we can find a set of vertex-disjoint paths joining y_i to y_i' in G such that together these paths cover all the vertices of G. Note that if G is (A, B)-Hamilton p-linked then it is balanced. In this subsection we show that, for p not too large, G is (A, B)-Hamilton p-linked when G is a balanced bipartite robust expander.

Given a balanced bipartite graph G with vertex classes A, B which contains a perfect matching M, we denote by G^* the *M-auxiliary digraph of G* obtained from G as follows. Let G^* have vertex set B. For each $v \in B$, we let v' be the unique vertex of A such that $vv' \in M$. Then, for all $x, v \in B$, we let $\overrightarrow{vx} \in E(G^*)$ if and only if $x \in N_G(v') \setminus \{v\}$. Note that the order of A and B matters here.

Lemma 6.10. *Let $n \in \mathbb{N}$ and $0 < 1/n \ll \nu \ll \tau \ll \eta < 1$. Let G be a balanced bipartite graph with vertex classes A, B so that $|A| = |B| = n$ and $\delta(G) \geq \eta n$. Suppose further that G is a bipartite robust (ν, τ)-expander (with bipartition A, B). Then*

(i) *G contains a perfect matching M;*
(ii) *the M-auxiliary digraph G^* of G is a robust (ν, τ)-outexpander with minimum degree at least $\eta n/2$.*

Proof. Observe that (i) follows immediately from Hall's Theorem. Write $M := \{xx' : x \in B, x' \in A\}$. To prove (ii), note that $\delta^0(G^*) \geq \delta(G) - 1 \geq \eta n/2$. Consider any $S \subseteq B$ with $\tau n \leq |S| \leq (1 - \tau)n$. Let $S_A := \{x' : x \in S\}$ and note that $RN^+_{\nu,G^*}(S) \supseteq RN_{\nu,G}(S_A)$. Thus

$$|RN^+_{\nu,G^*}(S)| \geq |RN_{\nu,G}(S_A)| \geq |S_A| + \nu|V(G)| \geq |S| + \nu|V(G^*)|,$$

and therefore G^* is a robust (ν, τ)-expander, proving (ii). □

We now prove an analogue of Lemma 6.6 for bipartite robust expanders.

Lemma 6.11. *Let $n \in \mathbb{N}$ and $0 < 1/n \ll \nu \ll \tau \ll \eta < 1$. Suppose that G is a bipartite graph on n vertices with vertex classes A, B, where $||A| - |B|| \leq \nu^2 n$. Suppose further that $\delta(G) \geq \eta n$ and G is a bipartite robust (ν, τ)-expander (with bipartition A, B). Then, given any distinct vertices $x, y \in V(G)$ there exists a path P between x and y in G such that $|V(P)| \leq 4/\nu$.*

Proof. Consider each $u \in \{x, y\}$. If $u \in B$, let u' be a neighbour of u which lies in A. If $u \in A$, let $u' := u$. Make these choices so that x', y' are distinct. So $\{x', y'\} \subseteq A$. Remove at most $||A| - |B|| \leq \nu|A|/2$ vertices from $A \cup B$ to obtain $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| = |B'|$ and $\{x', y'\} \subseteq A'$. Lemma 4.9(i) implies that $G' := G[A', B']$ is a bipartite robust $(\nu/2, 2\tau)$-expander and that $\delta(G') \geq \eta n'/2$ where $n' := |V(G')|$.

Let X_i be the set of vertices $v \in A'$ of distance at most $2i$ to x' in G'. Now Lemma 6.10(i) implies that G' contains a perfect matching M. So for all $i \geq 0$ we have $X_{i+1} \supseteq \{a \in A' : ab \in M, b \in N_{G'}(X_i)\}$. Thus $|X_1| \geq \eta n'/2$ and whenever $i \geq 1$ and $|X_i| < (1-\tau)n$ then

$$|X_{i+1}| \geq |N_{G'}(X_i)| \geq |RN_{\nu/2,G'}(X_i)| \geq |X_i| + \nu n'/2.$$

So certainly for $i' := \lfloor 2/\nu \rfloor - 4$ we have that $|X_{i'}| \geq (1-\tau)n'$. But since $\delta(G') \geq \eta n'/2 \geq \tau n'$ we have that $X_{i'+1} = A'$. In particular, this implies that there is a path of length at most $4/\nu - 5$ between x' and y' in G' and hence a path P with $|V(P)| \leq 4/\nu$ between x and y in G. □

The following is a bipartite analogue of Corollary 6.9. To prove it, we iterate Lemma 6.11 to find short paths between a small number of pairs of vertices. Then the graph obtained by deleting these paths is still a bipartite robust expander.

Lemma 6.12. *Let $n, p \in \mathbb{N}$, $0 < 1/n \ll \nu \ll \tau \ll \eta \leq 1$ and $p \leq \nu^4 n$. Suppose that G is a bipartite graph vertex classes A, B, so that $|A| = |B| = n$. Suppose further that G is a bipartite robust (ν, τ)-expander with $\delta(G) \geq \eta n$. Then G is (A, B)-Hamilton p-linked.*

Proof. Let $Y := \{y_1, y_1', y_2, y_2', \ldots, y_p, y_p'\}$ be a collection of distinct vertices in G such that $|Y \cap A| = |Y \cap B|$. For each $1 \leq i, j \leq p$, let $W_i := \{y_i, y_i'\}$ and let $W_{\geq j} := \bigcup_{j \leq i \leq p} W_i$. Suppose, for some $0 \leq \ell \leq p-2$, we have already obtained vertex-disjoint paths $R_1, \ldots, R_\ell$, where for each $1 \leq i \leq \ell$, R_i has endpoints y_i, y_i' and $|V(R_i)| \leq 8/\nu$. We obtain $R_{\ell+1}$ as follows. Let

$$G_\ell := G \setminus (V(R_1) \cup \ldots \cup V(R_{\ell-1}) \cup W_{\geq \ell+1})$$

and let $n_\ell := |V(G_\ell)|$. Note that

$$|V(G) \setminus V(G_\ell)| = \sum_{1 \leq i \leq \ell} |V(R_i)| + |W_{\geq \ell+1}| \leq 8p/\nu \leq \nu^2 n. \tag{6.1}$$

Let $A_\ell := A \cap V(G_\ell)$ and define B_ℓ analogously. Then Lemma 4.9(i) implies that G_ℓ is a bipartite robust $(\nu/2, 2\tau)$-expander with bipartition A_ℓ, B_ℓ, and $\delta(G_\ell) \geq \eta n_\ell/4$. Moreover $||A_\ell| - |B_\ell|| \leq |V(G) \setminus V(G_\ell)| \leq \nu^2 n \leq \nu^2 n_\ell$. Therefore we can apply Lemma 6.11 with $A_\ell, B_\ell, \nu/2, 2\tau, \eta/4$ playing the roles of A, B, ν, τ, η to see that G_ℓ contains a path $R_{\ell+1}$ between $y_{\ell+1}$ and $y_{\ell+1}'$ such that $|V(R_{\ell+1})| \leq 8/\nu$.

Therefore we can obtain vertex-disjoint paths $R_1, \ldots, R_{p-1}$ in $G \setminus \{y_p, y_p'\}$ such that $|V(R_i)| \leq 8/\nu$ and R_i joins y_i, y_i' for all $1 \leq i \leq p-1$. To obtain R_p, we now consider three cases depending on the classes in which y_p, y_p' lie. Let $V^* := \bigcup_{1 \leq i \leq p-1} V(R_i)$.

Case 1. *$y_p \in A$ and $y_p' \in B$.*

Using our assumption that $|Y \cap A| = |Y \cap B|$, it is easy to see that $|V^* \cap A| = |V^* \cap B|$. Let $G' := G \setminus \big(V^* \cup \{y_p, y_p'\}\big)$. Also let $A' := A \cap V(G')$ and define B' analogously. Then $|A'| = |B'| =: n'$. As above, G' is a bipartite robust $(\nu/2, 2\tau)$-expander with respect to A', B', and $\delta(G') \geq \eta(n'+1)/2$. Therefore G' contains a perfect matching M' by Lemma 6.10(i). Let $M'' := M' \cup \{y_p y_p'\}$. Then M'' is a perfect matching in the graph G^- obtained from $G \setminus V^*$ by adding the edge $y_p y_p'$ if necessary. Note that $|G^-| = 2(n'+1)$ and $\delta(G^-) \geq \eta(n'+1)/2$. Let G'' be the M''-auxiliary digraph of G^-. Then Lemma 6.10(ii) implies that G'' is a robust $(\nu/2, 2\tau)$-outexpander with minimum degree at least $\eta(n'+1)/4$. By Theorem 6.7, G'' contains a Hamilton cycle C. Then C corresponds to a Hamilton path R_p in $G \setminus V^*$ which joins y_p and y_p'. Thus $R_1, \ldots, R_p$ are vertex-disjoint from each

other, join y_i to y'_i, and together cover all the vertices of G. So G is (A, B)-Hamilton p-linked.

Case 2. $y_p, y'_p \in A$.

So it is easy to see that $|V^* \cap A| = |V^* \cap B| - 1$. Choose a neighbour z_p of y'_p in B which does not lie in V^*. Now delete y'_p from G and proceed as above with z_p playing the role of y'_p.

Case 3. $y_p, y'_p \in B$.

This is analogous to Case 2. □

6.4. **Proof of Lemma 6.2.** We are now ready to prove Lemma 6.2. Given a robust subpartition $\mathcal{U}$ in G and a $\mathcal{U}$-tour $\mathcal{P}$, we apply Corollary 6.9 within each robust expander component U of $\mathcal{U}$, with the endpoints of $\mathcal{P}$ which lie in U suitably ordered. Similarly, we apply Lemma 6.12 within each bipartite robust expander component Z of $\mathcal{U}$. In this way, we obtain a set $\mathcal{R}$ of 'joining paths'. Then together the paths in $\mathcal{P} \cup \mathcal{R}$ form a cycle containing every vertex of $\bigcup_{U \in \mathcal{U}} U$.

Proof of Lemma 6.2. Note that if $\nu' \leq \nu$, then any (bipartite) robust (ν, τ)-expander is also a (bipartite) robust (ν', τ)-expander. So without loss of generality, we may assume that $\nu \ll \tau$. Write $\mathcal{U} := \{U_1, \dots, U_k, Z_1, \dots, Z_\ell\}$ so that (D1′)–(D5′) are satisfied. Let $\mathcal{P}$ be a $\mathcal{U}$-tour with parameter γ, let $q := |\mathcal{P}|$ and $R := R_{\mathcal{U}}(\mathcal{P})$. So for each path $P \in \mathcal{P}$ there is a unique edge e_P in R. Without loss of generality, $e_{P_1} \dots e_{P_q}$ is the Euler tour guaranteed by (T2). This corresponds to an ordering $P_1, \dots, P_q$ of the paths in $\mathcal{P}$. Direct the edges of R so that $e_{P_1} \dots e_{P_q}$ is a directed tour. Direct the edges of (the paths in) $\mathcal{P}$ correspondingly, so that for all $1 \leq s \leq q$, if e_{P_s} has startpoint U and endpoint W, then P_s is a directed path from some vertex $x_s^- \in U$ to some vertex $x_s^+ \in W$. We thus obtain an ordering $x_1^+, x_2^-, x_2^+, \dots, x_q^-, x_q^+, x_1^-$ of the endpoints of $\mathcal{P}$. Note that for each $1 \leq i \leq q$, x_i^+, x_{i+1}^- lie in the same $X \in \mathcal{U}$, where the indices are considered modulo q.

Fix some $U \in \mathcal{U}$. Let $p := \mathrm{End}_{\mathcal{P}}(U)/2$. Thus $p \in \mathbb{N}$. Then there exists a subsequence $i_1, \dots, i_p$ of $1, \dots, q$ such that

$$K := (x_{i_1}^+, x_{i_1+1}^-, x_{i_2}^+, x_{i_2+1}^-, \dots, x_{i_p}^+, x_{i_p+1}^-)$$

is the subsequence of ordered endpoints of $\mathcal{P}$ which lie in U (where $x_{q+1}^- := x_1^-$). Let I be the (unordered) collection of internal vertices of $\mathcal{P}$ which lie in U. Let $U' := U \setminus I$. Note that each element of K lies in U'. Now (D4′) implies that $\delta(G[U]) \geq \eta n$. Furthermore, (T3) implies that $\mathrm{End}_{\mathcal{P}}(U) + \mathrm{Int}_{\mathcal{P}}(U) \leq \gamma n$. So

$$|U'| \overset{(T3)}{\geq} |U| - \gamma n \geq (\eta - \gamma)n \geq \eta n/2 \tag{6.2}$$

and hence

$$p = \mathrm{End}_{\mathcal{P}}(U)/2 \leq \gamma n/2 \leq \gamma |U'|/\eta \leq \sqrt{\gamma}|U'|; \tag{6.3}$$

$$\text{and } |I| = \mathrm{Int}_{\mathcal{P}}(U) \leq 2\gamma n \leq 4\gamma |U'|/\eta \leq \nu |U'|/5. \tag{6.4}$$

Suppose first that $U = U_i$ for some $1 \leq i \leq k$. Then U is a (ρ, ν, τ)-robust expander component. By (6.4), we may apply Lemma 4.8 with $U, U \setminus I$ playing the roles of U, U' to see that U' is a robust $(\nu/2, 2\tau)$-expander and $\delta(G[U']) \geq \eta n/2$. By (6.3) and Corollary 6.9,

$G[U']$ is Hamilton p-linked. So there is an ordered collection $\mathcal{R}_U$ of p vertex-disjoint paths in $G[U']$ spanning U' such that the jth path in $\mathcal{R}_U$ joins $x_{i_j}^+$ and $x_{i_j+1}^-$.

Suppose instead that $U = Z_i$ for some $1 \leq i \leq \ell$. Then there exists a bipartition A, B of U such that U is a bipartite (ρ, ν, τ)-robust expander component with bipartition A, B. Let $A' := A \setminus I$ and $B' := B \setminus I$. So A', B' is a bipartition of U'. Recall from (T4) that $\mathcal{P}$ is (A, B)-balanced. Thus $|A'| = |B'|$ and $\mathrm{End}_{\mathcal{P}}(A') = \mathrm{End}_{\mathcal{P}}(B') > 0$.

Let $n' := |A'|$. Note that (D5′) implies that $\delta(G[A, B]) \geq \eta n/2$. By (B1) and (C2) we have that $||A| - |B|| \leq \rho n$ and hence (6.2) implies that $|A| \geq 2|U'|/5$. Now (6.4) implies that $|I| \leq \nu|U'|/5 \leq \nu|A|/2$. So we may apply Lemma 4.9(i) to see that $G[U']$ is a bipartite robust $(\nu/2, 2\tau)$-expander with bipartition A', B', and that $\delta(G[A', B']) \geq \eta n'/4$. By (6.3) and Lemma 6.12, H is (A', B')-Hamilton p-linked. So there is an ordered collection $\mathcal{R}_U$ of p vertex-disjoint paths in H spanning U' such that the jth path in $\mathcal{R}_U$ joins $x_{i_j}^+$ and $x_{i_j+1}^-$.

Proceed in this way for each $U \in \mathcal{U}$ and let $\mathcal{R} := \bigcup_{U \in \mathcal{U}} \mathcal{R}_U$. Then for each $1 \leq i \leq q$, there exists exactly one path R_i in $\mathcal{R}$ which joins x_i^+ and x_{i+1}^- (with indices modulo q). Let

$$C := x_1^- P_1 x_1^+ R_1 x_2^- P_2 x_2^+ \dots x_p^- P_p x_p^+ R_p x_1^-.$$

Then C is a cycle in G which covers $\bigcup_{U \in \mathcal{U}} U$. □

7. The proof of Theorem 1.2

Our aim is to prove Theorem 1.2, i.e. that every sufficiently large 3-connected D-regular graph G on n vertices with $D \geq (1/4 + \varepsilon)n$ contains a Hamilton cycle. By Theorem 3.1 and Proposition 3.2(i), G has a robust partition $\mathcal{V}$ such that (k, ℓ) takes one of five values. By Corollary 6.3, to find a Hamilton cycle it suffices to find a $\mathcal{V}$-tour. We achieve this for each case. In the first subsection we consider the case $\ell = 0$ (so $1 \leq k \leq 3$), i.e. when G is a union of robust expander components. Then in Subsection 7.2 we prove some lemmas which are useful for the case when $\ell \geq 1$. Finally in Subsections 7.3 and 7.4 we consider the cases $(k, \ell) = (0, 1), (1, 1)$ respectively.

7.1. Finding $\mathcal{V}$-tours in a 3-connected graph with at most three robust expander components.

The main result of this section guarantees a $\mathcal{V}$-tour in a 3-connected graph G which has a robust partition $\mathcal{V}$ into at most three robust expander components.

Lemma 7.1. *Let $D, n \in \mathbb{N}$, let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \alpha < 1$ and let $D \geq \alpha n$. Suppose that G is a D-regular 3-connected graph on n vertices and that $\mathcal{V}$ is a robust partition of G with parameters $\rho, \nu, \tau, k, 0$ where $k \leq 3$. Then G contains a $\mathcal{V}$-tour with parameter $4/n$.*

We will use the following proposition which is an immediate consequence of Menger's Theorem.

Proposition 7.2. *Let $k \in \mathbb{N}$ and let G be a k-connected graph. Suppose that A is a subset of G with $|A|, |\overline{A}| \geq k$. Then there is a matching of size k between A and $\overline{A}$.*

Lemma 7.1 is an immediate corollary of the following lemma. To see this, note that (T4) is vacuous here.

Lemma 7.3. *Let G be a 3-connected graph and let $\mathcal{V}$ be a partition of $V(G)$ into at most three parts, where $|V| \geq 3$ for each $V \in \mathcal{V}$. Then G contains a path system $\mathcal{P}$ such that*

(i) *$e(\mathcal{P}) \leq 4$ and $\mathcal{P} \subseteq \bigcup_{V \in \mathcal{V}} G[V, \overline{V}]$;*

(ii) *$R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour;*
(iii) *for each $V \in \mathcal{V}$, if c_i is the number of vertices in V with degree i in $\mathcal{P}$ (for $i = 1, 2$), then $c_1 + 2c_2 \in \{2, 4\}$ and $c_2 \leq 1$.*

Proof. Suppose first that $|\mathcal{V}| = 1$. Let $\mathcal{P}$ consist of a single arbitrary edge. So (i) and (iii) are clear. Then $R_{\mathcal{V}}(\mathcal{P})$ is a loop, so (ii) holds.

Suppose instead that $|\mathcal{V}| = 2$ and write $\mathcal{V} := \{V, W\}$. Then Proposition 7.2 implies that G contains a matching $\mathcal{P}$ of size two between V and W. So (i) holds. In this case, $R_{\mathcal{V}}(\mathcal{P})$ consists of exactly two VW-edges, so (ii) holds. Moreover, for each $V \in \mathcal{V}$ we have $(c_1, c_2) = (2, 0)$, implying (iii).

Suppose finally that $|\mathcal{V}| = 3$ and write $\mathcal{V} := \{V_1, V_2, V_3\}$. We write M_{ij} for a matching between V_i and V_j. Given a path system $\mathcal{P}$ in G, we write c_i^j for the number of vertices in V_j with degree i in $\mathcal{P}$. Proposition 7.2 implies that there is a matching of size three between V_1 and $V_2 \cup V_3$. Without loss of generality, choose M_{12} such that $|M_{12}| = 2$. By Proposition 7.2 there is a matching of size three between V_3 and $V_1 \cup V_2$. Therefore there exist vertex-disjoint M_{13}, M_{23} such that $|M_{13}| + |M_{23}| = 3$. Throughout the remainder of the proof, we will let u_1, v_1, w_1, x_1 be distinct vertices in V_1 and we will label vertices in other classes similarly.

Case 1. $|M_{13}| = 3$.

If M_{13} contains two edges e, e' that are vertex-disjoint from M_{12}, then we let $\mathcal{P}$ have edge-set $\{e, e'\} \cup M_{12}$. So (i) holds. Note that $R_{\mathcal{V}}(\mathcal{P})$ consists of precisely two V_1V_2-edges and two V_1V_3-edges. Therefore (ii) holds. Moreover, $(c_1^1, c_2^1) = (4, 0)$ and $(c_1^j, c_2^j) = (2, 0)$ for $j = 2, 3$, implying (iii).

Otherwise, M_{13} contains exactly two edges that share endpoints with edges in M_{12}. Without loss of generality, let $M_{13} := \{u_1u_3, v_1v_3, w_1w_3\}$ and $M_{12} := \{u_1u_2, v_1v_2\}$. In this case, let $\mathcal{P} := \{u_1u_2, v_2v_1v_3, w_1w_3\}$. (i) is immediate, and $R_{\mathcal{V}}(\mathcal{P}) \cong C_3$ so (ii) holds. Moreover, $(c_1^1, c_2^1) = (2, 1)$ and $(c_1^j, c_2^j) = (2, 0)$ for $j = 2, 3$, implying (iii).

Case 2. *Without loss of generality, $|M_{13}| = 2$ and $|M_{23}| = 1$.*

Let v_2v_3 be the edge in M_{23}. Since $|M_{12}| = |M_{13}| = 2$ we can pick edges $w_1w_2 \in M_{12}$ and $x_1x_3 \in M_{13}$ so that $w_2 \neq v_2$ and $x_1 \neq w_1$. But M_{13} and M_{23} are vertex-disjoint, so $x_3 \neq v_3$. In this case, we let $\mathcal{P} := \{w_1w_2, v_2v_3, x_3x_1\}$. (i) is immediate, and $R_{\mathcal{V}}(\mathcal{P}) \cong C_3$ so (ii) holds. Moreover, $(c_1^j, c_2^j) = (2, 0)$ for all $V \in \mathcal{V}$, implying (iii). This completes the proof of the case $|\mathcal{V}| = 3$. □

7.2. **Finding an (A, B)-balanced path system in a bipartite robust expander.** In Section 6 we showed that, given a robust partition $\mathcal{V}$, 'the balancing property' (T4) was sufficient to extend a $\mathcal{V}$-tour into a Hamilton cycle. In this section we prove some lemmas which will be useful in finding a path system which satisfies (T4). We begin by observing the following crucial fact.

Proposition 7.4. *Let G be a D-regular graph with vertex partition A, B, V. Then*

(i) $2(e(A) - e(B)) + e(A, V) - e(B, V) = (|A| - |B|)D$.

In particular,

(ii) $2e(A) + e(A, V) \geq (|A| - |B|)D$;
(iii) *if $V = \emptyset$ then* $2(e(A) - e(B)) = (|A| - |B|)D$.

Proof. It suffices to prove (i) since (ii) and (iii) are then immediate. We have that

$$\sum_{x\in A} d_B(x) = e(A,B) = \sum_{y\in B} d_A(y).$$

Moreover, by counting degrees,

$$2e(A) + e(A,V) = \sum_{x\in A}(D - d_B(x)) = D|A| - \sum_{x\in A} d_B(x),$$

and similarly for B. So $2e(A) - 2e(B) + e(A,V) - e(B,V) = D(|A| - |B|)$, as desired. □

The following proposition follows immediately from Vizing's Theorem on edge-colourings.

Proposition 7.5. *Let H be a graph with $\Delta(H) \le \Delta$. Then H contains a matching of size $\lceil e(H)/(\Delta+1)\rceil$.*

Given a graph G, a collection $\mathcal{U}$ of disjoint subsets of $V(G)$ and a $\mathcal{U}$-anchored path system $\mathcal{P}$ in G, we say that a path system $\mathcal{P}'$ is a *$\mathcal{U}$-extension of $\mathcal{P}$* if

- every edge which lies in a path of $\mathcal{P}'$ but not a path of $\mathcal{P}$ lies in $\bigcup_{U\in\mathcal{U}} G[U]$;
- for every $P \in \mathcal{P}$ there is a unique $P' \in \mathcal{P}'$ such that $P \subseteq P'$;
- for every $P' \in \mathcal{P}'$ there is at most one $P \in \mathcal{P}$ such that $P \subseteq P'$.

If $U \subseteq V(G)$ we will write *U-extension* for $\{U\}$-extension. The next lemma shows that a $\mathcal{U}$-extension $\mathcal{P}'$ of $\mathcal{P}$ 'behaves similarly' to $\mathcal{P}$ in the reduced multigraph $R_{\mathcal{U}}$, and also that $R_{\mathcal{U}}$ is not affected by considering a slightly different partition.

Lemma 7.6. *Let $\mathcal{U}$ be a collection of disjoint vertex-subsets of a graph G and let $\mathcal{P}$ be a $\mathcal{U}$-anchored path system in G.*

(i) *Suppose that $\mathcal{P}'$ is a $\mathcal{U}$-extension of $\mathcal{P}$. Then $\mathcal{P}'$ is a $\mathcal{U}$-anchored path system.*
(ii) *Suppose that $\mathcal{P}'$ is an $\mathcal{X}$-extension of $\mathcal{P}$ for some $\mathcal{X} \subseteq \mathcal{U}$. Then $\mathcal{P}'$ is a $\mathcal{U}$-extension of $\mathcal{P}$.*
(iii) *Suppose that $\mathcal{P}'$ is a $\mathcal{U}$-extension of $\mathcal{P}$. Then $R_{\mathcal{U}}(\mathcal{P}')$ is an Euler tour if and only if $R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour.*
(iv) *Suppose that $\mathcal{U} := \{U_1,\dots,U_t\}$, $\mathcal{X} := \{X_1,\dots,X_t\}$, $X_i \subseteq U_i$ for all $1 \le i \le t$, and $\mathcal{P}$ is $\mathcal{X}$-anchored. Then $R_{\mathcal{X}}(\mathcal{P}) \cong R_{\mathcal{U}}(\mathcal{P})$.*

Proof. Note that (i), (ii) and (iv) are immediate. To prove (iii), let $\mathcal{R}$ be the subset of $\mathcal{P}'$ such that every $R \in \mathcal{R}$ contains some $P_R \in \mathcal{P}$. So $|\mathcal{R}| = |\mathcal{P}|$. Observe that P_R has endpoints in $U, U' \in \mathcal{U}$ if and only if R has endpoints in U, U'. So $R_{\mathcal{U}}(\mathcal{R}) \cong R_{\mathcal{U}}(\mathcal{P})$. Let $\mathcal{Q} := \mathcal{P}' \setminus \mathcal{R}$. Then every edge in a path in $\mathcal{Q}$ lies in $\bigcup_{U\in\mathcal{U}} G[U]$. So $R_{\mathcal{U}}(\mathcal{Q})$ consists entirely of loops. Therefore $R_{\mathcal{U}}(\mathcal{P}') = R_{\mathcal{U}}(\mathcal{R}) \cup R_{\mathcal{U}}(\mathcal{Q})$ is an Euler tour if and only if $R_{\mathcal{U}}(\mathcal{R})$ is, i.e. if and only if $R_{\mathcal{U}}(\mathcal{P})$ is. This proves (iii). □

Suppose that $A, B \subseteq V(G)$ are disjoint. The following lemma gives a sufficient condition which ensures that a path system $\mathcal{P}$ can be extended into an (A,B)-balanced path system which does not cover too much of $A \cup B$. Whenever we wish to find a balanced path system we need then only find a collection of paths which satisfy this condition.

Lemma 7.7. *Let $n \in \mathbb{N}$ and $0 < 1/n \ll \rho < 1$ and suppose that G is a graph on n vertices. Let $U \subseteq V(G)$ have bipartition A, B where $||A| - |B|| \le \rho n$ and $\delta(G[A,B]) > 9\rho n$. Let $\mathcal{P}$ be a path system in G such that $|V(\mathcal{P}) \cap U| \le \rho n$,*

$$2e_{\mathcal{P}}(A) - 2e_{\mathcal{P}}(B) + e_{\mathcal{P}}(A,\overline{U}) - e_{\mathcal{P}}(B,\overline{U}) = 2(|A| - |B|) \tag{7.1}$$

and $\mathcal{P}$ has at least one endpoint in U. Then G contains a path system $\mathcal{P}'$ such that

(α) *$\mathcal{P}'$ is a U-extension of $\mathcal{P}$;*
(β) *$\mathcal{P}'$ is (A,B)-balanced;*
(γ) *$|V(\mathcal{P}') \cap U| \leq 9\rho n$.*

Proof. Without loss of generality, suppose that $|A| \geq |B|$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be minimal such that $V(\mathcal{P}) \cap U \subseteq A_0 \cup B_0$ and

$$|A_0| - |B_0| = |A| - |B|. \tag{7.2}$$

Note that

$$|A_0| + |B_0| = |A| - |B| + 2|B_0| \leq ||A| - |B|| + 2|V(\mathcal{P}) \cap U| \leq 3\rho n. \tag{7.3}$$

For each $u \in A_0$, find a set N_u of $2 - d_{\mathcal{P}}(u)$ neighbours of u in $B \setminus B_0$. For each $v \in B_0$, find a set N_v of $2 - d_{\mathcal{P}}(v)$ neighbours of v in $A \setminus A_0$. Choose these sets to be disjoint and such that $(N_u \cup N_v) \cap V(\mathcal{P}) = \emptyset$. This is possible since for each $u \in A$ and $v \in B$ we have $d_B(u), d_A(v) > 3(|A_0| + |B_0|)$. Obtain $\mathcal{P}'$ from $\mathcal{P}$ by adding the edges xx' to (the paths in) $\mathcal{P}$ for each $x \in A_0 \cup B_0$ and for each $x' \in N_x$. It is clear that $\mathcal{P}'$ is a U-extension of $\mathcal{P}$, so (α) holds.

Note that the set of internal vertices of $\mathcal{P}'$ which lie in U is precisely $A_0 \cup B_0$. Then $\mathrm{Int}_{\mathcal{P}'}(A) - \mathrm{Int}_{\mathcal{P}'}(B) = |A_0| - |B_0| = |A| - |B|$ by (7.2). So to show (β), it is enough to check that $\mathrm{End}_{\mathcal{P}'}(A) = \mathrm{End}_{\mathcal{P}'}(B)$ and that this value is non-zero. Since

$$\sum_{u \in A} d_{\mathcal{P}'}(u) = 2e_{\mathcal{P}'}(A) + e_{\mathcal{P}'}(A,B) + e_{\mathcal{P}'}(A,\overline{U}) = 2e_{\mathcal{P}}(A) + e_{\mathcal{P}'}(A,B) + e_{\mathcal{P}}(A,\overline{U}),$$

and similarly for B, we have that

$$\begin{aligned} \sum_{u \in A} d_{\mathcal{P}'}(u) - \sum_{v \in B} d_{\mathcal{P}'}(v) &= 2e_{\mathcal{P}}(A) - 2e_{\mathcal{P}}(B) + e_{\mathcal{P}}(A,\overline{U}) - e_{\mathcal{P}}(B,\overline{U}) \\ &\overset{(7.1)}{=} 2(|A| - |B|). \end{aligned} \tag{7.4}$$

By construction, $\sum_{u \in A} d_{\mathcal{P}'}(u) = \mathrm{End}_{\mathcal{P}'}(A) + 2|A_0|$, and similarly for B. So, by (7.4),

$$\mathrm{End}_{\mathcal{P}'}(A) - \mathrm{End}_{\mathcal{P}'}(B) = 2(|A| - |B|) - 2(|A_0| - |B_0|) \overset{(7.2)}{=} 0. \tag{7.5}$$

Recall that $\mathcal{P}$ has at least one endpoint x lying in U. Then $|N_x| = 1$ and the vertex in N_x is an endpoint of a path in $\mathcal{P}'$. So $\mathrm{End}_{\mathcal{P}'}(A) = \mathrm{End}_{\mathcal{P}'}(B)$ is non-zero, proving (β).

Finally, note that every vertex in $V(\mathcal{P}') \cap U$ which does not lie in $A_0 \cup B_0$ is a neighbour of some $x \in A_0 \cup B_0$ in $\mathcal{P}'$. So (7.3) implies that

$$|V(\mathcal{P}') \cap U| \leq |A_0 \cup B_0| + |N_{\mathcal{P}'}(A_0 \cup B_0)| \leq 3(|A_0| + |B_0|) \leq 9\rho n,$$

proving (γ). □

The next lemma is essentially an iteration of Lemma 7.7. We will use it to successively extend a path system into one that is (A,B)-balanced for all appropriate A, B.

Lemma 7.8. *Let $n, k, \ell \in \mathbb{N}$ and $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \eta < 1$. Let G be a graph on n vertices and suppose that $\mathcal{U} := \{U_1, \dots, U_k, W_1, \dots, W_\ell\}$ is a weak robust subpartition in G with parameters $\rho, \nu, \tau, \eta, k, \ell$. For each $1 \leq j \leq \ell$, let A_j, B_j be the bipartition of W_j specified by* (D3′). *Let $\mathcal{P}$ be a $\mathcal{U}$-anchored path system such that for each $1 \leq j \leq \ell$,*

$$2e_{\mathcal{P}}(A_j) - 2e_{\mathcal{P}}(B_j) + e_{\mathcal{P}}(A_j, \overline{W_j}) - e_{\mathcal{P}}(B_j, \overline{W_j}) = 2(|A_j| - |B_j|). \tag{7.6}$$

Suppose further that $|V(\mathcal{P})\cap U| \le \rho n$ for all $U \in \mathcal{U}$, and that $R_{\mathcal{U}}(\mathcal{P})$ is a non-empty Euler tour. Then G contains a $\mathcal{U}$-extension $\mathcal{P}'$ of $\mathcal{P}$ that is a $\mathcal{U}$-tour with parameter 9ρ.

Proof. Let $\mathcal{P}_0 := \mathcal{P}$. Suppose that for some $0 \le i < \ell$, we have already defined a path system $\mathcal{P}_i$ such that

- (α_i) $\mathcal{P}_i$ is a $\{W_1, \dots, W_i\}$-extension of $\mathcal{P}$;
- (β_i) for all $1 \le j \le i$, $\mathcal{P}_i$ is (A_j, B_j)-balanced;
- (γ_i) for all $1 \le j \le i$, $|V(\mathcal{P}_i) \cap W_j| \le 9\rho n$.

Now we obtain $\mathcal{P}_{i+1}$ from $\mathcal{P}_i$ as follows. Note that (D3′) implies that (B1) and (C2) hold and hence that $||A_{i+1}| - |B_{i+1}|| \le \rho n$. Moreover, by (D5′) we have that $\delta(G[A_j, B_j]) \ge \eta n/2 > 9\rho n$. Also (α_i) implies that $|V(\mathcal{P}_i)\cap W_{i+1}| = |V(\mathcal{P})\cap W_{i+1}| \le \rho n$ and that (7.6) still holds with $i+1$ and $\mathcal{P}_i$ playing the roles of j and $\mathcal{P}$. Finally, $R_{\mathcal{U}}(\mathcal{P})$ is a non-empty Euler tour, so $\mathcal{P}$ contains at least one endpoint in W_{i+1}. Thus $\mathcal{P}_i$ contains at least one endpoint in W_{i+1} by (α_i). Therefore we can apply Lemma 7.7 with $W_{i+1}, A_{i+1}, B_{i+1}, \mathcal{P}_i, \rho$ playing the roles of $U, A, B, \mathcal{P}, \rho$. We thus obtain a path system $\mathcal{P}_{i+1}$ satisfying Lemma 7.7(α)–(γ). Now (α) and (α_i) imply that (α_{i+1}) holds. We obtain (β_{i+1}) and (γ_{i+1}) in a similar way.

Therefore we can obtain $\mathcal{P}' := \mathcal{P}_\ell$ that satisfies (α_ℓ)–(γ_ℓ). Now (α_ℓ) and Lemma 7.6(ii) imply that $\mathcal{P}'$ is a $\mathcal{U}$-extension of $\mathcal{P}$. It remains to show that (T1)–(T4) hold for $\mathcal{P}'$ with 9ρ playing the role of γ. Indeed, (T1) follows from Lemma 7.6(i) and the fact that $\mathcal{P}'$ is a $\mathcal{U}$-extension of $\mathcal{P}$. Since $\mathcal{R}_{\mathcal{U}}(\mathcal{P})$ is an Euler tour, Lemma 7.6(iii) implies that $\mathcal{R}_{\mathcal{U}}(\mathcal{P}')$ is an Euler tour, and hence (T2) holds. We have $|V(\mathcal{P}') \cap W_j| \le 9\rho n$ for all $1 \le j \le \ell$ by (γ_ℓ). Moreover, by (α_ℓ) we have that $|V(\mathcal{P}') \cap U_j| = |V(\mathcal{P}) \cap U_j| \le \rho n$ for all $1 \le j \le k$. So (T3) holds. Finally, (T4) is immediate from (β_ℓ). □

7.3. **Finding a $\mathcal{V}$-tour in a regular bipartite robust expander.** We now consider the case when G has a robust partition with $(k, \ell) = (0, 1)$, i.e. G is a regular bipartite robust expander. By Corollary 6.3, in order to find a Hamilton cycle in G it suffices to find a $\{V(G)\}$-tour with an appropriate parameter. This is guaranteed by the following lemma.

Lemma 7.9. *Let $D, n \in \mathbb{N}$ and let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \alpha < 1$. Let G be a D-regular graph on n vertices where $D \ge \alpha n$. Suppose that G has a robust partition $\mathcal{V}$ with parameters $\rho, \nu, \tau, 0, 1$. Then G contains a $\mathcal{V}$-tour with parameter 18ρ.*

Proof. Note (D3) implies that there exists a bipartition A, B of $V(G)$ such that G is a bipartite (ρ, ν, τ)-expander with bipartition A, B. By (D5) we have that $\delta(G[A, B]) \ge D/2$. Therefore

$$\Delta(G[A]), \Delta(G[B]) \le D/2. \tag{7.7}$$

Moreover, (B1) (which follows from (D3)) implies that G is ρ-close to bipartite with bipartition A, B. So (C2) holds, i.e.

$$||A| - |B|| \le \rho n. \tag{7.8}$$

Suppose first that $|A| = |B|$. Then let $\mathcal{P}$ consist of exactly one AB-edge. Note that $R_{\mathcal{V}}(\mathcal{P})$ is a loop and that $\mathcal{P}$ is (A, B)-balanced. All of (T1)–(T4) hold.

Let us now assume that $|A| > |B|$ (the case where $|B| > |A|$ is similar). Proposition 7.4(iii) implies that

$$e(A) \ge e(A) - e(B) = (|A| - |B|)D/2. \tag{7.9}$$

Proposition 7.5 implies that $G[A]$ contains a matching of size

$$\left\lceil \frac{e(A)}{\Delta(A)+1} \right\rceil \overset{(7.7),(7.9)}{\geq} \left\lceil \frac{(|A|-|B|)D/2}{D/2+1} \right\rceil = |A|-|B| - \left\lfloor \frac{|A|-|B|}{D/2+1} \right\rfloor$$
$$\overset{(7.8)}{\geq} |A|-|B|-\lfloor 2\rho/\alpha \rfloor = |A|-|B|.$$

So we can choose a matching M of size $|A|-|B|$ in $G[A]$.

Now Proposition 6.1(i) implies that $\mathcal{V}$ is a weak robust subpartition in G with parameters $\rho, \nu, \tau, \alpha^2/2, 0, 1$. Certainly M is $\mathcal{V}$-anchored and

$$2e_M(A) - 2e_M(B) + e_M(A, \overline{V(G)}) - e_M(B, \overline{V(G)}) = 2e_M(A) = 2(|A|-|B|).$$

We also have that $|V(M)| = 2(|A|-|B|) \leq 2\rho n$. Moreover, M is non-empty since $|A|-|B| > 0$. Thus $R_{\mathcal{V}}(M)$ is a non-empty collection of loops and hence a non-empty Euler tour. Therefore we can apply Lemma 7.8 with $\mathcal{V}, 0, 1, V(G), A, B, M, 2\rho, \alpha^2/2$ playing the roles of $\mathcal{U}, k, \ell, W_j, A_j, B_j, \mathcal{P}, \rho, \eta$ to obtain a path system $\mathcal{P}$ which is a $\mathcal{V}$-tour with parameter 18ρ. □

7.4. **Finding a $\mathcal{V}$-tour when there is exactly one component of each type.** We would like to find a Hamilton cycle when G is the union of a robust expander component V and a bipartite robust expander component W. By Corollary 6.3, it is sufficient to find a $\mathcal{V}$-tour for this robust partition $\mathcal{V}$. This is guaranteed by the following lemma.

Lemma 7.10. *Let $n, D \in \mathbb{N}$, $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \alpha < 1$ and let $D \geq \alpha n$. Suppose that G is a 3-connected D-regular graph on n vertices and that $\mathcal{V}$ is a robust partition of G with parameters $\rho, \nu, \tau, 1, 1$. Then G contains a $\mathcal{V}$-tour with parameter 36ρ.*

Let V, W be as above and let A, B be a bipartition of W such that W is a bipartite robust expander with respect to A, B. Suppose that $|A| \geq |B|$. To prove Lemma 7.10, our aim is to find a path system $\mathcal{P}$ to which we can apply Lemma 7.8 and hence obtain a $\mathcal{V}$-tour. Roughly speaking, $\mathcal{P}$ will consist of the union of two matchings, M_A in $G[A]$ and $M_{A,V}$ in $G[A, V]$ which together have the right size to 'balance' W.

Proof of Lemma 7.10. Let $\mathcal{V} := \{V, W\}$, where V is a (ρ, ν, τ)-robust expander component and W has bipartition A, B so that W is a bipartite (ρ, ν, τ)-robust expander component with respect to A, B. So (B1) and (C2) imply that

$$||A|-|B|| \leq \rho n. \tag{7.10}$$

Moreover, (D4) implies that $\delta(G[V]), \delta(G[W]) \geq D/2$ and therefore

$$D/2 \geq \Delta(G[W, V]) \geq \Delta(G[A, V]). \tag{7.11}$$

By (D5) we have

$$\Delta(G[A]) \leq D/2. \tag{7.12}$$

Claim 1. *It suffices to find a path system $\mathcal{P}$ in G such that the following hold:*

(i) $2e_{\mathcal{P}}(A) - 2e_{\mathcal{P}}(B) + e_{\mathcal{P}}(A, V) - e_{\mathcal{P}}(B, V) = 2(|A|-|B|)$;
(ii) $e(\mathcal{P}) \leq 2\rho n$;
(iii) *$\mathcal{P}$ has at least one VW-path.*

To see this, note that Proposition 6.1(i) implies that $\mathcal{V}$ is a weak robust subpartition in G with parameters $\rho, \nu, \tau, \alpha^2/2, 1, 1$. Clearly, $\mathcal{P}$ is a $\mathcal{V}$-anchored path system. Observe that (D5) implies that $\delta(G[A,B]) \geq D/4$. Let p be the number of VW-paths in $\mathcal{P}$. Then $R_{\mathcal{V}}(\mathcal{P})$ is an Euler tour if and only if p is positive and even. By (iii) we have $p > 0$. Now (i) implies that

$$e_{\mathcal{P}}(W,V) = e_{\mathcal{P}}(A,V) + e_{\mathcal{P}}(B,V) = 2(|A| - |B|) - 2e_{\mathcal{P}}(A) + 2e_{\mathcal{P}}(B) + 2e_{\mathcal{P}}(B,V)$$

is even. Note that any $P \in \mathcal{P}$ contains an odd number of VW-edges if P is a VW-path, and an even number otherwise. Therefore p is even and so $R_{\mathcal{V}}(\mathcal{P})$ is a non-empty Euler tour. Finally, for each $X \in \mathcal{V}$ we have $|V(\mathcal{P}) \cap X| \leq 2e(\mathcal{P}) \leq 4\rho n$ by (ii). Therefore we can apply Lemma 7.8 with $\mathcal{V}, 1, 1, W, A, B, \mathcal{P}, 4\rho, \alpha^2/2$ playing the roles of $\mathcal{U}, k, \ell, W_j, A_j, B_j, \mathcal{P}, \rho, \eta$ to find a $\mathcal{V}$-extension $\mathcal{P}'$ of $\mathcal{P}$ that is a $\mathcal{V}$-tour with parameter 36ρ, proving the claim.

So it remains to find a path system $\mathcal{P}$ as in Claim 1. Suppose first that $|A| = |B|$. Since G is 3-connected, Proposition 7.2 implies that $G[V,W]$ contains a matching of size three. We only consider the case when $G[A,V]$ contains a matching $M_{A,V}$ of size two. (The case when this holds for $G[B,V]$ is similar.) Now Proposition 7.4(i) implies that

$$2e(B) + e(B,V) = 2e(A) + e(A,V) \geq 2.$$

If $e(B) \geq 1$, let $\mathcal{P} := M_{A,V} \cup \{e\}$, where e is an edge in $G[B]$. Otherwise, $e(B) = 0$ and hence $e(B,V) \geq 2$. In this case we let $\mathcal{P}$ consist of two vertex-disjoint edges $e \in G[A,V]$ and $e' \in G[B,V]$. In both cases, (i)–(iii) clearly hold for $\mathcal{P}$ and we are done.

So let us assume that $|A| > |B|$. (The case when $|B| > |A|$ is similar.) Proposition 7.4(ii) implies that

$$2e(A) + e(A,V) \geq (|A| - |B|)D. \tag{7.13}$$

Suppose first that $e(A) < D/5$. Then (7.13) implies that $e(A,V) \geq (|A|-|B|)D - 2D/5$. Now Proposition 7.5 implies that $G[A,V]$ contains a matching of size at least

$$\begin{aligned} \left\lceil \frac{e(A,V)}{\Delta(G[A,V])+1} \right\rceil &\overset{(7.11)}{\geq} \left\lceil \frac{(|A|-|B|)D - 2D/5}{D/2+1} \right\rceil \\ &= 2(|A|-|B|) - \left\lfloor \frac{2(|A|-|B|) + 2D/5}{D/2+1} \right\rfloor \\ &\overset{(7.10)}{\geq} 2(|A|-|B|) - \left\lfloor \frac{D/2}{D/2+1} \right\rfloor = 2(|A|-|B|). \end{aligned} \tag{7.14}$$

Let $\mathcal{P}$ be a matching of size $2(|A|-|B|)$ in $G[A,V]$. Then $\mathcal{P}$ satisfies (i)–(iii) (indeed, (ii) follows from (7.10)).

Therefore we can assume that $e(A) \geq D/5$. Let

$$\ell := \min\left\{ \left\lceil \frac{e(A)}{D/2+1} \right\rceil, |A| - |B| \right\}. \tag{7.15}$$

Note that $\ell \geq 1$. Clearly $G[A]$ contains a matching of size ℓ by Proposition 7.5 and (7.12). We now consider two cases, depending on the value of ℓ.

Case 1. $\ell = |A| - |B|$.

Let M be a matching of size ℓ in $G[A]$. Since G is 3-connected, Proposition 7.2 implies that $G[V,W]$ contains a matching of size three. Suppose first that $G[A,V]$ contains a matching $M_{A,V}$ of size two. Write $V(M_{A,V}) \cap A := \{u, u'\}$. If uu' is an edge in M,

delete it to obtain M'. Otherwise delete an arbitrary edge from M to obtain M'. Let $\mathcal{P} := M' \cup M_{A,V}$. Then $\mathcal{P}$ is a path system satisfying (i). Also (ii) follows from (7.10). Moreover, u lies in a VW-path in $\mathcal{P}$, so (iii) holds.

So suppose that $G[A,V]$ does not contain a matching of size two. Then $G[B,V]$ contains a matching $M_{B,V}$ of size two. Moreover, there is at most one vertex in $A \cup V$ such that every edge in $G[A,V]$ is incident to this vertex. Therefore (7.11) implies that $e(A,V) \leq \Delta(G[A,V]) \leq D/2$. So

$$e(A) - |M| \overset{(7.13)}{\geq} (|A| - |B|)D/2 - D/4 - |M| \geq D/4 - 1 > 0,$$

where the penultimate inequality follows from the fact that $|M| = |A| - |B| > 0$. So we can find an edge e in $G[A]$ that is not contained in M. Let $\mathcal{P} := M_{B,V} \cup M \cup \{e\}$. Then $\mathcal{P}$ is a path system satisfying (i)–(iii). This completes the proof of Case 1.

Case 2. *$\ell < |A| - |B|$ and so $\ell = \lceil e(A)/(D/2+1) \rceil$.*

Claim 2. *Suppose that $G[A]$ contains no matching of size $\ell+1$. Then $G[A]$ contains a matching M^- of size $\ell - 1$ and a path $P := xyz$ which is vertex-disjoint from M^-.*

To see this, suppose first that $\Delta(G[A]) \leq D/8 - 1$. Then Proposition 7.5 implies that $G[A]$ contains a matching of size

$$\left\lceil \frac{e(A)}{D/8} \right\rceil = \left\lceil \frac{e(A)}{D/3} + \frac{5e(A)}{D} \right\rceil \geq \left\lceil \frac{e(A)}{D/3} + 1 \right\rceil \geq \ell + 1, \tag{7.16}$$

a contradiction. So $\Delta(G[A]) > D/8 - 1 > 2\ell$ by (7.10) and (7.15). Recall that $G[A]$ contains a matching M of size ℓ. Since M must be maximal, there is some $y \in V(M)$ such that $d_A(y) > 2\ell$. Let $x \in A$ be a neighbour of y such that $x \notin V(M)$. Let z be the neighbour of y in M. Let $M^- := M \setminus \{yz\}$ and $P := xyz$. This completes the proof of Claim 2.

Proposition 7.5 implies that $G[A,V]$ contains a matching of size

$$\begin{aligned}
\left\lceil \frac{e(A,V)}{\Delta(G[A,V])+1} \right\rceil &\overset{(7.11)}{\geq} \left\lceil \frac{e(A,V)}{D/2+1} \right\rceil + 2\left\lceil \frac{e(A)}{D/2+1} \right\rceil - 2\ell \\
&\geq \left\lceil \frac{2e(A)+e(A,V)}{D/2+1} \right\rceil - 2\ell \\
&\overset{(7.13)}{\geq} \left\lceil \frac{(|A|-|B|)D}{D/2+1} \right\rceil - 2\ell \geq 2(|A|-|B|-\ell),
\end{aligned}$$

where the final inequality follows in a similar way to (7.14). So we can choose a matching $M_{A,V}$ in $G[A,V]$ of size $2(|A| - |B| - \ell) > 0$.

Let E be any collection of ℓ edges in $G[A]$ and let $H := E \cup M_{A,V}$. Then

$$2e_H(A) - 2e_H(B) + e_H(A,V) - e_H(B,V) = 2|E| + |M_{A,V}| = 2(|A| - |B|). \tag{7.17}$$

Moreover,

$$e(H) = |M_{A,V}| + |E| = 2(|A| - |B|) - \ell \overset{(7.10)}{\leq} 2\rho n. \tag{7.18}$$

Suppose that $G[A]$ contains a matching M of size $\ell+1$. Then $\mathcal{P}^+ := M \cup M_{A,V}$ is a path system. If $\mathcal{P}^+$ contains a VW-path then obtain $\mathcal{P}$ from $\mathcal{P}^+$ by deleting an arbitrary edge of M. Otherwise there is an edge e in M which is incident to some edge in $M_{A,V}$. Let

$\mathcal{P} := \mathcal{P}^+ \setminus \{e\}$. Then at least one endpoint of e is an endpoint of a VW-path in $\mathcal{P}$. In both cases, (iii) holds. Also (i) and (ii) hold by (7.17) and (7.18).

Therefore we may assume that $G[A]$ contains no matching of size $\ell+1$. Let $M^-, P = xyz$ be as guaranteed by Claim 2. Then $M_1 := M^- \cup \{xy\}$ and $M_2 := M^- \cup \{yz\}$ are both matchings of size ℓ in $G[A]$. For $i = 1, 2$, let $\mathcal{P}_i := M_i \cup M_{A,V}$. These are both path systems. Now (7.17) and (7.18) imply that both of $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy (i) and (ii). If, for some $i = 1, 2$, $\mathcal{P}_i$ also satisfies (iii) then we are done by setting $\mathcal{P} := \mathcal{P}_i$, so suppose not. Then for each $i = 1, 2$ there exists $M_i' \subseteq M_i$ such that $V(M_i') = V(M_{A,V}) \cap A$. In particular, this implies that $M_1', M_2' \subseteq M^-$. Pick any edge $e \in M_1'$ and let $\mathcal{P} := P \cup (M^- \setminus \{e\}) \cup M_{A,V}$. Then both endpoints of e are endpoints of a VW-path in $\mathcal{P}$, so (7.17) and (7.18) imply that $\mathcal{P}$ satisfies (i)–(iii). □

7.5. **The proof of Theorem 1.2.** As already indicated at the beginning of the section, Theorem 1.2 now follows easily. Indeed, recall that we have a robust partition $\mathcal{V}$ with only five possible values of (k, ℓ). But Lemmas 7.1, 7.9 and 7.10 guarantee a $\mathcal{V}$-tour in each of these cases. Now Corollary 6.3 implies that G contains a Hamilton cycle.

Actually, we even prove the following stronger stability result of which Theorem 1.2 is an immediate consequence: if the degree of G is close to $n/4$ and G is not Hamiltonian, then G is either close to the union of four cliques, or two complete bipartite graphs, or the first extremal example discussed in Subsection 1.2.

Theorem 7.11. *For every $\varepsilon, \tau > 0$ with $2\tau^{1/3} \leq \varepsilon$ and every non-decreasing function $g : (0,1) \to (0,1)$, there exists $n_0 \in \mathbb{N}$ such that the following holds. For all 3-connected D-regular graphs G on $n \geq n_0$ vertices where $D \geq (1/5 + \varepsilon)n$, at least one of the following holds:*

- (i) *G has a Hamilton cycle;*
- (ii) *$D < (1/4 + \varepsilon)n$ and there exist ρ, ν with $1/n_0 \leq \rho \leq \nu \leq \tau$; $1/n_0 \leq g(\rho)$; $\rho \leq g(\nu)$, and $(k, \ell) \in \{(4,0), (2,1), (0,2)\}$ such that G has a robust partition $\mathcal{V}$ with parameters ρ, ν, τ, k, ℓ.*

Proof. Let $\alpha := 1/5 + \varepsilon$. Choose a non-decreasing function $f : (0,1) \to (0,1)$ with $f(x) \leq \min\{x, g(x)\}$ for all $x \in (0,1)$ such that the requirements of Proposition 3.2 (applied with $r := 5$), Corollary 6.3 and Lemmas 7.1, 7.9 and 7.10 (each applied with τ' playing the role of τ) are satisfied whenever $n, \rho, \gamma, \nu, \tau'$ satisfy

$$1/n \leq f(\rho), f(\gamma); \quad \rho \leq f(\nu), \varepsilon^3/8; \quad \gamma \leq f(\nu); \quad \nu \leq f(\tau'); \quad \tau' \leq f(\varepsilon), f(1/5), \tau \tag{7.19}$$

(and so $\tau' \leq f(\alpha)$). Choose τ', τ'' such that $0 < \tau' \leq f(\varepsilon), f(1/5), \tau$ and let $\tau'' := f(\tau')$. Apply Theorem 3.1 with $f/36, \alpha, \tau''$ playing the roles of f, α, τ to obtain an integer n_0. Let G be a 3-connected D-regular graph on $n \geq n_0$ vertices where $D \geq \alpha n$. Theorem 3.1 now guarantees ρ, ν, k, ℓ with $1/n_0 \leq \rho \leq \nu \leq \tau''$, $1/n_0 \leq f(\rho)$ and $36\rho \leq f(\nu)$ such that G has a robust partition $\mathcal{V}$ with parameters $\rho, \nu, \tau'', k, \ell$ (and thus also a robust partition with parameters $\rho, \nu, \tau', k, \ell$).

Let $\gamma := 36\rho$. Note that $n, \rho, \gamma, \nu, \tau'$ satisfy (7.19). So we can apply Proposition 3.2(ii) with $\tau', 5$ playing the roles of τ, r to see that (k, ℓ) is equal to (a) $(k, 0)$ for $1 \leq k \leq 3$; (b) $(0, 1)$; (c) $(1, 1)$; or (d) $(4, 0), (2, 1), (0, 2)$. Apply Lemmas 7.1, 7.9 and 7.10 (with τ' playing the role of τ) in the cases (a), (b), (c) respectively to obtain a $\mathcal{V}$-tour of G with parameter $36\rho = \gamma$. Then Corollary 6.3 (with τ' playing the role of τ) implies that G contains a Hamilton cycle so we are in case (i). If instead (d) holds, Proposition 3.2(i)

implies that $D < (1/4 + \varepsilon)n$. Since $f \leq g$ and $\mathcal{V}$ is a robust partition with parameters ρ, ν, τ, k, ℓ (as $\tau' \leq \tau$) we are in case (ii). □

Proof of Theorem 1.2. Let $\varepsilon > 0$. Choose a positive constant τ such that $2\tau^{1/3} \leq \varepsilon$. Apply Theorem 7.11 (with $g(x) = x$, say) to obtain an integer n_0. Let G be a 3-connected D-regular graph on $n \geq n_0$ vertices with $D \geq (1/4 + \varepsilon)n$. Then Theorem 7.11 implies that G has a Hamilton cycle. □

8. The proofs of Theorems 1.4 and 1.5

We first show that Theorem 1.4 is asymptotically best possible.

Proposition 8.1. *Let $t, r \in \mathbb{N}$ be such that $r \geq 2$. Then there are infinitely many $n \in \mathbb{N}$ for which there exists a t-connected D-regular graph G on n vertices with $D := (n-t)/(r-1)-1$ and circumference $c(G) \leq tn/(r-1) + t$.*

One can easily modify the construction to obtain a t-connected D-regular graph G with the same bound on $c(G)$ for smaller values of D (e.g. $D = n/r$).

Proof. We may suppose that $t \leq r - 1$. Pick any $k \in \mathbb{N}$ with $k \geq 2t$. Let

$$n := (r-1)(2k(r-1)+1) + t \quad \text{and} \quad D := \frac{n-t}{r-1} - 1 = 2k(r-1).$$

Construct a graph G on n vertices as follows. Let $X, U_1, \ldots, U_{r-1}$ be a partition of $V(G)$, where $|X| = t$ and the $|U_i| = D+1$. Add all edges within the U_i. So $G[U_i]$ is D-regular. Let M_i be a matching in $G[U_i]$ with $|V(M_i)| = tD/(r-1)$. Note that M_i exists since $tD/(r-1)$ is even, and at most D since $t \leq r - 1$. Add exactly one edge from each $y \in V(M_i)$ to X so that each $x \in X$ receives exactly $D/(r-1)$ edges from $V(M_i)$. Remove M_i from G.

Therefore G is t-connected (with vertex cut-set X) and D-regular. But any cycle in G traverses at most t of the U_i, so

$$c(G) \leq t|U_i| + |X| \leq tn/(r-1) + |X| = tn/(r-1) + t,$$

as required. □

The first part of the following proposition shows that the bound on the circumference in Theorem 1.5 is close to best possible. The second part of the proposition is a bipartite analogue of the extremal example in Figure 1(i). The proofs may be found in [30].

Proposition 8.2.

(i) *Let $t, r \in \mathbb{N}$ be such that $r \geq 4$ is even and $t \geq 2$. Then there are infinitely many $n \in \mathbb{N}$ for which there exists a t-connected D-regular bipartite graph G on n vertices with $D := (n-2)/(r-2)$ and circumference $c(G) \leq 2tn/(r-2) + t$;*

(ii) *For every $t \in \mathbb{N}$ with $t \geq 2$, there are infinitely many $D \in \mathbb{N}$ such that there exists a bipartite graph on $8D + 2$ vertices which is D-regular and t-connected but does not contain a Hamilton cycle.*

One can easily modify the construction to obtain a t-connected D-regular graph G with the same bound on $c(G)$ for smaller values of D.

The proof of Theorem 1.4 uses robust partitions as the main tool (Theorem 3.1). We show that, in a t-connected graph G with a robust partition, we can find a cycle that contains every vertex in the t largest robust components of G (or at least almost all

the vertices in the case of bipartite robust components). When G has degree slightly larger than n/r, its robust partition contains at most $r-1$ components. So the t largest components together contain at least $tn/(r-1)$ vertices, as required.

We let C_1 denote a loop and C_2 a double edge. The following result shows that, given any t-connected graph G and any collection $\mathcal{U}$ of t disjoint subsets of $V(G)$, we can find a path system $\mathcal{P}$ such that $R_{\mathcal{U}}(\mathcal{P}) \cong C_t$.

Proposition 8.3. *Let $t \in \mathbb{N}$, let G be a t-connected graph and let $\mathcal{U} := \{U_1, \dots, U_t\}$ be a collection of disjoint vertex-subsets of G with $|U_i| \geq 2t$ for each $1 \leq i \leq t$. Then there exists a $\mathcal{U}$-anchored path system $\mathcal{P}$ in G such that $R_{\mathcal{U}}(\mathcal{P}) \cong C_t$.*

Proof. For each i, let $\mathcal{U}_i := \{U_1, \dots, U_i\}$. Let P be a non-trivial path in G with both endpoints in U_1 and let $\mathcal{P}_1 := \{P\}$. Thus $R_{\mathcal{U}_1}(\mathcal{P}) \cong C_1$. Now suppose, for some $i < t$, we have obtained a $\mathcal{U}_i$-anchored path system $\mathcal{P}_i$ in G such that $R_{\mathcal{U}_i}(\mathcal{P}_i) \cong C_i$. Without loss of generality, we may assume that this cycle is $U_1U_2 \dots U_i$. So $\mathcal{P}_i$ consists of i paths $P_1, \dots, P_i$ where P_j has endpoints $x_j \in U_j, y_{j+1} \in U_{j+1}$ (with indices modulo i).

Suppose that there is some path $P_j \in \mathcal{P}_i$ with $|V(P_j) \cap U_{i+1}| \geq 2$. Let $u, v \in V(P_j) \cap U_{i+1}$ be distinct such that u is closer than v to x_j on P_j. Let $\mathcal{P}_{i+1}$ be the path system obtained from $\mathcal{P}_i$ be replacing P_j with the paths x_jP_ju, vP_jy_{j+1}.

So we may assume that $|V(\mathcal{P}_i) \cap U_{i+1}| = \sum_{1 \leq j \leq i} |V(P_j) \cap U_{i+1}| \leq i$. Let $U'_{i+1} := U_{i+1} \setminus V(\mathcal{P}_i)$. Note that $|U'_{i+1}| \geq 2t - i > t$. By Menger's Theorem, there exists a path system $\mathcal{R}$ consisting of $i+1$ paths which join $V(\mathcal{P}_i)$ to U'_{i+1} and have no internal vertices in $V(\mathcal{P}_i)$. By the pigeonhole principle, there exist $j \leq i$ and distinct paths $xRy, x'R'y' \in \mathcal{R}$ such that $x, x' \in V(P_j)$. Without loss of generality, x is closer to x_j on P_j than x'. Obtain $\mathcal{P}_{i+1}$ from $\mathcal{P}_i$ by replacing P_j with x_jP_jxRy, $y'R'x'P_jy_{j+1}$.

In both cases, $\mathcal{P}_{i+1}$ is a $\mathcal{U}_{i+1}$-anchored path system, and

$$R_{\mathcal{U}_{i+1}}(\mathcal{P}_{i+1}) = U_1 \dots U_jU_{i+1}U_{j+1} \dots U_i$$

is a cycle with vertex set $\mathcal{U}_{i+1}$. The path system $\mathcal{P}_t$ obtained in this way is as required in the proposition. □

Now we show that, if $R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour, we can discard suitable subpaths of each $P \in \mathcal{P}$ to ensure that $|V(\mathcal{P}) \cap U|$ is small for each $U \in \mathcal{U}$.

Proposition 8.4. *Let $\mathcal{U}$ be a collection of disjoint non-empty vertex-subsets of a graph G and let $\mathcal{P}$ be a $\mathcal{U}$-anchored path system in G containing t paths such that $R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour. Then there exists a $\mathcal{U}$-anchored path system $\mathcal{P}'$ in G such that $R_{\mathcal{U}}(\mathcal{P}')$ is an Euler tour, and for each $U \in \mathcal{U}$ we have that $|V(\mathcal{P}') \cap U| \leq 2t$.*

Proof. Let $s := |\mathcal{U}|$. Clearly, the proposition holds if $s = 1$. So we may assume that $s \geq 2$ and that no $P \in \mathcal{P}$ has both endpoints in the same $X \in \mathcal{U}$ (otherwise we could remove P from $\mathcal{P}$). Fix a path $P \in \mathcal{P}$ with endpoints $u \in U, v \in V$ where $U, V \in \mathcal{U}$ are distinct. We will define a sequence of path systems $\mathcal{R}_\ell$ with $E(\mathcal{R}_\ell) \subseteq E(P)$ as follows. Let $\mathcal{R}_0 := \{P\}$. Suppose, for some $0 \leq \ell < s$, we have already defined a path system $\mathcal{R}_\ell$ such that

- (α_ℓ) $\mathcal{R}_\ell$ is $\mathcal{U}$-anchored;
- (β_ℓ) if $\ell \geq 1$ then $E(\mathcal{R}_\ell) \subseteq E(\mathcal{R}_{\ell-1})$;
- (γ_ℓ) $E(R_{\mathcal{U}}(\mathcal{R}_\ell))$ forms a walk from U to V;
- (δ_ℓ) for at least ℓ of the X in $\mathcal{U}$, $|X \cap V(\mathcal{R}_\ell)| \leq 2$.

Now we obtain $\mathcal{R}_{\ell+1}$ from $\mathcal{R}_\ell$ as follows. We are done if there are at least $\ell+1$ sets X in $\mathcal{U}$ such that $|X \cap V(\mathcal{R}_\ell)| \leq 2$, so suppose not. Let $W \in \mathcal{U}$ be such that $|W \cap V(\mathcal{R}_\ell)| \geq 3$. By (γ_ℓ), there exists an integer $p \geq 1$ such that $R_{\mathcal{U}}(\mathcal{R}_\ell)$ equals the walk $U_1U_2 \dots U_{p+1}$ from $U_1 := U$ to $U_{p+1} := V$. So $\mathcal{R}_\ell$ consists of p paths $R_1, \dots, R_p$ such that R_j has endpoints $x_j \in U_j$ and $y_{j+1} \in U_{j+1}$. Choose $j \leq j'$ such that $W \cap V(R_j) \neq \emptyset \neq W \cap V(R_{j'})$ and $j' - j$ is maximal with this property. Let $w \in W$ be the vertex on R_j which is closest to x_j and let $w' \in W$ be the vertex on $R_{j'}$ which is closest to $y_{j'+1}$. Let $\mathcal{R}_{\ell+1} := \{R_1, \dots, R_{j-1}, x_jR_jw, w'R_{j'}y_{j'+1}, R_{j'+1}, \dots, R_p\}$. Certainly $\mathcal{R}_{\ell+1}$ satisfies $(\beta_{\ell+1})$ and $(\delta_{\ell+1})$ from the construction. $(\alpha_{\ell+1})$ follows from (α_ℓ). Since w, w' lie in the same set in $\mathcal{U}$, $(\gamma_{\ell+1})$ holds by (γ_ℓ).

Therefore we can obtain $\mathcal{P}_P := \mathcal{R}_s$ that satisfies (α_s)–(δ_s). We can obtain $\mathcal{P}_P$ independently for each $P \in \mathcal{P}$. Since the P are vertex-disjoint and (β_s) implies that $E(\mathcal{P}_P) \subseteq E(P)$, it follows that $\mathcal{P}' := \bigcup_{P\in\mathcal{P}} \mathcal{P}_P$ is a path system. Moreover $\mathcal{P}'$ is certainly $\mathcal{U}$-anchored by (α_s). We write $R := R_{\mathcal{U}}(\mathcal{P})$ and $R' := R_{\mathcal{U}}(\mathcal{P}')$. Note (γ_s) implies that one can obtain R' from R by replacing each edge UV of R with a walk joining U, V. Since R is an Euler tour we therefore have that R' is an Euler tour. Moreover, (δ_s) implies that for each $X \in \mathcal{U}$ we have $|V(\mathcal{P}') \cap X| = \sum_{P\in\mathcal{P}} |V(\mathcal{P}_P) \cap X| \leq 2t$ as required. □

In the following proposition, we show that, given a weak robust subpartition $\mathcal{U}$ in a t-connected graph G, we can adjust $\mathcal{U}$ slightly so that G contains a path system $\mathcal{P}$ which is a $\mathcal{U}$-tour. For this, we simply apply Propositions 8.3 and 8.4 to obtain a suitable $\mathcal{U}$-anchored path system and remove a small number of vertices from each bipartite robust component.

Proposition 8.5. *Let $t, n \in \mathbb{N}$ and let $0 < 1/n \ll \rho \ll \nu \ll \tau \ll \eta, 1/t \leq 1$. Suppose that G is a regular t-connected graph on n vertices. Let $\mathcal{U}$ be a weak robust subpartition in G with parameters $\rho, \nu, \tau, \eta, k, \ell$ where $k + \ell \leq t$. Then*

(i) *G has a weak robust subpartition $\mathcal{X}$ with parameters $6\rho, \nu/2, 2\tau, \eta/2, k, \ell$;*
(ii) *$|\bigcup_{X\in\mathcal{X}} X| \geq |\bigcup_{U\in\mathcal{U}} U| - 2\rho\ell n$;*
(iii) *G contains an $\mathcal{X}$-tour with parameter 54ρ.*

Proof. Write $\mathcal{U} = \{U_1, \dots, U_k, Z_1, \dots, Z_\ell\}$ satisfying (D1′)–(D5′). Apply Proposition 8.3 to $\mathcal{U}$ with $t' := k+\ell$ playing the role of t to obtain a $\mathcal{U}$-anchored path system $\mathcal{P}^*$ such that $R_{\mathcal{U}}(\mathcal{P}^*) \cong C_{t'}$. Since $\mathcal{P}^*$ contains at most t paths, we may apply Proposition 8.4 to $\mathcal{P}^*$ to obtain a $\mathcal{U}$-anchored path system $\mathcal{P}$ such that $R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour and $|V(\mathcal{P}) \cap U| \leq 2t$ for all $U \in \mathcal{U}$.

Consider any $1 \leq j \leq \ell$. Let A_j, B_j be the bipartition of Z_j guaranteed by (D3′). So Z_j is a bipartite (ρ, ν, τ)-robust expander component with respect to A_j, B_j. Moreover,

$$2e_{\mathcal{P}}(A_j) + e_{\mathcal{P}}(A_j, \overline{Z_j}) \leq \sum_{x\in Z_j} d_{\mathcal{P}}(Z_j) \leq 2|V(\mathcal{P}) \cap Z_j| \leq 4t.$$

A similar inequality holds for B_j. Now $||A_j| - |B_j|| \leq \rho n$ by (D3′), (B1) and (C2). Therefore we can remove at most $\rho n + 4t \leq 2\rho n$ vertices from $Z_j \setminus V(\mathcal{P})$ to obtain $A'_j \subseteq A_j$, $B'_j \subseteq B_j$ and $Z'_j := A'_j \cup B'_j$ such that

$$2e_{\mathcal{P}}(A'_j) - 2e_{\mathcal{P}}(B'_j) + e_{\mathcal{P}}(A'_j, \overline{Z'_j}) - e_{\mathcal{P}}(B'_j, \overline{Z'_j}) = 2(|A'_j| - |B'_j|). \tag{8.1}$$

To see this, it suffices to check that $e_{\mathcal{P}}(A'_j, \overline{Z'_j}) - e_{\mathcal{P}}(B'_j, \overline{Z'_j})$ (and thus the left-hand side of (8.1)) is even. To verify the latter note that, modulo two, $e_{\mathcal{P}}(Z'_j, \overline{Z'_j}) \equiv \mathrm{End}_{\mathcal{P}}(Z'_j) =$

$d_{R_{\mathcal{U}}(\mathcal{P})}(Z'_j)$. So

$$e_{\mathcal{P}}(A'_j, \overline{Z'_j}) - e_{\mathcal{P}}(B'_j, \overline{Z'_j}) = e_{\mathcal{P}}(Z'_j, \overline{Z'_j}) - 2e_{\mathcal{P}}(B'_j, \overline{Z'_j}) \equiv d_{R_{\mathcal{U}}(\mathcal{P})}(Z'_j) - 2e_{\mathcal{P}}(B'_j, \overline{Z'_j}) \equiv 0$$

where the final congruence follows since $R_{\mathcal{U}}(\mathcal{P})$ is an Euler tour. Therefore (8.1) can be satisfied.

Let $\mathcal{X} := \{U_1, \dots, U_k, Z'_1, \dots, Z'_\ell\}$. Clearly (ii) holds. To prove (i), first note that for each $1 \le j \le \ell$, we have $|A'_j \triangle A_j| + |B'_j \triangle B_j| = |Z_j \setminus Z'_j| \le 2\rho n$. Then Lemma 4.10(i) implies that $G[Z'_j]$ is a bipartite $(6\rho, \nu/2, 2\tau)$-robust expander component of G with bipartition A'_j, B'_j. So (D3′) holds. The remaining properties (D1′), (D2′), (D4′) and (D5′) are clear.

Finally, by (8.1) and the properties of $\mathcal{P}$ stated above, we can apply Lemma 7.8 with $\mathcal{X}, \mathcal{P}, 6\rho, \nu/2, 2\tau, \eta/2, k, \ell$ playing the roles of $\mathcal{U}, \mathcal{P}, \rho, \nu, \tau, \eta, k, \ell$ to obtain an $\mathcal{X}$-extension $\mathcal{P}'$ of $\mathcal{P}$ in G that is an $\mathcal{X}$-tour with parameter 54ρ. This proves (iii). □

We are now able to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\alpha := 1/r + \varepsilon$ and $\eta := 1/2r^2 \le \alpha^2/2$. Choose a non-decreasing function $f : (0,1) \to (0,1)$ with $f(x) \le x$ for all $x \in (0,1)$ such that the requirements of Propositions 3.2, 6.1 and 8.5 as well as Lemma 6.2 are satisfied whenever $n, \rho, \gamma, \nu, \tau$ satisfy the following:

$$1/n \le f(\rho); \quad \rho \le f(\nu), \varepsilon^3/8; \quad \nu \le f(\tau); \quad \tau \le f(\eta), f(1/t), f(1/r); \tag{8.2}$$

as well as $1/n \le f(\gamma)$ and $\gamma \le f(\nu)$. Choose τ, τ' so that

$$0 < \tau' \le \tau \le \frac{1}{2r^2}, \frac{\varepsilon}{2t}, \frac{\varepsilon^3}{8}, \frac{f(1/t)}{2}, \frac{f(\eta/2)}{2}, f(1/r) \quad \text{and } \tau' \le f(\tau). \tag{8.3}$$

Choose a non-decreasing function $f' : (0,1) \to (0,1)$ such that $54f'(x) \le f(x/2)$ for all $x \in (0,1)$. Apply Theorem 3.1 with f', α, τ' playing the roles of f, α, τ to obtain an integer n_0. Let G be a t-connected D-regular graph on $n \ge n_0$ vertices where $D \ge \alpha n$. Theorem 3.1 now guarantees ρ, ν, k', ℓ' with

$$1/n_0 \le \rho \le \nu \le \tau', \quad 1/n_0 \le f'(\rho) \quad \text{and} \quad \rho \le f'(\nu) \tag{8.4}$$

such that G has a robust partition $\mathcal{V}$ with parameters $\rho, \nu, \tau', k', \ell'$ (and thus also with parameters $\rho, \nu, \tau, k', \ell'$). Note that (8.3) and (8.4) together imply that (8.2) holds. Moreover,

$$2\rho \le 1/r^2 \quad \text{and} \quad 2\rho t \le \varepsilon. \tag{8.5}$$

Claim. *There are integers k, ℓ with $k + \ell \le t$ such that G has a weak robust subpartition $\mathcal{U}$ with parameters $\rho, \nu, \tau, \eta, k, \ell$ where*

$$\sum_{U \in \mathcal{U}} |U| \ge \min\left\{\frac{t}{r-1} + \frac{\ell}{r^2}, 1\right\} n. \tag{8.6}$$

To see this, recall that $\mathcal{V}$ is a robust partition in G with parameters $\rho, \nu, \tau, k', \ell'$. Let $m := k' + \ell'$. Suppose first that $m \le t$. Since by Proposition 6.1(i), $\mathcal{V}$ is a weak robust subpartition in G with parameters $\rho, \nu, \tau, \eta, k', \ell'$ we can take $\mathcal{U} := \mathcal{V}$ (and so $k = k'$ and $\ell = \ell'$). Therefore we may assume that $t \le m - 1$. Order the members of $\mathcal{V}$ as $X_1, \dots, X_m$ so that $|X_1| \ge \dots \ge |X_m|$. Let $\mathcal{U} := \{X_1, \dots, X_t\}$. Now by Proposition 6.1(i) and (ii) there exist integers k, ℓ so that $k + \ell = t$ and $\mathcal{U}$ is a weak robust subpartition in G with parameters $\rho, \nu, \tau, \eta, k, \ell$.

By averaging, we have that $\sum_{U\in\mathcal{U}}|U| \geq tn/m$. Note also that $m+\ell \leq m+\ell' = k'+2\ell' \leq r-1$ where the last inequality follows from Proposition 3.2. Therefore

$$\sum_{U\in\mathcal{U}}|U| \geq \frac{tn}{m} \geq \frac{tn}{r-1-\ell} = \frac{tn}{r-1}\left(1+\frac{\ell}{r-1-\ell}\right) \geq \frac{tn}{r-1}+\frac{\ell n}{r^2},$$

proving the claim.

Apply Proposition 8.5 to $G,\mathcal{U}$ to obtain a weak robust subpartition $\mathcal{X}$ with parameters $6\rho,\nu/2,2\tau,\eta/2,k,\ell$ in G and an $\mathcal{X}$-anchored path system $\mathcal{P}$ such that $\sum_{X\in\mathcal{X}}|X| \geq \sum_{U\in\mathcal{U}}|U|-2\rho\ell n$ and $\mathcal{P}$ is an $\mathcal{X}$-tour with parameter $\gamma := 54\rho$. Now (8.3) and (8.4) imply that

$$1/n \leq f(6\rho), f(\gamma); \quad 6\rho,\gamma \leq f(\nu/2); \quad \nu/2 \leq f(2\tau); \quad 2\tau \leq f(\eta/2). \tag{8.7}$$

Then Lemma 6.2 with $\mathcal{X},\mathcal{P},6\rho,\gamma,\nu/2,2\tau,\eta/2$ playing the roles of $\mathcal{U},\mathcal{P},\rho,\gamma,\nu,\tau,\eta$ implies that there is a cycle C in G which contains every vertex in $\bigcup_{X\in\mathcal{X}}X$. So

$$\begin{aligned}|V(C)| \quad &\geq \quad \sum_{U\in\mathcal{U}}|U|-2\rho\ell n \overset{(8.6)}{\geq} \min\left\{\frac{t}{r-1}+\frac{\ell}{r^2}-2\rho\ell, 1-2\rho\ell\right\}n\\ &\overset{(8.5)}{\geq} \quad \min\left\{\frac{t}{r-1}, 1-\varepsilon\right\}n,\end{aligned}$$

as required. □

Proof of Theorem 1.5 (Sketch). The proof is almost the same as that of Theorem 1.4. We proceed similarly as we did there to obtain a robust partition $\mathcal{V}$ with parameters ρ,ν,τ',k',ℓ'. Using that G is bipartite, it is easy to check that $k'=0$. Thus $\ell' \leq \lfloor (r-1)/2\rfloor = (r-2)/2$ by Proposition 3.2. Instead of the claim in the proof of Theorem 1.4, we now show that there exists an integer $\ell \leq t$ such that G has a weak robust subpartition $\mathcal{U}$ with parameters $\rho,\nu,\tau,\eta,0,\ell$ where $\sum_{U\in\mathcal{U}}|U| \geq \min\{2tn/(r-2), n\}$. (Using that $\ell' \leq (r-2)/2$, this follows as in the claim.) The remainder of the proof is now similar to that of Theorem 1.4. □

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