THE ORDER OF THE LARGEST COMPLETE MINOR IN A RANDOM GRAPH

NIKOLAOS FOUNTOULAKIS, DANIELA KÜHN AND DERYK OSTHUS

ABSTRACT. Let ccl(G) denote the order of the largest complete minor in a graph G (also called the contraction clique number) and let $G_{n,p}$ denote a random graph on n vertices with edge probability p. Bollobás, Catlin and Erdős [5] asymptotically determined ccl($G_{n,p}$) when p is a constant. Łuczak, Pittel and Wierman [10] gave bounds on ccl($G_{n,p}$) when p is very close to $1/n$, i.e. inside the phase transition. We show that for every $\varepsilon > 0$ there exists a constant C such that whenever $C/n < p < 1 - \varepsilon$ then asymptotically almost surely ccl($G_{n,p}$)$= (1 \pm \varepsilon)n/\sqrt{\log_b(np)}$, where $b := 1/(1 - p)$. If $p = C/n$ for a constant $C > 1$, then asymptotically almost surely ccl($G_{n,p}$)$= \Theta(\sqrt{n})$. This extends the results in [5] and answers a question of Krivelevich and Sudakov [9].

1. INTRODUCTION

1.1. Main results.

A graph H is a *minor* of G if for every vertex $h \in H$ there is a connected subset $B_h \subseteq V(G)$ such that all the B_h are disjoint and G contains an edge between B_h and $B_{h'}$ whenever hh' is an edge of H. The B_h's are called the *branch sets*. We denote by ccl(G) the order of the largest complete minor in G. The study of the largest complete minor contained in a given graph has its origins in Hadwiger's conjecture which states that if the chromatic number of a graph G is at least k, then G contains K_k minor. It has been proved for $k \leq 6$ (see for example [6, Chapter 7] for a discussion).

Bollobás, Catlin and Erdős [5] proved that Hadwiger's conjecture is true for almost all graphs. For this, they estimated the typical order of the largest complete minor in a graph on n vertices and compared it with the typical chromatic number of such a graph. In particular, they proved that for constant p and $\varepsilon > 0$ asymptotically almost surely ccl($G_{n,p}$) $= (1 \pm \varepsilon)n/\sqrt{\log_b n}$, where $b := 1/(1 - p)$. Here $G_{n,p}$ is a random graph on n vertices where the edges are present independently and with probability p. We say that an event occurs *asymptotically almost surely* (a.a.s.) if it occurs with probability tending to 1 as n tends to infinity.

Krivelevich and Sudakov [9] considered the order of the largest complete minor in a sparser random graph (and more generally in arbitrary pseudo-random and expanding graphs). They determined the order of magnitude of ccl($G_{n,p}$) as long as $p \geq n^{\varepsilon-1}$. Our first result determines ccl($G_{n,p}$) asymptotically as long as $p \geq C/n$ and $p = o(1)$.

Theorem 1. *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that if $pn \geq C$ and $p = o(1)$, then a.a.s.*

$$\text{ccl}(G_{n,p}) = (1 \pm \varepsilon)\sqrt{\frac{n^2 p}{\ln(np)}}.$$

One can combine Theorem 1 with [5] to obtain a single formula which allows for constant p as well. Indeed, let $b := 1/(1 - p)$. If $p = o(1)$ a series expansion gives $\ln b = -\ln(1 - p) =$

$p + O(p^2)$. Thus

$$\sqrt{\frac{n^2 p}{\ln(np)}} = \sqrt{\frac{n^2 p}{\ln b \log_b(np)}} = (1 + o(1)) \frac{n}{\sqrt{\log_b(np)}}.$$

Also if p is constant, then $\log_b n = (1+o(1))\log_b(np)$. So altogether we obtain the following.

Corollary 2. *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that if $C/n \leq p \leq 1-\varepsilon$, then a.a.s.*

$$\mathrm{ccl}(G_{n,p}) = (1 \pm \varepsilon)\frac{n}{\sqrt{\log_b(np)}}.$$

In the last section of the paper, we estimate $\mathrm{ccl}(G_{n,c/n})$ where $c > 1$ is fixed. Krivelevich and Sudakov [9] observed that there are constants c_1 and c_2 such that $c_1\sqrt{n/\log n} \leq \mathrm{ccl}(G_{n,c/n}) \leq c_2\sqrt{n}$ and asked what the correct order of magnitude is.

Theorem 3. *For every $c > 1$ there exists a constant $\delta = \delta(c)$ such that a.a.s. $\delta\sqrt{n} \leq \mathrm{ccl}(G_{n,c/n}) \leq 2\sqrt{cn}$.*

Note that the upper bound in Theorem 3 is immediate, since for any graph G, the number of edges in any minor of G is at most $e(G)$ (see the beginning of Section 4). The same argument shows that the condition that $p \geq c/n$ for some constant $c > 1$ is necessary to ensure a complete minor or order $\Theta(\sqrt{n})$ in $G_{n,p}$. This follows from the fact that if $pn \rightarrow 1$ the number of edges in any component is sublinear in n (see e.g. [3, Thm. 6.16]).

1.2. **Related results and open questions.** While the influence of the chromatic number on the existence of complete minors is far from clear, the corresponding extremal problem for the average degree has been settled for large complete minors: Thomason [14] asymptotically determined the smallest average degree $d(k)$ which guarantees the existence of a K_k minor in any graph of average degree at least $d(k)$. (The order of magnitude $k\sqrt{\log k}$ of $d(k)$ was determined earlier in [8, 12].) The extremal graphs are (disjoint copies of) dense random graphs. Recall that Theorem 3 shows that the behaviour of sparse random graphs is quite different: in that case $\mathrm{ccl}(G_{n,p})$ has the same order of magnitude as $\sqrt{e(G_{n,p})}$, a trivial upper bound which holds for any graph.

There are several results on large complete minors in pseudo-random graphs and expanding graphs. Thomason [13] introduced the following notion of pseudo-randomness: a graph G is (p,β)-*jumbled* if every induced subgraph H of G satisfies

$$\left| e(H) - p\binom{|H|}{2} \right| \leq \beta|H|.$$

He observed that if both pn and $(1-p)n$ tend to infinity, then $G_{n,p}$ is a.a.s. $(p, 2\sqrt{pn})$-jumbled. He also showed that $\mathrm{ccl}(G) \geq (1+o(1))n/\sqrt{\log_b n}$ for every (p,β)-jumbled graph G with constant p and $\beta = O(n^{1-\varepsilon})$. A result of Krivelevich and Sudakov [9] implies that $\mathrm{ccl}(G) \geq \Omega(\sqrt{n^2 p/\log(n\sqrt{p})})$ for (p,β)-jumbled graphs G with $\beta = o(np)$. Thus for $G_{n,p}$ their results only imply the lower bound in Theorem 1 up to a multiplicative constant if $p \geq n^{\varepsilon-1}$. It would be interesting to know whether their bound is best possible or whether (up to a multiplicative constant) the bound in Theorem 1 can be extended to jumbled graphs with appropriate parameters. Krivelevich and Sudakov [9] also considered minors in expanding graphs. Again, their results only imply the lower bound in Theorem 1 up to a multiplicative constant if $p \geq n^{\varepsilon-1}$. A property closely connected to expansion is that of having no small separator. Minors in such graphs were considered e.g. by Plotkin, Rao and Smith [11].

A question that is left open by our results is how the order of the largest complete minor behaves when p approaches the critical point $1/n$ of the appearance of the giant component. Results in this direction where proved by Łuczak, Pittel and Wierman [10], who determined the limiting probability $g(\lambda)$ that $G_{n,p}$ is planar when $np = (1 + \lambda n^{-1/3})$. Their results imply that if λ is bounded, then $g(\lambda)$ is bounded away from 0 and 1. (The likely obstacle to planarity in this case is the existence of a $K_{3,3}$ minor.) If $\lambda \to -\infty$, then $g(\lambda) \to 1$. If $\lambda \to \infty$, then $g(\lambda) \to 0$. In fact they proved the stronger result that a.a.s. $\mathrm{ccl}(G_{n,p})$ is unbounded in this case.

Another related problem is that of estimating the order $\mathrm{tcl}(G)$ of the largest topological clique in a graph G. (Recall that a topological clique of order k is obtained from a K_k by replacing its edges by internally disjoint paths.) The typical order of $\mathrm{tcl}(G)$ in a graph G was first estimated by Erdős and Fajtlowicz [7], who used their bounds to prove that the conjecture of Hajós is false for almost all graphs. Their estimates were improved by Bollobás and Catlin [4], who showed that for almost all graphs $\mathrm{tcl}(G)$ it is close to $2\sqrt{n}$. Later Ajtai, Komlós and Szemerédi [1] proved that as long as the expected degree $(n-1)p$ is at least $1+\varepsilon$ and is $o(\sqrt{n})$, then a.a.s. $\mathrm{tcl}(G_{n,p})$ is almost as large as the maximum degree. Again the likely order of $\mathrm{tcl}(G_{n,p})$ when p is close to $1/n$ is not known.

Finally, note that our results do not cover the case where $p \to 1$. Usually, the investigation of $G_{n,p}$ for such p is not particularly interesting. However, any counterexamples to Hadwiger's conjecture are probably rather dense, so in this case it might be worthwile to investigate the values of the chromatic number and the order of the largest complete minor of such a random graph (though it seems rather unlikely that this approach will yield any counterexamples).

1.3. **Strategy of the proofs.** As in [5], the upper bound in Theorem 1 is proved by a first moment argument. The main difference between the arguments is that in our case, we need to make use of the fact that the branch sets of a minor have to be connected, whereas this was not necessary in [5].

For the lower bound, let $k := n/\sqrt{\log_b(np)}$ be the function appearing in Corollary 2. The proof in [5] for the case when p is a constant proceeds as follows. One first shows that a.a.s. there are k large pairwise disjoint connected sets B_i in $G_{n,p}$. These are used as candidates for the branch sets. The number U_0 of pairs of B_i which are not connected by an edge is then shown to be $o(k)$. So by discarding a comparatively small number of candidate branch sets, one can obtain the desired minor. For small p, the main problem is that U_0 will be much larger than k. However, we can show that U_0 is at most a small fraction of n. We make use of this as follows. We first find a path P whose length satisfies $U_0 \ll |P| \ll n$ and which is disjoint from the B_i. We will divide this path into disjoint subpaths. Our aim is to join most of those pairs of B_i which are not yet joined by an edge via one of these subpaths. More precisely, we are looking for a matching of size $U_0 - \varepsilon k$ in the auxiliary bipartite random graph G^* whose vertex classes consist of the unjoined pairs of candidate branch sets and of the subpaths and where a subpath is adjacent to such an unjoined pair if it sends an edge to both of the candidate branch sets in this unjoined pair. There are two difficulties to overcome in order to find such a matching. Firstly, some of the B_i are involved in several unjoined pairs, so the edges G^* are not independent. Secondly, if we make the subpaths too short, then the density of G^* is not large enough to guarantee a sufficiently large matching, while if we make the subpaths too long, then there will not be enough of them. We overcome this by using paths of very different lengths together with a greedy matching algorithm which starts off by using short paths to try and join the unjoined pairs. Then in the later stages the algorithm uses successively longer paths to try and join those

pairs which were not joined in the previous stages until $U_0 - \varepsilon k$ of the pairs have been joined. To ensure that the dependencies between the existence of edges in G^* are not too large, we also remove some of the unjoined pairs from future consideration after each stage (namely those containing a candidate branch set that is involved in comparatively many pairs which are still unjoined).

2. THE UPPER BOUND OF THEOREM 1

To prove the upper bound, let

$$k := (1 + \varepsilon)\sqrt{\frac{n^2 p}{\ln(np)}}.$$

We use a first moment argument to show that a.a.s. there are no k subsets of V that can serve as branch sets of a K_k minor, where V denotes the vertex set of $G_{n,p}$. Let $B_1, \ldots, B_k$ denote any collection of disjoint subsets of V and let $s_1, \ldots, s_k$ be their sizes. We are not assuming that the B_i's cover V and hence we let s be the number of vertices that do not belong to any B_i. So altogether the B_i's contain $n - s$ vertices. We will estimate the probability that these sets form the branch sets of a K_k minor.

The probability that for all $i < j$, B_i is joined to B_j by at least one edge is

$$\prod_{i<j} \left(1 - (1-p)^{s_i s_j}\right) \le \exp\left(-\sum_{i<j}(1-p)^{s_i s_j}\right).$$

An easy argument shows that the sum in the exponent is minimized when the sizes of the branch sets are as equal as possible. In other words,

$$\sum_{i<j}(1-p)^{s_i s_j} \ge \binom{k}{2}(1-p)^{\left(\frac{n-s}{k}\right)^2}.$$

Therefore,

$$(1) \qquad \mathbb{P}(\forall i < j, \ B_i \text{ is joined to } B_j) \le \exp\left(-\binom{k}{2}(1-p)^{\left(\frac{n}{k}\right)^2}\right).$$

For all i, the probability that B_i is connected is at most the expected number of spanning trees induced on B_i, which is $s_i^{s_i-2} p^{s_i-1}$. Since all these events are independent for different B_i, we have

$$(2) \qquad \mathbb{P}\left(\text{all } B_i\text{'s are connected}\right) \le \prod_{i=1}^{k} s_i^{s_i-2} p^{s_i-1} \le p^{n-s-k} \prod_{i=1}^{k} s_i^{s_i}.$$

Note that for fixed $s_1, \ldots, s_k$, there are $\frac{n!}{s! s_1! \cdots s_k!}$ ways of choosing the candidate branch sets $B_1, \ldots, B_k$. Since $s_i! \ge (s_i/e)^{s_i}$ by Stirling's formula, we have that

$$\frac{n!}{s! s_1! \cdots s_k!} = \binom{n}{s}\frac{(n-s)!}{\prod_{i=1}^{k} s_i!} \le 2^n \frac{n^{n-s}}{\prod_{i=1}^{k}\left(\frac{s_i}{e}\right)^{s_i}} \le \frac{e^{2n} n^{n-s}}{\prod_{i=1}^{k} s_i^{s_i}}.$$

Together with (1) and (2) this shows that for fixed $s_1, \ldots, s_k$ the expected number of families of k disjoint subsets of V forming the branch sets of a K_k minor is at most
$$(3)$$
$$e^{2n} n^{n-s} p^{n-s-k} \exp\left(-\binom{k}{2}(1-p)^{\left(\frac{n}{k}\right)^2}\right) = e^{2n}(np)^{n-s}\exp\left(-k\ln p - \binom{k}{2}(1-p)^{\left(\frac{n}{k}\right)^2}\right).$$

But using that $p = o(1)$, we have $1 - p \geq e^{-p-p^2} \geq e^{-(1+\varepsilon)p}$ (for the first inequality, see [3, p. 5]). Together with our assumption that $pn \geq C = C(\varepsilon)$ this implies that

$$\binom{k}{2}(1-p)^{\left(\frac{n}{k}\right)^2} \geq \frac{n^2 p}{2\ln(np)} e^{-(1+\varepsilon)\ln(np)/(1+\varepsilon)^2} = \frac{n}{2\ln(np)}(np)^{1-\frac{1}{1+\varepsilon}} \geq 3n\ln(np).$$

Also, using $p \geq C/n \geq 1/n$, we have

$$-k\ln p \leq 2\ln n \sqrt{\frac{n^2 p}{\ln(np)}} = 2\sqrt{n}\ln n \ln(np)\sqrt{\frac{np}{(\ln(np))^3}} \leq 2\sqrt{n}\ln n \ln(np)\sqrt{\frac{n}{(\ln n)^3}}$$
$$\leq n\ln(np),$$

provided that C is large enough. Hence, by (3) for fixed $s_1, \ldots, s_k$ the expected number of families of k disjoint subsets of V forming the branch sets of a K_k minor is at most

$$e^{2n}(np)^{n-s}\exp(n\ln(np) - 3n\ln(np)) \leq e^{2n}(np)^{-n}.$$

There are $\binom{n}{k}$ ways of choosing nonnegative integers $s_1, \ldots, s_k$ such that $\sum_{i=1}^{k} s_i \leq n$. Since $e^{2n}(np)^{-n}\binom{n}{k} = o(1)$ this shows that a.a.s. $G_{n,p}$ does not contain a K_k minor.

3. The lower bound of Theorem 1

We set

$$k := (1-\varepsilon)\sqrt{\frac{n^2 p}{\ln(np)}},$$

and we will show that there exists a constant $C = C(\varepsilon)$ such that if $np > C$, then a.a.s. $G_{n,p}$ contains a K_k minor.

As before, we let V denote the vertex set of $G_{n,p}$. Let V' be any subset of V containing $n' := (1 - \varepsilon/4)n$ vertices and put $V'' := V \setminus V'$. First, we construct a family $B_1, \ldots, B_{k'}$ of $k' := (1 + \varepsilon/4)k$ mutually disjoint and connected candidate branch sets in $G_{n,p}[V']$, each having size $t := (1 - \varepsilon/4)n'/k'$. It will turn out that a.a.s. there are k of these which can be extended into the branch sets of a K_k minor.

3.1. Constructing k' candidate branch sets.

To construct the k' candidate branch sets $B_1, \ldots, B_{k'}$ we will choose a long path in $G_{n,p}[V']$ and divide it into consecutive subpaths, each consisting of t vertices. We will obtain this path via a two-round exposure of $G_{n,p}[V']$. More precisely, let p', p'' be such that $p = p' + p'' - p'p''$. In our two-round exposure of $G_{n,p}[V']$ we first generate a random graph $G_{n,p'}[V']$ and then independently $G_{n,p''}[V']$. We then take their union, ignoring multiple edges (i.e. regarding them as a single edge).

In our case, we will take p' to be a small proportion of p, whereas p'' will be almost equal to p. However, we have to make sure that p' is not too small either, so that $G_{n,p'}[V']$ contains a long path. Ajtai, Komlós and Szemerédi [2] and de la Vega [15] independently proved that there exists a constant $C' = C'(\varepsilon)$ such that if $p'|V'| = C'$ then $G_{n,p'}[V']$ contains a path P' on $(1 - \varepsilon/4)|V'|$ vertices (see also Theorem 8.1 in [3]). It is easily seen that if we choose C sufficiently large compared to C' and $1/\varepsilon$ then

$$(4) \qquad p'' = \frac{p - p'}{1 - p'} \geq p - p' \geq (1 - \varepsilon^2)p.$$

We divide the path P' into consecutive disjoint subpaths $B_1, \ldots, B_{k'}$, each containing t vertices. We shall use these B_i's as candidate branch sets. In what follows, we assume that C was chosen to be sufficiently large compared to $1/\varepsilon$ and n was chosen to be sufficiently large compared to C for our estimates to hold.

3.2. **Estimating the number of unjoined pairs of candidate branch sets.** Our aim now is to show that k of the candidate branch sets $B_1, \ldots, B_{k'}$ are joined pairwise to each other (by edges or paths through V'') and thus form a K_k minor. First, we will estimate the number of the pairs B_i, B_j that are not joined by an edge in the second round of the two-round exposure of $G_{n,p}[V']$. So let $\mathcal{U}_0$ denote the set of all those pairs B_i, B_j which are not joined by an edge in $G_{n,p''}[V']$ and let $U_0 := |\mathcal{U}_0|$. Consider the auxiliary graph G_0' whose vertices are the B_i's and whose edges correspond to the pairs in $\mathcal{U}_0$. So G_0' is a binomial random graph on k' vertices whose edges occur independently with probability $q := (1 - p'')^{t^2}$. Note that if ε is small enough then

$$t^2 = \frac{(1 - \varepsilon/4)^4}{(1 + \varepsilon/4)^2 (1 - \varepsilon)^2} \frac{\ln(np)}{p} \geq \left(\frac{1 - \varepsilon/2}{(1 + \varepsilon/4)(1 - \varepsilon)} \right)^2 \frac{\ln(np)}{p} \geq (1 + \varepsilon/4)^2 \frac{\ln(np)}{p}$$

$$(5) \qquad \geq (1 + \varepsilon/2) \frac{\ln(np)}{p}$$

and so

$$q = (1 - p'')^{t^2} \leq e^{-p'' t^2} \overset{(4)}{\leq} e^{-(1 - \varepsilon^2) p t^2} \leq \left(\frac{1}{np} \right)^{1 + \varepsilon/3}.$$

Thus

$$(6) \qquad \mathbb{E}(U_0) = \binom{k'}{2} q \leq \frac{n^2 p}{\ln(np)} \frac{1}{(np)^{1 + \varepsilon/3}} \leq \frac{n}{(np)^{\varepsilon/3}}.$$

Suppose first that $\mathbb{E}(U_0) \leq n^{2/5}$. Then Chebyshev's inequality implies that a.a.s. $U_0 \leq 2n^{2/5} \leq \varepsilon^2 k'$. Thus by deleting one candidate branch set from every pair in $\mathcal{U}_0$ we obtain a K_k minor. So we may assume that $\mathbb{E}(U_0) \geq n^{2/5}$. (In fact, a straightforward but tedious calculation shows that this is always the case.) In particular, $\mathbb{E}(U_0) \to \infty$ as $n \to \infty$ and so a.a.s.

$$(7) \qquad U_0 = (1 \pm \varepsilon) \mathbb{E}(U_0)$$

by Chebyshev's inequality. Note that in the case when $p \geq 1/(\ln n)^2$ inequalities (6) and (7) together imply that a.a.s

$$\varepsilon^2 k' \geq \frac{\varepsilon^2 n}{2} \sqrt{\frac{p}{\ln n}} \geq \frac{\varepsilon^2 n}{(\ln n)^2} \geq 2n^{1 - \varepsilon/3} (\ln n)^{2\varepsilon/3} \geq \frac{2n^{1 - \varepsilon/3}}{p^{\varepsilon/3}} \geq U_0.$$

Thus by deleting one candidate branch set from every pair in $\mathcal{U}_0$ we obtain a K_k minor.

So in what follows, we may assume that $p \leq 1/(\ln n)^2$. We will show that most of the unjoined pairs of candidate branch sets (i.e. most of the pairs in $\mathcal{U}_0$) can be joined using V''. More precisely, for every pair $u \in \mathcal{U}_0$ we will try to find a path in $G_{n,p}[V'']$ which sends an edge to both candidate branch sets belonging to u. Of course these paths have to be disjoint for different pairs. So they will be chosen as follows: we consider a longest path in $G_{n,p}[V'']$, divide it into consecutive disjoint subpaths of various lengths and use them to successively join the pairs of candidate branch sets in $\mathcal{U}_0$. Here the fact that by (6) and (7) a.a.s. the number $U_0 = |\mathcal{U}_0| = e(G_0')$ of unjoined pairs is much smaller than $|V''|$ is crucial.

More formally, we proceed as follows. Let

$$i^* := \frac{\ln U_0 - \ln(n^{1/3})}{\ln 8} \overset{(7)}{\leq} \ln n.$$

For each $i = 1, \ldots, i^*$ let

$$(8) \qquad U_i := U_0 / 8^i.$$

Thus $U_{i^*} = n^{1/3}$. The results in [2, 15] imply that $G_{n,p}[V'']$ contains a path on $|V''|/2 \geq \varepsilon^2 n$ vertices. We divide this path into disjoint consecutive subpaths $Q_1, Q_2, \ldots, Q_{i^*}$ such that

$$(9) \qquad |Q_i| = \frac{\varepsilon^2 n}{2^i \ln(np)} \qquad \text{and so} \qquad |Q_i| = \frac{|Q_1|}{2^{i-1}}.$$

For each $i = 1, \ldots, i^*$ we divide Q_i further into a set $\mathcal{P}_i$ of U_{i-1} disjoint smaller consecutive paths each containing $\ell_i := |Q_i|/U_{i-1}$ vertices. Thus

$$(10) \qquad \ell_i = \frac{|Q_i|}{U_{i-1}} = \frac{8^{i-1}|Q_1|}{2^{i-1}U_0} = \frac{4^{i-1}|Q_1|}{U_0} \qquad \text{and so} \qquad \ell_i = 4^{i-1}\ell_1.$$

Note that ℓ_i and $|Q_i|$ are large. So in particular viewing them as integers does not affect the calculations. Given a path P in $G_{n,p}[V'']$ and a candidate branch set B we write $P \sim B$ if some vertex in B sends an edge to some vertex on P. Given a pair $u \in \mathcal{U}_0$ consisting of candidate branch sets A and B, we write $P \sim u$ if $P \sim A$ and $P \sim B$.

Roughly speaking, for each $i = 1, 2, \ldots, i^*$ in turn, we will use a greedy algorithm to join all but precisely U_i of all those pairs in $\mathcal{U}_0$ that are still unjoined, using one path in $\mathcal{P}_i$ for each such pair $u \in \mathcal{U}_0$, i.e. we want to find a path $P \in \mathcal{P}_i$ such that $P \sim u$ (see Sections 3.3 and 3.4 for details). We call each iteration of our algorithm a *stage*. Thus after the ith stage a.a.s. precisely U_i pairs in $\mathcal{U}_0$ will still be unjoined and we aim to join precisely $7U_i/8 = U_i - U_{i+1}$ of these during the next stage. Since $U_{i^*} = n^{1/3}$ this means that after i^* stages we obtain a complete minor of order $k' - n^{1/3} \geq k$ by removing one candidate branch set for every of the U_{i^*} pairs in $\mathcal{U}_0$ which are still unjoined.

3.3. A greedy algorithm for joining the candidate branch sets.
Let $\hat{p}$ denote the probability that a given vertex $v \in V''$ is joined to a given candidate branch set. Thus $\hat{p} = 1 - (1-p)^t$. But

$$(11) \qquad 1 - pt \leq (1-p)^t \leq 1 - pt + (pt)^2/2.$$

Similarly as in (5) one can show that $pt^2 = O(\ln(np))$. So our assumption that $p \leq 1/(\ln n)^2$ implies $pt = o(1)$. Thus

$$(12) \qquad \hat{p} = (1 + o(1))pt.$$

Suppose that $1 \leq i < i^*$ and after the $(i-1)$th stage of our algorithm we have a set $\mathcal{U}_{i-1} \subseteq \mathcal{U}_0$ of unjoined pairs such that $|\mathcal{U}_{i-1}| = U_{i-1}$ and that we now wish to run the ith stage of our algorithm. (If $i = 1$ then $\mathcal{U}_0$ is the set defined at the beginning of Section 3.2.) So we now wish to use the paths in $\mathcal{P}_i$ to join precisely $7U_{i-1}/8$ of the pairs in $\mathcal{U}_{i-1}$. (The set of remaining pairs in $\mathcal{U}_{i-1}$ will then be $\mathcal{U}_i$.) However, before running the actual algorithm we will first discard all those pairs in $\mathcal{U}_{i-1}$ which contain a candidate branch set lying in too many pairs from $\mathcal{U}_{i-1}$. This will reduce the dependencies between the successes of the individual steps of the algorithm within a stage. More precisely, let G'_{i-1} denote the spanning subgraph of G'_0 whose edge set corresponds to $\mathcal{U}_{i-1}$. (G'_0 is the graph defined in Section 3.2.) For all $i = 1, \ldots, i^*$ put

$$(13) \qquad \Delta_{i-1} := \frac{8\mathbb{E}(U_0)}{\varepsilon^{3/2}4^{i-1}k'}.$$

Let $\mathcal{B}_{i-1}$ be the set of all those candidate branch sets whose degree in G'_{i-1} is greater than Δ_{i-1}. In other words, $\mathcal{B}_{i-1}$ consists of all those candidate branch sets which lie in more than Δ_{i-1} pairs from $\mathcal{U}_{i-1}$. As $|\mathcal{B}_{i-1}|\Delta_{i-1} \leq 2e(G'_{i-1}) = 2U_{i-1}$, we have that

$$\frac{|\mathcal{B}_{i-1}|}{k'} \overset{(13)}{\leq} \frac{\varepsilon^{3/2}4^{i-1}U_{i-1}}{4\mathbb{E}(U_0)} \overset{(7)}{\leq} \frac{\varepsilon^{3/2}4^{i-1}U_{i-1}}{2U_0} \overset{(8)}{=} \frac{\varepsilon^{3/2}}{2^i}.$$

Thus

$$\sum_{j \geq 0} |\mathcal{B}_j| \leq \varepsilon^{3/2} k' \sum_{j=0}^{\infty} 2^{-(j+1)} = \varepsilon^{3/2} k'. \tag{14}$$

In the ith stage of our algorithm, we discard all those pairs in $\mathcal{U}_{i-1}$ which contain at least one candidate branch set from $\mathcal{B}_{i-1}$, i.e. we will consider all these pairs as being joined (even if they are not joined). After running all the i^* stages of our algorithm we will make up for this by deleting all the candidate branch sets in $\bigcup_{j \geq 0} \mathcal{B}_j$. By (14) we do not loose too many of the candidate branch sets in this way. More formally, in the ith stage of our algorithm we proceed as follows:

(i) Consider all those pairs in $\mathcal{U}_{i-1}$ which contain at least one candidate branch set from $\mathcal{B}_{i-1}$. If there are at least $7U_{i-1}/8$ such pairs, delete precisely $7U_{i-1}/8$ of them and stop the ith stage of the algorithm.

(ii) If there are less than $7U_{i-1}/8$ such pairs, let $\mathcal{U}_{i-1}^*$ denote the set of all these pairs, let $\mathcal{U}_{i-1}' := \mathcal{U}_{i-1} \setminus \mathcal{U}_{i-1}^*$ and $U_{i-1}' := |\mathcal{U}_{i-1}'|$. So we are aiming to join precisely $7U_{i-1}/8 - |\mathcal{U}_{i-1}^*| =: U_{i-1}''$ pairs in $\mathcal{U}_{i-1}'$ by using paths in $\mathcal{P}_i$. To do this, we proceed as follows:

   (a) Let $u_1, \ldots, u_{U_{i-1}'}$ be an ordering of the pairs in $\mathcal{U}_{i-1}'$ and let $P_1, \ldots, P_{U_{i-1}}$ be an ordering of the paths in $\mathcal{P}_i$.

   (b) For each P_j in turn, consider the set of all remaining pairs in $\mathcal{U}_{i-1}'$. If there are precisely $U_{i-1}' - U_{i-1}''$ such pairs, stop the ith stage of the algorithm. (This means that we have successfully joined the desired number U_{i-1}'' of pairs in $\mathcal{U}_{i-1}'$.) If there are more than $U_{i-1}' - U_{i-1}''$ such pairs, then among all these pairs let u_ℓ be the one with the smallest index such that $P_j \sim u_\ell$ (if such an u_ℓ exists), remove u_ℓ and continue with P_{j+1}. If there is no such u_ℓ, then continue with P_{j+1}. In the latter case we say that P_j *fails*.

$\mathcal{U}_i$ will be the subset of $\mathcal{U}_{i-1}$ obtained after the ith stage of our algorithm. So our aim is to show that a.a.s. $|\mathcal{U}_i| = U_i$ for all $i = 1, \ldots, i^*$. To do this, it suffices to show that, for all $i = 1, \ldots, i^*$, if we are in Part (ii) of the algorithm, then in Part (b) a.a.s. to U_{i-1}'' paths in $\mathcal{P}_i$ we assign pairs in $\mathcal{U}_{i-1}'$. As $|\mathcal{P}_i| - U_{i-1}'' = U_{i-1} - U_{i-1}'' \geq U_{i-1}/8$ we have that

$$\mathbb{P}(\geq |\mathcal{P}_i| - U_{i-1}'' \text{ paths in } \mathcal{P}_i \text{ failed}) \leq \mathbb{P}(\geq U_{i-1}/8 \text{ paths in } \mathcal{P}_i \text{ failed})$$

and so it suffices to show that the latter probability is sufficiently small. Note that this probability is determined by the bipartite subgraph G^* of $G_{n,p}$ whose first vertex class is V' and whose second vertex class consists of all the vertices on the paths in $\mathcal{P}_i$. Moreover, note that none of the events considered in Sections 3.1 and 3.2 affects the existence of edges in G^*. Furthermore, the success of different stages of the algorithm depends on edge-disjoint subgraphs of G^*. So by the principle of deferred decisions, when analysing the success of the ith stage, we may assume that $\mathcal{U}_{i-1}$ is fixed (and thus also $\mathcal{U}_{i-1}'$) and that the bipartite graph G_i^* between the vertices in the candidate branch sets belonging to $\mathcal{U}_{i-1}$ and the vertices in the paths in $\mathcal{P}_i$ is a binomial random graph with edge probability p. Moreover, when analysing the success of the ith stage the events we consider can be viewed as sets of bipartite graphs having the same vertex classes as G_i^*.

3.4. **Analysis of the greedy algorithm.** Recall that all the paths in $\mathcal{P}_i$ consist of ℓ_i vertices. Let $\hat{p}_i$ denote the probability that a given path $P \in \mathcal{P}_i$ is joined to a given candidate branch set B, i.e. that $P \sim B$. Thus

$$\hat{p}_i = 1 - (1 - \hat{p})^{\ell_i} \leq 1 - (1 - \ell_i \hat{p}) = \ell_i \hat{p}. \tag{15}$$

Moreover, if $\hat{p}\ell_i \leq 1$ then as in (11)

$$(16) \qquad \hat{p}_i = 1 - (1 - \hat{p})^{\ell_i} \geq \ell_i \hat{p} - \frac{(\ell_i \hat{p})^2}{2} \geq \frac{\ell_i \hat{p}}{2}$$

while if $\hat{p}\ell_i \geq 1$ then

$$(17) \qquad \hat{p}_i = 1 - (1 - \hat{p})^{\ell_i} \geq 1 - (1 - 1/\ell_i)^{\ell_i} \geq 1/2.$$

As $|\mathcal{P}_i| = U_{i-1}$ we have that

$$\mathbb{P}(\geq U_{i-1}/8 \text{ paths in } \mathcal{P}_i \text{ failed}) = \sum_{A \subseteq [U_{i-1}], \ |A| \geq U_{i-1}/8} \mathbb{P}(P_j \text{ fails iff } j \in A)$$

$$(18) \qquad \qquad \leq 2^{U_{i-1}} \max_{A \subseteq [U_{i-1}], \ |A| \geq U_{i-1}/8} \mathbb{P}(P_j \text{ fails iff } j \in A).$$

So let us now consider any $A \subseteq [U_{i-1}]$ with $|A| \geq U_{i-1}/8$. For all $j \in A$ put $A_j := A \cap [j-1]$. Then

$$(19) \qquad \mathbb{P}(P_j \text{ fails iff } j \in A) \leq \mathbb{P}(P_j \text{ fails } \forall j \in A) = \prod_{j \in A} \mathbb{P}(P_j \text{ fails } \mid P_r \text{ fails } \forall r \in A_j).$$

To estimate the latter probability, we split up the event that P_r fails for all $r \in A_j$ further. Let $\mathcal{W}_j$ be the set of all bipartite graphs whose first vertex class V^* consists of all the vertices in the candidate branch sets belonging to $\mathcal{U}_{i-1}$ and whose second vertex class is $V(P_1) \cup \cdots \cup V(P_{j-1})$. We view each $W \in \mathcal{W}_j$ as the event that the bipartite subgraph of G_i^* induced by V^* and $V(P_1) \cup \cdots \cup V(P_{j-1})$ equals W (G_i^* was defined at the end of Section 3.3). Note that for every $W \in \mathcal{W}_j$ the intersection of the event W and the event that P_r fails for all $r \in A_j$ is either W or empty. Let $\mathcal{W}_j^*$ be the set of all those $W \in \mathcal{W}_j$ for which the former holds. Then

$$\mathbb{P}(P_j \text{ fails } \mid P_r \text{ fails } \forall r \in A_j) =$$

$$= \sum_{W \in \mathcal{W}_j^*} \mathbb{P}(P_j \text{ fails } \mid (P_r \text{ fails } \forall r \in A_j) \cap W) \mathbb{P}(W \mid P_r \text{ fails } \forall r \in A_j)$$

$$\leq \left(\max_{W \in \mathcal{W}_j^*} \mathbb{P}(P_j \text{ fails } \mid (P_r \text{ fails } \forall r \in A_j) \cap W) \right) \sum_{W \in \mathcal{W}_j^*} \mathbb{P}(W \mid P_r \text{ fails } \forall r \in A_j)$$

$$(20) \qquad = \max_{W \in \mathcal{W}_j^*} \mathbb{P}(P_j \text{ fails } \mid W).$$

Now note that for each $W \in \mathcal{W}_j^*$ there is a set $\mathcal{U}_W \subseteq \mathcal{U}_{i-1}'$ such that whenever the event W occurs, the set of remaining pairs in $\mathcal{U}_{i-1}'$ after we have considered $P_1, \ldots, P_{j-1}$ in the ith stage of our algorithm is precisely $\mathcal{U}_W$ (i.e. the set of remaining pairs in $\mathcal{U}_{i-1}'$ is the same for every bipartite graph belonging to the event W). Thus

$$\mathbb{P}(P_j \text{ fails } \mid W) = \frac{\mathbb{P}(P_j \text{ fails } \cap W)}{\mathbb{P}(W)} = \frac{\mathbb{P}((P_j \not\sim u \ \forall u \in \mathcal{U}_W) \cap W)}{\mathbb{P}(W)}$$

$$(21) \qquad = \frac{\mathbb{P}(P_j \not\sim u \ \forall u \in \mathcal{U}_W) \mathbb{P}(W)}{\mathbb{P}(W)} = \mathbb{P}(P_j \not\sim u \ \forall u \in \mathcal{U}_W).$$

To estimate the latter probability, we may assume without loss of generality that $\mathcal{U}_W$ consists of $u_1, \ldots, u_{|\mathcal{U}_W|}$. Note that

$$(22) \qquad |\mathcal{U}_W| > U_{i-1}/8$$

as otherwise we would have stopped the ith stage of our algorithm. Moreover,

$$\mathbb{P}(P_j \not\sim u \; \forall u \in \mathcal{U}_W) = \prod_{r=1}^{|\mathcal{U}_W|} \mathbb{P}(P_j \not\sim u_r \mid P_j \not\sim u_s \; \forall s < r)$$

$$= \prod_{r=1}^{|\mathcal{U}_W|} (1 - \mathbb{P}(P_j \sim u_r \mid P_j \not\sim u_s \; \forall s < r))$$

$$(23) \qquad = \prod_{r=1}^{|\mathcal{U}_W|} \left(1 - \frac{\mathbb{P}(P_j \sim u_r, \; P_j \not\sim u_s \; \forall s < r)}{\mathbb{P}(P_j \not\sim u_s \; \forall s < r)} \right).$$

Note that the definition of $\hat{p}_i$ implies that

$$\mathbb{P}(P_j \sim u_1) = \hat{p}_i{}^2.$$

So if all of the above events were independent, then the value of each bracket would be equal to $1 - \hat{p}_i^2$. Our next aim is to show that this is approximately true (see (28) below). Hence we now consider any $2 \le r \le |\mathcal{U}_W|$. Let B and B' denote the candidate branch sets belonging to u_r. Let $\mathcal{U}_{r,1}$ be the set of all those u_s with $s < r$ which contain B or B'. Let $\mathcal{U}_{r,2}$ consist of those u_s with $s < r$ not in $\mathcal{U}_{r,1}$ for which at least one of their candidate branch sets belongs to a pair in $\mathcal{U}_{r,1}$. Finally, we let $\mathcal{U}_{r,3} := \{u_1, \dots, u_{r-1}\} \setminus (\mathcal{U}_{r,1} \cup \mathcal{U}_{r,2})$ be the set of the remaining pairs. Note that

$$\begin{aligned} \mathbb{P}(P_j \sim u_r, \; P_j \not\sim u_s \; \forall s < r) &= \mathbb{P}(P_j \sim u_r, \; P_j \not\sim u \; \forall u \in \mathcal{U}_{r,1} \cup \mathcal{U}_{r,2} \cup \mathcal{U}_{r,3}) \\ &= \mathbb{P}(P_j \sim u_r, \; P_j \not\sim u \; \forall u \in \mathcal{U}_{r,1} \cup \mathcal{U}_{r,3}) \\ (24) \qquad &= \mathbb{P}(P_j \sim u_r, \; P_j \not\sim u \; \forall u \in \mathcal{U}_{r,1}) \mathbb{P}(P_j \not\sim u \; \forall u \in \mathcal{U}_{r,3}). \end{aligned}$$

Indeed, to see the second equality let $\mathcal{C}_r$ denote the set of all those candidate branch sets C with $(B, C) \in \mathcal{U}_{r,1}$ or $(B', C) \in \mathcal{U}_{r,1}$. Recall that $P_j \sim u_r$ means $P_j \sim B$ and $P_j \sim B'$. So if $P_j \sim u_r$ and $P_j \not\sim u$ for all $u \in \mathcal{U}_{r,1}$ then $P_j \not\sim C$ for all $C \in \mathcal{C}_r$. But this means that automatically $P_j \not\sim u$ for any $u \in \mathcal{U}_{r,2}$. The last equality holds since the events $P_j \sim u_r$, $P_j \not\sim u \; \forall u \in \mathcal{U}_{r,1}$ and $P_j \not\sim u \; \forall u \in \mathcal{U}_{r,3}$ are independent as they involve disjoint families of candidate branch sets. Now

$$(25) \quad \mathbb{P}(P_j \sim u_r, \; P_j \not\sim u \; \forall u \in \mathcal{U}_{r,1}) = \mathbb{P}(P_j \sim B, \; P_j \sim B', \; P_j \not\sim C \; \forall C \in \mathcal{C}_r) = \hat{p}_i^2 (1 - \hat{p}_i)^{|\mathcal{C}_r|}.$$

We wish to show that the right hand side of (25) is at least $\hat{p}_i^2/2$. So we have to show that $\hat{p}_i|\mathcal{C}_r|$ is small. To do this, note that $B, B' \notin \mathcal{B}_{i-1}$ since $u_r = (B, B') \in \mathcal{U}_W \subseteq \mathcal{U}'_{i-1}$. Thus $|\mathcal{C}_r| \le d_{G'_{i-1}}(B) + d_{G'_{i-1}}(B') \le 2\Delta_{i-1}$ and so

$$\hat{p}_i|\mathcal{C}_r| \quad \le \quad 2\hat{p}_i\Delta_{i-1} \overset{(15),(13)}{\le} 2\hat{p}\ell_i \frac{8\mathbb{E}(U_0)}{\varepsilon^{3/2}4^{i-1}k'} \overset{(12),(10)}{\le} 3pt\frac{|Q_1|}{U_0} \cdot \frac{8\mathbb{E}(U_0)}{\varepsilon^{3/2}k}$$

$$(26) \qquad \overset{(9),(7)}{\le} \quad \frac{3pn}{k} \cdot \frac{\varepsilon^2 n}{2\ln(np)} \cdot \frac{10}{\varepsilon^{3/2}k} \le 20\sqrt{\varepsilon}.$$

Together with (25) and the fact that $1 - \hat{p}_i \ge e^{-\hat{p}_i - \hat{p}_i^2} \ge e^{-2\hat{p}_i}$ (see e.g. [3, p. 5]) this implies that

$$(27) \qquad \mathbb{P}(P_j \sim u_r, \; P_j \not\sim u \; \forall u \in \mathcal{U}_{r,1}) \ge \hat{p}_i^2 e^{-2\hat{p}_i|\mathcal{C}_r|} \ge \hat{p}_i^2/2.$$

Thus

$$
\frac{\mathbb{P}(P_j \sim u_r, \ P_j \not\sim u_s \ \forall s < r)}{\mathbb{P}(P_j \not\sim u_s \ \forall s < r)} = \frac{\mathbb{P}(P_j \sim u_r, \ P_j \not\sim u_s \ \forall s < r)}{\mathbb{P}(P_j \not\sim u \ \forall u \in \mathcal{U}_{r,1} \cup \mathcal{U}_{r,2} \cup \mathcal{U}_{r,3})}
$$

$$
\geq \frac{\mathbb{P}(P_j \sim u_r, \ P_j \not\sim u_s \ \forall s < r)}{\mathbb{P}(P_j \not\sim u \ \forall u \in \mathcal{U}_{r,3})}
$$

$$
(28) \qquad\qquad \overset{(24)}{=} \ \mathbb{P}(P_j \sim u_r, \ P_j \not\sim u \ \forall u \in \mathcal{U}_{r,1}) \overset{(27)}{\geq} \hat{p}_i^2/2
$$

and hence

$$
\mathbb{P}(P_j \not\sim u \ \forall u \in \mathcal{U}_W) \overset{(23),(28)}{\leq} \left(1 - \hat{p}_i^2/2\right)^{|\mathcal{U}_W|} \overset{(22)}{\leq} \exp\left(-\hat{p}_i^2 U_{i-1}/16\right).
$$

Together with (19)–(21) this in turn implies that

$$
\mathbb{P}(P_j \text{ fails iff } j \in A) \leq \exp\left(-\hat{p}_i^2 U_{i-1}|A|/16\right) \leq \exp\left(-\hat{p}_i^2 (U_{i-1})^2/2^7\right)
$$

for any subset $A \subseteq [U_{i-1}]$ with $|A| \geq U_{i-1}/8$. Together with (18) this gives

(29)
$$
\mathbb{P}(\geq U_{i-1}/8 \text{ paths in } \mathcal{P}_i \text{ failed}) \leq 2^{U_{i-1}} \exp\left(-\hat{p}_i^2 (U_{i-1})^2/2^7\right) \leq \exp\left(U_{i-1}(1 - \hat{p}_i^2 U_{i-1}/2^7)\right).
$$

Together with (17) this implies that $\mathbb{P}(\geq U_{i-1}/8 \text{ paths in } \mathcal{P}_i \text{ failed}) \leq e^{-U_{i-1}}$ if $\hat{p}\ell_i \geq 1$. So we may assume that $\hat{p}\ell_i \leq 1$ and thus $\hat{p}_i \geq \ell_i \hat{p}/2$ by (16). But then

$$
\hat{p}_i^2 U_{i-1} \geq \left(\frac{\ell_i \hat{p}}{2}\right)^2 U_{i-1} \overset{(10)}{=} \frac{4^{i-1}\ell_1 |Q_i| \hat{p}^2}{4} \overset{(12)}{\geq} \frac{4^{i-1}|Q_i| p^2 t^2}{5} \overset{(9)}{\geq} \frac{\varepsilon^2 n}{2\ln(np)} \cdot \frac{p^2 n^2}{6k^2}
$$

$$
(30) \qquad\qquad\qquad \geq \frac{\varepsilon^2 pn}{12} \geq 2^8.
$$

Together with (29) this implies that in both cases we have

(31)
$$
\mathbb{P}(\geq U_{i-1}/8 \text{ paths in } \mathcal{P}_i \text{ failed}) \leq e^{-U_{i-1}} = o(1/\ln n).
$$

The final bound follows from the fact that $U_{i-1} \geq U_{i^*} = n^{1/3}$. Note that the derivation of (30) made essential use of the fact that the probability $\hat{p}_i^2$ that a given path joins a given pair of candidate branch sets grows quadratically in ℓ_i in each stage. This is the reason why it is worthwhile to use fewer but longer paths in each stage.

As $i^* \leq \ln n$, (31) implies that a.a.s. for all $i = 1, \ldots, i^*$ our algorithm produces sets $\mathcal{U}_i$ with $|\mathcal{U}_i| = U_i$. Recall that each pair of candidate branch sets in $\mathcal{U}_0$ which has not been joined by the algorithm either lies in $\mathcal{U}_{i^*}$ or contains some candidate branch set belonging to $\bigcup_{j \geq 0} \mathcal{B}_j$. So by deleting all the candidate branch sets in $\bigcup_{j \geq 0} \mathcal{B}_j$ as well as one candidate branch set from every pair in $\mathcal{U}_{i^*}$ we obtain a complete graph of order $k' - U_{i^*} - |\bigcup_{j \geq 0} \mathcal{B}_j| \geq k$ as minor. (The last inequality follows from (14) and the fact that $U_{i^*} = n^{1/3}$.)

4. Proof of Theorem 3

To verify the upper bound of Theorem 3 note that a.a.s. $G_{n,c/n}$ has at most cn edges. But this implies that a.a.s. the order k of the largest complete minor in $G_{n,c/n}$ satisfies $\binom{k}{2} \leq cn$, i.e. $k \leq 2\sqrt{cn}$.

To prove the lower bound of Theorem 3 we are aiming to show that for every constant $c > 1$ there exists a constant $\delta = \delta(c) > 0$ such that a.a.s. $G_{n,c/n}$ contains a K_k minor where

$$
k := \delta\sqrt{n}.
$$

(So in what follows, we will assume that δ is sufficiently small compared to $c - 1$ for our estimates to hold.) To show this, we will modify our argument from Section 3. Instead of

partitioning the vertex set of $G_{n,c/n}$ into two sets V' and V'' and considering a long path in each of them, this time we will work with a single long path in two stages. We will obtain this path in the first stage of a two-round exposure of $G_{n,c/n}$. More precisely, let $c_1 := (c+1)/2$. Thus $1 < c_1 < c$. Put $c_2 := (c - c_1)/(1 - c_1/n)$. So $c/n = c_1/n + c_2/n - c_1 c_2/n^2$. In our two-round exposure of $G_{n,c/n}$ we first generate a random graph $G_{n,c_1/n}$ and then independently $G_{n,c_2/n}$. We then take their union, ignoring multiple edges again. The choice of c_1 ensures that there exists an $\alpha = \alpha(c) > 0$ such that a.a.s. $G_{n,c_1/n}$ contains a path P on $2\alpha n$ vertices (see [2]). We split P into two consecutive disjoint paths P' and P'' each having αn vertices. We split P' further into $k' := 2k = 2\delta\sqrt{n}$ consecutive disjoint subpaths $B_1, \ldots, B_{k'}$ each containing $t := \alpha\sqrt{n}/(2\delta)$ vertices. The B_i's will be our candidate branch sets.

We will now use the greedy algorithm described in Section 3.3 to join k of these candidate branch sets into a K_k minor in the second round of our two-round exposure of $G_{n,c/n}$. Thus similarly as before, let $\mathcal{U}_0$ denote the set of all those pairs B_i, B_j of candidate branch sets which are not joined by an edge in $G_{n,c_2/n}$. Let $U_0 := |\mathcal{U}_0|$ and let G'_0 denote the auxiliary random graph whose vertices are the B_i and whose edges correspond to the pairs in $\mathcal{U}_0$. So the edges of G'_0 occur independently with probability $q := (1 - c_2/n)^{t^2}$. Note that if δ is sufficiently small compared to $c - 1$ then

$$q = (1 - c_2/n)^{(\alpha\sqrt{n}/(2\delta))^2} \le \exp(-c_2\alpha^2/(4\delta^2)) \le \delta^4$$

and so

$$\mathbb{E}(U_0) = \binom{k'^2}{2}q \le 2\delta^2 nq \le \delta^5 n/2.$$

Again, we may assume that $\mathbb{E}(U_0) \ge n^{2/5}$. (In fact, an easy calculation shows that $\mathbb{E}(U_0)$ is linear in n.) Thus

$$(32) \qquad\qquad \mathbb{E}(U_0)/2 \le U_0 \le 2\mathbb{E}(U_0) \le \delta^5 n$$

by Chebyshev's inequality. The path P'' will play the role of the long path inside $G_{n,p}[V'']$ considered in Section 3.2. We define i^* and $U_1, \ldots, U_{i^*}$ as there and divide P'' into disjoint consecutive subpaths $Q_1, \ldots, Q_{i^*}$ where

$$(33) \qquad\qquad |Q_i| = \frac{\delta^3 \alpha n}{2^i} \quad \text{and so} \quad |Q_i| = \frac{|Q_1|}{2^{i-1}}.$$

We then split each Q_i into a set $\mathcal{P}_i$ of smaller paths, each containing ℓ_i vertices, where $\ell_i = |Q_i|/U_{i-1}$ as before. So by (32) we have that

$$(34) \qquad\qquad \ell_1 = \frac{|Q_1|}{U_0} \ge \frac{\alpha}{2\delta^2} \quad \text{and} \quad \ell_i = 4^{i-1}\ell_1 = \frac{4^{i-1}|Q_1|}{U_0}.$$

Similarly as in Section 3.3, our aim is to apply the greedy algorithm for each $i = 1, \ldots, i^*$ in turn to join all but precisely U_i of the pairs in $\mathcal{U}_0$ that are still unjoined after the $(i-1)$th stage of the algorithm, using one path in $\mathcal{P}_i$ for each pair. So suppose that $1 \le i < i^*$ and after the $(i-1)$th stage of our algorithm we have a set $\mathcal{U}_{i-1} \subseteq \mathcal{U}_0$ of unjoined pairs such that $|\mathcal{U}_{i-1}| = U_{i-1}$ and suppose that we now wish to run the ith stage of our algorithm. As before, let G'_{i-1} denote the spanning subgraph of G'_0 whose edge set corresponds to $\mathcal{U}_{i-1}$. Again, in the ith stage of the algorithm we will first discard all those pairs in $\mathcal{U}_{i-1}$ which contain a candidate branch set lying in too many pairs from $\mathcal{U}_{i-1}$. This time, for all $i = 1, \ldots, i^*$ we put

$$(35) \qquad\qquad \Delta_{i-1} = \frac{8\mathbb{E}(U_0)}{\delta^{1/2}4^{i-1}k'}$$

and let $\mathcal{B}_{i-1}$ be the set of all those candidate branch sets whose degree in G'_{i-1} is greater than Δ_{i-1}. We now run the algorithm described in Section 3.3 and have to show that $\mathbb{P}(\geq U_{i-1}/8$ paths in $\mathcal{P}_i$ failed) is small.

So as before, let $\hat{p}$ denote the probability that a particular vertex is joined to a particular candidate branch set in $G_{n,c_2/n}$. Thus $\hat{p} = 1 - (1 - c_2/n)^t$. Since $1 - c_2 t/n \leq (1 - c_2/n)^t \leq 1 - c_2 t/(2n)$ by (11) we have that

$$(36) \qquad \frac{c_2\alpha}{4\delta\sqrt{n}} = \frac{c_2 t}{2n} \leq \hat{p} \leq \frac{c_2 t}{n} = \frac{c_2\alpha}{2\delta\sqrt{n}}.$$

Then as before, the probability $\hat{p}_i$ that a given path from $\mathcal{P}_i$ is joined to a particular candidate branch set satisfies (15)–(17). Thus

$$\hat{p}_i\Delta_{i-1} \overset{(15),(35)}{\leq} \hat{p}\ell_i \frac{8\mathbb{E}(U_0)}{\delta^{1/2}4^{i-1}k'} \overset{(36),(34)}{\leq} \frac{c_2\alpha}{2\delta\sqrt{n}} \cdot \frac{|Q_1|}{U_0} \cdot \frac{4\mathbb{E}(U_0)}{\delta^{3/2}\sqrt{n}} \overset{(33),(32)}{\leq} \frac{2c_2\alpha^2\delta^3 n}{\delta^{5/2}n} \leq \delta^{1/3}$$

if δ is sufficiently small compared to $c-1$. This gives an analogue to (26) and thus as before we can show that (29) holds. Again, if $\hat{p}\ell_i \geq 1$, this implies that $\mathbb{P}(\geq U_{i-1}/8$ paths in $\mathcal{P}_i$ failed) $\leq e^{-U_{i-1}}$. So suppose that $\hat{p}\ell_i < 1$ and thus $\hat{p}_i \geq \ell_i\hat{p}/2$ by (16). Hence

$$\hat{p}_i^2 U_{i-1} \geq \left(\frac{\ell_i\hat{p}}{2}\right)^2 U_{i-1} \overset{(34)}{=} \frac{4^{i-1}\ell_1|Q_i|}{4}\hat{p}^2 \overset{(36)}{\geq} \frac{4^{i-1}\ell_1|Q_i|}{4} \cdot \frac{c_2^2\alpha^2}{16\delta^2 n} \overset{(33)}{\geq} \frac{c_2^2\delta\alpha^3\ell_1}{2^7} \overset{(34)}{\geq} \frac{c_2^2\alpha^4}{2^8\delta} \geq 2^8$$

if δ is sufficiently small compared to $c-1$. So (30) still holds and as before this shows that

$$\mathbb{P}(\geq U_{i-1}/8 \text{ paths in } \mathcal{P}_i \text{ failed}) \leq e^{-U_{i-1}} = o(1/\ln n)$$

holds in this case too. As before this implies that a.a.s. our algorithm produces sets $\mathcal{U}_i$ with $|\mathcal{U}_i| = U_i$ for all $i = 1, \ldots, i^*$. Similarly as in (14) one can show that $\sum_{j\geq 0}|\mathcal{B}_j| \leq \delta^{1/2}k'$. Thus by deleting all the candidate branch sets in $\bigcup_{j\geq 0}\mathcal{B}_j$ as well as one candidate branch set from every pair in $\mathcal{U}_{i^*}$ we obtain a complete graph of order $k' - U_{i^*} - |\bigcup_{j\geq 0}\mathcal{B}_j| \geq k$ as a minor. This completes the proof of the lower bound in Theorem 3.

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Nikolaos Fountoulakis, Daniela Kühn & Deryk Osthus
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

E-mail addresses: {nikolaos,kuehn,osthus}@maths.bham.ac.uk