Topological minors in graphs of large girth

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Abstract

We prove that every graph of minimum degree at least r and girth at least 186 contains a subdivision of K_{r+1} and that for $r \geq 435$ a girth of at least 15 suffices. This implies that the conjecture of Hajós that every graph of chromatic number at least r contains a subdivision of K_r (which is false in general) is true for graphs of girth at least 186 (or 15 if $r \geq 436$). More generally, we show that for every graph H of maximum degree $\Delta(H) \geq 2$, every graph G of minimum degree at least $\max\{\Delta(H), 3\}$ and girth at least $166 \frac{\log |H|}{\log \Delta(H)}$ contains a subdivision of H. This bound on the girth of G is best possible up to the value of the constant and improves a result of Mader, who gave a bound linear in $|H|$.

1 Introduction

A classical result of Mader (see e.g. [5]) states that for every r there exists a smallest number $d(r)$ such that every graph G of minimum degree larger than $d(r)$ has a topological K_r minor, i.e. contains a subdivision of the complete graph K_r on r vertices. Jung [8] observed that the complete bipartite graph $K_{s,s}$ with $s = \lfloor r^2/8 \rfloor$ shows that $d(r) \geq \lfloor r^2/8 \rfloor$, and Bollobás and Thomason [3] as well as Komlós and Szemerédi [10] independently proved that r^2 is the correct order of magnitude for $d(r)$. Recently, Mader [15] showed that if we restrict our attention to graphs G of large girth, then a minimum degree of at least r already guarantees the existence of a subdivision of K_{r+1} in G. His bound on the girth required is linear in r, and in [16] he asked whether a girth of at least five might even suffice (for all r). Here, we show that a large but constant girth suffices:

Theorem 1 *Let $r \in \mathbb{N}$. Then every graph G of minimum degree at least r and girth at least 186 contains a subdivision of K_{r+1}.*

For large r, we can significantly reduce the bound on the required girth:

Theorem 2 *Let $r \in \mathbb{N}$ with $r \geq 435$. Then every graph G of minimum degree at least r and girth at least 15 contains a subdivision of K_{r+1}.*

In [14] we show that a girth of at least five 'nearly' suffices: every C_4-free graph of minimum degree at least r contains a subdivision of a complete graph of order $r^{1-o(1)}$. Also, a result of [11] states that if we only seek an ordinary K_{r+1} minor instead of a topological one, then a girth of at least five does suffice for large enough r. In fact in [12], we prove the following stronger result: every

$K_{s,s}$-free graph of average degree at least r contains a complete graph of order $r^{1+\frac{1}{2(s-1)}-o(1)}$ as minor. (See [11] for a more precise bound in the case $s=2$.)

Returning to topological minors, a well-known conjecture of Hajós (see e.g. [7]) states that every graph of chromatic number at least r contains a subdivision of K_r. This conjecture was disproved by Catlin [4], and Erdős and Fajtlowicz [6] proved that it fails even for almost all graphs. On the other hand, since every graph of chromatic number at least r has a subgraph of minimum degree at least $r-1$, Theorem 1 shows that the conjecture does hold for all graphs of sufficiently large girth:

Corollary 3 *Let $r \in \mathbb{N}$. Then every graph of chromatic number at least r and girth at least 186 contains a subdivision of K_r. If $r \geq 436$ then a girth of at least 15 suffices.* □

Mader [15] also extended his results about subdivisions of complete graphs to subdivisions of arbitrary graphs. He proved that for every graph H there exists a smallest number $f(H)$ such that every graph G of minimum degree at least $\max\{\Delta(H), 3\}$ and girth at least $f(H)$ contains a subdivision of H, and he showed that $f(H)$ is at most linear in $|H|$. Our next result gives a much better bound on $f(H)$:

Theorem 4 *Let H be a graph of maximum degree $\Delta(H) \geq 2$. Then every graph G of minimum degree at least $\max\{\Delta(H), 3\}$ and girth at least*

$$166\,\frac{\log|H|}{\log\Delta(H)}$$

contains a subdivision of H.

In Proposition 12, we show that for every graph H this bound on the girth is best possible up to the value of the constant 166. The following reformulation of Theorem 4 shows that for any family of graphs whose maximum degree is at least polynomial in the number of vertices, large but constant girth (and the appropriate minimum degree) suffices to force a subdivision of any graph in this family:

Corollary 5 *Let $\varepsilon > 0$ and let H be a graph satisfying $\Delta(H) \geq \max\{|H|^{\varepsilon}, 2\}$. Then every graph G of minimum degree at least $\max\{\Delta(H), 3\}$ and girth at least $166/\varepsilon$ contains a subdivision of H.* □

In [13] we prove the following analogue of Theorem 4 for graphs G of large average degree: There exists a constant c such that for all $\varepsilon > 0$ there is an integer $r = r(\varepsilon)$ such that for every graph H with $\Delta(H) \geq r$ every graph G of average degree at least $\Delta(H) - 1 + \varepsilon$ and girth at least $c \log|H| / \log\Delta(H)$ contains a subdivision of H. The proof is based on that of a related result of Mader [17] which does not require $\Delta(H)$ to be large at the expense of a bound on the girth which is at least linear in $|H|$ and also depends on ε.

An important tool in our proofs will be a result of Bollobás and Thomason [2] that every $22c$-connected graph G is c-linked. As is well-known, the graph obtained from K_{3c-1} by deleting c independent edges shows that it is not possible to replace the function $22c$ by anything smaller than $3c-2$. On the other hand, Mader used the result that every $2c$-connected graph which contains a subdivision of $K_{2c,2c}$ is c-linked [15, Lemma 1] to show that a connectivity of at least $2c$ suffices if G has sufficiently large girth [15, Cor. 1]. His bound on the girth required is linear in c. Furthermore, he proved that for $c \geq 2$ one cannot replace $2c$ by $2c-1$. Theorem 4 with $H := K_{2c,2c}$ combined with [15, Lemma 1] immediately implies a bound on the girth which is independent of c:

Corollary 6 *Let $c \in \mathbb{N}$. Then every $2c$-connected graph G of girth at least 250 is c-linked.*

The fact the the girth in Corollary 6 can be chosen to be a constant was observed independently of us by Kawarabayashi [9] and Thomason [18]. In [12] we show that if c is large enough compared to s, then even every $2c$-connected $K_{s,s}$-free graph is c-linked. Corollary 6 combined with the argument in the proof of [15, Thm. 3] implies the following result:

Corollary 7 *Every $2\binom{r}{2}$-connected graph G of girth at least 250 contains a topological K_r minor with prescribed branch vertices.* $\square$

For general graphs G the best known asymptotic bound on the connectivity of G that guarantees a topological K_r minor with prescribed branch vertices is $11r^2$ [2, Cor. 1].

This paper is organized as follows. In the next section, we give a rough outline of the strategy of our proofs. We also introduce the necessary definitions and collect several tools which we will need later on. In Section 3, we prove Theorems 1 and 4 and Corollary 6. In Section 4, we then prove Theorem 2 using more involved arguments.

2 Strategy of the proof and preliminaries

The strategy of the proof of our main results is as follows. Given a graph G of large girth and minimum degree r, it is easy to see that we can partition the vertices of G into rooted induced subtrees of G so that (amongst others) the roots of the trees have degree at least r and each tree sends many edges to many other trees (see Definition 10 and Proposition 11). Thus if we contract each of these trees, the resulting graph G' has large minimum degree. Suppose now that in G we are seeking a subdivision TH of our given graph H of maximum degree at most r. A first attempt to find it might be as follows. G' contains a highly connected subgraph. This subgraph is highly linked (Theorem 8) and hence contains a subdivision TH which corresponds to an ordinary minor of H in G, but unfortunately not necessarily to a topological one.

To deal with this, we choose our highly linked substructure of G' more carfully. We consider a set A of vertices of G' together with their neighbours B having the property that the graph G^* obtained from $G'[A \cup B]$ by contracting an independent set E of A–B edges is highly connected (Lemma 9). We then find $|H|$ disjoint stars in $G'[A \cup B]$ (with centres in A) which correspond both to disjoint stars in G^* as well as subdivided stars in G. As G^* is highly linked, we can link the leaves of the stars to obtain a subdivision of H in G^*. Since each star in G^* corresponds to a subdivided star in G, this subdivision of H in G^* will then correspond to one in G.

The strategy described above is similar to the one employed by Mader [15]. The improvements we obtain are mainly due to a different construction of the stars in $G'[A \cup B]$ and a more careful analysis of the properties of the rooted trees that partition the vertices of G.

We will now collect some definitions and results which we will need later. All logarithms in this paper are base e, where e denotes the Euler number. We denote the maximum degree of a graph G by $\Delta(G)$ and its minimum degree by $\delta(G)$. The *girth* $g(G)$ of a graph G is the length of the shortest cycle in G. A *subdivision* TG of a graph G is a graph obtained from G by replacing the edges of G with internally disjoint paths. The *branch vertices* of TG are all those vertices that correspond to vertices of G. We say that G is a *topological minor* of a graph H if H contains a subdivision of G as a subgraph. An r-star is a star with r leaves. Given a set A of vertices of a graph G, we write $N_G(A)$ for the set of all those neighbours of vertices in A that lie outside A. Given $c \in \mathbb{N}$, we say that a graph G is *c-linked* if $|G| \geq 2c$ and for every $2c$ distinct vertices $x_1, \dots, x_c$ and $y_1, \dots, y_c$ of G there exist disjoint paths $P_1, \dots, P_c$ such that P_i joins x_i to y_i. In the proof of Theorem 2 we will use the simple fact that if G is c-linked then such paths $P_1, \dots, P_c$ also exist when we allow $x_i = y_i$ for some indices $i \leq c$. (P_i will then be trivial for each such i.)

An important tool in our proofs is the following theorem of Bollobás and Thomason [3].

Theorem 8 *Let $c \in \mathbb{N}$. Then every $22c$-connected graph is c-linked.*

The following lemma is essentially due to Mader [15], who proved a slightly stronger result for triangle-free graphs. As the proof is short, we include it here for completeness. The 'moreover' part of Lemma 9 will only be used in [13].

Lemma 9 *Let $c \geq 1$ be an integer and let G be a graph of minimum degree at least $2c$. Then there exist disjoint sets $A, B \subseteq V(G)$ and a set E of $|B|$ independent A–B edges such that $|A| > c$, $|B| < c$, $N_G(A) \subseteq B$ and so that the graph G^* obtained from $G[A \cup B]$ by contracting the edges in E is $\lceil c/3 \rceil$-connected. Moreover, every vertex in B has at least two neighbours in A.*

Proof. Choose $A \subseteq V(G)$ minimal such that A is non-empty and $|N_G(A)| < c$, and let $B := N_G(A)$. Clearly $|A| > c$ and every vertex in B has at least two neighbours in A. Moreover, note that there exists a set E of $|B|$ independent A–B edges in G. Indeed, if not then by Menger's theorem there is a set S of less than $|B|$ vertices in $A \cup B$ separating A from B in $G[A \cup B]$. But then $A \setminus S$ is a non-empty proper subset of A and $|N_G(A \setminus S)| \leq |S| < |B| < c$, contradicting the choice of A.

Let us now show that the graph G^* obtained from $G[A \cup B]$ by contracting the edges in E is $\lceil c/3 \rceil$-connected. Let S be a separator of G^*. Let E^* be the set of vertices in G^* corresponding to the edges in E and let C^* be a component of $G^* - S$ with $|V(C^*) \cap E^*| \leq |E^* - S|/2$. Let $C \subseteq V(G)$ be the set of all those vertices in A whose images in G^* belong to C^*. Then

$$\begin{aligned}|N_G(C)| &\leq |S| + |E^* \cap S| + |V(C^*) \cap E^*| \leq |S| + |E^* \cap S| + |E^* - S|/2 \\ &= |S| + |E^* \cap S|/2 + |E^*|/2 \leq 3|S|/2 + c/2.\end{aligned}$$

On the other hand, since C is a non-empty proper subset of A, we have $|N_G(C)| \geq c$ and therefore $|S| \geq c/3$, as required. $\square$

Given two vertices x and y of a graph G, the length of the shortest x–y path in G is called the *distance of x and y* and denoted by $d_G(x,y)$ (here the *length* of a path is the number of its edges). The *ℓ-ball* around a vertex x of a graph G is the subgraph of G induced by all vertices in G of distance at most ℓ from x (including x itself). We denote this subgraph by $B^\ell_G(x)$. A set X of vertices of a graph G is *ℓ-independent* if the vertices in X have pairwise distance at least $\ell+1$ from each other. So the 1-independent subsets of $V(G)$ are precisely the sets of independent vertices.

We say that a graph G is *partitioned into trees* $T_1, \ldots, T_k$ if the T_i are disjoint induced subtrees of G and $V(G) = V(T_1) \cup \ldots V(T_k)$. We often denote by T_x a tree with root x.

Definition 10 Given $\ell \in \mathbb{N}$ and a set X of vertices of a graph G, a family $(T_x)_{x\in X}$ of rooted subtrees of G is called *a family of ℓ-trees assigned to X* if it satisfies the following properties for all $x \in X$.

(i) G is partitioned into T_x, $x \in X$.

(ii) In T_x every vertex has distance at most 2ℓ from x.

(iii) $B^\ell_G(x)$ is a (rooted) subtree of T_x. In particular, T_x has at least $\delta(G)(\delta(G)-1)^{\ell-1}$ leaves and therefore it sends at least $\delta(G)(\delta(G)-1)^\ell$ edges to vertices outside T_x.

Proposition 11 *Let $\ell \geq 1$ be an integer and let X be a maximal 2ℓ-independent set of vertices of a graph G. If the girth of G is at least $4\ell+2$, then there exists a family of ℓ-trees assigned to X.*

Proof. Since $g(G) \geq 2\ell+2$, for every $x \in X$ the ℓ-ball $B^\ell_G(x)$ is a tree. Since X is 2ℓ-independent, these trees are disjoint for different $x \in X$. Extend these trees to connected subgraphs T_x $(x \in X)$ by adding first every vertex of distance $\ell+1$ from X to one of the trees to which it is adjacent, then adding all vertices of distance $\ell+2$ from X to one of the subgraphs constructed in the previous step to which it is adjacent etc. By the choice of X, each vertex of G has distance at most 2ℓ from X. Thus, for every $x \in X$, each vertex in T_x has distance at most 2ℓ from x. Since $g(G) \geq 4\ell+2$, each T_x is an induced subtree of G. So $(T_x)_{x\in X}$ is a family of ℓ-trees assigned to X. $\square$

3 Subdivisions of arbitrary graphs

Let us first observe that Theorem 4 is best possible up to the value of the constant:

Proposition 12 *Let H be a graph of maximum degree $\Delta(H) \geq 2$ and put*

$$g := \left\lfloor \frac{\log |H|}{\log \Delta(H)} \right\rfloor.$$

Then either $g \leq 2$ or there exists a graph G with minimum degree at least $\max\{\Delta(H), 3\}$ and girth at least g which does not contain a subdivision of H.

Note that if $g \leq 2$ and if there is any graph G with $\delta(G) \geq \max\{\Delta(H), 3\}$ which does not contain a subdivision of H, then we also cannot replace the constant 166 in Theorem 4 by anything less than one. (The example where H is a star shows that such a graph G need not exist.)

Proposition 12 is an immediate consequence of the following upper bound on the minimum order of a graph of large girth and large minimum degree which is due to Sauer (who proved a slightly sharper bound), for a proof see [1, Ch.III Thm. 1.4].

Theorem 13 *Let $k \geq 3$ and $g \geq 3$ be integers. Then there exists a graph G of order less than $(k-1)^g$ whose minimum degree is at least k and whose girth is at least g.*

Proof of Proposition 12. Suppose that $g \geq 3$ and let $k := \Delta(H)+1$. Apply Theorem 13 to obtain a graph G as described there. Then

$$|G| < (k-1)^g \leq \exp\left(\frac{\log(k-1)\log|H|}{\log \Delta(H)}\right) = |H|.$$

So G cannot contain a subdivision of H. □

Given two vertices v and w of a rooted tree T_x, we say that v *lies above* w if w lies on the path in T_x joining v to the root x of T_x. We denote by $\mathrm{br}_{T_x}(w)$ the subtree of T_x induced by all its vertices above w (including w itself).

Instead of proving Theorems 1 and 4 directly, we will derive them from the following slightly stronger result.

Theorem 14 *Given a graph H, put $r := \max\{\Delta(H), 3\}$ and let $\ell \geq 2$ be an integer satisfying*

$$r(r-1)^\ell \geq 6 \cdot (22e(H) + |H|). \tag{1}$$

Then every graph G of minimum degree at least r and girth at least $20\ell + 6$ contains a subdivision of H.

Proof. Put $h := |H|$, $m := e(H)$ and $c := 3 \cdot (22m + h)$. Let X be a maximal 2ℓ-independent subset of $V(G)$. Since $g(G) \geq 4\ell + 2$, we may apply Proposition 11 to obtain a family $(T_x)_{x \in X}$ of ℓ-trees assigned to X. Let G' be the graph obtained from G by contracting each T_x. For each $x \in X$, we denote by x' the vertex of G' corresponding to the (contracted) tree T_x. Note that condition (ii) of Definition 10 implies that

$$g(G') \geq \frac{g(G)}{4\ell + 1} > 5.$$

In particular, G' does not contain multiple edges, i.e. each pair of distinct T_x is joined by at most one edge in G. Together with condition (iii) of Definition 10 this implies that

$$\delta(G') \geq r(r-1)^\ell \overset{(1)}{\geq} 2c. \tag{2}$$

Apply Lemma 9 to G' to obtain sets $A, B \subseteq V(G')$ and a set E of $|B|$ edges as described there. So in particular, $N_{G'}(a) \subseteq A \cup B$ for all $a \in A$ and $|B| < c$. Together with (2) this implies that $G'[A]$ contains a star S with h leaves, $x'_1, \ldots, x'_h$ say. Let x'_0 be the centre of S and let $u_1, \ldots, u_h$ be an enumeration of the vertices of H. Let d_i be the degree of u_i in H. We claim that for all $i = 1, \ldots, h$ there is a d_i-star S'_i in $G' - x'_0$ whose centre is x'_i and which satisfies the following two properties.

(a) No edge of S'_i belongs to E.

(b) T_{x_i} contains a subdivided d_i-star S_i with centre x_i and leaves $v_{i1}, \ldots, v_{id_i}$ such that for each v_{ij} there exists a leaf x'_{ij} of S'_i such that v_{ij} sends an edge to $T_{x_{ij}}$ and $x'_{ij} \neq x'_{ik}$ for all $j \neq k$.

Before proving the existence of such stars S'_i, let us explain why they are useful. Our aim is to show that G contains a subdivision of H in which x_i is the branch vertex corresponding to u_i. The subdivided edge corresponding to an edge $u_i u_j$ of H will consist of a path in S_i joining x_i to some v_{is}, a path in S_j joining x_j to some v_{jt} and a suitable v_{is}–v_{jt} path P_{ij} in G. How can we find disjoint such paths P_{ij}? By our choice of A, B and E, the graph G^* obtained from $G'[A \cup B]$ by contracting the edges in E is highly connected. Moreover, our definition of the stars S'_i will imply that all the x'_i and all the x'_{ij} correspond to distinct vertices in G^*. So in G^* we may suitably link the leaves of the S'_i to obtain a subdivision of H. If P^*_{ij} is the path in G^* obtained in this way that joins the leaf x'_{is} of S'_i to the leaf x'_{jt} of S'_j, then (since v_{is} is adjacent to $T_{x_{is}}$ and v_{jt} is adjacent to T_{jt} by (b)) this path will correspond to a path $P_{ij} \subseteq G$ with the desired properties.

So let us now show that such stars S'_i exist. First note that by condition (iii) of Definition 10 each x_i has $r \geq d_i$ neighbours in T_{x_i} and, for each of these neighbours w_{ij}, the subtree $\mathrm{br}_{T_{x_i}}(w_{ij})$ has at least $(r-1)^{\ell-1} \geq r-1 \geq 2$ leaves. Since no two of them send an edge to the same T_x, there must be a leaf v_{ij} of $\mathrm{br}_{T_{x_i}}(w_{ij})$ which sends no edge to T_{x_0}. Moreover, since T_{x_i} is an induced subtree of G, v_{ij} sends at least $r-1 \geq 2$ edges to vertices outside T_{x_i}, i.e. to (distinct) other trees T_x and at most one of these edges can correspond to an edge in E (since the edges in E are independent). Thus v_{ij} must be incident to some tree $T_{x_{ij}} \neq T_{x_0}$ with $x'_i x'_{ij} \notin E$. Since G has at most one edge joining a pair of distinct T_x, the x_{ij} are distinct for different neighbours w_{ij} of x_i and

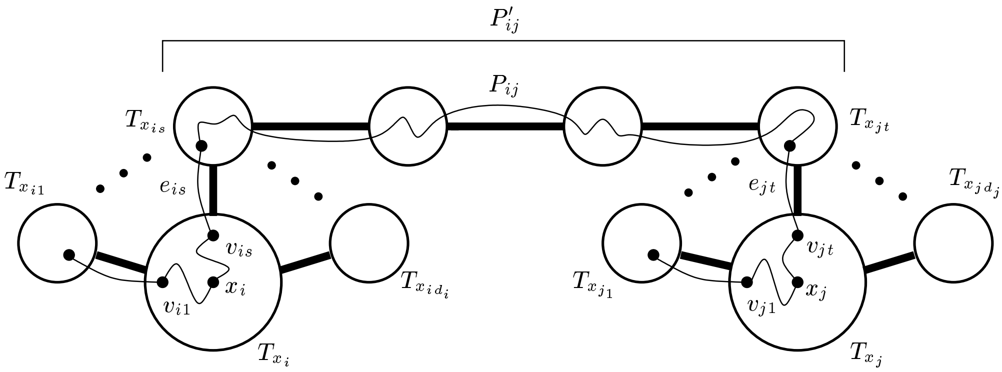

Figure 1: Replacing the path P'_{ij} in G' by a path P_{ij} in G

so we can take for S'_i the d_i-star in $G' - x'_0$ whose centre is x'_i and whose leaves are $x'_{i1}, \dots, x'_{id_i}$.

As all neighbours of x'_i in G' lie in $A \cup B$, the S'_i are d_i-stars in $G'[A \cup B]$. Moreover, since $g(G') \geq 6$, the x'_i have distance at least four in $G' - x'_0$. Thus no edge in E joins vertices belonging to distinct S'_i and, since G' is triangle-free, no edge in E joins two leaves of the same S'_i. Let G^* be the graph obtained from $G'[A \cup B]$ by contracting the edges in E. Then (a) together with the above two properties implies that each S'_i is still a d_i-star in G^* and that all these stars are disjoint in G^*. As G^* is $\lceil c/3 \rceil$-connected, Theorem 8 implies that $G^* - \{x'_1, \dots, x'_h\}$ is m-linked. So for every edge $u_i u_j \in H$ we can find a path P^*_{ij} in $G^* - \{x'_1, \dots, x'_h\}$ joining some leaf x'_{is} of S'_i to some leaf x'_{jt} of S'_j such that all these paths are pairwise disjoint. This gives a subdivision of H in G^* with branch vertices $x'_1, \dots, x'_h$. As P^*_{ij} corresponds to an x'_{is}–x'_{jt} path P'_{ij} in G', this subdivision corresponds to a subdivision of H in G'. But as indicated above, (b) implies that the latter subdivision corresponds to a subdivision of H in G with branch vertices $x_1, \dots, x_h$. Indeed, we only have to replace each S'_i with the subdivided star obtained from S_i by adding a v_{ij}–$T_{x_{ij}}$ edge e_{ij} for every $j = 1, \dots, d_i$ and replace each path P'_{ij} joining a leaf x'_{is} of S'_i to a leaf x'_{jt} of S'_j with some path P_{ij} that contracts to P'_{ij} and joins the endvertex of e_{is} in $T_{x_{is}}$ to the endvertex of e_{jt} in $T_{x_{jt}}$ (Fig. 1). $\square$

Let us now derive Theorems 1 and 4 from Theorem 14.

Proof of Theorem 1. Clearly, we may assume that $r \geq 3$. Setting $\ell := 9$ we have $g(G) \geq 20\ell + 6$. So Theorem 1 follows from Theorem 14 once we have checked that (1) holds with $\ell = 9$. But this is the case since

$$\begin{aligned} 6 \cdot (22e(K_{r+1}) + r + 1) &\leq 6 \cdot 12r(r+1) \leq 2 \cdot 6 \cdot 12r(r-1) \\ &\leq 2^8 r(r-1) \leq r(r-1)^\ell. \end{aligned}$$

$\square$

Note that the calculation in the proof of Theorem 1 immediately shows that for $r \geq 75$ we can take $\ell := 2$ (as in that case $6 \cdot 12(r+1) \leq (r-1)^2$) and thus girth at least 46 suffices.

Proof of Theorem 4. We may assume that $|H| \geq 4$ since otherwise H is a path or a cycle on three vertices and the theorem holds. Let $h := |H|$, $\Delta := \Delta(H)$, $r := \max\{\Delta, 3\}$ and $\ell := \lfloor 8 \log h / \log \Delta \rfloor$. Since $\log h / \log \Delta \geq 1$, this means that $\ell \geq 2$ and

$$g(G) \geq 20 \cdot 8 \log h / \log \Delta + 6 \geq 20\ell + 6.$$

So again Theorem 4 will follow from Theorem 14 once we have checked that (1) holds. For this, first note that (using $e(H) \leq \binom{h}{2}$ and then that $h \geq 4$) we have

$$6 \cdot (22e(H) + h) \leq 6 \cdot 11 \cdot h^2 \leq 4^{3.03} h^2 \leq h^{5.03}.$$

On the other hand

$$r(r-1)^\ell \geq (r-1)^{\ell+1} \geq h^{8 \log(r-1)/\log \Delta} \geq h^{8 \log 2 / \log 3} \geq h^{5.04}.$$

Thus (1) follows. $\square$

4 Subdivisions of cliques

Recall that in the proof of Theorem 14 the girth of G was large enough to guarantee that G could be partitioned into trees so that the graph G' obtained from G by contracting these trees still had girth at least six. We used this to show that G' has large minimum degree and to construct the stars S'_i whose centres became the branch vertices of the subdivision of H in G'. The following lemma will help us to carry out these two steps even though in the proof of Theorem 2 the requirement on the girth of G is weakened to such an extent that we may have multiple edges between the trees.

For the proof of the lemma we need one further definition. Given $k \in \mathbb{N}$ and a vertex w of a rooted tree T_x, we call each vertex above w that has distance k from w in T_x a *k-successor* of w in T_x. A 1-successor of w is also called a *successor* of w.

Lemma 15 *Let $r \geq 60$ and let G be a graph of minimum degree at least r and girth at least 15. Let X be a maximal 4-independent subset of $V(G)$ and let $(T_x)_{x \in X}$ be a family of 2-trees assigned to X. Let $x \in X$ and let $x_1, \dots, x_r$ be a choice of r distinct neighbours of x in T_x. Then for every $i = 1, \dots, r$ there exists a set $Y_i \subseteq X \setminus \{x\}$ satisfying the following conditions.*

(i) $|Y_i| \geq (r-1)^2/6$.

(ii) $Y_i \cap Y_j = \emptyset$ *for all* $j \neq i$.

(iii) *For every $y \in Y_i$ there is an edge in G joining T_y to a vertex in $\mathrm{br}_{T_x}(x_i)$.*

Proof. Recall from Definition 10 that $B^2_G(x)$ is a rooted subtree of T_x and that each vertex of T_x has distance at most four from x in T_x. Let T'_x be a rooted subtree of T_x (with root x) in which $x_1, \dots, x_r$ are the successors of x, every other vertex of T'_x has at most $r-1$ successors and which is maximal with these properties. So each x_i has precisely $r-1$ successors. Let A_i be the set of all these $r-1$ successors of x_i in T'_x.

For all $i = 1, \ldots, r$, we will now assign a set Y_i to x_i such that these sets Y_i satisfy conditions (i)–(iii). To do this, we will distinguish several cases according to the structure of $\mathrm{br}_{T'_x}(x_i)$. Let A_i^* be the set of all those vertices in A_i that have at least $(r-1)/2$ successors in T'_x. Let I^* be the set of all those indices $i \in \{1, \ldots, r\}$ for which $|A_i^*| \geq 2|A_i|/3 = 2(r-1)/3$ and put $I := \{1, \ldots, r\} \setminus I^*$. We will first define the sets Y_i for all $i \in I$.

Case 1. $i \in I$.

Since $\delta(G) \geq r$, the choice of T'_x implies that each vertex in $A_i \setminus A_i^*$ sends at least $r-1-(r-1)/2 = (r-1)/2$ edges to vertices outside T_x. Let E_i denote the set of all edges of G joining some vertex in A_i to some vertex outside T_x and let Y'_i be the set of all those vertices $y \in X \setminus \{x\}$ for which T_y contains an endvertex of some edge in E_i (Fig. 2). Since each such endvertex has distance two from x_i and since $g(G) \geq 13$, no two edges of E_i are incident to the same tree T_y with $y \in X \setminus \{x\}$. Thus $|Y'_i| = |E_i| \geq |A_i \setminus A_i^*|(r-1)/2 \geq (r-1)^2/6$.

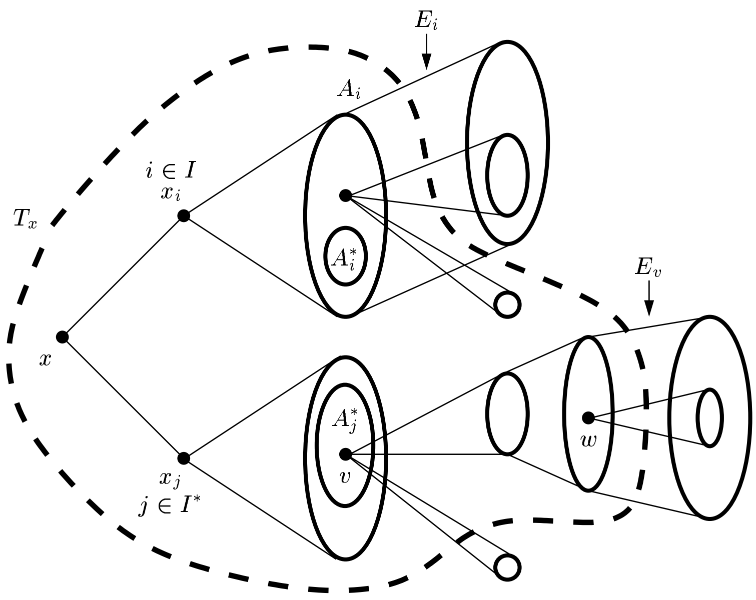

Figure 2: Illustrating Case 1 and Case 2.1 of Lemma 15

Moreover, note that for distinct $i, j \in I$ the sets Y'_i, Y'_j are disjoint. Indeed, since $g(G) \geq 15$ and since each vertex in A_i (respectively A_j) has distance two from x, it follows that no edges $e \in E_i$ and $e' \in E_j$ are incident to the same tree T_y with $y \in X \setminus \{x\}$. For each $i \in I$, let Y_i be any set of $\lceil (r-1)^2/6 \rceil$ vertices in Y'_i.

Case 2. $i \in I^*$.

For all indices $i \in I^*$, we shall now define a set Y'_i of at least $r\lceil (r-1)^2/6 \rceil$ vertices which satisfies condition (iii). The surplus in the size of the Y'_i will give us enough room to ensure that we can then find sets $Y_i \subseteq Y_i'$ also satisfying (i) and (ii).

Case 2.1. *There exists a vertex $v \in A_i^*$ for which the number of 2-successors in T_x' is at least $(r-1)^2/5$.*

As the distance from x in T_x of each such 2-successor w of v is four and thus each w is a leaf of T_x, w sends at least $r-1$ edges to vertices outside T_x. Let E_v be the set of all edges of G joining a 2-successor of v to a vertex outside T_x. So $|E_v| \geq (r-1)^3/5$. Since each 2-successor of v has distance two from v and $g(G) \geq 15$, it follows that each tree T_y $(y \in X \setminus \{x\})$ is incident to at most one edge in E_v. Let Y_i' be the set of all $y \in X \setminus \{x\}$ for which T_y is incident to an edge in E_v. Then $|Y_i'| = |E_v| \geq (r-1)^3/5 \geq r\lceil (r-1)^2/6 \rceil$ (the latter inequality holds since $r \geq 60$).

Case 2.2. *For every vertex in A_i^* the number of 2-successors in T_x' is less than $(r-1)^2/5$.*

Consider any vertex $v \in A_i^*$. Since v has at least $(r-1)/2$ successors in T_x', there are at least $(r-1)^2/2 - (r-1)^2/5 \geq r((r-1)/4+1)$ edges of G joining a successor of v to a vertex outside T_x. Let E_i be the set consisting of all these edges for all $v \in A_i^*$. We take Y_i' to be the set of all those vertices $y \in X \setminus \{x\}$ for which T_y contains an endvertex of an edge in E_i. Since each such endvertex has distance at most three from x_i and since $g(G) \geq 15$, no two edges in E_i are incident to the same tree T_y with $y \in X \setminus \{x\}$. Thus

$$|Y_i'| = |E_i| \geq |A_i^*| \cdot r((r-1)/4+1) \geq \frac{2}{3}(r-1) \cdot r((r-1)/4+1) \geq r\lceil (r-1)^2/6 \rceil.$$

For all $i \in I^*$ in turn, we choose a set Y_i of $\lceil (r-1)^2/6 \rceil$ vertices in Y_i' such that $Y_i \cap Y_j = \emptyset$ for all $j \in I$ and all $j \in I^*$ with $j < i$. Then the sets $Y_1, \dots, Y_r$ are as desired. $\square$

Proof of Theorem 2. Let $c := 3(22\binom{r+1}{2} + r + 1)$ and let X be a maximal 4-independent subset of $V(G)$. Since $g(G) \geq 10$ we may apply Proposition 11 to obtain a family $(T_x)_{x \in X}$ of rooted 2-trees assigned to X. Let G' be the graph obtained from G by contracting each T_x (and deleting multiple edges). For every $x \in X$ we denote the vertex of G' corresponding to the (contracted) tree T_x by x'. Lemma 15 together with the fact that $r \geq 435$ implies that the minimum degree of G' is at least $r(r-1)^2/6 \geq 6 \cdot 12r(r+1) \geq 2c$. (Indeed, let $x \in X$ and let $x_1, \dots, x_r$ be distinct successors of x in T_x. Apply Lemma 15 to obtain sets $Y_1, \dots, Y_r$. Then every vertex in $\bigcup_{i=1}^r Y_i$ is a neighbour of x' in G'.) Apply Lemma 9 to G' to obtain $A, B \subseteq V(G')$ and $E \subseteq E(G')$ as described there. So in particular, E is independent.

We will now inductively construct r-stars $S_0', \dots, S_r'$ in G' satisfying conditions (a)–(g) stated below. Conditions (a)–(e) ensure that in the graph G^* obtained from $G'[A \cup B]$ by contracting the edges in E each S_i' is still an r-star and that in G^* these stars have distinct centres and meet pairwise in at most one leaf, and furthermore that every leaf lies in at most two stars. Since our choice of A, B and E implies that G^* is highly connected, we may link the leaves of these stars to obtain a subdivision of K_{r+1} in G^* whose branch vertices are the centres of the stars. Condition (g) ensures that each S'_i corresponds to a subdivided star S_i in G and therefore the subdivision of K_{r+1} in G^* will correspond to one in G.

(a) The centre x'_i of S'_i lies in A and does not lie in another star S'_j $(j \neq i)$.

(b) No edge in E joins two vertices of S'_i.

(c) No edge in E joins the centre of S'_i to a vertex of another star S'_j $(j \neq i)$.

(d) No vertex $x' \in G'$ lies in more than two S'_i and if x' lies in two such stars (and thus x' is a leaf of both of them) then no edge in E joins x' to a third star.

(e) If $j \neq i$ then S'_i and S'_j have at most one leaf in common. Moreover, at most one edge in E joins a leaf of S'_i to a leaf of S'_j and if this happens, then S'_i and S'_j are disjoint.

(f) There are at most 15 indices $j < i$ for which there is either an edge in E joining S'_i to S'_j or for which $S'_i \cap S'_j \neq \emptyset$.

(g) T_{x_i} contains a subdivided r-star S_i with center x_i and leaves $v_{i1}, \dots, v_{ir}$ such that for each v_{ij} there exists a leaf y'_{ij} of S'_i such that v_{ij} sends an edge to $T_{y_{ij}}$ and $y'_{ij} \neq y'_{ik}$ for all $j \neq k$.

So let us inductively define r-stars $S'_0, \dots, S'_r$ satisfying conditions (a)–(g) for all $i = 0, \dots, r$. For the centre of S'_0 we take any vertex x'_0 in A. Let $x_{01}, \dots, x_{0r}$ be distinct successors of x_0 in T_{x_0} and apply Lemma 15 to obtain disjoint sets $Y_{01}, \dots, Y_{0r}$. Thus each Y_{0i} consists of neighbours of x'_0 in G'. For all $i = 1, \dots, r$ in turn choose a vertex $y'_{0i} \in Y_{0i}$ such that $x'_0 y'_{0i} \notin E$ and $y'_{0j} y'_{0i} \notin E$ for all $j < i$. This can be done since the edges in E are independent and $|Y_{0i}| \geq (r-1)^2/6 \geq r+2$. Let S'_0 be the r-star in G' whose centre is x'_0 and whose leaves are $y'_{01}, \dots, y'_{0r}$. Then S'_0 satisfies condition (b). Moreover, Lemma 15 (iii) implies that G contains an edge joining $\mathrm{br}_{T_{x_0}}(x_{0i})$ to $T_{y_{0i}}$, and so S'_0 also satisfies condition (g) (and thus conditions (a)–(g)).

Suppose that $1 \leq k \leq r$ and that we have already constructed stars $S'_0, \dots, S'_{k-1}$ satisfying conditions (a)–(g). Let $X_k \subseteq X$ be the set of all vertices belonging to S'_ℓ for some $\ell < k$ together with the set of endvertices of all those edges in E that are incident to a vertex of S'_ℓ for some $\ell < k$. Thus, by Lemma 9,

$$|X_k| \leq 2k(r+1) \leq 2r(r+1) < c < |A|. \tag{3}$$

As centre of S'_k we choose any vertex x'_k in $A \setminus X_k$. Let $x_{k1}, \dots, x_{kr}$ be r distinct successors of x_k in T_{x_k} and apply Lemma 15 to obtain sets $Y_{k1}, \dots, Y_{kr}$. As leaves of S'_k we will now successively choose one vertex in each Y_{ki}. When choosing the first leaf y'_{k1} in Y_{k1} we have to avoid a certain set of vertices in Y_{k1} in order to ensure that our new star S'_k satisfies conditions (a)–(d). The subset of $V(G')$ which is forbidden by these conditions is precisely the set X'_k that consists of firstly all the centres of S'_ℓ for $\ell < k$, secondly all those vertices that send an edge in E to the centre of S'_ℓ for some $\ell \leq k$ (where the centre of 'S'_k' is x'_k), thirdly all the leaves belonging to two stars S'_ℓ, $S'_{\ell'}$ for some $\ell < \ell' < k$ together with all those vertices that send an edge in E to such a leaf, and finally of the endvertices of all those edges in E that join S'_ℓ to another star $S'_{\ell'}$ for some $\ell < \ell' < k$. Condition (f) for $S'_0, \dots, S'_{k-1}$ shows that the number of pairs ℓ, ℓ' with $\ell < \ell' < k$ occuring either in the third or the final type of forbidden vertices is at most $15k$. Together with condition (e) this implies that

$$|X'_k| \leq k + k + 1 + 2 \cdot 15k \leq 33r < |Y_{k1}|. \tag{4}$$

Let y'_{k1} be any vertex in $Y_{k1} \setminus X'_k$ which lies outside X_k whenever this is possible. As $X'_k \setminus X_k$ is either empty or a singleton (in the latter case $X'_k \setminus X_k$ consists of the unique vertex of G' sending an edge in E to x'_k) it follows that $y'_{k1} \notin X_k$ whenever $|Y_{k1} \setminus X_k| \geq 2$.

Now suppose that $1 < \ell \leq r$ and that $y'_{k1} \in Y_{k1}, \dots, y'_{k(\ell-1)} \in Y_{k(\ell-1)}$ have already been defined so that $y'_{ki} \notin X_k$ whenever $|Y_{ki} \setminus X_k| \geq i+1$ for all $i < \ell$. Let us now choose $y'_{k\ell} \in Y_{k\ell}$. As in the choice of y'_{k1}, we forbid the vertices in X'_k when choosing $y'_{k\ell}$ in order to avoid conflicts with the stars S'_j with $j < k$. However, we now have to forbid further vertices in order to avoid conflicts with those leaves of the star S'_k that are already determined, i.e. with $y'_{k1}, \dots, y'_{k(\ell-1)}$. So we have to forbid the at most $\ell - 1$ vertices that send an edge in E to y'_{ki} for some $i < \ell$ in order to satisfy condition (b). We call these vertices the *new vertices forbidden by condition* (b). Moreover, to satisfy condition (e), we may have to avoid even more vertices: if for some $i < \ell$ and $j < k$ the vertex y'_{ki} was either already a leaf of S'_j or an endvertex of an edge in E adjacent to S'_j, then all leaves of S'_j are now forbidden when choosing $y'_{k\ell}$, and so are all those vertices that send an edge in E to a leaf of S'_j. However, note that for each such $i < \ell$ we must have $y'_{ki} \in X_k$ and thus $|Y_{ki} \setminus X_k| \leq i$, i.e. Y_{ki} contains at least

$$|Y_{ki}| - i \geq (r-1)^2/6 - r \geq r^2/7$$

vertices from X_k. As the Y_{ki} are disjoint, (3) implies that the above situation can occur for at most $7|X_k|/r^2 \leq 15$ indices $i < \ell$. Moreover, since $y'_{ki} \notin X'_k$, for each such i there is exactly one index $j < k$ such that y'_{ki} is already a leaf of S'_j or sends an edge in E to S'_j. Besides taking care of condition (f), this shows that there are at most $15 \cdot 2r$ additional vertices which are forbidden by condition (e). So when choosing $y'_{k\ell}$ we have at most

$$|X'_k| + \ell - 1 + 30r \overset{(4)}{\leq} 64r < |Y_{k\ell}|$$

forbidden vertices in $Y_{k\ell}$ (using $r \geq 400$). Let $y'_{k\ell}$ be a vertex in $Y_{k\ell}$ which is not forbidden and which lies outside X_k if possible. Note that all forbidden vertices in $Y_{k\ell}$ are contained in X_k except the new vertices forbidden by condition (b) and the vertex of G' which sends an edge in E to x'_k (if such a vertex exists). Thus if $|Y_{k\ell} \setminus X_k| \geq \ell + 1$ then there exists a vertex in $Y_{k\ell} \setminus X_k$ which is not forbidden and which we therefore can choose as $y'_{k\ell}$. This completes the inductive definition of $y'_{k1}, \dots, y'_{kr}$. These vertices are the leaves of a star S'_k in G' whose centre is x'_k. Our choice of x'_k and $y'_{k1}, \dots, y'_{kr}$ implies that

$S'_0, \dots, S'_k$ satisfy conditions (a)–(g), which completes also the induction step for the definition of the stars S'_i.

Now consider the graph G^* obtained from $G'[A \cup B]$ by contracting all edges in E. Since the centre of each S'_i lies in A and since $A \cup B$ contains all neighbours of A in G', it follows that S'_i is an r-star in $G'[A \cup B]$. Condition (b) now implies that S'_i is still an r-star in G^*. By conditions (a), (c) and (e) two such stars have distinct centres and meet at most in one leaf. Moreover, from condition (d) it follows that no vertex of G^* lies in more than two stars. Since A, B and E are as described as in Lemma 9, the graph G^* is $\lceil c/3 \rceil$-connected. Thus by Theorem 8 the graph obtained from G^* by deleting the centres of the S'_i is $\binom{r+1}{2}$-linked. Therefore we can join the leaves of the S'_i by disjoint paths to each other to obtain a subdivision of K_{r+1} in G^* whose branch vertices are the centres of the S'_i. (Here each vertex which is a leaf of two stars is already a trivial such path.)

As each such path P^* joining two leaves of the S'_i corresponds to a path P' in G', this subdivision corresponds to a subdivision of K_{r+1} in G'. Again, the branch vertices are the centres of the S'_i and each subdivided edge corresponds to a path joining two leaves of distinct S'_i, S'_j (this path is trivial if S'_i and S'_j have a leaf in common). Condition (g) now implies that the latter subdivision corresponds to one in G. Indeed, we only have to replace each S'_i with the subdivided star obtained from S_i by adding an v_{ij}–$T_{y_{ij}}$ edge e_{ij} for all $j = 1, \dots, r$ and replace each path P' joining a leaf y'_{is} of S'_i to a leaf y'_{jt} of S'_j with some path in G which contracts to P' and joins the endvertex of e_{is} in $T_{y_{is}}$ to the endvertex of e_{jt} in $T_{y_{jt}}$. □

At the expense of more involved calculations, the bounds on r in Theorem 2 and on the girth in Theorem 14 can be improved a little. However, we believe that the primary question is whether the bound on the girth in Theorem 2 can be reduced.

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