Improved bounds for topological cliques in graphs of large girth

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Abstract

We prove that every graph of minimum degree at least r and girth at least 27 contains a subdivision of K_{r+1}. This implies that the conjecture of Hajós, that every graph of chromatic number at least r contains a subdivision of K_r, is true for graphs of girth at least 27. This conjecture is known to be false in general.

1 Introduction

It is well known that the existence of a subdivision of the complete graph K_r is forced by large but constant minimum degree: let $d(r)$ be the smallest number such that every graph G of minimum degree at least $d(r)$ contains a subdivided K_r. The existence of $d(r)$ was proved by Mader (see e.g. [1]). Bollobás and Thomason [2, 3] as well as Komlós and Szemerédi [8] independently proved that $d(r)$ is quadratic in r. As observed by Jung [7] earlier on, complete bipartite graphs show that $d(r)$ is at least quadratic in r.

On the other hand, Mader [15] proved that the situation is rather different for graphs which have large girth, i.e. which do not contain short cycles: he showed that there is a function $g(r)$ so that every graph of minimum degree at least r and girth at least $g(r)$ contains a subdivided K_{r+1}. At first, this might seem rather surprising since the condition on the minimum degree only ensures that every vertex has sufficiently many neighbours to be a candidate for a branch vertex. Mader's bound on the $g(r)$ was linear in r. He asked about the growth of $g(r)$ and pointed out that it might even be true that $g(r) = 5$ for $r \geq 4$ (see [17]). The complete bipartite graph $K_{r,r}$ provides the lower bound $g(r) \geq 5$ for $r \geq 4$. In [9], we showed that $g(r) \leq 186$ and that $g(r) \leq 15$ for all $r \geq 435$. In this paper, we prove that $g(r) \leq 27$.

Theorem 1 *Let $r \geq 1$ be a natural number. Every graph of minimum degree at least r and girth at least 27 contains a subdivision of K_{r+1}.*

In [12] we proved the related result that if we relax the condition of having girth at least 27 to being C_4-free, then at least we can find a subdivision of a complete graph whose order is almost linear in the minimum degree of the host graph.

Theorem 1 has an immediate application to the well known conjecture of Hajós (see [6]) which states that every graph of chromatic number r contains a subdivision of K_r. Catlin [4] found several counterexamples to this. A little later, Erdős and Fajtlowicz [5] proved that the conjecture fails even for almost all graphs. On the other hand, since every graph of chromatic number at least r has a subgraph of minimum degree at least $r-1$, Theorem 1 shows that the conjecture does hold for all graphs whose girth is at least 27:

Corollary 2 *Let $r \geq 2$ be a natural number. Every graph of chromatic number r and girth at least 27 contains a subdivision of K_r.*

Thomassen [21] asked whether the conjecture of Hajós might even be true for all triangle-free graphs. Note that there is a difference to the minimum degree condition here: while it may be that Theorem 1 remains true if one replaces the condition of 'girth ≥ 27' by 'girth ≥ 5', one cannot replace it by 'girth ≥ 4'. Also, no counterexamples to the conjecture of Hajós are known for $r = 5, 6$.

Based on a result of Mader [18], in [11] we proved an analogue of Theorem 1 with the minimum degree condition replaced by one on the average degree: for every $\varepsilon > 0$ there exists an integer $f(\varepsilon)$ such that for all $r \geq 2$ every graph G of average degree at least $r + \varepsilon$ and girth at least $f(\varepsilon)$ contains a subdivision of K_{r+2}. (In [18], the same result is proved with the difference that the function f also depends on r.)

Apart from the obvious question of whether the girth bound in Theorem 1 and its relatives can be improved, one could also try to strengthen the above results by asking for an *induced* subdivision. In particular, Shi (personal communication) posed the following question:

Problem 3 *Is there a function $h(r)$ such that every graph of minimum degree at least $r \geq 3$ and girth at least $h(r)$ contains a subdivision of K_{r+1} as an induced subgraph?*

This question was motivated by our earlier result [13] that for all integers $s \geq 2$ and $r \geq 3$ there exists an integer d such that every $K_{s,s}$-free graph of minimum degree at least d contains an induced subdivision of K_r. (Clearly, the condition of being $K_{s,s}$-free cannot be omitted here.) Thus the above problem has an affirmative answer if we assume that the minimum degree is sufficiently large compared to r.

The assumption of large girth also makes a major difference if one asks for ordinary minors instead of subdivisions. The first result in this direction was proved by Thomassen [20], who showed that there is a function $q(r)$ so that every graph of minimum degree at least 3 and girth at least $q(r)$ contains a K_r minor. In [10], we proved much more precise bounds on the existence of large complete minors in graphs of given minimum degree and given girth. Also in [14] we proved that one obtains surprisingly large complete minors if one relaxes the condition of having large girth to being $K_{s,s}$-free.

This paper is organized as follows. In the next section, we give a brief outline of the proof of Theorem 1. In Section 3 we then present the necessary definitions and some tools which we will need later on. Section 4 is devoted to the proof itself and is divided into two subsections: one for the easier case when we seek a subdivision of K_{r+1} for $r \geq 5$ and one for the case where we seek a subdivision of K_5. In the final section, we then very briefly discuss possible approaches for further improvements and the obstacles to these.

2 Outline of the proof

Suppose that we are given a graph G of minimum degree r and girth at least 27. First we choose a maximal set X of vertices whose pairwise distance is at least 7. We then extend these vertices into rooted induced subtrees of G such that each tree sends many edges to different other trees. (The vertices in X are the roots of these trees.) Then the minor G' of G which is obtained by contracting each of the trees into a single vertex has large minimum degree (at least $r(r-1)^3$). One could then show that G' contains an $\binom{r+1}{2}$-linked subgraph G^* (provided that r is not too small). Thus G^*, and therefore also G', contains a subdivision of a K_{r+1}. But unfortunately, this need not correspond to a subdivision of a K_{r+1} in our original graph G.

Thus we have to find a highly linked substructure G^* of G' which allows us more control on what the subdivision of K_{r+1} in G^* (and in G') looks like in order to guarantee that this subdivision will correspond to one in G. Indeed, we find a set A of vertices of G' together with their neighbours B having the property that the graph G^* obtained from $G'[A \cup B]$ by contracting an independent set F of A–B edges is highly connected (Lemma 5). At first it might not seem to be a good idea to consider a highly connected minor of G' instead of a highly connected subgraph. But the advantage of this is that at least the vertices in A are now guaranteed to 'keep' all their neighbours in the connected substructure. In the case $r \geq 5$ this property can be used to find $r+1$ disjoint r-stars in $G'[A \cup B]$ (with centres in A) which correspond both to disjoint r-stars in G^* as well as subdivided r-stars in G. As G^* is highly connected and thus highly linked, we can link the leaves of the stars to obtain a subdivision of K_{r+1} in G^*. Since each star in G^* corresponds to a subdivided star in G, this subdivision of K_{r+1} in G^* will then correspond to one in G.

This strategy was first used by Mader [15] and subsequently also in [9]. The improvements we obtain here are partly due to a more economical construction of the stars in $G'[A \cup B]$. In particular, in order to find stars with the desired properties, the graph G' was required to have a girth which was linear in r in [15] and to have a girth of least 6 in [9] where the weaker bound of 186 on the girth of G was proved. Dropping this requirement on G' immediately leads to a significantly lower requirement on the girth of G.

The strategy described above does not work when $r = 4$ since in this case we cannot guarantee that the graph G^* is sufficiently linked. So we have to work harder here (see the beginning of Section 4.2 for more details).

3 Notation and tools

We will now collect some definitions and results which we will need later. We denote the minimum degree of a graph G by $\delta(G)$. The *girth* $g(G)$ of G is the length of the shortest cycle in G. A *subdivision* of a graph G is a graph TG obtained from G by replacing the edges of G with internally disjoint paths. The *branch vertices* of TG are all those vertices that correspond to the vertices of G. An *r-star* is a star with r leaves. Given a set A of vertices of a graph G, we write $N_G(A)$ for the set of all those neighbours of vertices in A that lie outside A. Given $\ell \in \mathbb{N}$, the *ℓ-ball around a vertex x* of a graph G is the subgraph of G induced by all its vertices of distance at most ℓ from x (including x itself). We denote this subgraph by $B^\ell_G(x)$. Given integers $r \geq 3$ and $\ell \geq 1$, we define an *r-uniform tree of radius ℓ* to be the rooted tree in which all leaves have distance ℓ from the root and all other vertices have degree r. Thus an r-uniform tree of radius ℓ has precisely $r(r-1)^{\ell-1}$ leaves. Given a tree T with root x and a vertex $v \in T$, we say that a vertex $u \in T$ *lies above* v if v lies on the subpath of T which joins x to u. The *branch above* v is the subtree of T which is spanned by all the vertices lying above v.

Given $k \in \mathbb{N}$, we say that a graph G is k-linked if $|G| \geq 2k$ and for every $2k$ distinct vertices $x_1, \dots, x_k$ and $y_1, \dots, y_k$ of G there exist disjoint paths $P_1, \dots, P_k$ such that P_i joins x_i to y_i. We will make essential use of the following recent result of Thomas and Wollan [19].

Theorem 4 *Let $k \geq 1$ be a natural number. Every $2k$-connected graph of average degree at least $10k$ is k-linked.*

The first linear bound on the necessary average degree was established by Bollobás and Thomason [2], who obtained the same result but with the $10k$ replaced by $22k$.

The following lemma is essentially due to Mader [15], who proved a slightly stronger result for triangle-free graphs. A proof of the version below is contained in [9].

Lemma 5 *Let $c \geq 1$ be an integer and let G be a graph of minimum degree at least $2c$. Then there exist disjoint sets $A, B \subseteq V(G)$ and a set F of $|B|$ independent A–B edges such that $|A| > c > |B|$, $N_G(A) \subseteq B$ and so that the graph G^* obtained from $G[A \cup B]$ by contracting the edges in F is $\lceil c/3 \rceil$-connected.*

4 Proofs

Note that Theorem 1 is trivial for $r \leq 2$. Moreover, it is easy to check that every graph of minimum degree at least 3 contains a subdivision of K_4 (this was first observed by Dirac, see e.g. [1, Ch. VII, Thm. 2.2]). Thus Theorem 1 also holds when $r = 3$ (even with no assumption on the girth). Pelikán showed that every graph of minimum degree at least 4 contains a subdivision of the graph obtained from K_5 by deleting one edge (see e.g. [1, Ch. VII, Thm. 2.5]). However, as mentioned in the introduction, an additional assumption on the girth is needed to guarantee a subdivision of K_5. (Alternatively, instead of increasing the girth, one can also guarantee a subdivided K_5 by increasing the minimum degree to 6. This follows from the result of Mader [16] that $3|G| - 5$ edges force a TK_5.) As the proof of the K_5-case of Theorem 1 needs some special arguments, it will be considered separately. So let us first prove Theorem 1 for the case $r \geq 5$.

4.1 Finding a subdivision of K_{r+1} for $r \geq 5$

Let G be a graph of minimum degree $r \geq 5$ and girth at least 27. Consider a maximal set X of vertices of G such that every two vertices in X have distance at least 7 from each other. Since the girth of G is at least 27, the 3-ball $B^3_G(x)$ around any vertex $x \in X$ is a tree. Since the vertices in X have distance at least 7 from each other, all these trees must be disjoint. We now extend these trees to connected subgraphs T_x ($x \in X$) by adding first every vertex of distance 4 from X to one of the trees to which it is adjacent, then adding all vertices of distance 5 from X to one of the subgraphs constructed in the previous step and then those of distance 6. By the maximality of X, every vertex in $V(G) \setminus X$ has distance at most 6 from X and is thus contained in some T_x. Our assumption that the girth of G is at least 27 now implies that the T_x ($x \in X$) satisfy the following properties:

(i) T_x is an induced subtree of G. We will view x as the root of T_x.

(ii) Each leaf of T_x has in T_x distance at most 6 from the root x of T_x.

(iii) The 3-ball $B^3_G(x) = B^3_{T_x}(x)$ around x contains an r-uniform tree of radius 3. Thus T_x has at least $r(r-1)^2$ leaves and so there are at least $r(r-1)^3$ edges in G emanating from T_x. All these edges go to different other trees T_y, i.e. between every pair T_x and T_y of trees there exists at most one edge in G.

We now consider the graph G' obtained from G by contracting each T_x ($x \in X$) into a single vertex. For each $x \in X$, we denote by x' the vertex of G' corresponding to the (contracted) tree T_x. Note that (iii) implies that

$$\delta(G') \geq r(r-1)^3.$$

Put

$$c := 6\binom{r+1}{2} + 3(r+1). \tag{1}$$

Using that $r \geq 5$, it is easy to check that

$$\delta(G') \geq 2c. \tag{2}$$

Thus we can apply Lemma 5 to G' to obtain sets A and B and a set F of $|B|$ independent A–B edges as described there. The edges of F are called *F-edges*.

Given a vertex $v \in T_x$ and a neighbour y' of x' in G', we say that y' *lies above* v if v lies on the subpath of T_x which joins x to the unique T_x–T_y edge of G (Fig.1). Our next aim is to find disjoint r-stars $S_1, \dots, S_{r+1}$ in $G'[A \cup B]$ such that, writing x'_i for the centre of S_i, they satisfy the following properties:

(a) No edge of S_i belongs to F. No F-edge joins 2 leaves of S_i.

(b) There are no F-edges joining two different S_i.

(c) The leaves of S_i lie above different neighbours of x_i in T_{x_i}.

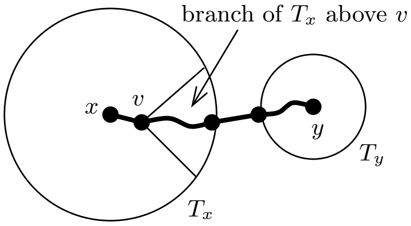

Figure 1: The neighbour y' of x' lies above v.

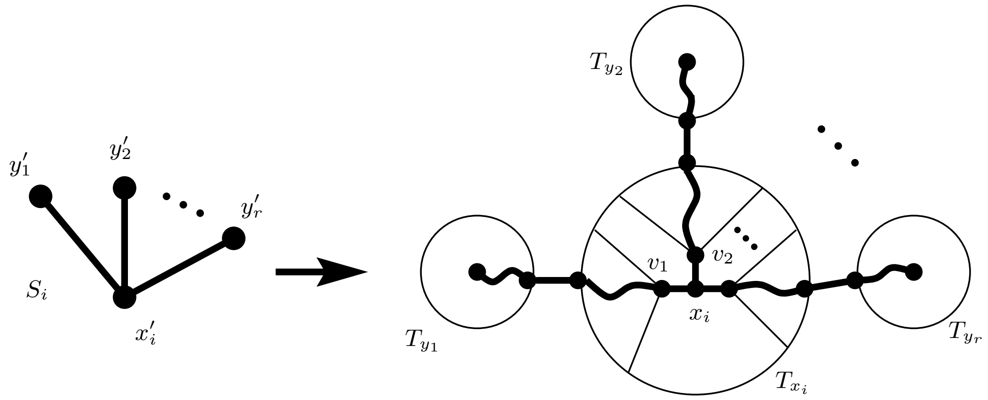

Figure 2: The star S_i corresponds to a subdivided star in G.

The vertices x_i will be the branch vertices of our subdivided K_{r+1}. Property (c) ensures that each S_i corresponds to a subdivided r-star in G (Fig. 2). The stars $S_1, \dots, S_{r+1}$ can be found greedily as follows. Suppose that we have already chosen $S_1, \dots, S_{i-1}$ for some $i \leq r+1$. Let W be the set consisting of all those vertices in $G'[A \cup B]$ which send an F-edge to some vertex in $V(S_1) \cup \cdots \cup V(S_{i-1}) =: W'$. Since the F-edges are independent, we have $|W \cup W'| \leq 2|W'| \leq 2r(r+1)$. Take x'_i to be any vertex in $A \setminus (W \cup W')$. To see that such a vertex exists, we have to check that $|A| > |W \cup W'|$. But this holds since all the vertices in A have their neighbours in $A \cup B$ and thus

$$|A| \geq \delta(G') - |B| > \delta(G') - c \overset{(2)}{\geq} c \geq 2r(r+1) \geq |W \cup W'|,$$

as desired. Now we have found the centre x'_i of S_i. So next we will choose its leaves. Let $v_1, \dots, v_r$ be distinct neighbours of x_i in the tree T_{x_i} (Fig. 2). (So the v_j are vertices of G.) Let V_j be the set of all those neighbours of x'_i in G' which lie above v_j. Let $U_j := V_j \cap (W \cup W')$. We assume that the v_j are ordered descendingly according to the size of U_j, i.e. if $k > j$, then $|U_k| \leq |U_j|$. For each v_j in turn we have to choose a neighbour y'_j of x'_i in G' such that (I) $y'_j \in V_j$, (II) $y'_j \notin U_j$, (III) $x'_i y'_j \notin F$, (IV) no F-edge joins y'_j to any $y'_1, \dots, y'_{j-1}$ chosen previously for $v_1, \dots, v_{j-1}$ and such that (V) $y'_j \neq y'_k$ for $k < j$. Then (a)–(c) are satisfied and $y'_1, \dots, y'_r$ can be taken as leaves of S_i. We call every neighbour of x'_i in V_j which violates one of the properties (II)–(IV) a *forbidden neighbour*.

We will show that there exists one such neighbour y'_j which is not forbidden. For this, first note that $V_j \cap V_k = \emptyset$ for $j \neq k$. (Indeed, if $y' \in V_j \cap V_k$ then G would contain at least two edges between T_{x_i} and T_y, contradicting (iii). In particular, this shows that (V) will automatically be satisfied.) Thus also $U_j \cap U_k = \emptyset$ for $j \neq k$. By our assumption on the ordering of the vertices $v_1, \dots, v_r$, this implies that $|U_j| \leq |W \cup W'|/j$. Since the number of neighbours which are forbidden by (III) and (IV) is clearly at most j, this implies that the total number of forbidden neighbours is at most

$$j + |U_j| \leq j + 2r(r+1)/j \leq 1 + 2r(r+1).$$

The final inequality is due to the fact that $j+2r(r+1)/j$ is a decreasing function in j for $1 \leq j \leq r$.

On the other hand, property (iii) implies that the branch of T_{x_i} above v_j has at least $(r-1)^2$ leaves and thus sends out at least $(r-1)^3$ edges going to other trees T_y. So there are at least $(r-1)^3$ neighbours of x'_i which lie above v_j, i.e. $|V_j| \geq (r-1)^3$, which is greater than $1 + 2r(r+1)$ for $r \geq 5$. Thus there exists a neighbour y'_j of x'_i lying above v_j which is not forbidden, as desired. This proves the existence of the stars $S_1, \dots, S_{r+1}$.

We now consider the graph G^* obtained from $G'[A \cup B]$ by contracting every edge in F. Conditions (a) and (b) ensure that the images of $S_1, \dots, S_{r+1}$ in G^* are still disjoint r-stars. Our aim now is to delete the centres of these stars and then to link the leaves of them by $\binom{r+1}{2}$ disjoint paths in such a way that (after adding the centres again) we obtain a subdivision of K_{r+1} in G^* whose branch vertices are the centres. Since by condition (c) each S_i corresponds to a subdivided r-star in G, it is easy to see that this subdivision of K_{r+1} in G^* would then correspond to one in G (with branch vertices $x_1, \dots, x_{r+1}$).

Thus it suffices to show that the graph obtained from G^* by deleting the centres of the images of $S_1, \dots, S_{r+1}$ is $\binom{r+1}{2}$-linked. By Theorem 4 this is the case if the minimum degree of this graph is at least $10\binom{r+1}{2}$ and if its connectivity is at least $2\binom{r+1}{2}$. Thus is suffices to show that the minimum degree of G^* is at least $10\binom{r+1}{2}+r+1$ and its connectivity is at least $2\binom{r+1}{2}+r+1$. The second condition is satisfied since Lemma 5 implies that G^* is $\lceil c/3 \rceil$-connected and $c/3 = 2\binom{r+1}{2} + r + 1$. Moreover, when contracting F, each vertex $a \in A$ looses one neighbour for each F-edge to which a is either incident or which has the property that a is joined to both of its endvertices. Thus

$$\delta(G^*) \geq \delta(G') - |B| > r(r-1)^3 - c \geq 10\binom{r+1}{2} + r + 1,$$

as desired (use that $r \geq 5$). This completes the proof of Theorem 1 for the case when $r \geq 5$.

4.2 Finding a subdivision of K_5

Let G be a graph of minimum degree 4 and girth at least 27. Set

$$c := 51 = 3\left(2 \cdot \left(\binom{5}{2} - 4\right) + 5\right).$$

Thus c is a little smaller than the number obtained by substituting $r = 4$ in (1). Proceed similarly as in the first part of Section 4.1 to obtain trees T_x $(x \in X)$, a graph G', vertex sets $A, B \subseteq V(G')$ and a set F of $|B|$ independent A–B edges. Again, denote by G^* the graph obtained from $G'[A \cup B]$ by contracting each edge in F. We now would like to choose stars $S_1, \ldots, S_5$ in $G'[A \cup B]$ and continue as in Section 4.1. However, c is now smaller and thus the graph obtained from G^* by deleting any 5 vertices (e.g. the star centres) will only be $(\binom{5}{2} - 4)$-linked in Case 1 below instead of $\binom{5}{2}$-linked as in Section 4.1 and in Case 2 it will only be $(\binom{5}{2} - 5)$-linked. So we have to argue more carefully to ensure that when applying the linkedness we only have to find 6 of the 10 subdivided edges of our TK_5 in Case 1 and 5 subdivided edges in Case 2. The reason why we choose c to be smaller now is that when passing from G' to G^*, the minimum degree may decrease by $|B| \leq c - 1$, and thus we can guarantee a larger minimum degree of G^* in this way.

The idea is to distinguish 2 cases according to the minimum degree of G^*.

Case 1: The minimum degree of G^* is at least 65.

The strategy here is to choose the stars $S_1, \ldots, S_5$ in such a way that their centres form a path of length 5. Then we can use the edges of this path to find 4 of the subdivided edges of our TK_5 and thus we only need the linkedness of G^* to find the 6 remaining edges. More formally, we wish to find stars $S_1, \ldots, S_5$ in G' satisfying the following properties:

(α) Both S_1 and S_5 are 3-stars and each of S_2, S_3, S_4 is a 2-star. All the S_i are disjoint from each other. The centres $x'_1 \ldots x'_5$ of $S_1, \ldots, S_5$ form a path in $G'[A]$.

(β) No edge of S_i belongs to F. No F-edge joins 2 leaves of S_i.

(γ) There are no F-edges joining two different S_i. In particular, none of the edges on the path $x'_1 \ldots x'_5$ lies in F.

(δ) Let H' denote the union of all the stars S_i and the path $x'_1 \ldots x'_5$. (Thus every x'_i has degree 4 in H'.) The neighbours of x'_i in H' lie above different neighbours of x_i in T_{x_i} (Fig. 3).

This can be achieved as follows. We first choose the path $x'_1 \ldots x'_5$. This can be done greedily. Indeed, suppose that we have already found $x'_1 \ldots x'_j$ for some $1 \leq j < 5$. For x'_{j+1} we can take any neighbour y' of x'_j which lies in $A \setminus \{x'_1, \ldots, x'_j\}$, does not send an F-edge to any of $x'_1, \ldots, x'_j$ and is such that y' and x'_{j-1} lie above different neighbours of x_j in T_{x_j}. But by (iii) there are at least $3 \cdot 3^3 = 81$ neighbours of x'_j satisfying the latter property and at most $|B| + j \leq c + 4 = 55$ of them lie in $B \cup \{x'_1, \ldots, x'_j\}$. Moreover, at most $j \leq 4$ of the remaining neighbours send an F-edge to any of $x'_1, \ldots, x'_j$. Thus we can find the path $x'_1 \ldots x'_5$ consisting of the centres of $S_1, \ldots, S_5$.

We now have to choose the leaves of the S_i. As before, this can be done greedily. It is easy to check that one can find leaves for each of $S_1, \ldots, S_4$ in turn. So let us now choose the 3 leaves for S_5. Let v_1, v_2, v_3 be 3 neighbours of x_5 in T_{x_5} such that x'_4 does not lie above v_i for each $i = 1, 2, 3$. Let W be the

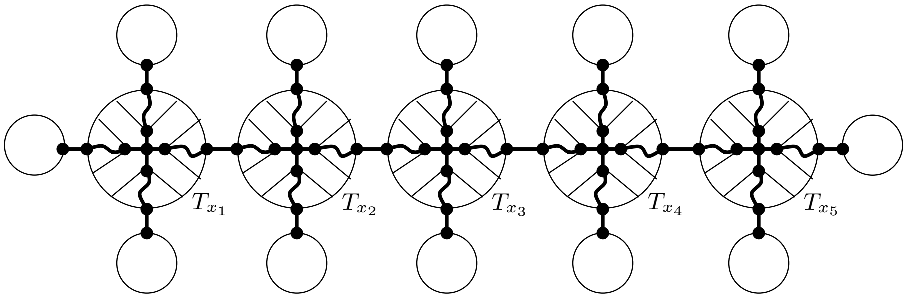

Figure 3: The subgraph of G corresponding to graph $H' \subseteq G'$ defined in condition (δ).

set consisting of all those vertices in $G'[A \cup B]$ which send an F-edge to some vertex in $V(S_1) \cup \cdots \cup V(S_4) \cup \{x'_5\}$. Set $W' := (V(S_1) \cup \cdots \cup V(S_4)) \setminus \{x'_4\}$. Given $i \leq 3$, we say that a vertex y' is a *candidate for the* i*th leaf of* S_5 if y' lies above v_i and if $y' \notin W \cup W'$. We are looking for 3 candidates, one for the first leaf, one for the second and one for the third, such that no F-edge joins two of them. (Then these candidates can be taken as leaves of S_5. Indeed, note that the candidates are automatically distinct from x'_4 since otherwise G would contain at least two T_{x_4}–T_{x_5} edges.) There are at least 27 neighbours of x'_5 which lie above v_i. Since at most $|W \cup W'| \leq 26$ of these neighbours lie in $W \cup W'$, there is at least one candidate for the ith leaf. Moreover, this argument also shows that for at most one index i there are less than 3 candidates for the ith leaf (indeed, suppose there are at most two candidates for the first leaf, say. Then at most one of the vertices above v_2 or v_3 can be in $W \cup W'$, as G' contains no multiple edges). Thus there must be 3 candidates, one for each of the 3 leaves, such that no F-edge joins two of them, as required.

Similarly as in Section 4.1 we now consider the graph G^* obtained from $G'[A \cup B]$ by contracting each edge in F. Again, conditions (β) and (γ) ensure that the image of H' in G^* is still isomorphic to H', i.e. nothing in H' will be contracted. As in Section 4.1 we now delete the images of the centres $x'_1, \dots, x'_5$ in G^* and then wish to link the images of the leaves of the S_i to obtain a subdivision of K_5 in G^* whose branch vertices are the centres. However, since the centres lie on a path of length 4 in G^* we can link adjacent branch vertices via an edge of the path. Thus we now only have to find the $\binom{5}{2} - 4 = 6$ remaining subdivided edges of our TK_5 in G^*. For this, it suffices that the graph obtained from G^* by deleting the 5 branch vertices is 6-linked. By Theorem 4 this is the case if the minimum degree this graph is at least 60 and if its connectivity is at least 12. This in turn holds if the minimum degree of G^* is at least 65 and its connectivity is at least 17. The first requirement holds by our assumption. The latter is satisfied since by Lemma 5 the graph G^* is $\lceil c/3 \rceil$-connected and $c/3 = 17$. Condition (δ) now implies that our subdivision of K_5 in G^* corresponds to a subdivision of K_5 in G whose branch vertices are $x_1, \dots, x_5$.

Case 2: The minimum degree of G^* is at most 64.

Recall that, as in Section 4.1, when contracting the edge set F to obtain G^*

from $G'[A \cup B]$, each vertex $x' \in A$ loses one neighbour for each F-edge which is incident with a or of which a sees both endvertices. Thus we obtain the lower bound

$$\delta(G^*) \geq \delta(G') - |B| > 4 \cdot 3^3 - c = 57. \tag{3}$$

This is too small for us to be able to proceed as in Case 1, since with this bound we can now only guarantee that the graph obtained by deleting 5 vertices of G^* is 5-linked instead of 6-linked as it was in Case 1. Our solution is to find in G' a suitable subgraph Q' consisting of a triangle and a path attached to it. The branch vertices and 5 of the 10 subdivided edges of our subdivision will be contained in the subgraph of G which corresponds to Q'. The existence of some triangle in G' is easy to show: our assumption on the minimum degree of G^* implies that there exists a triangle $x'y'z'$ in $G'[A \cup B]$ which is formed by some vertex $x' \in A$ and an F-edge $y'z'$. (In fact, some $x' \in A$ must form such a triangle with at least $108 - 64 - 1 = 43$ of the F-edges, but we will not make use of this fact.)

Given a triangle $u'v'w'$ in G', we denote by $C(uvw)$ its corresponding cycle in G, i.e. the unique cycle which consists of the T_u–T_v edge, the T_v–T_w edge, the T_u–T_w edge as well as the unique paths in T_u, T_v and T_w joining endvertices of these edges. The *turning point of* $C(uvw)$ *in* T_u is that vertex in $C(uvw) \cap T_u$ whose distance to u is minimal. The turning points in T_v and T_w are defined similarly. Consider all triangles as described in the previous paragraph, i.e. all triangles in $G'[A \cup B]$ which meet A in precisely 2 vertices and contain an F-edge. Among all these triangles choose $a'b'c'$ such that the distance of a to the turning point t of $C(abc)$ in T_a is minimal, where $a', c' \in A$ and $b' \in B$. We have to distinguish 2 cases.

Case 2.1: The distance between a and t is at most 1.

Let us start with a definition. Given $v \in T_x$, we say that distinct neighbours $y'_1, \ldots, y'_i$ of x' in G' *form a v-join* if all the subpaths of T_x which join v to the unique T_x–T_{y_i} edges meet in v and are otherwise disjoint from each other. This means that if x' is the centre of an i-star S in G' whose leaves form a v-join for some vertex $v \in T_x$, then S corresponds to a subdivided i-star in G whose centre is v. Moreover, note that neighbours $y'_1, \ldots, y'_i$ of x' form an x-join if and only if they lie above different neighbours of x in T_x.

Now let v_1 and v_2 denote the neighbours of t on $C(abc)$. The vertices t, v_1, v_2 will be 3 branch vertices of our subdivision of K_5. We will use the cycle $C(abc)$ to join these 3 branch vertices pairwise. Before this we will choose for each v_i two neighbours y'_i and z'_i of a' in G' which will serve as 'docking points' when using the linkedness of G^* to join v_i to the remaining 2 branch vertices of the subdivided K_5 (Fig. 4). (These 'docking points' play the same role for v_i as the leaves of S_i do for x_i in both Section 4.1 and Case 1.) The desired 'docking points' y'_i and z'_i have to satisfy the following properties:

- No F-edge joins a' or c' to any of the y'_i or z'_i.
- There is no F-edge between any of the four vertices y'_i and z'_i.

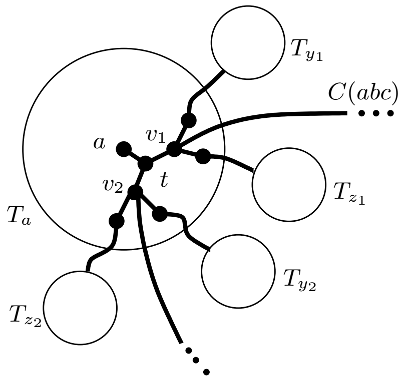

Figure 4: The trees corresponding to the 'docking points' for v_1 and v_2.

- The vertices y_i' and z_i' are neighbours of a' in $G' - \{b', c'\}$ which lie above v_i. Moreover, y_i', z_i', b', c' form a v_i-join.

Note that the third condition together with the fact that either $b'a' \in F$ or $b'c' \in F$ implies that neither y_i' nor z_i' can be joined to b' by an F-edge. Let us now show that such 'docking points' y_1', y_2', z_1' and z_2' exist. (For this, without mentioning it explicitly, we will make frequent use of the fact that every vertex sends out at most one F-edge and that G contains at most one edge between every pair T_x, T_y of trees.) First fix two different neighbours v_i^y and v_i^z of v_i in T_a which lie above v_i and avoid $C(abc)$. Such neighbours exist since v_i is not the turning point of $C(abc)$ in T_a and thus only one of the at least 3 neighbours of v_i lying above v_i in T_a lies on $C(abc)$. Since v_i has distance at most 2 from the root a of T_a, the branches of T_a above v_i^y and v_i^z both send out at least 3 edges. (Note that the definition of v_i^y and v_i^z implies that none of these edges can go to T_b or T_c.) Thus there are at least 3 neighbours of a' in $G' - \{b', c'\}$ which lie above v_i^y and the same holds for v_i^z. This means that for each of y_1', y_2', z_1', z_2' there are at least 3 candidates which satisfy the third of the above properties. Without loss of generality we may assume that no F-edge joins a' to a candidate for z_1', y_2' or z_2' (relabel if necessary to achieve this). Moreover, we will only consider the case when no F-edge joins c' to a candidate for z_1' or z_2'. In this case we will first choose y_1' and then y_2'. The remaining cases are analogous. Since there are at least 3 candidates for y_1', one of them sends no F-edge to a' or c'. Denote it by y_1'. Similarly, for y_2' take any candidate which does not send an F-edge to y_1' or c'. Note that $a'y_2' \notin F$ by our assumption. Next we have to choose z_1' and z_2'. We will only consider the case when there are at least 2 candidates for z_2' which are joined to neither y_1' nor y_2' by an F-edge. (The case when this holds for z_1' instead of z_2' is analogous.) Take z_1' to be any candidate which is joined to neither y_1' nor y_2' by an F-edge. Finally, we can choose z_2' since by assumption there is at most 1 candidate for it which is joined to y_1' or y_2' by an F-edge and at most 1 of the remaining ≥ 2 candidates sends an F-edge to z_1'. (Recall that our earlier assumptions imply that neither a' nor c' send an F-edge to z_1' or z_2'.)

Next we will choose one 'docking point' for t. Let v_t^y and v_t^z be two neighbours of t in T_a which are distinct from v_1 and v_2. If $t \neq a$, we choose v_t^z to be a. Note that the branch of T_a above v_t^y sends out at least 9 edges and none of them goes to T_b, T_c, T_{y_1}, T_{y_2}, T_{z_1} or T_{z_2}. Thus there are at least 9 neighbours of a' in $G' - \{b', c', y_1', y_2', z_1', z_2'\}$ which lie above v_t^y. Let y_t' be one such neighbour which sends an F-edge to none of a', c', y_1', y_2', z_1', z_2'.

For $i = 1, 2$ let S_i denote the 2-star in $G'[A \cup B]$ whose centre is a' and whose leaves are y_i' and z_i'. Let S_3 be the 1-star with centre a' and leaf y_t'. Thus the stars S_1, S_2 and S_3 meet in a' and are disjoint otherwise. We will now choose a 2-star S_4 and a 3-star S_5 satisfying the following properties:

(α') S_4 and S_5 are disjoint and avoid $V(S_1) \cup V(S_2) \cup V(S_3) \cup \{b', c'\}$. The centres x_4' and x_5' of S_4 and S_5 lie in A. Moreover, either $a'x_4'x_5'$ forms a path or else there is a vertex $z_t' \in B \setminus (V(S_1 \cup \cdots \cup S_5) \cup \{b', c'\})$ such that $a'z_t'x_4'x_5'$ is a path and $z_t'x_4' \in F$.

(β') No edge of S_4 or S_5 belongs to F. No F-edge joins 2 leaves of S_4 or of S_5.

(γ') There is no F-edge joining S_4 to a vertex in $V(S_1 \cup S_2 \cup S_3 \cup S_5) \cup \{a', b', c'\}$. The analogous condition holds for S_5.

(δ') Let P' denote the path $a'x_4'x_5'$ or $a'z_t'x_4'x_5'$ guaranteed in (α'). Let H' be the union of P', S_4 and S_5. (Thus both x_4' and x_5' have degree 4 in H'.) For each $i = 4, 5$ the 4 neighbours of x_i' in H' form an x_i-join. Moreover, the neighbour of a' on P' lies above v_t^z.

(The vertices x_4 and x_5 will be the 2 remaining branch vertices of our subdivision of K_5 in G.) We will only show that we can either find a path $a'x_4'x_5'$ or a path $a'z_t'x_4'x_5'$ with the desired properties. The existence of the leaves of S_4 and S_5 then follows by an argument analogous to the one in Case 1 (but with more room to spare this time). So let us consider all those at least 27 neighbours of a' that lie above v_t^z. Clearly, none of them lies in $V(S_1 \cup S_2 \cup S_3) \cup \{b', c'\}$. Moreover, at most 8 of them are joined by an F-edge to a vertex in $V(S_1 \cup S_2 \cup S_3) \cup \{b', c'\}$. If one of the remaining neighbours lies in A, we take it to be x_4'. Thus we may assume that each of the at least 19 remaining neighbours lies in B. Take z_t' to be any such neighbour which has the property that the endvertex in A of the unique F-edge incident to z_t' does not lie in $V(S_1 \cup S_2 \cup S_3) \cup \{a', b', c'\}$. Take x_4' to be this endvertex. The vertex x_5' can now be found in the same way as in Case 1.

Having chosen our stars $S_1, \ldots, S_5$, we proceed similarly as in Case 1. Again, we consider the graph G^* obtained from $G'[A \cup B]$ by contracting each edge in F. The stars $S_1, \ldots, S_5$ have been chosen is such a way that in the subgraph of G' induced by all the stars S_i, the path P' and the triangle $a'b'c'$ at most two edges are contracted, namely the F-edge lying on $a'b'c'$ and the F-edge $z_t'x_4'$ (if z_t' exists). Otherwise nothing changes in this subgraph when passing over to G^*.

We will now choose our subdivision of K_5 in G. As indicated before, the branch vertices will be t, v_1, v_2, x_4 and x_5. Five of the pairs of branch vertices can be linked directly. Indeed, we can use the cycle $C(abc)$ to link t, v_1 and v_2 pairwise. Morover, x_4 will be linked to x_5 by the path consisting of the unique

T_{x_4}–T_{x_5} edge together with the paths in T_{x_4} and T_{x_5} joining x_4 and x_5 to this edge. In the case when a' and x'_4 are joined by an edge, we can link t to x'_4 in a similar way. If z'_t exists, i.e. if $a'z'_tx'_4$ is a path of length 2, then the path linking t to x_4 will run through T_{z_t}. We now have to find paths joining the remaining 5 pairs of branch vertices. Similarly as in Case 1, this will be done by using the linkedness of G^* to connect the 'docking points', i.e. the images of the leaves of the stars $S_1, \dots, S_5$. Each of the 5 paths we are looking for has to avoid the star centres as well as the image of c' in G^*. Thus we delete these 4 vertices. We can now link the images of the star leaves if the subgraph of G^* thus obtained is 5-linked. So by Theorem 4 we are done if the minimum degree this graph is at least 50 and if its connectivity is at least 10. This in turn holds if the minimum degree of G^* is at least 54 and its connectivity is at least 14. Inequality (3) shows that the first condition holds. The latter condition is satisfied since by Lemma 5 the graph G^* is $\lceil c/3 \rceil$-connected and $c/3 = 17$.

Case 2.2: The distance between a and t is at least 2.

Similarly as in Case 2.1, also in this case we will choose the branch vertices in such a way that 3 of them lie on the cycle $C(abc)$ and thus this cycle can be used to link them pairwise. The 2 remaining branch vertices will also be chosen in an analogous way, i.e. they will be attached to $C(abc)$ by a path. However, this time we cannot choose the first 3 branch vertices in $T_a \cap C(abc)$ as $C(abc)$ stays too far away from the centre when passing through T_a. Instead, we will only choose two vertices in $T_a \cap C(abc)$ and one in $T_c \cap C(abc)$.

Since by assumption the distance from a to the turning point of $C(abc)$ in T_a is at least 2 and since by the choice of $a'b'c'$ the analogue also holds for the turning point of $C(abc)$ in T_c, the cycle $C(abc)$ meets both T_a and T_c in at most 9 vertices (use condition (ii) to see this). Since $C(abc)$ meets T_b in at most 13 vertices and the girth of G is at least 27, it follows that $C(abc)$ has to meet each of T_a and T_c in at least 5 vertices. Moreover, it follows that $C(abc)$ meets at least one of T_a and T_b in at least 7 vertices. Thus without loss of generality we may assume that $C(abc)$ meets T_a in at least 7 and at most 9 vertices and that $C(abc)$ meets T_c in at least 5 and at most 9 vertices (otherwise relabel a' and c').

Let us now prove that there exists a vertex $c_0 \in T_c$ and neighbours y'_c and z'_c of c' in $G' - \{a', b'\}$ satisfying the following properties (the 'moreover' part of the first property will not be used until Case 2.2.2):

- The vertex c_0 is either the first, the second or the third vertex on $C(abc)$ in T_c when coming from T_a. Moreover, c_0 is not the turning point of $C(abc)$ in T_c.
- The vertices y'_c, z'_c, a', b' form a c_0-join.
- F does not contain $y'_cz'_c$.
- Neither y'_c nor z'_c sends an F-edge to any of a', b', c'.

The vertex c_0 will be one of the branch vertices of our subdivision. Both y'_c and z'_c will serve as 'docking points' for c_0. So let us now show that there exist such vertices c_0, y'_c and z'_c. Choose any two vertices c_1 and c_2 such that they are among the first 3 vertices on $C(abc)$ in T_c when coming from T_a and such that none of them equals the turning point. (These two vertices exist since $C(abc)$ meets T_c in at least 5 vertices.) Thus c_1 and c_2 are candidates for c_0 in the sense that they satisfy the first property above.

Before we continue, we need some more notation which generalizes the notion of 'lying above'. Given a subtree $T \subseteq T_x$ and a neighbour z' of x' in G', we say that z' *belongs to* T if T contains an endvertex of the unique T_x–T_z edge. Thus if $v \in T_x$ then z' lies above v if and only if z' belongs to the branch of T_x above v.

Next we show that, for each $i = 1, 2$, one can find neighbours y'_i and z'_i of c' in $G' - \{a', b'\}$ such that a', b', y'_i, z'_i form a c_i-join and $y'_i z'_i \notin F$. To do this, we first consider the case that all neighbours of c_i in T_c are contained in $C(abc)$. This implies that c_i sends out at least two edges to vertices in $G-(T_a \cup T_b \cup T_c)$, i.e. there are (at least) two neighbours of c' in $G' - \{a', b'\}$ which form a c_i-join together with a' and b'. Take y'_i and z'_i to be two such neighbours. Suppose that $y'_i z'_i \in F$. We may assume that $y'_i \in A$ and $z'_i \in B$ (relabel if necessary). Since the triangle $c' y'_i z'_i$ was a candidate for the choice of $a'b'c'$, it follows that the turning point of $C(cy_iz_i)$ in T_{y_i} has distance at least 2 from y_i. Thus $C(cy_iz_i)$ meets T_{y_i} in at most 9 vertices. However, it meets T_{z_i} in at most 13 vertices and T_c in precisely one vertex. Thus the length of $C(cy_iz_i)$ is at most 23, contradicting the fact that the girth of G is at least 27. So we may turn to the case that c_i has a neighbour v in $T_c - C(abc)$. In the case when v is the only such neighbour, c_i has to send out an edge to $G - (T_a \cup T_b \cup T_c)$. Take y'_i to be the vertex for which the corresponding tree T_{y_i} contains the endvertex of that edge. If c_i has another neighbour w in $T_c - C(abc)$, take for y'_i any neighbour of c' which belongs to the component of $T_c - c_i$ containing w. To find z'_i, consider those at least 3 neighbours of c' which belong to the component of $T_c - c_i$ containing v (Fig. 5). At most one of them sends an F-edge to y'_i. Take

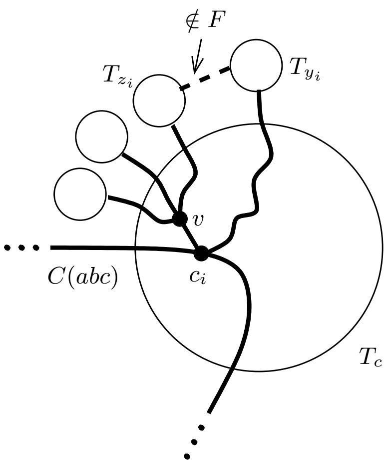

Figure 5: Choosing y'_i and z'_i in the case when c_i has a neighbour v in T_c–$C(abc)$.

z'_i to be any other such neighbour. Thus in each case we have found vertices y'_i

and z'_i as required. Note that the sets $\{y'_1, z'_1\}$ and $\{y'_2, z'_2\}$ are disjoint. (Indeed, if for example $y'_1 = y'_2$ then G would contain at least two T_c–T_{y_1} edges.)

Let us now show that for some i we have $a'y'_i, a'z'_i, c'y'_i, c'z'_i \notin F$. (Note that $b'y'_i, b'z'_i \notin F$ holds automatically since either $a'b' \in F$ or $b'c' \in F$ and F consists of independent edges.) If this is not the case, then for example we have that $a'y'_1, c'y'_2 \in F$. But this contradicts the fact that either a' or c' already sends an F-edge to b'. The vertices c_i, y'_i, z'_i which for which $a'y'_i, a'z'_i, c'y'_i, c'z'_i \notin F$ are a good choice for c_0, y'_c, z'_c.

Having found one branch vertex c_0 in $C(abc) \cap T_c$, we will choose two other branch vertices in $C(abc) \cap T_a$. So let $a_1, a_2, \dots, a_j$ denote all those vertices on $C(abc)$ which lie in T_a in the order in which they appear when coming from T_c. Thus $7 \le j \le 9$ by our assumption. For each $i = 1, \dots, j$ we choose neighbours y'_i and z'_i of a' in $G - \{b', c'\}$ which satisfy the following properties (Fig. 6):

(A) The vertices y'_i, z'_i, b', c' form an a_i-join.

(B) F does not contain $y'_i z'_i$.

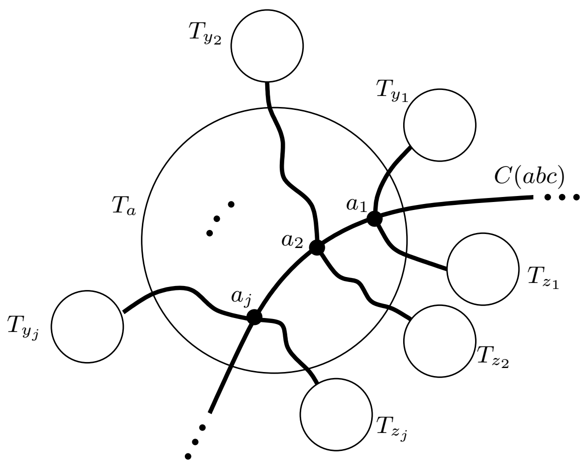

Figure 6: The neighbours y'_i and z'_i chosen for a_i.

Such neighbours can be found in the same way as before. (Recall that the fact that c_0 was not a turning point was not used when showing the existence of y_c and z_c.) Among all possible choices, we choose each pair y'_i, z'_i such that $|\{y'_i, z'_i\} \cap \{y'_c, z'_c\}|$ is minimal. Thus we try to keep the pair y'_i, z'_i as disjoint as possible from y'_c, z'_c. Again, the sets $\{y'_i, z'_i\}$ are automatically disjoint for different indices $i \le j$.

Next we show that there exists an index $i_0 \le j$ for which we can take a_{i_0} to be the second branch vertex and both y'_{i_0} and z'_{i_0} to be 'docking points' for a_{i_0}. So i_0 has to satisfy the following properties:

(C) Neither y'_{i_0} nor z'_{i_0} sends an F-edge to any of a', b', c', y'_c, z'_c.

(D) Neither y'_{i_0} nor z'_{i_0} lies in $\{y'_c, z'_c\}$.

Since either $a'b' \in F$ or $b'c' \in F$, at most one of a', b', c' sends an F-edge to a vertex in $G' - \{a', b', c'\}$. Thus altogether the vertices a', b', c', y'_c, z'_c send out at most 3 F-edges to vertices in $G' - \{a', b', c'\}$. This implies that at most 5 of the indices $i \leq 7$ violate (C) or (D). Hence there exists an index i_0 as required. Put $I := \{1, \ldots, j\} \setminus \{i_0\}$.

Our next aim is to show that there is another a_i which we can take as a branch vertex (and such that y'_i and z'_i can serve as 'docking points' for a_i). Thus we have to show that there exists an index $i_1 \in I$ which satisfies the following properties:

(C') Neither y'_{i_1} nor z'_{i_1} sends an F-edge to any of $a', b', c', y'_c, z'_c, y'_{i_0}, z'_{i_0}$.

(D') Neither y'_{i_1} nor z'_{i_1} lies in $\{y'_c, z'_c, y'_{i_0}, z'_{i_0}\}$.

As before, it follows that at most 5 indices $i \in I$ violate (C'). Since $\{y'_{i_0}, z'_{i_0}\}$ is disjoint from $\{y'_i, z'_i\}$ for each $i \in I$, at most 2 indices $i \in I$ violate (D'). Hence altogether at most 7 of the indices $i \in I$ are forbidden. Thus i_1 exists if $j = 9$ (since then $|I| = j - 1 = 8$).

So we only need to consider the case when $7 \leq j \leq 8$. Moreover, we may assume that for some index $k \in I$ either y'_k or z'_k lies in $\{y'_c, z'_c\}$. So let us assume that $y'_k = y'_c =: y'$ (relabel if necessary to achieve this). Furthermore, if $j = 8$, we may assume that there is some index $\ell \in I \setminus \{k\}$ for which either y'_ℓ or z'_ℓ equals z'_c. So let us assume that $y'_\ell = z'_c$ in the case when $j = 8$. To show the existence of the index i_1, we will now prove that these assumptions lead to a contradiction. To do this, we distinguish the following two cases.

Case 2.2.1: The vertex a_k is not an endvertex of the unique T_a–T_y edge.

Let u denote the neighbour of a_k on the subpath of T_a joining a_k to the endvertex of the unique T_a–T_y edge (Fig. 7). Thus there are at least 3 neighbours of a' which belong to the component of $T_a - a_k$ containing u. So each of these 3 neighbours was a candidate for y'_k when choosing the pair y'_k, z'_k. Since y'_k, z'_k was chosen to be as disjoint as possible from y'_c, z'_c, this implies there are precisely 3 such neighbours: one of which (namely $y' = y'_k$) is equal to y'_c, one of which, w' say, is equal to z'_c and the third one, v' say, sends an F-edge to z'_k. (Otherwise the pair consisting of z'_k together with one such neighbour would have been a better choice for y'_k, z'_k.) Since there are precisely 3 such neighbours, u has to be an endvertex of all the 3 edges joining T_a to each of T_v, T_w and T_y. Moreover, the choice of y'_k, z'_k implies that a_k is an endvertex of the unique T_a–T_{z_k} edge (Fig. 7). Let us consider the cycle $C(avz_k)$ corresponding to the triangle $a'v'z'_k$. This cycle meets T_a in precisely 2 vertices. Since the girth of G is at least 27, it meets both T_v and T_{z_k} in at least $27 - 2 - 13 = 12$ vertices. Thus the turning points of $C(avz_k)$ in T_v and T_{z_k} have to be v and z_k. But this shows that the triangle $a'v'z'_k$ would have been a better choice for $a'b'c'$, a contradiction.

Case 2.2.2: The vertex a_k is an endvertex of the unique T_a–T_y edge.

This time consider the cycle $C(acy)$ corresponding to the triangle $a'c'y'$. Let us first estimate the number of vertices in $C(acy) \cap T_c$. Note that the distance from the root c of T_c to c_0 is as least one larger than the distance from c_0 to the turning point of $C(abc)$ in T_c (this holds since c_0 is not this turning point).

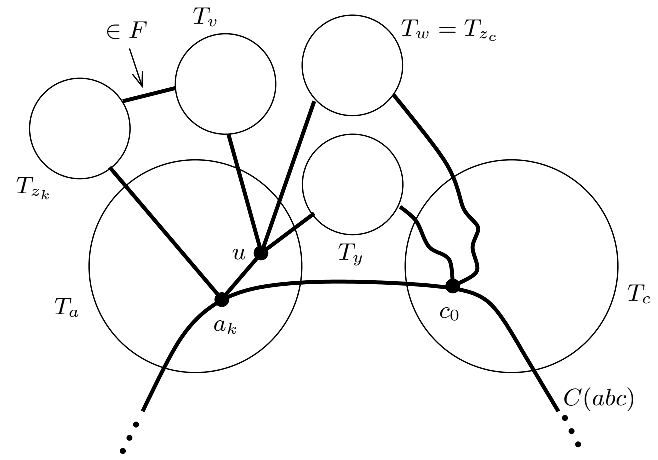

Figure 7: The 3 neighbours of a' lying above u.

Since the latter distance is at least 2, it follows that the segment of $C(acy) \cap T_c$ which joins c_0 to the endvertex of the T_c–T_y edge consists of at most 4 vertices. Since c_0 was one of the first 3 vertices on $C(abc)$ in T_c when coming from T_a, it follows that the segment of $C(acy) \cap T_c$ which joins c_0 to the endvertex of the T_a–T_c edge consists of at most 3 vertices. Thus $C(acy)$ contains at most 6 vertices in T_c (we have counted c_0 twice). Since $C(acy)$ contains at most 13 vertices in T_y and the girth of G is at least 27, it follows that $C(acy)$ meets T_a in at least 8 vertices. On the other hand, since by assumption a_k is an endvertex of the unique T_a–T_y edge, it follows that $C(acy)$ meets T_a precisely in the vertices $a_1, \dots, a_k$. Thus we may assume that $k = 8$ and therefore $j = 8$. (Recall that we had previously ruled out all possibilities for j except 7 and 8.) As shown before Case 2.2.1, this in turn implies that we may assume that $y'_\ell = z'_c =: z'$ for some index $\ell \in I \setminus \{k\}$. Thus $\ell \leq 7$. The case when a_ℓ is not an endvertex of the unique T_a–T_z edge can be dealt with as in Case 2.2.1. Thus we may assume that a_ℓ is an endvertex of the unique T_a–T_z edge. But just as for $C(acy)$, one can show that the cycle $C(acz)$ corresponding to the triangle $a'c'z'$ has to meet T_a in at least 8 vertices. But this is a contradiction since it meets T_a precisely in $a_1, \dots, a_\ell$ and $\ell \leq 7$. This completes the proof of the existence of the index i_1.

So far, we have chosen 3 branch vertices c_0, a_{i_0} and a_{i_1} on $C(abc)$ together with 2 docking points for each of them. We will now find our subdivision of K_5 similarly as in Case 2.1. As there, the remaining 2 branch vertices will be attached to $C(abc)$ by a path. So we have to find a suitable path in $G'[A \cup B]$ of length 2 or 3 which starts in either a' or c'. As in Case 2.1, the last two vertices on this path will be the centres of the stars S_4 and S_5. The remaining two branch vertices of our subdivision will be the roots of the two trees corresponding to the centres of S_4 and S_5. Again, S_4 will be a 2-star and S_5 a 3-star.

More precisely, we proceed as follows. First consider the case that one of the 'docking points' $y'_c, y'_{i_0}, y'_{i_1}, z'_c, z'_{i_0}, z'_{i_1}$ lies in A. Suppose for example that $z'_{i_0} \in A$. (The other cases are analogous.) Then we take z'_{i_0} to be the centre x'_4 of S_4. S_1 will be the 2-star in $G'[A \cup B]$ whose centre is c' and whose leaves are y'_c and z'_c. S_2 will be the 2-star in $G'[A \cup B]$ whose centre is a' and whose leaves are y'_{i_1} and z'_{i_1}. S_3 will be the 1-star in $G'[A \cup B]$ whose centre is a' and whose leaf is y'_{i_0}.

So let us now consider the case that all the 6 'docking points' lie in B. Then at least one of them, z'_{i_0} say, sends an F-edge to a vertex outside $\{a', c'\}$. Let x'_4 denote this vertex. Since $x'_4 \in A$, x'_4 is automatically distinct from b' and from all the 'docking points'. Moreover, x'_4 sends neither an F-edge to any of a', b', c' nor to any 'docking point' other than z'_{i_0}. We take x'_4 to be the centre of S_4. The stars S_1, S_2, S_3 are defined similarly as before.

In both cases, the centre x'_5 of the star S_5 as well as the 2 leaves of S_4 and the 3 leaves of S_5 can now be found similarly as in Case 2.1. Having chosen $S_1, \dots, S_5$, the subdivision of K_5 can also be found as in Case 2.1. This completes the proof of Case 2.2 and thus of the case when $r = 4$.

5 Further improvements: approaches and obstacles

Below, we briefly discuss three rather natural modifications to the proof strategy which one might try out in order to improve the bound of 27 on the girth in Theorem 1.

Recall that throughout this paper, the trees which were contracted to yield the auxiliary graph G' have radius at least 3. It would of course be desirable if we could adapt our strategy to work also for trees of radius at least two. In this case a girth of at least 19 would now suffice to ensure that the auxiliary graph G' graph has no multiple edges. However, the obvious drawback is that the minimum degree of G' may now be smaller. In particular, in the case $r = 4$ we can only guarantee $\delta(G') \geq 36$ instead of $\delta(G') \geq 108$, so this does not seem feasible for small r.

Another approach is of course to work with trees of radius at least three as before but now to relax the girth requirement a little. This leads to the possibility of multiple edges between pairs of trees. One can show that this does not reduce the minimum degree by more than a constant factor (e.g. by a factor of six if one assumes a girth of at least 21 instead of a girth of 27, see Lemma 15 in [9] for the argument). However, the resulting bound on the minimum degree is again too small for our techniques to apply if r is small. (Moreover, multiple edges cause other problems, too.)

Thirdly, one might hope for an improvement on the bound of $10k$ on the necessary average degree in order to ensure k-linkedness in Theorem 4. An example in [19] shows that the best one can hope for is to replace the constant 10 by 4. But even with such an optimal bound our proof would not give a better bound on the girth. However, maybe combining this with the allowance of multiple edges and some new ideas would work.

On the other hand, one might wonder how much of our improvement from a girth of 186 in [9] to 27 here is due to the fact that we did not use Theorem [19] in [9] but the previous bound of Bollobás and Thomason [2] where the $10k$ is replaced by $22k$: it is easy to check that our proofs would still work using the latter bound if one requires the girth to be at least 35 (and considers trees which now have radius at least 4 instead of radius at least 3).

Finally, recall that there is a lot of room to spare in the estimates in Section 4.1 when r is large. However, even if we assume r to be large, the arguments in Section 4.1 no longer work for trees of radius at least 2 instead of radius at least 3. (So they do not provide a short proof of the fact that the girth bound can be reduced to 19 instead of 27 for large r.) It is still an open question if the bound of 15 on the girth for large r [9] can be reduced.

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