Forcing a K_r minor by high external connectivity

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Abstract

We investigate the minimal structure a graph H is required to have if H is to force a large complete minor in any graph in which it has high external connectivity. We observe that such graphs H must contain a large binary tree with some small additions, and prove that some canonical instances of this structure are also sufficient to force a large complete minor.

1 Introduction

A fundamental result of Robertson and Seymour [6] states that a graph has large tree-width if and only if it contains a large grid minor. In their short proof of this theorem, Diestel et al. [2] introduced the concept of externally connected sets. A set $X \subseteq V(G)$ is *externally k-connected* in G if $|X| \geq k$ and, for all subsets $Y, Z \subseteq X$ with $|Y| = |Z| \leq k$, there are $|Y|$ disjoint Y-Z paths in G without inner vertices or edges in $G[X]$. We say that a subgraph H of G is *externally k-connected* in G if $V(H)$ is externally k-connected in G. For example the bottom row of a $k \times k$ grid G is externally k-connected in G. A result from [2] is that a graph G has large tree-width if and only if it contains a large externally highly connected set X. Thus such a set X forces a large grid minor in G, even if $G[X]$ consists of isolated vertices.

One of the central tools in the proof of the Graph Minor Theorem of Robertson and Seymour is the observation that every large externally highly connected grid forces a large complete minor (and thus so do the graphs with sufficiently large tree-width). Indeed, if we take a large grid H and add $\binom{r}{2}$ independent edges in such a way that any endvertex of such an edge has horizontal distance at least r from every other such endvertex as well as distance at least r from the boundary of the grid, then in the resulting graph we can contract suitable (zig-zag) paths in H to vertices of a K_r whose edges are the edges added to H. Now if H is externally highly connected in some other graph G, these additional edges can be found (subdivided) as paths through G. In this paper, we address the question raised by Grohe [5] whether thinner structures than grids can still force large complete minors in the same way.

Given a graph property $\mathcal{P}$, let us say that a class $\mathcal{H}$ of graphs *forces large minors from $\mathcal{P}$* if for every $r \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that whenever a graph $H \in \mathcal{H}$ is externally k-connected in another graph G, then G has a minor of order at least r in $\mathcal{P}$. In this terminology, our observation above says that grids force large complete minors, while the result from [2] cited earlier says that the class of all edgeless finite graphs forces large grids. And Grohe's question is whether a class $\mathcal{H}$ of graphs 'substantially thinner' than grids can still force large complete minors. For instance:

Problem 1 *Is there a class $\mathcal{H}$ of bounded tree-width that forces large complete minors?*

We shall see that the answer to this question is yes. So the next problem will be to find, if possible, the 'thinnest' such class $\mathcal{H}$. To make this precise, let us write $\mathcal{H} \preccurlyeq \mathcal{H}'$ for classes $\mathcal{H}$ and $\mathcal{H}'$ of finite graphs if for every $H \in \mathcal{H}$ there exists an $H' \in \mathcal{H}'$ such that H is a minor of H'. If both $\mathcal{H} \preccurlyeq \mathcal{H}'$ and $\mathcal{H}' \preccurlyeq \mathcal{H}$ then $\mathcal{H}$ and $\mathcal{H}'$ are *equivalent*. (For example, the class of grids is the unique least element, up to equivalence, among the classes of unbounded tree-width.) The Graph Minor Theorem implies that there are no infinite strictly descending chains of graph properties $\mathcal{H}$ with respect to $\preccurlyeq$ (see e.g. [3, Lemma 1.5]). Thus even if there is no least class forcing large complete minors, we can still try to find the minimal ones:

Problem 2 *Determine the $\preccurlyeq$-minimal classes of graphs that force large complete minors.*

In this paper, we shall settle Problem 1 in the affirmative by constructing four inequivalent classes $\mathcal{H}'$, $\mathcal{H}^4$, $\mathcal{H}^{2,3}$ and $\mathcal{H}^{3,2}$ of graphs such that each of them has bounded tree-width but forces large complete minors (Theorem 1). All these classes are $\preccurlyeq$-minimal with respect to the property of forcing large complete minors (Theorem 2). Indeed, I conjecture that, up to equivalence, $\mathcal{H}'$, $\mathcal{H}^4$, $\mathcal{H}^{2,3}$ and $\mathcal{H}^{3,2}$ are the only such $\preccurlyeq$-minimal classes, which would also settle Problem 2. But this conjecture remains open.

2 Preliminary observations and statement of results

First recall that, as was observed in Section 1, every class $\mathcal{H}$ of finite graphs that has unbounded tree-width forces large complete minors. So we can restrict our attention to classes $\mathcal{H}$ whose tree-width is bounded. If even the path-width of the graphs in $\mathcal{H}$ is bounded, say $\mathrm{pw}(H) < \ell$ for all $H \in \mathcal{H}$, it turns out that we can join the vertices of any $H \in \mathcal{H}$ bijectively to the bottom row of a grid to obtain a graph that has no K_r minor for any $r > 3\ell + 4$ but in which H is externally $|H|$-connected. (See [4, Lemma 2.3] for a proof.)

Similarly, the vertices of every outerplanar graph H can be joined bijectively to the bottom row of a grid to obtain a planar graph in which H is externally $|H|$-connected. Thus outerplanar graphs do not even force a K_5 minor. (A graph is *outerplanar* if it can be drawn in the plane so that all its vertices lie on the boundary of the outer face.) Hence if all graphs in $\mathcal{H}$ can be made outerplanar by deleting ℓ vertices in each of them, then the graphs in $\mathcal{H}$ do not force a $K_{\ell+5}$ minor.

Now the graphs of unbounded path-width are precisely those that contain arbitrarily large binary trees as minors (see e.g. [1, p. 260]), while the outerplanar graphs are precisely those that contain neither K_4 nor $K_{2,3}$ as a minor (see e.g. [9]). So if the graphs in $\mathcal{H}$ are to force arbitrarily large complete minors, they must contain unboundedly large binary trees and at the same time unboundedly many copies of K_4 or $K_{2,3}$ as minors.

Our main result in this paper says that a natural combination of these conditions is also sufficient. Let T_n^2 denote the binary tree of height n. Let H'_n be the disjoint union of n graphs each of which is obtained from T_n^2 by adding a new vertex and joining it to the leaves of T_n^2. Let H_n^4 be the graph obtained from T_n^2 by identifying each of its leaves with a vertex of a K_4 (where the K_4's glued to different leaves of T_n^2 are disjoint from each other and from the rest of T_n^2). Let $H_n^{2,3}$ be the graph obtained from T_n^2 by identifying each of its leaves with a vertex of a $K_{2,3}$ having degree two, and let $H_n^{3,2}$ be the graph obtained from T_n^2 by identifying each of its leaves with a vertex of a $K_{2,3}$ having degree three (where the $K_{2,3}$'s glued to different leaves of T_n^2 are disjoint from each other and from the rest of T_n^2). Let $\mathcal{H}'$ be the class consisting of all H'_n, and define $\mathcal{H}^4$, $\mathcal{H}^{2,3}$ and $\mathcal{H}^{3,2}$ similarly. It is easy to show that these classes are incomparable with respect to $\preccurlyeq$. The following theorem states that each of them forces large complete minors.

Theorem 1 *Given an integer r, there are integers $k = k(r)$ and $n = n(r)$ with the following property. Whenever a graph G contains an externally k-connected set X such that $G[X]$ has a minor isomorphic to any of H'_n, H_n^4, $H_n^{2,3}$ or $H_n^{3,2}$, there is a K_r minor in G.*

Moreover, each of the four classes $\mathcal{H}'$, $\mathcal{H}^4$, $\mathcal{H}^{2,3}$ and $\mathcal{H}^{3,2}$ is $\preccurlyeq$-minimal with the property of forcing large complete minors:

Theorem 2 *If $\mathcal{H}$ is a class of finite graphs which forces large complete minors and if $\mathcal{H} \preccurlyeq \mathcal{H}^*$, where $\mathcal{H}^*$ is one of the classes $\mathcal{H}'$, $\mathcal{H}^4$, $\mathcal{H}^{2,3}$ and $\mathcal{H}^{3,2}$, then $\mathcal{H}$ is equivalent to $\mathcal{H}^*$.*

Our proof of Theorem 1 uses methods as developed in [2]. An alternative approach would have been to show that any graph containing H'_n, H_n^4, $H_n^{2,3}$ or $H_n^{3,2}$ for sufficienly large n as an externally highly connected subgraph cannot be nearly embedded in a given surface (see e.g. [4] for definitions). Theorem 1 would then follow from the theorem of Robertson and Seymour [7], that characterizes the structure of graphs without a K_r minor (r fixed).

The paper is organized as follows. In Section 3 we show that the graphs in $\mathcal{H}'$ force arbitrarily large complete minors, while in Section 4 we prove the same for the graphs in $\mathcal{H}^4$, $\mathcal{H}^{2,3}$ and $\mathcal{H}^{3,2}$. Theorem 2 is proved in the final section of this paper.

3 Trees attached to stars

Our terminology follows [1]. All trees considered in this paper will have a root. The *binary tree of height* $n \geq 1$ is the tree in which the root has degree two, all leaves have distance n from the root, and all other vertices have degree three.

Let $H_n^\sharp$ be the graph obtained from the binary tree T of height n by adding a new vertex x and joining it to all leaves of T. Thus H'_n is the disjoint union of n copies of $H_n^\sharp$. We call T the *binary tree in* $H_n^\sharp$. The leaves of T will be called *leaves of* $H_n^\sharp$, and x will be its *new vertex*.

Theorem 3 *Given an integer r, there exist integers d, f, n with the following property. Whenever a graph G contains an externally d-connected set X such that $G[X]$ has the graph consisting of f disjoint copies of $H_n^\sharp$ as a minor, there is a K_r minor in G.*

For the proof of the theorem we will need the following definitions and lemmas. Given two vertices x and y of a tree T, we say that x is *above* y if y lies on the path from x to the root of T. A vertex x is called *successor* of y, if x is a neighbour of y and lies above y. Two vertices of T are *incomparable* if none of them lies above the other. The *branch above* x is the subtree of T induced by the set of all vertices above x (including x itself). If e is an edge of T, then the *branch above* e is the branch above the heighest endvertex of e. A *branch strictly above* x is a branch above a successor of x. If S is a subtree of a tree T, we take the unique lowest vertex of S in T as the root of S. In what follows, we assume that for any given tree T we have chosen a linear ordering σ_T of its vertices in such a way that for every incomparable pair x, y of vertices in T the vertices of the branch above x either all precede or all succeed those of the branch above y; and if x is above y, then x succeeds y. Thus such an ordering may be obtained by considering a drawing of T. For a subtree of T or a subdivision of T we choose the ordering induced by σ_T.

If $P = x_1 \dots x_n$ is a (directed) path and $1 \leq i \leq n$, we write $Px_i := x_1 \dots x_i$, $x_iP := x_i \dots x_n$, $P\mathring{x}_i := x_1 \dots x_{i-1}$ and $\mathring{x}_iP := x_{i+1} \dots x_n$ for the appropriate subpaths of P.

Lemma 4 *Let T be the binary tree of height $n \geq 2$ and A a set of leaves of T. Let $h \leq n$ be a positive integer. If $|A| \geq n^h$, then T contains a subdivision S of a binary tree of height h such that all leaves of S are contained in A and the root of S (when S is viewed as the subdivision of a binary tree) can be taken as the lowest vertex of S in T.*

Proof. Induction on h. If $h = 1$ the assertion holds. Assume that $h > 1$ and the statement is true for smaller values of h. Since $2 + (n-1)|A|/n \leq |A|$, there is a vertex x in T such that each of the branches strictly above x contains $\geq \lfloor |A|/n \rfloor \geq n^{h-1}$ leaves in A. The result follows by taking x as the root of S and applying the induction hypothesis to each of the two branches strictly above x. □

An $r \times t$ *pseudogrid* is a graph consisting of r disjoint directed paths $W_1, \dots, W_r$ and t disjoint directed paths $V_1, \dots, V_t$ such that each W_i consists of t consecutive (vertex disjoint) segments, every V_j meets every W_i exactly in its jth segment, and V_j meets W_i before it meets W_{i+1} (for all $1 \leq i < r$). The W_i are the *horizontal* and the V_j the *vertical paths* of the pseudogrid. A tree T is *s-attached to a pseudogrid* G if there is a set A of s leaves of T such that in each of them there begins a vertical path of G, and G meets T only in A. A is the set of *attached leaves* of T. T is *tidily s-attached to a pseudogrid G* if T is s-attached to G and the order of the leaves in A (in the restriction of the ordering σ_T on A) corresponds to the order of the vertical paths in G starting in these leaves. Let $T_1, \dots, T_k$ be disjoint trees. We say that $T_1, \dots, T_k$ are *[tidily] s-attached to a pseudogrid G* if each T_i is [tidily] s-attached to G and the vertical paths of G starting in T_i either all precede or all succeed those starting in T_{i+1} (for all $1 \le i < k$).

A family $\mathcal{P} = \{P_1, \dots, P_k\}$ of directed paths in a tree T is *nested* if the P_i are disjoint, each of them joins two leaves of T, and for all $1 \le i, j \le k$, the first vertex of P_i precedes the last vertex of P_j in the ordering σ_T.

Lemma 5 *Let G be a graph that contains $\binom{r}{2}$ disjoint subgraphs $G_1, \dots, G_{\binom{r}{2}}$ such that each G_i contains a subdivision T_i of the binary tree of height $2r-1$ and $G_i - T_i$ has a component C_i that is joined (by edges) to every leaf of T_i. Let C be the union of the C_i. If $T_1, \dots, T_{\binom{r}{2}}$ are tidily 2^{2r-1}-attached to an $r \times \binom{r}{2} 2^{2r-1}$ pseudogrid in $G - C$, then G contains a K_r minor.*

Proof. Note that every T_i has a set of r nested paths. We may join the nested paths of all T_i using suitable paths in the pseudogrid to obtain a set $\mathcal{P}$ of r disjoint paths such that each of them meets every T_i (Fig. 1).

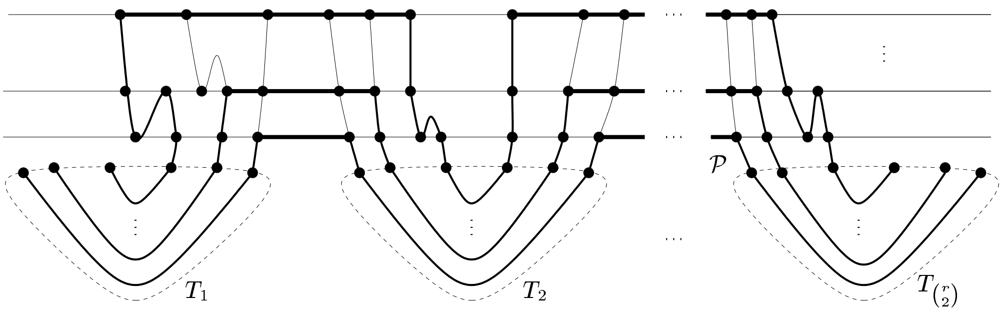

Figure 1: Finding a set $\mathcal{P}$ of disjoint paths

The paths in $\mathcal{P}$ are the branch sets of a (subdivided) K_r minor, as any two of them may be joined by a path through one of the C_i. □

Various proofs of the following result of Erdős and Szekeres can be found in [8].

Lemma 6 *Every sequence of n distinct integers contains a monotone subsequence of length at least $\sqrt{n}$.*

Lemma 7 *Let r, n, s be positive integers such that $n \ge 2$ and $\sqrt{s} \ge n^{2r-1}$, and let $k := \binom{r}{2}$. Let G be a graph that contains k disjoint copies $H_1, \dots, H_k$ of $H_n^\sharp$. For all $1 \le i \le k$ let T_i be the binary tree in H_i and x_i its new vertex. Suppose that $T_1, \dots, T_k$ are s-attached to an $r \times sk$ pseudogrid G' in $G - \{x_1, \dots, x_k\}$. Then G contains a K_r minor.*

Proof. Lemma 6 applied to the sets of attached leaves of every T_i shows that $T_1, \ldots, T_k$ are tidily $\lceil\sqrt{s}\,\rceil$-attached to an $r \times \lceil\sqrt{s}\,\rceil k$ subpseudogrid of G'. Now Lemma 7 follows immediately by first applying Lemma 4 and then Lemma 5. $\square$

The next lemma is proved in [2, Lemma 6].

Lemma 8 *Let $G = (A, B)$ be a bipartite graph, $|A| = a$, $|B| = b$, and let $c \leq a$ and $d \leq b$ be positive integers. Suppose that G has at most $(a - c)(b - d)/d$ edges. Then there exist $C \subseteq A$ and $D \subseteq B$ such that $|C| = c$ and $|D| = d$ and $C \cup D$ is independent in G.*

For a set I of vertices in a graph G let $N(I)$ denote its neighbourhood. We will also make use of the following easy consequence of Hall's theorem.

Lemma 9 *Suppose that $G = (A, B)$ is a bipartite graph such that $s|A| = |B|$ for some positive integer s and $|N(I)| \geq s|I|$ for all subsets $I \subseteq A$. Then G contains $|A|$ disjoint stars, each of them having s edges and their centre in A.*

Proof. Form a new bipartite graph $G' = (A', B)$ by replacing each vertex $a \in A$ by s new vertices and joining each of them to all the neighbours of a. Then G' satisfies Hall's condition, and a matching in G' yields the required disjoint stars. $\square$

Proof of Theorem 3. It suffices to show the following assertion.

> *Let $c := 2^{r^5} r^{r+4}$, $s := c^{\lfloor r^2/2 \rfloor - 2}$ and $n := r^2 c^{r^4}$. Let H be the disjoint union of r copies of $H^\sharp_{\lceil \log_2 n + \log_2 s \rceil}$ and sr copies of $H^\sharp_{\lceil \log_2 n \rceil}$. Let G be a graph containing H as an externally nrs-connected subgraph. Then G contains a K_r minor.* $(*)$

We may assume that $r \geq 4$. Let A be the set consisting of the r copies of $H^\sharp_{\lceil \log_2 n + \log_2 s \rceil}$ in H, and let B be the set consisting of the sr copies of $H^\sharp_{\lceil \log_2 n \rceil}$ in H. Choose ns leaves of every graph in A, and let Z denote the set consisting of all these leaves. Similarly, choose n leaves of every graph in B, and let Z' denote the set consisting of all these leaves. As H is externally nrs-connected in G, there is a set $\mathcal{Q}$ of $|Z| = nrs$ disjoint Z-Z' paths having no inner vertices in H. Lemma 9 implies that we may label the graphs in A by $H_1, \ldots, H_r$ and the graphs in B by H_{ik}, where $1 \leq i \leq r$ and $1 \leq k \leq s$, such that for all i, k the number of H_i-H_{ik} paths in $\mathcal{Q}$ is $\geq n/r^2 s$. Indeed, consider the bipartite graph (A, B) in which $S \in A$ is joined to $T \in B$ if there are $\geq n/r^2 s$ S-T paths in $\mathcal{Q}$. We have to check that the assumption of Lemma 9 holds. Suppose not, and choose $I \subseteq A$ such that $|N(I)| < s|I|$. Then there are $\geq |I|ns - |N(I)|n \geq n$ paths in $\mathcal{Q}$ which join a graph in I to a graph in $B \backslash N(I)$. Then $\geq n/rs$ of these paths all have one endvertex in the same graph of $B \backslash N(I)$, T say, and $\geq n/r^2 s$ of those paths all have the other endvertex in the same graph of I, S say. But then T is a neighbour of S in (A, B), a contradiction.

For all $i = 1, \ldots, r$ let $\mathcal{Q}_i$ be a set of n/r^2 paths from $\mathcal{Q}$ such that for all $k = 1, \ldots, s$ exactly $n/r^2 s$ of them join H_i to H_{ik}. Choose $n/r^2 s c^{\lfloor r^2/2 \rfloor}$ leaves of H_{ik} that are disjoint from the endvertices of paths from $\mathcal{Q}_i$, and let Y_i be the union (over $k = 1, \ldots, s$) of all these leaves. Since H is externally nrs-connected in G, for all pairs ij with $1 \le i < j \le r$ there is a set $\mathcal{P}_{ij}$ of disjoint Y_i-Y_j paths in G having no inner vertices in H. To show $(*)$, we will try to find single paths $P_{ij} \in \mathcal{P}_{ij}$ that are both disjoint for different pairs ij and disjoint from the paths in any $\mathcal{Q}_i$, and thus link up the graphs consisting of H_i together with H_{ik} for all $k = 1, \ldots, s$ and the paths in $\mathcal{Q}_i$ to form a K_r minor in G. If that is not possible, there will be either two sets $\mathcal{P}_{pq}$ and $\mathcal{P}_{ij}$ such that many paths of $\mathcal{P}_{pq}$ meet many paths of $\mathcal{P}_{ij}$, and we shall then use this 'intersection property' to find a K_r minor within the graph consisting of the H_{pk} $(1 \le k \le s)$ together with the paths in $\mathcal{P}_{pq} \cup \mathcal{P}_{ij}$, or there will be a $\mathcal{P}_{pq}$ and a $\mathcal{Q}_i$ such that many paths of $\mathcal{P}_{pq}$ meet many paths of $\mathcal{Q}_i$, and in this case we will find a K_r minor within the graph consisting of the H_{ik} $(1 \le k \le s)$ together with the paths in $\mathcal{P}_{pq} \cup \mathcal{Q}_i$.

Let $\sigma : \ \{ij \mid 1 \le i < j \le r\} \to \{0, 1, \ldots, \binom{r}{2} - 1\}$ be any bijection. Starting with $\ell = 0$, for successive ℓ and $pq := \sigma^{-1}(\ell)$, we will try to find a path $P \in \mathcal{P}_{pq}$ that is disjoint from the previous selected paths and replace both the later sets $\mathcal{P}_{ij}$ and all sets $\mathcal{Q}_i$ by smaller sets of paths disjoint from P. More precisely, let $\ell^* \le \binom{r}{2}$ be maximal such that for all $0 \le \ell < \ell^*$, $1 \le i \le r$, $i < j \le r$ (if $i < r$) there exist sets $\mathcal{P}^\ell_{ij}$ and $\mathcal{Q}^\ell_i$ satisfying the following conditions.

(i) $\mathcal{P}^\ell_{ij}$ is a non-empty set of disjoint Y_i-Y_j paths having no inner vertices in H.

(ii) $\mathcal{Q}^\ell_i$ is a subset of $\mathcal{Q}_i$ of size $|\mathcal{Q}^\ell_i| = n/r^2c^{2\ell}$, and each H_{ik} $(1 \le k \le s)$ contains endvertices of $\le n/r^2sc^\ell$ paths from $\mathcal{Q}^\ell_i$.

As soon as $\mathcal{P}^\ell_{ij}$ and $\mathcal{Q}^\ell_i$ are defined, let H^ℓ_{ij} be the graph consisting of all paths in $\mathcal{P}^\ell_{ij}$, and F^ℓ the graph consisting of all paths contained in some $\mathcal{Q}^\ell_i$. Furthermore, let Y^ℓ_{ij} be the set of all endvertices of paths from $\mathcal{P}^\ell_{ij}$ in Y_i and Y^ℓ_{ji} the set of those in Y_j.

(iii) If $\sigma(ij) < \ell$, then $\mathcal{P}^\ell_{ij}$ has exactly one element P^ℓ_{ij}, and P^ℓ_{ij} is disjoint from any path belonging to a set $\mathcal{P}^\ell_{ab}$ with $ab \ne ij$ and any path belonging to a set $\mathcal{Q}^\ell_a$ (for all a).

(iv) If $\sigma(ij) = \ell$, then $|\mathcal{P}^\ell_{ij}| = n/r^2c^{\ell r^2 + \lfloor r^2/2 \rfloor}$ and each H_{ik} $(1 \le k \le s)$ contains endvertices of $\le n/r^2sc^{\ell r^2 + \lfloor r^2/2 \rfloor - \ell}$ paths from $\mathcal{P}^\ell_{ij}$. Moreover, for every edge $e \in E(H^\ell_{ij}) \backslash E(F^\ell)$ there are no $|\mathcal{P}^\ell_{ij}|$ disjoint Y^ℓ_{ij}-Y^ℓ_{ji} paths in the graph $(H^\ell_{ij} \cup F^\ell) - e$.

(v) If $\sigma(ij) > \ell$, then $|\mathcal{P}^\ell_{ij}| = n/r^2c^{(\ell+1)r^2}$ and each H_{ik} $(1 \le k \le s)$ contains endvertices of $\le n/r^2sc^{(\ell+1)r^2 - \ell}$ paths from $\mathcal{P}^\ell_{ij}$.

(vi) If $\ell = \sigma(pq) < \sigma(ij)$, then for every edge $e \in E(H^\ell_{ij}) \backslash E(H^\ell_{pq})$ there are no $|\mathcal{P}^\ell_{ij}|$ disjoint Y^ℓ_{ij}-Y^ℓ_{ji} paths in the graph $(H^\ell_{ij} \cup H^\ell_{pq}) - e$.

If $\ell^* = \binom{r}{2}$, then by (i)–(iii) we have a (subdivided) K_r minor (the branch sets are the graphs consisting of H_i together with H_{ik} for all $k = 1, \ldots, s$ and all

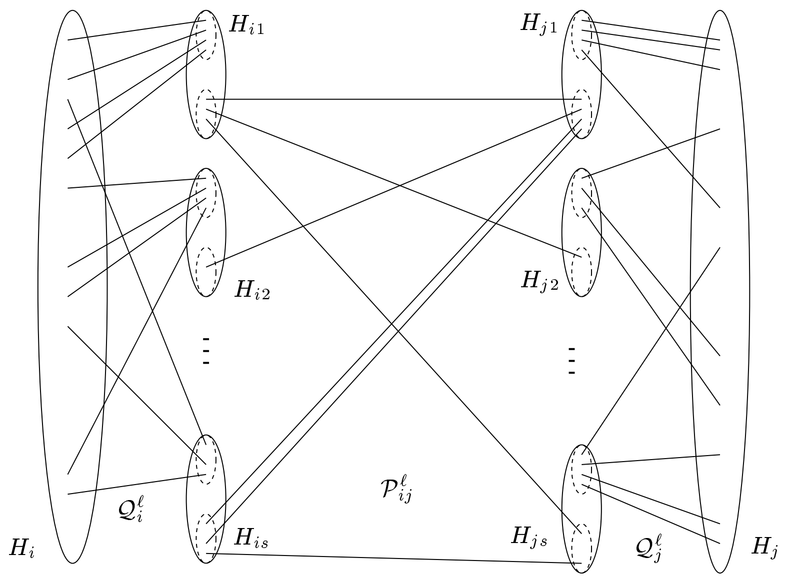

Figure 2: The set-up of the proof of Theorem 3

paths in $\mathcal{Q}_i^{\binom{r}{2}-1}$). Thus we may assume that $\ell^* < \binom{r}{2}$. To see that $\ell^* > 0$, first note that condition (ii) holds with $\mathcal{Q}_i^0 := \mathcal{Q}_i$. Let $pq := \sigma^{-1}(0)$. Then setting $\mathcal{P}_{pq}^0 := \mathcal{P}_{pq}$ would satisfy condition (i) and the first half of (iv), but may not satisfy its second half. If so, let H_{pq} be the graph consisting of all paths in $\mathcal{P}_{pq}$, and choose $I \subseteq E(H_{pq})\backslash E(F^0)$ maximal such that there are $|\mathcal{P}_{pq}|$ disjoint Y_p-Y_q paths in the graph $(H_{pq} \cup F^0) - I$; then let $\mathcal{P}_{pq}^0$ be such a set of paths. It is easily checked that this choice of $\mathcal{P}_{pq}^0$ satisfies (i) and (iv). For all ij with $\sigma(ij) > 0$, choose a subset of $\mathcal{P}_{ij}$ containing $n/r^2c^{r^2}$ paths such that from every H_{ik} there start exactly $n/r^2sc^{r^2}$ of them. Then the obtained set of paths satisfies condition (v) and may be modified similarly as before to obtain a good choice for $\mathcal{P}_{ij}^0$.

Let $\ell := \ell^* - 1$. Hence conditions (i)–(vi) are satisfied for ℓ, but cannot be satisfied for $\ell + 1$. Let $pq := \sigma^{-1}(\ell)$. We first show that there is no path P in $\mathcal{P}_{pq}^{\ell}$ that avoids $\geq |\mathcal{P}_{ij}^{\ell}|/c$ of the paths in $\mathcal{P}_{ij}^{\ell}$ for all ij with $\sigma(ij) > \ell$ as well as $|\mathcal{Q}_i^{\ell}|/c$ of the paths in $\mathcal{Q}_i^{\ell}$ for all i. Suppose there is such a path P. We will show that then we can satisfy conditions (i)–(vi) for $\ell + 1$. Indeed, condition (ii) implies that for all i we may discard paths from $\mathcal{Q}_i^{\ell}$ avoiding P to find a set $\mathcal{Q}_i^{\ell+1}$ of $|\mathcal{Q}_i^{\ell}|/c^2$ paths in $\mathcal{Q}_i^{\ell}$ such that each of them avoids P and each H_{ik} contains endvertices of $\leq n/r^2sc^{\ell+1}$ paths from $\mathcal{Q}_i^{\ell+1}$. Thus $\mathcal{Q}_i^{\ell+1}$ satisfies condition (ii). Similarly, for every ij with $\sigma(ij) > \ell$ we can find a set $\mathcal{P}_{ij}'$ of $|\mathcal{P}_{ij}^{\ell}|/c^2 = n/r^2c^{(\ell+1)r^2+2}$ paths from $\mathcal{P}_{ij}^{\ell}$ such that each H_{ik} contains endvertices of $\leq n/r^2sc^{(\ell+1)r^2-\ell+1}$ of these paths and P avoids all of them. If $\sigma(ij) = \ell + 1$, we choose a set $\mathcal{P}_{ij}''$ of $|\mathcal{P}_{ij}'|/c^{\lfloor r^2/2\rfloor-2} = n/r^2c^{(\ell+1)r^2+\lfloor r^2/2\rfloor}$ paths in $\mathcal{P}_{ij}'$ such that each H_{ik} contains endvertices of $\leq n/r^2sc^{(\ell+1)r^2+\lfloor r^2/2\rfloor-(\ell+1)}$ of them. $\mathcal{P}_{ij}''$ can be modified as before to yield a good choice for $\mathcal{P}_{ij}^{\ell+1}$. In a similar way, one can now define $\mathcal{P}_{ij}^{\ell+1}$ for all ij with $\sigma(ij) > \ell+1$, contradicting the maximality of ℓ^*.

Thus for every path P in $\mathcal{P}_{pq}^{\ell}$ there is either a pair ij with $\sigma(ij) > \ell$ such that P avoids $< |\mathcal{P}_{ij}^{\ell}|/c$ paths in $\mathcal{P}_{ij}^{\ell}$, or there is an i such that P avoids $< |\mathcal{Q}_i^{\ell}|/c$ paths in $\mathcal{Q}_i^{\ell}$. Hence there is a set $\mathcal{P}$ of $\geq |\mathcal{P}_{pq}^{\ell}|/r^2$ paths from $\mathcal{P}_{pq}^{\ell}$ together with either a pair ij such that each path in $\mathcal{P}$ avoids $< |\mathcal{P}_{ij}^{\ell}|/c$ paths in $\mathcal{P}_{ij}^{\ell}$ or an i such that each path in $\mathcal{P}$ avoids $< |\mathcal{Q}_i^{\ell}|/c$ paths in $\mathcal{Q}_i^{\ell}$.

Case 1. Each path in $\mathcal{P}$ avoids less than $|\mathcal{Q}_i^{\ell}|/c$ paths in $\mathcal{Q}_i^{\ell}$.

We first use Lemma 8 to find a set $\mathcal{V}$ of $\geq \lceil |\mathcal{Q}_i^{\ell}|/2 \rceil = |\mathcal{Q}_i^{\ell}|/2$ paths from $\mathcal{Q}_i^{\ell}$ and a set $\mathcal{W}$ of r paths from $\mathcal{P}$ such that every path in $\mathcal{V}$ meets every path in $\mathcal{W}$. Indeed, to show the existence of such sets we have to check that the bipartite graph $(\mathcal{Q}_i^{\ell}, \mathcal{P})$, in which $Q \in \mathcal{Q}_i^{\ell}$ is joined to $P \in \mathcal{P}$ if P avoids Q, has not too many edges. But this follows, since the number of edges of this bipartite graph is at most

$$|\mathcal{Q}_i^{\ell}||\mathcal{P}|/c \;\leq\; |\mathcal{Q}_i^{\ell}||\mathcal{P}|/4r \;\leq\; (|\mathcal{Q}_i^{\ell}| - |\mathcal{Q}_i^{\ell}|/2)(|\mathcal{P}| - r)/r.$$

We now find a set $\mathcal{V}'$ of $\geq |\mathcal{V}|/r^r$ paths from $\mathcal{V}$ and a labelling of the paths from $\mathcal{W}$ by $W_1, \ldots, W_r$ such that on its way from $\bigcup_{k=1}^{s} H_{ik}$ to H_i every path from $\mathcal{V}'$ meets W_a before it meets W_{a+1} (for all $1 \leq a < r$). Recall that, by condition (ii), each H_{ik} $(k = 1, \ldots, s)$ contains endvertices of $\leq n/r^2 s c^{\ell}$ paths from $\mathcal{Q}_i^{\ell}$, and, note that $r \geq 4$ implies that $\binom{r}{2} \leq \lfloor r^2/2 \rfloor - 2$. Together with

$$|\mathcal{V}'|/2 \;\geq\; |\mathcal{Q}_i^{\ell}|/4r^r \;=\; \frac{n}{4r^{r+2}c^{2\ell}} \;\geq\; \frac{n}{c^{\ell+\binom{r}{2}}} \;\geq\; r^2 \frac{n}{r^2 s c^{\ell}}$$

and

$$|\mathcal{V}'|/2 \;\geq\; \frac{n}{c^{\ell r^2+1}} \;\geq\; (s - r^2)c\frac{n}{r^2 c^{\ell r^2 + \lfloor r^2/2 \rfloor}} \;=\; (s - r^2)c|\mathcal{P}_{pq}^{\ell}|,$$

this implies that there are r^2 of the H_{ik}, without loss of generality $H_{i1}, \ldots, H_{ir^2}$, such that each of them contains endvertices of $\geq c|\mathcal{P}_{pq}^{\ell}|$ paths from $\mathcal{V}'$. For all $k = 1, \ldots, r^2$, let $\mathcal{V}_k$ be the set of all paths from $\mathcal{V}'$ begining at H_{ik}. We shall now prove that the paths from $\mathcal{W}$ together with many paths from each of $\mathcal{V}_k$ form a pseudogrid (the paths from $\mathcal{W}$ will be its horizontal paths and those from the $\mathcal{V}_k$ its vertical paths). The result will then follow from Lemma 7.

Direct W_1 from Y_{pq}^{ℓ} to Y_{qp}^{ℓ}. Let e^1 be the first edge of W_1 such that the initial component W^1 of $W_1 - e^1$ meets $\geq c|\mathcal{P}_{pq}^{\ell}|/r^2$ paths from $\mathcal{V}_k$ for some k, and so that e^1 does not lie on one of these paths. Without loss of generality we may assume that $k = 1$. Note that $e^1 \notin E(F^{\ell})$, since $\mathcal{V}_1 \subseteq \mathcal{Q}_i^{\ell}$ and the paths in $\bigcup_{a=1}^{r} \mathcal{Q}_a^{\ell}$ are disjoint. Let e^2 be the first edge of $W_1 - W^1 - e^1 =: W'$ such that the initial component W^2 of $W' - e^2$ meets $\geq c|\mathcal{P}_{pq}^{\ell}|/r^2$ paths from $\mathcal{V}_k$ for some $k \geq 2$, and so that e^2 does not lie on one of these paths. Continuing in this fashion, define $e^1, \ldots, e^{r^2-1}$ and $W^1, \ldots, W^{r^2-1}$, and let W^{r^2} be the final component of $W_1 - e^{r^2-1}$. Without loss of generality we may assume that each W^k meets $\geq c|\mathcal{P}_{pq}^{\ell}|/r^2$ paths from $\mathcal{V}_k$.

Let $m := \lfloor \sqrt{c}/r^2 \rfloor$. For every $k = 1, \ldots, r^2$ let e_1^k be the first edge of W^k such that the initial component W_1^k of $W^k - e_1^k$ meets $\geq \lfloor \sqrt{c} \rfloor |\mathcal{P}_{pq}^{\ell}|$ paths from

$\mathcal{V}_k$, and so that e_1^k does not lie on one of these paths. Let e_2^k be the first edge of $W^k - W_1^k - e_1^k =: R^k$ such that the initial component W_2^k of $R^k - e_2^k$ meets $\geq \lfloor\sqrt{c}\rfloor|\mathcal{P}_{pq}^\ell|$ paths from $\mathcal{V}_k$, and so that e_2^k does not lie on one of these paths. Continuing in this fashion, define $e_1^k, \dots, e_{m-1}^k$ and $W_1^k, \dots, W_{m-1}^k$, and let W_m^k denote the final component of $W^k - e_m^k$. Thus each W_a^k meets $\geq \lfloor\sqrt{c}\rfloor|\mathcal{P}_{pq}^\ell|$ paths from $\mathcal{V}_k$, and $e_a^k \notin E(F^\ell)$. For $k < r^2$, let $e_m^k := e^k$. Menger's theorem and condition (iv) now imply that for each e_a^k there is a set S_a^k of $< |\mathcal{P}_{pq}^\ell|$ vertices separating Y_{pq}^ℓ from Y_{qp}^ℓ in the graph $(H_{pq}^\ell \cup F^\ell) - e_a^k$. Let S be the union of all these S_a^k. Then

$$|S| \leq (r^2m - 1)(|\mathcal{P}_{pq}^\ell| - 1) \leq \lfloor\sqrt{c}\rfloor(|\mathcal{P}_{pq}^\ell| - 1).$$

Hence each W_a^k meets at least one path $V_a^k \in \mathcal{V}_k$ that avoids S. Clearly, S_a^k must consist of exactly one vertex $v_a^k(P)$ on each path $P \in \mathcal{P}_{pq}^\ell \backslash \{W_1\}$. For all P and $1 \leq k < r^2$ let $e_0^{k+1} := e_m^k$ and $v_0^{k+1}(P) := v_m^k(P)$. Let $v_0^1(P)$ be the endvertex of P in Y_{pq}^ℓ and $v_m^{r^2}(P)$ that in Y_{qp}^ℓ. Note that V_a^k meets P neither in the initial component of $P - v_{a-1}^k(P)$ nor in the final component of $P - v_a^k(P)$ (here both the initial component of $P - v_0^1(P)$ and the final component of $P - v_m^{r^2}(P)$ are defined to be the empty set)—otherwise there would be a Y_{pq}^ℓ-Y_{qp}^ℓ path in the graph $(H_{pq}^\ell \cup F^\ell) - e_{a-1}^k$ or $(H_{pq}^\ell \cup F^\ell) - e_a^k$ avoiding S. This implies that $v_a^k(P)$ precedes $v_{a+1}^k(P)$ when P is directed from Y_{pq}^ℓ to Y_{qp}^ℓ. Thus for all $1 \leq a \leq m$, $1 < b \leq r$ and $1 \leq k \leq r^2$ the path V_a^k meets W_b exactly in the segment of W_b strictly between $v_{a-1}^k(W_b)$ and $v_a^k(W_b)$, and V_a^k meets W^1 exactly in the segment W_a^k. That means that the binary trees in the H_{ik} $(1 \leq k \leq r^2)$ are m-attached to an $r \times mr^2$ pseudogrid whose vertical paths are the V_a^k and whose horizontal paths are those obtained from $W_1, \dots, W_r$ by deleting their endvertices. Thus the horizontal paths are disjoint from all H_{ik}, and the vertical paths meet the H_{ik} only in their first vertices. Since

$$\begin{aligned}
\lceil \log_2 n \rceil^{2r-1} &= \lceil \log_2 (2^{r^9} r^{r^5+4r^4+2}) \rceil^{2r-1} \\
&= \lceil r^9 + (r^5 + 4r^4 + 2)\log_2 r \rceil^{2r-1} \\
&\leq (2r^9)^{2r-1} \\
&\leq 2^{r^5/4} \\
&\leq \sqrt{m},
\end{aligned}$$

we can apply Lemma 7 with n replaced by $\lceil \log_2 n \rceil$ to find the desired K_r minor in G.

Case 2. All paths in $\mathcal{P}$ avoid less than $|\mathcal{P}_{ij}^\ell|/c$ paths in $\mathcal{P}_{ij}^\ell$.

As in Case 1, we first apply Lemma 8 to find a set $\mathcal{V}$ of $\geq |\mathcal{P}|/2$ paths from $\mathcal{P}$ and a set $\mathcal{W}$ of r paths from $\mathcal{P}_{ij}^\ell$ such that every path in $\mathcal{V}$ meets every path in $\mathcal{W}$. Again, we then find a set $\mathcal{V}'$ of $\geq |\mathcal{V}|/r^r$ paths from $\mathcal{V}$ and a labelling of the paths from $\mathcal{W}$ by $W_1, \dots, W_r$ such that on its way from Y_{pq}^ℓ to Y_{qp}^ℓ every path from $\mathcal{V}'$ meets W_a before it meets W_{a+1} (for all $1 \leq a < r$). Recall that by condition (ii) each H_{pk} $(k = 1, \dots, s)$ contains endvertices of $\leq n/r^2sc^{\ell r^2 + \lfloor r^2/2 \rfloor - \ell}$ paths from $\mathcal{P}^{\ell}_{pq}$, and $\lfloor r^2/2\rfloor - 2 \geq \binom{r}{2}$ since $r \geq 4$. Together with

$$|\mathcal{V}'|/2 \;\geq\; |\mathcal{P}^{\ell}_{pq}|/4r^{r+2} \;=\; \frac{n}{4r^{r+4}c^{\ell r^2+\lfloor r^2/2\rfloor}} \;\geq\; \frac{n}{c^{\ell r^2+\lfloor r^2/2\rfloor+1}} \;\geq\; r^2\frac{n}{r^2 s c^{\ell r^2+\lfloor r^2/2\rfloor-\ell}}$$

and

$$|\mathcal{V}'|/2 \geq (s-r^2)c\frac{n}{r^2c^{(\ell+1)r^2}} \;=\; (s-r^2)c|\mathcal{P}^{\ell}_{ij}|,$$

this implies that there are r^2 of the H_{pk}, without loss of generality $H_{p1},\dots,H_{pr^2}$, such that each of them contains endvertices of $\geq c|\mathcal{P}^{\ell}_{ij}|$ paths from $\mathcal{V}'$. For all $k = 1,\dots,r^2$, let $\mathcal{V}_k$ be the set of all paths begining at H_{pk}. Similarly as in Case 1, the paths in $\mathcal{W}$ together with $\lfloor\sqrt{c}/r^2\rfloor$ paths from each $\mathcal{V}_k$ form a pseudogrid, and we can apply Lemma 7 to find a K_r minor in G. $\square$

4 A tree attached to many non-outerplanar graphs

Let $\mathcal{H}_n$ be the class of all graphs H which can be obtained from the binary tree T of height n by identifying each leaf v of T with a vertex of a connected non-outerplanar graph $K(v)$ (where the $K(v)$ are disjoint from each other and from the rest of T). T is called the binary tree in H, the leaves of T are called *leaves of* H, and $K(v)$ is said to be the non-outerplanar graph *glued to* v.

Theorem 10 *Given an integer r, there exist integers d and n with the following property. Whenever a graph G contains an externally d-connected set X such that $G[X]$ has some graph in $\mathcal{H}_n$ as a minor, there is a K_r minor in G.*

Lemma 4 together with the fact that every non-outerplanar graph contains a subdivision of K_4 or $K_{2,3}$ implies that for $n \geq 2$ every graph in $\mathcal{H}_n$ contains H^4_k, $H^{2,3}_k$ or $H^{3,2}_k$ as a minor, where $k := \lfloor (n-2)/\log_2 n\rfloor$. Thus in the statement of Theorem 10 one could have alternatively required that $G[X]$ contains either H^4_n, $H^{2,3}_n$ or $H^{3,2}_n$ as a minor.

Actually, the property that will be used in the proof of Theorem 10 is not that every graph $K(v)$ is non-outerplanar, but that each $K(v)$ has three distinct vertices $x, y, z \neq v$ such that any two vertices of x, y, z can be joined by a path P while the third can be joined to v by a path not meeting P. Indeed, since every non-outerplanar graph contains a subdivision of K_4 or $K_{2,3}$, such vertices x, y, z can be found.

Conversely, note that every graph K containing distinct vertices v, x, y, z satisfying the above property cannot be outerplanar. Indeed, suppose that K is outerplanar. Then adding a new vertex a to K and joining it to v, x, y, z yields a planar graph K'. Consider a drawing of K', and let aa_1, aa_2, aa_3, aa_4 be the edges of K' incident to a in clockwise order (thus $\{a_1, a_2, a_3, a_4\} = \{v, x, y, z\}$). Then $K' - a$ contains an a_1-a_3 path Q_1 and an a_2-a_4 path Q_2 such that Q_1 and Q_2 are disjoint, contradicting the planarity of K'.

For every leaf v of $H \in \mathcal{H}_n$ choose a set $B(v)$ consisting of three such vertices x, y, z of $K(v)$. For the proof of Theorem 10 we shall need the following lemmas.

Lemma 11 *Let n, r, s be positive integers such that $n \geq 2$ and $s^{1/4} \geq n^{2r-1+\lceil \log_2 \binom{r}{2} \rceil}$. Let G be a graph that contains a binary tree T of height n together with a path P such that T and P are disjoint and there is a set A of s leaves of T which are joined (by edges) injectively to the vertices of P. Suppose that T is s-attached in $G - P$ to an $r \times s$ pseudogrid G' so that A is precisely the set of attached leaves of T. Then G contains a K_r minor.*

Proof. First apply Lemma 6 to obtain a set A' of $\geq \sqrt{s}$ leaves in A such that their order in T corresponds to the order of their neighbours on P. Apply Lemma 6 once more to obtain a set A'' of $\geq s^{1/4}$ leaves in A' such that their order in T corresponds to the order of the vertical paths in G' which begin in A''. Lemma 4 now gives us a subdivision S of the binary tree of height $2r - 1 + \lceil \log_2 \binom{r}{2} \rceil$ in T such that all leaves of S are contained in A''. S contains $\binom{r}{2}$ disjoint subdivisions $S_1, \ldots, S_{\binom{r}{2}}$ of the binary tree of height $2r-1$ such that each of them is a branch of S. Note that $S_1, \ldots, S_{\binom{r}{2}}$ are tidily 2^{2r-1}-attached in $G-P$ to an $r \times 2^{2r-1}\binom{r}{2}$ subpseudogrid of G', and that there are $\binom{r}{2}$ disjoint segments of P, each containing all the neighbours of leaves of some S_i on P. Lemma 11 now follows from Lemma 5. □

Lemma 12 *Let $\mathcal{M}$ and $\mathcal{H} = \{H_1, H_2, H_3\}$ be sets of disjoint directed paths such that $|\mathcal{M}| = 3$ and every path from $\mathcal{M}$ meets every path from $\mathcal{H}$. Then there are vertices x_1, x_2, x_3 on H_1, H_2, H_3 respectively, and a labelling of the paths in $\mathcal{M}$ as M_1, M_2, M_3, such that, for all $i = 1, 2, 3$, the vertex x_i lies on M_i and $M_i \mathring{x}_i$ does not meet any $H_j x_j$ $(j = 1, 2, 3)$.*

Proof. For every $i = 1, 2, 3$, let y_i^1 be the first vertex of H_i that lies on a path from $\mathcal{M}$. Given $k \geq 1$, assume inductively that for every $i = 1, 2, 3$ we have constructed a sequence $y_i^1, \ldots, y_i^k$ of vertices on H_i satisfying the following conditions.

(i) If $k > 1$, then $y_i^k \in y_i^{k-1} H_i$, and $y_i^k \neq y_i^{k-1}$ for at least one $i \in \{1, 2, 3\}$.

(ii) The vertex y_i^k lies on some path $M_i^k \in \mathcal{M}$.

(iii) For every $M \in \mathcal{M}$, the initial component of $M - \{y_1^k, y_2^k, y_3^k\}$ meets none of the paths $H_j y_j^k$ $(j = 1, 2, 3)$.

If the paths M_1^k, M_2^k, M_3^k are all distinct, then $x_i := y_i^k$ and $M_i := M_i^k$ satisfy the assertion of the lemma. We show that if M_1^k, M_2^k, M_3^k are not distinct then there are vertices y_i^{k+1} extending our three sequences $y_i^1, \ldots, y_i^k$; by (i), this can happen only finitely often.

If M_1^k, M_2^k, M_3^k are not distinct, then there exists a path $M \in \mathcal{M}$ containing more than one of the vertices y_1^k, y_2^k, y_3^k (let y_i^k be the last of these on M), as well as a path $M' \in \mathcal{M}$ avoiding $\{y_1^k, y_2^k, y_3^k\}$. By (iii), M' avoids $H_i y_i^k$ and hence meets H_i in $\mathring{y}_i^k H_i$. So $\mathring{y}_i^k H_i$ has a first vertex in $\bigcup \mathcal{M}$; we choose this vertex as y_i^{k+1} and put $y_j^{k+1} := y_j^k$ for $j \neq i$. Then conditions (i) and (ii) hold for $k + 1$. Condition (iii) for $k + 1$ holds for our M because $M \mathring{y}_i^k$ contains another

$y_j^k = y_j^{k+1}$. Condition (iii) for the other two paths in $\mathcal{M}$ is again clear: as they do not contain y_i^k, their initial components in (iii) did not get longer when y_i^k was replaced by y_i^{k+1}, so they satisfy (iii) for $k+1$ because they did for k. $\square$

Lemma 13 *Let n, r, s be positive integers such that $n \geq 2$ and $\sqrt{s} \geq n^{2r-1+\lceil \log_2 \binom{r}{2} \rceil}$, and let G be a graph containing a graph $H \in \mathcal{H}_n$. Let $A = \{v_1, \ldots, v_s\}$ be a set of leaves of H, and let $B := \bigcup_{v \in A} B(v)$. Suppose that there is a set $\mathcal{V}$ of $3s$ disjoint directed paths in G starting in B and having no other vertices in H, and a set $\mathcal{W} = \{W_0, \ldots, W_r\}$ of disjoint directed paths in $G - H$ such that every path in $\mathcal{V}$ meets each W_i, and it does so before it meets W_{i+1}. Suppose furthermore that each path $W_i \in \mathcal{W}$ consists of s consecutive (vertex disjoint) segments such that, for all $j = 1, \ldots, s$, every path from $\mathcal{V}$ that starts in $B(v_j)$ meets W_i exactly in its jth segment. Then G contains a K_r minor.*

Proof. First apply Lemma 6 to obtain a set A' of $\geq \sqrt{s}$ leaves in A such that their order in the binary tree T of H corresponds to the order of the paths from $\mathcal{V}$ that begin in $\bigcup_{v \in A'} B(v)$, i.e. if $v, w \in A'$ and v is the successor of w in the ordering σ_T restricted to A', then no path from $\mathcal{V}$ that starts in $B(u)$ for some $v, w \neq u \in A'$ lies between paths from $\mathcal{V}$ starting in $B(v)$ and $B(w)$. Moreover, reversing the orientation of the paths from $\mathcal{W}$ if necessary, we may assume that for each W_i the segment of W_i belonging to the paths from $\mathcal{V}$ starting in $B(w)$ precedes that belonging to the paths from $\mathcal{V}$ starting in $B(v)$. By Lemma 4, there is a subdivision S of the binary tree of height $2r - 1 + \lceil \log_2 \binom{r}{2} \rceil$ in T such that all leaves of S are contained in A'.

Let $S_1, \ldots, S_{\binom{r}{2}}$ be disjoint subdivisions of the binary tree of height $2r - 1$ in S such that each of them is a branch of S. Each S_ℓ has a set $\mathcal{P}_\ell$ of r nested paths. Let G' be the graph consisting of the paths from $\mathcal{W}$ together with all paths from $\mathcal{V}$ starting in non-outerplanar graphs glued to leaves of the S_ℓ.

We now construct the branch sets for our K_r minor. Each of these branch sets will contain a path Q_i running alternately through an S_ℓ and G' in a similar way as in the proof of Lemma 5. In particular, each Q_i will contain exactly one path from each $\mathcal{P}_\ell$. For the edges of the K_r we need disjoint paths, one between any two of the Q_i. Each of these paths we will find in one of the trees $S_1, \ldots, S_{\binom{r}{2}}$. Indeed, in each S_ℓ we can join two neighbouring Q_i (i.e. two Q_i containing paths from $\mathcal{P}_\ell$ lying next to each other), and we will show that we can also use a non-outerplanar graph glued to a leaf of S_ℓ to 'switch' two neighbouring Q_i (Fig. 3). Together this will imply that for every edge of the K_r we can find a path connecting the corresponding Q_i.

More precisely, suppose that we have partially constructed such paths $Q_1, \ldots, Q_r$, which have their current endvertices on different paths from $\mathcal{W} \setminus \{W_0\}$, and that next we want to let them run through S_ℓ, and that furthermore we want to switch Q_i and Q_j, where Q_j currently runs along W_k and Q_i along W_{k+1} for some $k > 0$. Let A_ℓ be the set of all those leaves of S_ℓ in which there begins a path from $\mathcal{P}_\ell$. Let v be the $(k+1)$th leaf in A_ℓ (in the ordering σ_T restricted to A_ℓ), and let w be the predecessor of v in A_ℓ. We will use the non-outerplanar graph $K(v)$ to switch Q_i and Q_j.

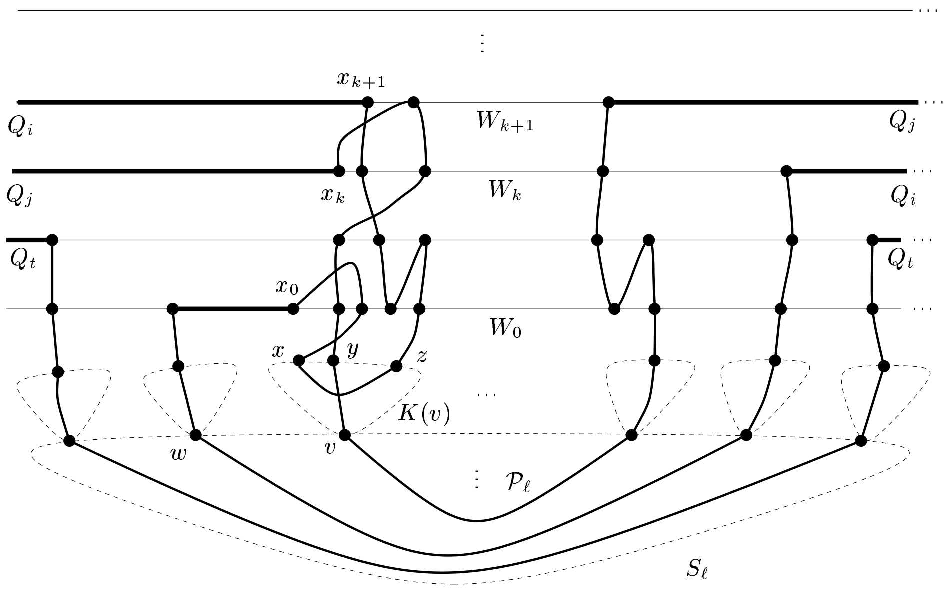

Figure 3: Switching Q_i and Q_j

Let V_x, V_y, V_z denote the paths from $\mathcal{V}$ starting in $B(v) =: \{x, y, z\}$. Apply Lemma 12 to $\mathcal{H} := \{W_0, W_k, W_{k+1}\}$ and the set $\mathcal{M}$ consisting of the subpaths of V_x, V_y, V_z between their endvertices in $B(v)$ and their first vertices on W_{k+1} to obtain vertices x_0, x_k, x_{k+1} and a labelling M_0, M_k, M_{k+1} of the paths from $\mathcal{M}$. Then for all $i = 0, k, k+1$, traversing M_i from $B(v)$ to x_i, and then moving backwards along W_i gives disjoint paths. Thus we may extend Q_j by traversing W_k as far as x_k, and then moving along M_k to $B(v)$ and further through $K(v)$ to v, and then along the path P_v from $\mathcal{P}_\ell$ that begins in v and up to W_{k+1} along a path from $\mathcal{V}$ starting in the non-outerplanar graph glued to the endvertex of P_v. Extend Q_i by traversing W_{k+1} as far as x_{k+1}, then moving along M_{k+1} to $B(v)$ and through $K(v)$ to the endvertex of M_0, then along M_0 to x_0, then backwards along W_0 and down through $K(w)$, and along the path P_w from $\mathcal{P}_\ell$ starting in w, and then up to W_k along a path from $\mathcal{V}$ starting in the non-outerplanar graph glued to the endvertex of P_w. From the definition of $B(v)$, it follows that we may choose the subpaths of Q_i and Q_j running through $K(v)$ so that Q_i and Q_j remain disjoint.

Moreover, note that $\binom{r}{2}$ switchings suffice to ensure that for any two Q_i there is a tree S_ℓ in which they lie next to each other (and thus can be joined by a path). Indeed, first switch the lowest Q_i with all higher ones, then the one which is now the lowest with all but the highest. Continuing in this fashion, we need $(r-1)+(r-2)+\cdots+1 = \binom{r}{2}$ switchings. Thus the Q_i can be constructed to form the branch sets of a (subdivided) K_r minor in G. $\square$

Lemma 14 *Let T be a tree with $\Delta(T) \leq 3$. Suppose that $B_1, \ldots, B_k$ are disjoint sets of leaves of T such that for all $1 \leq i < k$ the leaves of T in B_i either all precede or all succeed those in B_{i+1} in the ordering σ_T. Then there are $\lceil k/3 \rceil$*

disjoint subtrees of T such that each of them contains all vertices of some B_i.

Proof. Induction on k. We may assume that $k \geq 4$. Let e be an edge of T such that the branch T' above e contains all vertices of some B_i, and such that T' is as small as possible. As $\Delta(T) \leq 3$, the endvertex x of e in T' has degree at most two in T'. It is easy to see that the minimality of T' now implies that T' meets at most three of the B_i. T' will be one of the desired subtrees of T, and applying the induction hypothesis on $T - T' - e$ and the B_i's not meeting T' gives the remaining $\lceil (k-3)/3 \rceil = \lceil k/3 \rceil - 1$ subtrees. $\square$

Lemma 15 *Let n, r, s be positive integers such that $n \geq 2$ and $s^{1/4} \geq n^{2r-1+\lceil \log_2 3\binom{r}{2} \rceil}$. Let G be a graph containing two disjoint copies T_1 and T_2 of the binary tree of height n, and let A be a set of s leaves of T_1. Suppose that there is a set $\mathcal{P}$ of s disjoint paths joining the vertices in A to leaves of T_2 and meeting $T_1 \cup T_2$ only in their endvertices. Let G_0 be the graph obtained from G by deleting T_2 and all inner vertices of paths from $\mathcal{P}$. Furthermore, suppose that T_1 is s-attached in G_0 to an $r \times s$ pseudogrid G' such that A is precisely the set of attached leaves of T_1. Then G contains a K_r minor.*

Proof. By Lemma 6, there exists a set A' of $\geq s^{1/4}$ leaves in A such that their order in σ_{T_1} corresponds (or is opposite) to both the order of the leaves of T_2 joined to A' by paths from $\mathcal{P}$ and the order of the vertical paths in G' that begin in A'. Let $k := 3\binom{r}{2}$. Lemma 4 implies that T_1 contains a subdivision S of the binary tree of height $2r - 1 + \lceil \log_2 k \rceil$ such that all leaves of S are contained in A'. Let $S_1, \ldots, S_k$ be disjoint subdivisions of the binary tree of height $2r-1$ in S such that each S_i is a branch of S. Then $S_1, \ldots, S_k$ are tidily 2^{2r-1}-attached to an $r \times k2^{2r-1}$ subpseudogrid of G'. For all $i = 1, \ldots k$, let B_i be the set of all leaves of T_2 which are endvertices of those paths from $\mathcal{P}$ that start in a leaf of S_i. From the choice of A', it follows that for all $1 \leq i < k$, the leaves in B_i either all precede or all succeed those in B_{i+1}. Lemma 14 now implies that there are $\lceil k/3 \rceil = \binom{r}{2}$ disjoint subtrees of T_2 such that each of them contains all vertices of some B_i. Lemma 15 thus follows from Lemma 5. $\square$

Lemma 16 *Let W be a directed path. Let $\ell_1, \ldots, \ell_s$ and $r_1, \ldots, r_s$ be vertices on W such that $\ell_i \in W r_i$ for all $i = 1, \ldots, s$. Let $S_1, \ldots, S_s$ be non empty disjoint segments of W such that S_i precedes S_{i+1} for all $1 \leq i < s$. Let $t := \lfloor (s/4)^{1/3} \rfloor$. Then there exists $I \subseteq \{1, \ldots, s\}$ with $|I| \geq t$ such that one of the following conditions holds.*

(a) Either $S_{\max I} \subseteq W\mathring{r}_i$ for all $i \in I$ or $S_{\min I} \subseteq \mathring{\ell}_i W$ for all $i \in I$.

(b) For all $i \in I$, there is a segment A_i of W such that each A_i contains ℓ_i, r_i and S_i, and $A_i \cap A_j = \emptyset$ for all $i, j \in I$ with $i \neq j$.

Proof. We may assume that $t \geq 2$. Moreover, let us first assume that for a set I_1 of $\geq s/2$ elements $i \in \{1, \ldots, s\}$ either $r_i \in S_i$ or $S_i \subseteq W\mathring{r}_i$. If for $\geq t$

elements $j \in I_1$ the segment S_j precedes the first r_i with $i \in I_1$ on W, then the subset of I_1 consisting of all these j satisfies condition (a). Thus, denoting the first of the r_i with $i \in I_1$ on W by r_{i_1}, we may assume that $S_j \subseteq \mathring{r}_{i_1}W$ for a set I_1' of $\geq |I_1| - t$ elements $j \in I_1$. Note that $i_1 \notin I_1'$. As before, we are done if $\geq t$ segments S_j with $j \in I_1'$ precede the first r_i with $i \in I_1'$ on W, r_{i_2} say. Thus we may assume that $S_j \subseteq \mathring{r}_{i_2}W$ for a set I_1'' of $\geq |I_1'| - t$ elements $j \in I_1'$. Continuing in this fashion, we may assume that there is a set $I_2 \subseteq I_1$ with $|I_2| \geq |I_1|/t$ such that $S_j \subseteq \mathring{r}_iW$ for all $i < j$ in I_2. (Indeed, let I_2 be the set consisting of $i_1, i_2, \ldots$.)

Let S_i' be the smallest segment of W containing S_i and r_i. Then for all $i \in I_2$ the segments S_i' are pairwise disjoint. If for a set I_3 of $\geq |I_2|/2 \geq t$ elements $i \in I_2$ either $\ell_i \in S_i$ or $S_i \subseteq W\mathring{\ell}_i$, then I_3 satisfies condition (b) (since $\ell_i \in Wr_i$ we may take A_i to be S_i'). Thus we may assume that there is a set I_3 of $\geq |I_2|/2$ elements of I_2 such that $S_i \subseteq \mathring{\ell}_iW$ for all $i \in I_3$. Considering the vertices ℓ_i and the segments S_i' for all $i \in I_3$ and arguing similarly as before, we may assume that there is a set I of $\geq |I_3|/t \geq t$ elements of I_3 such that $S_i' \subseteq W\mathring{\ell}_j$ for all $i < j$ in I. Thus I satisfies condition (b). The case that $S_i \subseteq \mathring{r}_iW$ for $\geq s/2$ elements $i \in \{1, \ldots, s\}$ is similar. $\square$

Proof of Theorem 10. It suffices to show the following assertion.

> *Let* $c := r^{16r^7 3^{2r+2}}$ *and* $n := c^{2\binom{r}{2}}$. *Suppose* G *contains a graph* H *consisting of* r *disjoint graphs* $H_1, \ldots, H_r \in \mathcal{H}_{\lceil \log_2 n \rceil}$ *as an externally* $3n$*-connected subgraph. Then* G *contains a* K_r *minor.* $(**)$

We may assume that $r \geq 4$. The first part of the proof of $(**)$ is similar to (but much easier than) that of $(*)$, and we will only sketch it. For every $i = 1, \ldots, r$, choose n leaves of H_i and let Y_i denote the union of the sets $B(v)$ over all chosen leaves v. Since H is externally $3n$-connected in G, for all pairs $1 \leq i < j \leq r$ there is a set $\mathcal{P}_{ij}$ of $|Y_i| = 3n$ disjoint Y_i-Y_j paths in G which have no inner vertices in H.

As in the proof of $(*)$, we will try to find single paths $P_{ij} \in \mathcal{P}_{ij}$ that are disjoint for different pairs ij, and thus link up the graphs H_i to form a K_r minor in G. If that is not possible, there will be two sets $\mathcal{P}_{pq}$ and $\mathcal{P}_{ij}$, such that many paths of $\mathcal{P}_{pq}$ together with many paths of $\mathcal{P}_{ij}$ form a pseudogrid, which we shall then use to find a K_r minor within the graph consisting of H_p, H_q and the paths in $\mathcal{P}_{pq} \cup \mathcal{P}_{ij}$. Let $\sigma:\ \{ij \mid 1 \leq i < j \leq r\} \to \{0, 1, \ldots, \binom{r}{2} - 1\}$ be any bijection. Let $\ell^* \leq \binom{r}{2}$ be maximal such that for all $0 \leq \ell < \ell^*$ and $1 \leq i < j \leq r$ there exist sets $\mathcal{P}_{ij}^\ell$ satisfying the following conditions.

(i) $\mathcal{P}_{ij}^\ell$ is a non-empty set of disjoint Y_i-Y_j paths having no inner vertices in H.

As soon as $\mathcal{P}_{ij}^\ell$ is defined, let H_{ij}^ℓ be the graph consisting of all paths in $\mathcal{P}_{ij}^\ell$. Furthermore, let Y_{ij}^ℓ be the set of all endvertices of paths from $\mathcal{P}_{ij}^\ell$ in Y_i and Y_{ji}^ℓ the set of those in Y_j.

(ii) If $\sigma(ij) < \ell$, then $\mathcal{P}_{ij}^\ell$ has exactly one element P_{ij}^ℓ, and P_{ij}^ℓ is disjoint from any path belonging to a set $\mathcal{P}_{ab}^\ell$ with $ab \neq ij$.

(iii) If $\sigma(ij) \geq \ell$ and v is a leaf of H_i, then either each of the three vertices in $B(v)$ belongs to Y^ℓ_{ij} or non of them does. In the first case we will say that the three paths of $\mathcal{P}^\ell_{ij}$ that start from the vertices in $B(v)$ form a *bundle*. Thus (iii) says that $\mathcal{P}^\ell_{ij}$ consists only of bundles.

(iv) If $\sigma(ij) = \ell$, then $|\mathcal{P}^\ell_{ij}| = 3n/c^{2\ell}$.

(v) If $\sigma(ij) > \ell$, then $|\mathcal{P}^\ell_{ij}| = 3n/c^{2\ell+1}$.

(vi) If $\ell = \sigma(pq) < \sigma(ij)$, then for every edge $e \in E(H^\ell_{ij}) \backslash E(H^\ell_{pq})$ there are no $|\mathcal{P}^\ell_{ij}|$ disjoint Y^ℓ_{ij}-Y^ℓ_{ji} paths in the graph $(H^\ell_{ij} \cup H^\ell_{pq}) - e$.

If $\ell^* = \binom{r}{2}$, then by (i) and (ii) we have a (subdivided) K_r minor with branch sets H_i. Thus suppose that $\ell^* < \binom{r}{2}$, and note that as in the previous section $\ell^* > 0$. Let $\ell := \ell^* - 1$ and $pq := \sigma^{-1}(\ell)$. Similarly as in the proof of $(*)$, for every path $P \in \mathcal{P}^\ell_{pq}$ there exists a pair ij with $\sigma(ij) > \ell$ such that P avoids $\leq |\mathcal{P}^\ell_{ij}|/3c$ bundles from $\mathcal{P}^\ell_{ij}$ (where P *avoids a bundle* if it avoids every path in it). Thus there is a set $\mathcal{P}$ of $\geq |\mathcal{P}^\ell_{pq}|/3^2\binom{r}{2}$ paths from $\mathcal{P}^\ell_{pq}$ for which this pair ij can be chosen to be the same, and such that any two paths from $\mathcal{P}$ belong to different bundles of $\mathcal{P}^\ell_{pq}$ and end in different non-outerplanar graphs glued to leaves of H_q. Again, we now use Lemma 8 to find a set $\mathcal{V}'$ of $\geq |\mathcal{P}|/2$ paths from $\mathcal{P}$ and a set $\mathcal{W}'$ of $t := (2r)^4 + 2r$ bundles from $\mathcal{P}^\ell_{ij}$ such that no path in $\mathcal{V}'$ avoids a bundle in $\mathcal{W}'$. Then there is a set $\mathcal{V}''$ of $\geq |\mathcal{V}'|/3^t$ paths from $\mathcal{V}'$ and a set $\mathcal{W}''$ of t paths, one from each bundle in $\mathcal{W}'$, such that every path in $\mathcal{V}''$ meets every path in $\mathcal{W}''$. We now find a set $\mathcal{V}'''$ of $\geq |V''|/t^t \geq |\mathcal{P}^\ell_{ij}|c/3^{t+2}t^tr^2$ paths from $\mathcal{V}''$ and a labelling of the paths in $\mathcal{W}''$ by $W''_1, \ldots, W''_t$ such that on its way from H_p to H_q every path in $\mathcal{V}'''$ meets W''_a before it meets W''_{a+1} (for all $1 \leq a < t$). Similarly as in the proof of $(*)$, condition (vi) now yields a set $\mathcal{V}^*$ of $\geq \lfloor \sqrt{c}/3^{(t+2)/2}rt^{t/2} \rfloor$ paths from $\mathcal{V}'''$ which form a pseudogrid together with the paths in $\mathcal{W}''$. To make the horizontal paths of the pseudogrid disjoint from $H_p \cup H_q$, we direct each path in $\mathcal{W}''$ from H_i to H_j, and let $\mathcal{W}$ be the set of all (directed) paths obtained from paths in $\mathcal{W}''$ by deleting their endvertices. Let

$$s := \left\lceil \frac{\sqrt{c}}{t^{2t}} \right\rceil \leq \left\lfloor \frac{\sqrt{c}}{3^{(t+2)/2}rt^{t/2}} \right\rfloor - 2.$$

Discarding also the leftmost and the rightmost path from $\mathcal{V}^*$ in the pseudogrid (as well as any $|\mathcal{V}^*| - s - 2$ other paths), we have thus found sets $\mathcal{V} = \{V_1, \ldots, V_s\} \subseteq \mathcal{V}^* \subseteq \mathcal{P}^\ell_{pq}$ and $\mathcal{W} = \{W_1, \ldots, W_t\} \subseteq \mathcal{P}^\ell_{ij}$ satisfying the following conditions.

- No two paths from $\mathcal{V}$ belong to the same bundle of $\mathcal{P}^\ell_{pq}$. Furthermore, no two paths of $\mathcal{V}$ end in the same non-outerplanar graph glued to a leaf of H_q.
- Every path in $\mathcal{W}$ is disjoint from $H_p \cup H_q$.
- The paths in $\mathcal{V}$ and $\mathcal{W}$ together form an $s \times t$ pseudogrid; where the paths in $\mathcal{V}$ are its vertical and those in $\mathcal{W}$ its horizontal paths, every path

from $\mathcal{V}$ on its way from H_p to H_q meets W_a before it meets W_{a+1} (for all $1 \le a < t$), and for every $i = 1, \dots, t$ the segment of W_i belonging to V_a precedes that belonging to V_{a+1} (for all $1 \le a < s$).

For all $a = 1, \dots, s$, denote the two paths from $\mathcal{P}^\ell_{pq}$ that are in the same bundle as V_a by V'_a and V''_a, where we may assume that V''_a meets as least as many paths from $\mathcal{W}$ as V'_a.

Case 1. There are at least $s/2$ of the V'_a such that each of them avoids at least r paths in $\mathcal{W}$.

Then there is a set $I \subseteq \{1, \dots, s\}$ with $|I| \ge s/2t^r$ such that each V'_a with $a \in I$ avoids the same r paths in $\mathcal{W}$, $W_{b_1}, \dots, W_{b_r}$ say (where $b_1 < \cdots < b_r$). For all $a \in I$, let V^*_a be the subpath of V_a between its endvertex in H_p and its first vertex on W_{b_r}. Note that $\lceil \log_2 3\binom{r}{2} \rceil \le r+1$ since $r \ge 4$, and thus

$$\lceil \log_2 n \rceil^{2r-1+\lceil \log_2 3\binom{r}{2}\rceil} \;\le\; \lceil r^2 \log_2 c \rceil^{3r} \;\le\; \lceil 16 r^9 3^{2r+2} \log_2 r \rceil^{3r} \;\le\; r^{r^7}. \qquad (1)$$

Together with $t + r \le r^7$ this implies

$$|I|^{1/4} \;\ge\; \left(\frac{\sqrt{c}}{2t^{2t+r}}\right)^{1/4} \;\ge\; \left(\frac{\sqrt{c}}{r^{14r^7}}\right)^{1/4} \;\ge\; r^{r^7} \;\overset{(1)}{\ge}\; \lceil \log_2 n \rceil^{2r-1+\lceil \log_2 3\binom{r}{2}\rceil}.$$

Thus we may apply Lemma 15 (with n replaced by $\lceil \log_2 n \rceil$) to the minor of G obtained by contracting every $K(v)$ to find a K_r minor in G. (The binary tree in H_p plays the role of T_1 in Lemma 15, the binary tree in H_q that of T_2, the set $\{V'_a \mid a \in I\}$ that of $\mathcal{P}$, and the pseudogrid formed by the V^*_a (for all $a \in I$) and $W_{b_1}, \dots, W_{b_r}$ that of G'.)

Case 2. There are at least $s/2$ of the V'_a such that each of them meets at least $t - r$ paths in $\mathcal{W}$.

Note that if V'_a meets $\ge t - r$ paths in $\mathcal{W}$, then so does V''_a. Thus there is a set $I \subseteq \{1, \dots, s\}$ with $|I| \ge s/2t^{2r}$ such that all V'_a and V''_a with $a \in I$ meet the same $t - 2r = (2r)^4$ paths of $\mathcal{W}$. Noting that there are $\le t^t$ permutations of a $(t-2r)$-element set and using Lemma 6 twice, we can find a set $I' \subseteq I$ with $|I'| \ge s/2t^{2t+2r}$ and paths $W_{b_1}, \dots, W_{b_{2r}}$ with $b_1 < \cdots < b_{2r}$ such that

- either, on its way from H_p to H_q, every V'_a with $a \in I'$ meets W_{b_k} before it meets $W_{b_{k+1}}$ or every V'_a with $a \in I'$ meets $W_{b_{k+1}}$ before it meets W_{b_k} (for all $1 \le k < 2r$), and
- the analogous condition holds for all V''_a with $a \in I'$.

Case 2.1. Every V'_a with $a \in I'$ meets $W_{b_{k+1}}$ before it meets W_{b_k} (for all $1 \le k < 2r$).

For all $a \in I'$, let V^*_a be the subpath of V_a between its endvertex in H_p and its first vertex on W_{b_r}. Using that $t + r \le r^7$ since $r \ge 4$, we have

$$|I'|^{1/4} \;\ge\; \left(\frac{\sqrt{c}}{2t^{4t+2r}}\right)^{1/4} \;\ge\; \left(\frac{\sqrt{c}}{r^{28r^7}}\right)^{1/4} \;\ge\; r^{r^7} \;\overset{(1)}{\ge}\; \lceil \log_2 n \rceil^{2r-1+\lceil \log_2 \binom{r}{2}\rceil}.$$

Thus we may apply Lemma 11 to (a minor of) G to find a K_r minor in G. (The binary tree in H_p plays the role of T in Lemma 11, $W_{b_{2r}}$ that of P, and the pseudogrid formed by all the V_a^* with $a \in I'$ and $W_{b_1}, \ldots, W_{b_r}$ that of G'.)

Case 2.2. Every V_a'' with $a \in I'$ meets $W_{b_{k+1}}$ before it meets W_{b_k} (for all $1 \leq k < 2r$).

The proof of this case is the same as that of Case 2.1.

Case 2.3. Neither Case 2.1 nor Case 2.2 hold.

For all $a \in I'$, let ℓ_a' (respectively ℓ_a'') be the first vertex of W_{b_1} that lies on the subpath of V_a' (respectively V_a'') between its endvertex in H_p and its first vertex on $W_{b_{r+1}}$. Similarly define r_a' and r_a'' to be the respective last vertices. Let S_a be the segment of W_{b_1} belonging to V_a in the pseudogrid formed by the paths in $\mathcal{V}$ and $\mathcal{W}$ (i.e. S_a is the segment between the first and the last vertex of W_{b_1} on V_a). Let $J \subseteq I'$ be obtained by applying Lemma 16 to ℓ_a', r_a' and S_a (for all $a \in I'$). Thus $|J| \geq \lfloor (|I'|/4)^{1/3} \rfloor \geq (|I'|/32)^{1/3}$.

Suppose first that J satisfies condition (a) of Lemma 16, say $S_{\max J} \subseteq W \mathring{r}_a'$ for all $a \in J$. Then the binary tree T in H_p is $|J|$-attached to an $r \times |J|$ pseudogrid whose vertical paths are the V_a with $a \in J$ together with (for each of these V_a) a path in the non-outerplanar graph glued to a leaf of H_p which joins this leaf to the endvertex of V_a in H_p, and whose horizontal paths are $W_{b_{r+1}}, \ldots, W_{b_{2r}}$. Note that

$$\begin{aligned} \left(\frac{|I'|^{1/3^{2r+2}}}{32} \right)^{1/4} &\geq \left(\frac{1}{32} \left(\frac{\sqrt{c}}{2t^{4t+2r}} \right)^{1/3^{2r+2}} \right)^{1/4} \\ &\geq \frac{c^{1/(8 \cdot 3^{2r+2})}}{32^{1/4} (r^{28r^7})^{1/(4 \cdot 3^{2r+2})}} \geq r^{r^7} \overset{(1)}{\geq} \lceil \log_2 n \rceil^{2r-1+\lceil \log_2 \binom{r}{2} \rceil}. \end{aligned} \tag{2}$$

Hence in particular

$$|J|^{1/4} \geq \left(\frac{|I'|}{32} \right)^{1/3 \cdot 1/4} \overset{(2)}{\geq} \lceil \log_2 n \rceil^{2r-1+\lceil \log_2 \binom{r}{2} \rceil},$$

and we may apply Lemma 11 to a minor of G to find a K_r minor in G. (The role of P in Lemma 11 is played by the subpath of W_{b_1} that starts in the first r_a' on W_{b_1} with $a \in J$.)

Hence we may assume that J satisfies condition (b) of Lemma 16. For all $a \in J$, let A_a be as in condition (b). Now let $J' \subseteq J$ be obtained by applying Lemma 16 to ℓ_a'', r_a'' and A_a (for all $a \in J$). Thus $|J'| \geq \lfloor (|J|/4)^{1/3} \rfloor \geq |I'|^{1/3^2}/32$. As before, since $|J'|^{1/4} \geq \lceil \log_2 n \rceil^{2r-1+\lceil \log_2 \binom{r}{2} \rceil}$ by (2), we may assume that J' satisfies condition (b). Applying the same argument to every W_{b_k} with $k \leq r+1$, we may assume that there exists $J'' \subseteq J'$ such that

$$|J''| \geq \frac{|I'|^{1/3^{2r+2}}}{32^{(3^0+3^1+\cdots+3^{2r+1})/3^{2r+2}}} \geq \frac{|I'|^{1/3^{2r+2}}}{32},$$

and that to every $a \in J''$ there belongs a segment B_{ak} on every W_{b_k} $(k \leq r+1)$ that contains not only every vertex of V_a on W_{b_k} but also all vertices of W_{b_k} that lie on the subpaths of V'_a and V''_a between their endvertices in H_p and their first vertices on $W_{b_{r+1}}$, and such that the B_{ak} are disjoint for different $a \in J''$. (Indeed, as $|J''|^{1/4} \geq \lceil \log_2 n \rceil^{2r-1+\lceil \log_2 \binom{r}{2} \rceil}$ by (2), we may assume that each application of Lemma 16 yields a subset of J' satisfying condition (b).)

But since $|J''|^{1/2} \geq \lceil \log_2 n \rceil^{2r-1+\lceil \log_2 \binom{r}{2} \rceil}$ by (2), we may now apply Lemma 13 to find the desired K_r minor in G. $\square$

5 Proof of Theorem 2

In the proof of Theorem 2 we use the notion of a k-embedding of a given graph in the plane with one vortex. (It is a special case of the definition of a near-embedding of a graph in a surface introduced by Robertson and Seymour.) Let S be the surface obtained from the sphere by removing the interior of a closed disc. So the boundary of S consists of one component D, which is homeomorphic to the unit circle. We say that a graph G is *k-embeddable in the plane with one vortex* if G has a set X of at most k vertices such that $G - X$ can be written as $G^0 \cup G^1$ and the following conditions hold:

- there is an embedding of G^0 in S with $V(G^0) \cap D = V(G^0) \cap V(G^1) =: \{v_1, \dots, v_\ell\}$,
- G^1 has a path-decomposition $(p_1 \dots p_\ell, (X_{p_i})_{i=1}^{\ell})$ of width at most k and such that $v_i \in X_{p_i}$ for all $i = 1, \dots, \ell$, and the points $v_1, \dots, v_\ell$ occur on D in this order (for one of the two orientations of D).

The idea for the proof of Theorem 2 is to show that if $\mathcal{H}$ and $\mathcal{H}^*$ are inequivalent, then there exists $k \in \mathbb{N}$ such that the vertices of each graph $H \in \mathcal{H}$ can be joined bijectively to the bottom row of a grid to obtain a graph G_H in which H is externally $|H|$-connected but which is k-embeddable in the plane with one vortex. Lemma 2.3 from [4] then implies that no graph G_H $(H \in \mathcal{H})$ contains a large complete minor. Hence $\mathcal{H}$ does not force large complete minors, contradicting our assumption.

Let us first prove a simple lemma, which implies that if T, T' are trees and T is a minor of T', then T does not lie 'upside down' in T'.

Lemma 17 *Suppose that T, T' are trees and T is a minor of T'. Let $X \subseteq V(T')$ be the branch set corresponding to a leaf x of T. Then T' contains a path which joins X to a leaf of T' but does not meet another branch set.*

Proof. Suppose that $|T| \geq 2$, let x be a leaf of T and let y be the neighbour of x in T. Let e be the unique edge of T' corresponding to xy (i.e. e is the edge joining the branch set X of x to that of y). Write C for the component of $T - e$ which contains X. Then the (connected) subgraph of T' which contracts to $T - x$ is disjoint from C. Thus joining x to any leaf of T contained in C yields a path as desired. $\square$

Proof of Theorem 2. We consider only the case where $\mathcal{H}^* = \mathcal{H}'$, the other three cases are similar. Let $H_n^\sharp$ denote the graph obtained from the binary tree T of height n by adding a new vertex x and joining it to all leaves of T. Thus H_n' is the disjoint union of n copies of $H_n^\sharp$. The vertices in H_n' of the form x are called the *new vertices* of H_n'.

Since $\mathcal{H} \preccurlyeq \mathcal{H}'$, we find for every $H \in \mathcal{H}$ an $n \in \mathbb{N}$ such that H is a minor of H_n'. Let X_H be the set of all vertices of H whose branch sets contain a new vertex of H_n'. Thus $H - X_H$ is a forest, every component of which is joined to at most one vertex in X_H by edges of H. For every $x \in X_H$, let F_x be the forest consisting of all paths in $H - X_H$ between neighbours (in H) of x. So we have that

(i) the F_x are disjoint for different $x \in X_H$,

(ii) for each $x \in X_H$ the neighbours of x in H all lie in F_x,

(iii) for each $x \in X_H$ every component C of F_x is a subtree of some component of $H - X_H$, and this component meets F_x precisely in C and does not meet any $F_{x'}$ with $x' \neq x$.

Case 1. For every $k \in \mathbb{N}$ there is an $H \in \mathcal{H}$ such that at least k of the forests F_x ($x \in X_H$) have path-width at least k.

We will show that $\mathcal{H}' \preccurlyeq \mathcal{H}$. Our assumption implies that for every $n \in \mathbb{N}$ there exists a graph $H \in \mathcal{H}$ and distinct $x_1, \ldots, x_n \in X_H$ such that $\mathrm{pw(F_{x_i})} \geq \mathrm{2^{n+1}}$ for every $i = 1, \ldots, n$ (where $\mathrm{pw(F_{x_i})}$ denotes the path-width of F_{x_i}). Thus each F_{x_i} contains the binary tree of height n as a minor (see e.g. [1, Thm. 12.4.5]). Lemma 17 now says that every branch set corresponding to a leaf of this binary tree can be reached from a leaf of F_{x_i} by a path in F_{x_i} not meeting another branch set. Clearly, these paths are disjoint for different leaves of the binary tree. As in H every leaf of F_{x_i} is a neighbour of x_i, it follows that $H_n^\sharp$ is a minor of the graph obtained from F_{x_i} by including x_i and joining it to all its neighbours in H. Thus (i) implies that H_n' is a minor of H, and so $\mathcal{H}' \preccurlyeq \mathcal{H}$, as desired.

Case 2. There exists a $k \in \mathbb{N}$ such that for every $H \in \mathcal{H}$ all but at most k of the forests F_x ($x \in X_H$) have path-width less than k.

For every $H \in \mathcal{H}$, let X_H' denote the set of all vertices $x \in X_H$ for which $\mathrm{pw(F_x) < k}$, and let $X_H'' := X_H \setminus X_H'$. Put

$$G_H^1 := H[X_H' \cup \bigcup_{x \in X_H'} V(F_x)].$$

Let us first show that the vertices of H can be joined bijectively to the bottom row of the $|H| \times |H|$ grid in such a way that the graph G_H thus obtained can be k-embedded in the plane with one vortex, namely as $G_H - X_H'' = G_H^0 \cup G_H^1$ where X_H'' is the deleted set and G_H^0 is the subgraph of $G_H - X_H''$ obtained by deleting all edges of G_H^1. Indeed, since $H - X_H$ is a forest, the graph H' obtained from $H - X_H''$ by deleting all vertices of G_H^1 is also a forest. Since H' is obtained from $H - X_H$ by deleting $\bigcup_{x \in X'_H} F_x$, it follows from (ii) and (iii) that each component of H' is joined to G^1_H by at most one edge of H. Moreover, (i)–(iii) imply that G^1_H is obtained from the disjoint union of all F_x with $x \in X'_H$ by including every vertex $x \in X'_H$ and joining it to all its neighbours in F_x. Hence $\mathrm{pw(G^1_H)} \le \mathrm{k}$. Put $\ell := |G^1_H|$. Then there exists a path-decomposition $(p_1 \ldots p_\ell, (X_{p_i})_{i=1}^{\ell})$ of G^1_H of width at most k and such that to each X_{p_i} one can assign a vertex $v_i \in X_{p_i}$ with $v_i \neq v_j$ for $i \neq j$. So $H - X''_H$ can be thought of as a 'wide path' (consisting of G^1_H) with trees added, each of them joined to the 'wide path' by at most one edge. Now join the vertices of $H - X''_H$ injectively to the bottom row R of the $|H| \times |H|$ grid so that the ordering of the neighbours of $V(G^1_H) = \{v_1, \ldots, v_\ell\}$ in R corresponds to $v_1, \ldots, v_\ell$, and so that the vertices of every component C of H' are joined without crossings to vertices of R, and if C is joined to v_i by an edge in H then these vertices of R lie between the neighbours of v_{i-1} and v_{i+1} in R (Fig. 4). Joining the vertices in X''_H to the

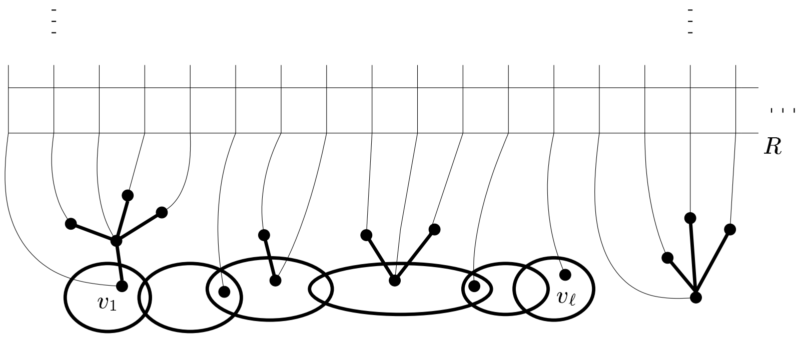

Figure 4: Joining the vertices of $H - X''_H$ to the bottom row of a grid.

remaining vertices of R then yields a graph G_H as desired.

The first part of the proof of Lemma 2.3 from [4] now implies that none of the graphs G_H $(H \in \mathcal{H})$ contains a K_{4k+8} minor. But since the bottom row of an $s \times s$ grid G is externally s-connected in G, it follows that every $H \in \mathcal{H}$ is externally $|H|$-connected in G_H, contradicting our assumption that $\mathcal{H}$ forces large complete minors. $\square$

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