Extremal connectivity for topological cliques in bipartite graphs

Daniela Kühn Deryk Osthus

Abstract

Let $d(s)$ be the smallest number such that every graph of average degree $> d(s)$ contains a subdivision of K_s. So far, the best known asymptotic bounds for $d(s)$ are $(1+o(1))9s^2/64 \leq d(s) \leq (1+o(1))s^2/2$. As observed by Łuczak, the lower bound is obtained by considering bipartite random graphs. Since with high probability the connectivity of these random graphs is about the same as their average degree, a connectivity of $(1+o(1))9s^2/64$ is necessary to guarantee a subdivided K_s. Our main result shows that for bipartite graphs this gives the correct asymptotics. We also prove that in the non-bipartite case a connectivity of $(1+o(1))s^2/4$ suffices to force a subdivision of K_s. Moreover, we slightly improve the constant in the upper bound for $d(s)$ from $1/2$ (which is due to Komlós and Szemerédi) to $10/23$.

1 Introduction

Given a natural number s, let $d(s)$ be the smallest number such that every graph of average degree $> d(s)$ contains a subdivision of the complete graph K_s of order s. The existence of $d(s)$ was proved by Mader [13]. As first observed by Jung [10], the complete bipartite graph $K_{t,t}$ with $t := \lfloor s^2/8 \rfloor$ shows that $d(s) \geq \lfloor s^2/8 \rfloor$. Bollobás and Thomason [5] as well as Komlós and Szemerédi [12] showed that s^2 is the correct order of magnitude for $d(s)$. More precisely, it is known that

$$(1+o(1))\frac{9s^2}{64} \leq d(s) \leq (1+o(1))\frac{s^2}{2}. \qquad (1)$$

The upper bound is due to Komlós and Szemerédi [12]. As observed by Łuczak, the lower bound is obtained by considering a random subgraph of a complete bipartite graph with edge probability $3/4$ (see Proposition 16). It is widely believed that the lower bound gives the correct constant, i.e. that random graphs provide the extremal graphs. If true, this would mean that the situation is similar as for ordinary minors. Indeed, Thomason [17] was recently able to prove that random graphs are extremal for minors and Myers [16] showed that all extremal graphs are essentially disjoint unions of pseudo-random graphs.

In this paper we show that the lower bound in (1) is correct if we restrict our attention to bipartite graphs whose connectivity is close to their average degree:

Theorem 1 *Given $s \in \mathbb{N}$, let $c_{bip}(s)$ denote the smallest number such that every $c_{bip}(s)$-connected bipartite graph contains a subdivision of K_s. Then*

$$c_{bip}(s) = (1 + o(1))\frac{9s^2}{64}.$$

In Theorem 1 the condition of being bipartite can be weakened to being H-free for some arbitrary but fixed 3-chromatic graph H (Theorem 19). For arbitrary graphs, the best current upper bound on the extremal connectivity is the same as in (1). The proof of Theorem 1 yields the following improvement.

Theorem 2 *Given $s \in \mathbb{N}$, let $c(s)$ denote the smallest number such that every $c(s)$-connected graph contains a subdivision of K_s. Then*

$$(1 + o(1))\frac{9s^2}{64} \leq c(s) \leq (1 + o(1))\frac{s^2}{4}.$$

The lower bounds in Theorems 1 and 2 are provided by the random bipartite graphs mentioned above (since their connectivity is close to their average degree). Thus at least in the case of highly connected bipartite graphs they are indeed extremal.

By using methods as in the proof of Theorem 1, we also obtain a small improvement for the constant in the upper bound in (1).

Theorem 3 *Given $s \in \mathbb{N}$, let $d(s)$ denote the smallest number such that every graph of average degree $> d(s)$ contains a subdivision of K_s. Then*

$$(1 + o(1))\frac{9s^2}{64} \leq d(s) \leq (1 + o(1))\frac{10s^2}{23}.$$

The example of Łuczak mentioned above only gives us extremal graphs for Theorem 1 whose connectivity is about $3n/8$, i.e. whose connectivity is rather large compared to the order n of the graph. However, in Proposition 18 we show that there are also extremal graphs whose order is arbitrarily large compared to their connectivity. In contrast to this, the situation for ordinary minors is quite different. In general a connectivity of order $s\sqrt{\log s}$ is needed to force a K_s minor, but the connectivity of the known extremal graphs is linear in their order. In fact, confirming a conjecture of Thomason [18], Böhme, Kawarabayashi, Maharry and Mohar [1] proved that there is a constant c such that for all integers s there is an integer $n_0 = n_0(s)$ such that every graph of order at least n_0 and connectivity at least cs contains the complete graph K_s as minor. Thus a linear connectivity suffices to force a K_s minor if we only consider sufficiently large graphs.

Our paper is organized as follows. In Section 2 we introduce the necessary notation and tools which we need in the proof of the upper bound of Theorem 1. In Section 3 we provide the (sparse) extremal examples for the lower bound. The proof of the upper bound of Theorems 1, 2 and 3 is contained in Section 4. It builds on results and methods of Komlós and Szemerédi [12]. Finally, in the last section we then briefly discuss the difficulties which arise if one tries to extend Theorem 1 to arbitrary graphs.

2 Notation and tools

Throughout this paper we omit floors and ceilings whenever this does not affect the argument. Given constants $0 < \alpha, \beta < 1$, we write $\alpha \ll \beta$ if α is sufficiently small compared with β, i.e. there will always exist a positive $\alpha_0 = \alpha_0(\beta)$ such that the assertion in question holds for all $\alpha \leq \alpha_0$ and $\alpha \ll \beta$ means that $\alpha \leq \alpha_0$. We write $e(G)$ for the number of edges of a graph G, $|G|$ for its order, $\delta(G)$ for its minimum degree and $d(G) := 2e(G)/|G|$ for its average degree. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and the set of its neighbours by $N_G(x)$. Given disjoint $A, B \subseteq V(G)$, an *A–B edge* is an edge of G with one endvertex in A and the other in B, the number of these edges is denoted by $e_G(A, B)$. We write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all A–B edges in G. More generally, we often write (A, B) for a bipartite graph with vertex classes A and B. A *subdivision* of a graph H is a graph TH obtained from H by replacing the edges of H with internally disjoint paths. The *branch vertices* of TH are all those vertices that correspond to vertices of H.

Our proof of Theorem 1 is based on Szemerédi's Regularity lemma. We will now collect all the information we need about it (see [11] for a survey). Let us start with some more notation. The *density* of a bipartite graph $G = (A, B)$ is defined to be

$$d(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

Given $\varepsilon > 0$, we say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A, B) - d(X, Y)| < \varepsilon$.

We will often use the following simple fact.

Proposition 4 *Given an ε-regular bipartite graph (A, B) of density at least d and a set $X \subseteq A$ with $|X| \geq \varepsilon|A|$, there are less than $\varepsilon|B|$ vertices in B which have at most $(d - \varepsilon)|X|$ neighbours in X.*

We will work with the following degree form of the Regularity lemma which can be easily derived from the classical version. Proofs of the latter are for example included in [3] and [6].

Lemma 5 (Regularity lemma) *For all $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for every number $d \in [0, 1]$ and for every graph G there are a partition of $V(G)$ into $V_0, V_1, \ldots, V_k$ and a spanning subgraph G' of G such that the following holds:*

- $1/\varepsilon \leq k \leq N$,
- $|V_0| \leq \varepsilon|G|$,
- $|V_1| = \cdots = |V_k| =: L$,
- $d_{G'}(x) > d_G(x) - (d + \varepsilon)|G|$ *for all vertices* $x \in G$,
- *for all* $i \geq 1$ *the graph* $G'[V_i]$ *is empty,*

- *for all* $1 \leq i < j \leq k$ *the graph* $(V_i, V_j)_{G'}$ *is* ε*-regular and has density either* 0 *or* $> d$.

The sets V_i $(i \geq 1)$ are called *clusters*. Given clusters $V_1, \ldots, V_k$ and G' as in Lemma 5, the *reduced graph* R is the graph whose vertices are $V_1, \ldots, V_k$ and in which V_i is joined to V_j whenever $(V_i, V_j)_{G'}$ is ε-regular and has density $> d$. Thus V_iV_j is an edge of R if and only if G' has an edge between V_i and V_j. Given an edge $V_iV_j \in R$, for convenience we will refer to the density of $(V_i, V_j)_{G'}$ as the *density of the edge* V_iV_j.

In Propositions 6, 7 and 9 as well as in Lemmas 8 and 10 R will denote the reduced graph obtained by applying the Regularity lemma (Lemma 5) with parameters ε and d to the given graph G. G' and L will be as defined in this lemma.

Proposition 6 *Suppose that* ε*,* c *and* d *are positive numbers such that* $2\varepsilon \leq d < c/2$ *and suppose that* G *is a graph of minimum degree at least* $c|G|$*. Let* μ *be the maximum density of an edge in the reduced graph* R*. Then the minimum degree of* R *is at least* $(c - 2d)|R|/\mu$*.*

Proof. Set $n := |G|$. Consider a cluster V and let $U \subseteq V(G)$ be the union of all those clusters which are neighbours of V in R. Then there exists a vertex $v \in V$ which has at most $\mu|U|$ neighbours in U (in the graph G'). Indeed, suppose not. Then there exists a cluster $W \in N_R(V)$ such that $e_{G'}(V, W) > \mu|V||W|$. This contradicts the fact that, by definition of μ, the density of $(V, W)_{G'}$ is at most μ. But for every vertex $v \in V$ with at most $\mu|U|$ neighbours in U, we have

$$\mu \cdot d_R(V)L = \mu|U| \geq d_{G'}(v) - |V_0| > d_G(v) - (d + 2\varepsilon)n.$$

Therefore

$$d_R(V) > \frac{(c - d - 2\varepsilon)n}{\mu L} \geq \frac{(c - 2d)|R|}{\mu},$$

as required. □

Proposition 7 *Let* V *be a vertex of the reduced graph* R *and let* A *be a set of neighbours of* V *in* R*. Then, given* $\ell \in \mathbb{N}$*, there are at most* $\ell\varepsilon L$ *vertices* $v \in V$ *which have at most* $(d - \varepsilon)L$ *neighbours in at least* $|A|/\ell$ *clusters belonging to* A *(in the graph* G'*).*

Proof. We say that a vertex $z \in V$ is *bad* for a cluster $W \in A$ if $|N_{G'}(z) \cap W| \leq (d - \varepsilon)L$. Let Z be the set of all those vertices $z \in V$ which are bad for at least $|A|/\ell$ clusters $W \in A$. By summing the number of vertices $z \in Z$ which are bad for W over the clusters $W \in A$, it is easy to see that for some cluster $W \in A$ there are at least $|Z|/\ell$ vertices in Z which are bad for W. Together with Proposition 4 this implies that $|Z|/\ell < \varepsilon L$. □

The following lemma is a special case of the well-known 'Embedding lemma' (see e.g. [3, Chapter IV, Thm. 31], [6, Lemma 7.3.2] or [11, Thm. 2.1] for a proof).

Lemma 8 *For every $0 < d \leq 1$ and every 3-chromatic graph H there exists a positive constant $\varepsilon_0 = \varepsilon_0(d, H)$ such that for each $0 < \varepsilon \leq \varepsilon_0$ there is an integer $n_0 = n_0(\varepsilon, d, H)$ for which the following holds. Let G be a graph of order at least n_0 and suppose that R contains a triangle. Then G contains a copy of H.*

The next two simple observations (Proposition 9 and Lemma 10) will only be used in the proof of Theorem 3. Given $d \in [0, 1]$, we say that G is (ε, d)-*super-regular* if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) > d$ and, furthermore, if $d_G(a) > d|B|$ for all $a \in A$ and $d_G(b) > d|A|$ for all $b \in B$. The following well-known proposition shows that, given a subgraph H of R of bounded maximum degree, one can slightly modify the clusters belonging to H such that each edge of H corresponds to a super-regular graph. It can be easily proved by using Proposition 4.

Proposition 9 *Let H be a subgraph of the reduced graph R with $\Delta(H) \leq \Delta \leq 1/2\varepsilon$. Then each vertex V_i of H contains a subset V_i' of size $(1-\varepsilon\Delta)L$ such that for every edge V_iV_j of H the graph $(V_i', V_j')_{G'}$ is $(2\varepsilon, d-(1+\Delta)\varepsilon)$-super-regular.*

Lemma 10 *Given positive constants ε and d with $5\varepsilon \leq d < 1$, suppose that P is a path in R with endvertices U and W. Then G' contains at least $(1 - 2\varepsilon)L$ disjoint paths such that each of them starts in some vertex belonging to U, ends in some vertex belonging to W and contains only vertices belonging to clusters in $V(P)$ (precisely one vertex in each of these clusters).*

Proof. Suppose that $P = V_1 \dots V_r$. First apply Proposition 9 to P to obtain a $(1 - 2\varepsilon)L$-element subset V_i' of each cluster $V_i \in V(P)$ such that $(V_i', V_{i+1}')_{G'}$ is $(2\varepsilon, d - 3\varepsilon)$-super-regular for all $1 \leq i < r$. It is easily checked that the super-regularity of $(V_i', V_{i+1}')_{G'}$ implies that this graph satisfies Hall's matching condition and thus contains a perfect matching. The union of all these matchings forms a set of paths as required. □

Given a bipartite graph (U, W), a set $S \subseteq U$ and numbers $\ell \leq |S|$ and $1/\ell \leq \beta \leq 1$, we say that S is (ℓ, β)-*dense for* W if for each ℓ-element subset S' of S there are at most $\beta|W|$ vertices in W which have less than $\beta|S'|$ neighbours in S'. If $U = S$, this notion can be viewed as a weakening of ε-regularity. Indeed, if (U, W) is ε-regular of density at least $\beta + \varepsilon$, $\beta \geq \varepsilon$ and $\ell \geq \varepsilon|U|$, then by Proposition 4 the set U is (ℓ, β)-dense for W. The following even weaker notion will also be convenient. S is called (ℓ, β)-*attached to* W if for each ℓ-element subset S' of S all but at most $\beta|W|$ vertices in W have a neighbour in S'.

Roughly speaking, the next lemma implies that with high probability the ε-regularity of a bipartite graph (U, W) is not lost completely when passing over to a subgraph (S, W) where S is a random subset of U. The point here is that $|S|$ need not be linear in $|U|$. A similar statement was proved independently of us by Gerke et al. [8] in the context of extremal subgraphs of random graphs.

Lemma 11 *Given constants $0 < \alpha, \beta, \varepsilon, d < 1$ with $\varepsilon \ll \alpha \le 1/2$, $\varepsilon \le \beta$ and $\beta \ll d$, there exists a natural number $s_0 = s_0(\varepsilon, \beta, \alpha, d)$ such that the following is true for all $s \ge s_0$. Set $\ell := \alpha s$ and suppose that $G = (U, W)$ is an ε-regular bipartite graph of density at least d such that $|U|, |W| \ge s$. Let S be a subset of U which is obtained by successively selecting s vertices in U uniformly at random without repetitions. Then with probability at least $1 - \mathrm{e}^{-s}$ the set S is (ℓ, β)-dense for W.*

Proof. Consider a subset S' of U which is obtained by successively selecting ℓ vertices in U uniformly at random without repetitions. We call S' a *failure* if there are at least $\beta|W|$ vertices in W which have less than $\beta|S'|$ neighbours in S'. We will first show that the probability that S' is a failure is very small. This will be done by grouping the vertices in S' into successive 'epochs' and by analyzing one such 'epoch' at a time. Set $r := d/(8\beta)$; r will be the number of such 'epochs' and so each 'epoch' will contain $\ell/r = 8\beta\ell/d$ vertices. We call the subset of S' which consists of the first ℓ/r vertices chosen for S' the *first epoch* of S' and denote it by S'_1. Similarly, given $2 \le i \le r$, we define the *ith epoch* S'_i of S'. Given $1 \le i \le r$, let W_i be the set of all those vertices $w \in W$ which have at least $\beta\ell$ neighbours in S'_i. For all $i \le r+1$ set $W'_i := W \setminus \bigcup_{j<i} W_j$. Thus W'_i contains all those vertices for which, after $i-1$ epochs, we cannot guarantee that they have enough neighbours in S'. We say that the ith epoch S'_i is *successful* if either $|W'_i| < \beta|W|$ or if at least half of the vertices in S'_i have at least $d|W'_i|/2$ neighbours in W'_i.

The aim now is to show that if the ith epoch is successful and W'_i is still large, then a significant proportion of the vertices in W'_i will belong to W_i. Since the probability that an epoch is successful will turn out to be quite large, this will then imply that with large probability the set W'_{r+1} is small and thus with large probability S' is not a failure. Set $\mu_i := |W'_i \cap W_i|/|W'_i|$ and suppose that the ith epoch S'_i is successful but $|W'_i| \ge \beta|W|$. By counting the edges between W'_i and S'_i and recalling that $|S'_i| = 8\beta\ell/d$, we get

$$\mu_i|W'_i| \cdot \frac{8\beta\ell}{d} + |W'_i| \cdot \beta\ell \ge e_G(W'_i, S'_i) \ge \frac{4\beta\ell}{d} \cdot \frac{d|W'_i|}{2}.$$

Hence

$$\mu_i \ge d/8.$$

We now bound the probability that an epoch is not successful. Since (U, W) is ε-regular and has density at least d, Proposition 4 implies that if $|W'_i| \ge \beta|W| \ge \varepsilon|W|$ then at most $\varepsilon|U|$ vertices in U have less than $d|W'_i|/2$ neighbours in W'_i. So in this case, for every $s \in S'_i$, the probability that s has less than $d|W'_i|/2$ neighbours in W'_i is at most $\varepsilon|U|/(|U| - \ell) \le 2\varepsilon$. Thus for any event $\mathcal{A}_{i-1}$ depending only on the outcome of the first $i-1$ epochs, we have

$$\mathbb{P}(S'_i \text{ is not successful} \mid \mathcal{A}_{i-1}) \le 2^{|S'_i|}(2\varepsilon)^{|S'_i|/2} = (8\varepsilon)^{4\beta\ell/d}.$$

Hence

$$\mathbb{P}(\text{at least } r/2 \text{ epochs are not successful}) \le 2^r(8\varepsilon)^{2\beta\ell r/d} \le (16\varepsilon)^{\ell/4}.$$

Let N denote the number of successful epochs. Then $|W'_{N+1}| \le \max\{\beta|W|, (1-d/8)^N|W|\}$. But if $N \ge r/2$ we have

$$(1-d/8)^N|W| \le (1-d/8)^{d/(16\beta)}|W| \le \mathrm{e}^{-d^2/(128\beta)}|W| \le \beta|W|.$$

This shows that with probability at most $(16\varepsilon)^{\ell/4}$, a random ℓ-set S' is a failure.

Now suppose that S is a random set as given by the lemma. Then, since every ℓ-element subset S' of S is again a random set whose distribution is uniform amongst all ℓ-element subsets of U,

$$\begin{aligned}
\mathbb{P}(S \text{ is not } (\ell,\beta)\text{-dense for } W) &\le \sum_{S' \in S^{(\ell)}} \mathbb{P}(S' \text{ is a failure}) \\
&\le \binom{s}{\ell}(16\varepsilon)^{\ell/4} \\
&\le \left(\frac{\mathrm{e}s}{\ell}\right)^{\ell}(16\varepsilon)^{\ell/4} \\
&= \mathrm{e}^{\ell\log(\mathrm{e}/\alpha)}\mathrm{e}^{-(\ell/4)\log(1/(16\varepsilon))} \\
&\le \mathrm{e}^{-(\ell/5)\log(1/\varepsilon)} \le \mathrm{e}^{-s},
\end{aligned}$$

as required. (The third inequality is a weak form of Stirling's formula, see e.g. [4, p. 4].) □

The following special case of Lemma 11 was already proved by Komlós and Szemerédi [12]. A result which is slightly stronger than Corollary 12 was also proved earlier by Duke and Rödl [7].

Corollary 12 *Under the conditions of Lemma 11, the set S is (ℓ,β)-attached to W with probability at least $1-\mathrm{e}^{-s}$.*

Given a positive number ε and sets $A, Q \subseteq T$, we say that A *is split ε-fairly by Q* if

$$\left|\frac{|A\cap Q|}{|Q|} - \frac{|A|}{|T|}\right| \le \varepsilon.$$

Thus, if ε is small and A is split ε-fairly by Q, then the proportion of all those elements of T which lie in A is almost equal to the proportion of all those elements of Q which lie in A. We will use the following version of the well-known fact that if Q is random then it tends to split large sets ε-fairly.

Proposition 13 *For each $0 < \varepsilon < 1$ there exists an integer $q_0 = q_0(\varepsilon)$ such that the following holds. Given $t \ge q \ge q_0$ and a set T of size t, let Q be a subset of T which is obtained by successively selecting q elements uniformly at random without repetitions. Let $\mathcal{A}$ be a family of at most q^{10} subsets of T such that $|A| \ge \varepsilon t$ for each $A \in \mathcal{A}$. Then with probability at least $1/2$ every set in $\mathcal{A}$ is split ε-fairly by Q.*

To prove Proposition 13 we will use the following large deviation bound for the hypergeometric distribution (see e.g. [9, Thm. 2.10 and Cor. 2.3]).

Lemma 14 *Given $q \in \mathbb{N}$ and sets $A \subseteq T$ with $|T| \geq q$, let Q be a subset of T which is obtained by successively selecting q elements of T uniformly at random without repetitions. Let $X := |A \cap Q|$. Then for all $0 < \varepsilon < 1$ we have*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2\mathrm{e}^{-\frac{\varepsilon^2}{3}\mathbb{E}X}.$$

Proof of Proposition 13. Given $A \in \mathcal{A}$, Lemma 14 implies that

$$\begin{aligned}\mathbb{P}(A \text{ is not split } \varepsilon\text{-fairly by } Q) &\leq \mathbb{P}(|\,|A \cap Q| - q|A|/t\,| \geq \varepsilon q|A|/t) \\ &\leq 2\mathrm{e}^{-\frac{\varepsilon^2}{3}\frac{q|A|}{t}} \leq 2\mathrm{e}^{\frac{-\varepsilon^3 q}{3}}.\end{aligned}$$

Hence, if q_0 is sufficiently large compared with ε, the probability that there is an $A \in \mathcal{A}$ which is not split ε-fairly is at most $2q^{10}\mathrm{e}^{-\varepsilon^3 q/3} < 1/2$, as required. □

Finally, we need the following result of Mader [14]. (A proof is also included in [2].)

Theorem 15 *Every graph G contains a $\lceil d(G)/4 \rceil$-connected subgraph.*

3 Proof of Theorem 1 – extremal graphs

As mentioned in [12], the following example of Łuczak shows that the function $c_{bip}(s)$ defined in Theorem 1 is at least $(1 + o(1))9s^2/64$ (and thus also the functions $c(s)$ and $d(s)$ defined in Section 1).

Proposition 16 *For every positive λ and each integer κ_0 there exists a bipartite graph G such that G is κ-connected for some $\kappa \geq \kappa_0$ and does not contain a subdivision of a clique of order at least $(1 + \lambda)8\sqrt{\kappa}/3$.*

We include the proof of Proposition 16 here firstly for completeness and secondly because we will build on the argument in the proof of Proposition 18 below. In both proofs, the next simple and well-known fact (see e.g. [4, Ch. II, Thm. 2.1]) will be rather useful.

Theorem 17 *Let $n \in \mathbb{N}$ and let $0 < \varepsilon, p < 1$ be fixed. Let B_{np} be a bipartite random graph whose vertex classes A and B both have size n and where the edges between these classes are included with probability p independently of each other. Then, with probability tending to 1 as $n \to \infty$,*

$$(1 - \varepsilon)p|U||W| \leq e_{B_{np}}(U, W) \leq (1 + \varepsilon)p|U||W|$$

for all sets $U \subseteq A$ and $W \subseteq B$ with $|U|, |W| \geq (\log n)^2$.

Proof of Proposition 16. Throughout the proof we assume that λ is sufficiently small and n is sufficiently large for our estimates to hold. Let $\kappa := (1 - \lambda/2)3n/4$ and $s := (1 + \lambda)8\sqrt{\kappa}/3$. Put $p := 3/4$ and let B_{np} be a bipartite random graph as in Theorem 17. Using the lower bound in Theorem 17, one can easily show that B_{np} is κ-connected with probability tending to 1 as $n \to \infty$. We will show that, with probability tending to 1, there will not be any sets $U \subseteq A$ and $W \subseteq B$ such that $U \cup W$ can serve as the set of branch vertices of a subdivided K_s in B_{np}. Without loss of generality we assume that $|U| \geq |W|$. Clearly, if $|W| \leq (\log n)^2$, then B_{np} cannot contain a subdivided edge for all the pairs of vertices in U since each such edge must have an inner vertex in B and $|B| < \binom{s-(\log n)^2}{2}$. But Theorem 17 implies that with probability tending to 1 we have

$$e_{B_{np}}(U, W) \leq (1 + \lambda/30)p|U||W| \tag{2}$$

for all U, W with $|U|, |W| \geq (\log n)^2$. However, if $U \cup W$ is the set of branch vertices of a TK_s, then B contains an inner vertex of each subdivided edge joining a pair of vertices in U as well as an inner vertex of each subdivided edge which joins some $a \in U$ to some $b \in W$ with $ab \notin B_{np}$. Thus, if (2) holds, then the number of all these subdivided edges is

$$\binom{|U|}{2} + |U||W| - e_{B_{np}}(U, W) > n.$$

This shows that with probability tending to 1 the graph B_{np} does not contain a subdivided K_s. Thus, with probability tending to 1, we can take B_{np} for the graph G in Proposition 16. $\square$

If we take a sequence of disjoint copies of the graph given by Proposition 16 and attach successive copies by inserting κ independent edges, then the next proposition shows that we obtain arbitrarily large κ-connected bipartite graphs which do not contain a subdivision of a large clique (and the density of these graphs is arbitrarily small).

Proposition 18 *For every positive λ and every integer κ_0 there exists an integer $\kappa \geq \kappa_0$ and arbitrarily large bipartite graphs G which are κ-connected and do not contain a subdivision of a clique of order at least $(1 + \lambda)8\sqrt{\kappa}/3$.*

Proof. Throughout the proof we assume that λ is sufficiently small and n is sufficiently large for our estimates to hold. Let $G = (A, B)$ be the bipartite (random) graph given by the proof of Proposition 16. Thus $|A| = |B| = n$, G is κ-connected where $\kappa := (1 - \lambda/2)3n/4$ and all sets $U \subseteq A$ and $W \subseteq B$ with $|U|, |W| \geq (\log n)^2$ satisfy

$$e(U, W) \leq (1 + \lambda/30)3|U||W|/4. \tag{3}$$

Moreover, G does not contain a subdivision of K_s where $s := (1 + \lambda)8\sqrt{\kappa}/3$. Given an integer k, let G^* denote the graph obtained from k disjoint copies $G_1 = (A_1, B_1), \ldots, G_k = (A_k, B_k)$ of G by inserting κ independent edges between B_i and A_{i+1} (for all $1 \leq i < k$). Thus G^* is κ-connected and bipartite. We will show that G^* does not contain a subdivided K_s. So suppose not and choose a TK_s in G^*. For each $i \leq k$ let X_i be the set of all branch vertices of TK_s in $G_1 \cup \cdots \cup G_i$ and let Y_i be the set of all branch vertices in $G_i \cup \cdots \cup G_k$. Since each subdivided edge joining a branch vertex in X_i to a branch vertex in Y_{i+1}

must contain one of the κ edges between B_i and A_{i+1}, we have $\kappa \geq |X_i||Y_{i+1}| = |X_i|(s - |X_i|)$. This implies that for each i either $|X_i| \leq 0.17s$ or $|X_i| \geq 0.83s$. Let i be the first index for which the latter holds. Thus $x := |X_{i-1}| \leq 0.17s$ and $y := |Y_{i+1}| \leq 0.17s$. Let S_A be the set of all branch vertices in A_i and let S_B be the set of all branch vertices in B_i. Put $X := X_{i-1}$, $Y := Y_{i+1}$, $s_A := |S_A|$ and $s_B := |S_B|$.

Let us now estimate the number of all those vertices in A_i which are contained in the TK_s. Firstly, since all the B_{i-1}–A_i edges are independent, A_i contains at least $x(s-x)$ vertices on subdivided edges joining a branch vertex in X to a branch vertex in $S_A \cup S_B \cup Y$. Secondly, at most $s_B/2$ subdivided edges joining two branch vertices in S_B begin and end with an S_B–A_{i+1} edge. But all the $\binom{s_B}{2} - s_B/2$ remaining such subdivided edges have an inner vertex in A_i. (Note that this also shows that $s_A \geq (\log n)^2$ since otherwise $\binom{s_B}{2} - s_B/2 > n$. Similarly, we have that $s_B \geq (\log n)^2$.) Thirdly, at most s_B subdivided edges joining some branch vertex $a \in S_A$ to some branch vertex $b \in S_B$ with $ab \notin G_i$ end with an S_B–A_{i+1} edge. Again all the at least $s_A s_B - e_{G_i}(S_A, S_B) - s_B$ remaining such subdivided edges must have an inner vertex in A_i. Since $s_A, s_B \geq (\log n)^2$, together with (3) this implies that

$$n = |A_i| \geq x(s-x) + \binom{s_B}{2} - \frac{s_B}{2} + \left(\frac{1}{4} - \frac{\lambda}{40}\right) s_A s_B - s_B. \tag{4}$$

Similarly, we arrive at an analogous inequality where A and B are interchanged and x is replaced by y. Adding (4) and this second inequality gives

$$x(s-x) + y(s-y) + \binom{s_A}{2} + \binom{s_B}{2} + \frac{s_A s_B}{2} - \frac{3}{2}(s_A + s_B) - \frac{\lambda s_A s_B}{20} \leq 2n. \tag{5}$$

But $\binom{s_A}{2} + \binom{s_B}{2} + s_A s_B/2$ is minimized if $s_A = s_B$, i.e. if $s_A = s_B = (s-x-y)/2$. Thus (5) implies that

$$x(s-x) + y(s-y) + 2\binom{\frac{s-x-y}{2}}{2} + \frac{1}{2}\left(\frac{s-x-y}{2}\right)^2 - \frac{\lambda s^2}{16} \leq 2n.$$

This shows that

$$2xs + 2ys - 5(x^2 + y^2) + 6xy + \lambda s^2 \leq 0. \tag{6}$$

However, recall that $x, y \leq 0.17s$. It is easy to check that (6) has no solution for such numbers x and y. □

4 Proof of Theorem 1 – upper bound

Clearly, it suffices to prove the following stronger statement. It implies that in Theorem 1 the condition of being bipartite can be weakened to being H-free where H is any fixed 3-chromatic graph.

Theorem 19 *For every $0 < \lambda < 1$ and every 3-chromatic graph H there exists $\kappa_0 = \kappa_0(\lambda, H)$ such that for every natural number $\kappa \geq \kappa_0$ each κ-connected H-free graph G_0 contains a subdivision of a clique of order at least $8\sqrt{(1-\lambda)\kappa}/3$.*

For the proof of Theorem 19 we need the following consequence of Theorem 2.1 and Corollary 2.1 in Komlós and Szemerédi [12].

Theorem 20 *For all $\varepsilon^* > 0$ there are positive constants $c_0 = c_0(\varepsilon^*)$ and $d_0 = d_0(\varepsilon^*)$ such that every graph G^* of average degree at least $d^* \geq d_0$ either contains a subdivided clique of order at least $8\sqrt{d^*}/3$ or a subgraph G whose average degree d satisfies both $d \geq c_0|G|$ and $d \geq d^*/(1+\varepsilon^*)$.*

Very roughly, the strategy of the proof of Theorem 1 is as follows. By Theorem 20, we may assume that our given graph G_0 contains a dense subgraph G. We then apply the Regularity lemma to G to obtain a reduced graph R. If R contains a vertex X of rather large degree (Case 1), we choose the set of our branch vertices randomly inside X. In this case—similarly as in the proof of Komlós and Szemerédi [12]—most of the branch vertices can be joined by a path of length two whose midpoint lies in some cluster which is adjacent to X in R. The main difference is that here we need the more powerful Lemma 11 instead of Corollary 12 (which was sufficient in [12]). The left-over pairs of branch vertices are then joined by suitable paths of length four using special sets of vertices which we set aside earlier for this purpose.

If we cannot guarantee a vertex of large degree in R (Case 2) we proceed as follows. Let XY be an edge in R of maximum density. Proposition 6 implies that in Case 2 this density must be large. The branch vertices are now chosen within both X and Y. This has the advantage that many pairs of branch vertices can be connected directly by edges between them. The subdivided edges connecting two branch vertices in X (respectively two branch vertices in Y) are selected similarly as in Case 1. The main difficulty of the proof is that now we have to use the connectivity of G_0 in order to find subdivided edges joining every branch vertex $x \in X$ to all those branch vertices $y \in Y$ for which $xy \notin G_0$.

Proof of Theorem 19. Choose

$$\varepsilon^* \ll \lambda. \tag{7}$$

Let $c_0(\varepsilon^*)$ be as defined in Theorem 20 and choose constants

$$0 < \varepsilon \ll \beta \ll \alpha \ll \xi \ll \tau \ll d \ll \min\{c_0(\varepsilon^*), \varepsilon^*\}. \tag{8}$$

We will prove Theorem 19 for every κ_0 which is sufficiently large compared to each of $d_0(\varepsilon^*)$, $N(\varepsilon)$, $q_0(\varepsilon)$, $q_0((80N(\varepsilon))^{-1})$, $n_0(\varepsilon, d, H)$, $s_0(2\varepsilon, \beta, \alpha, d/2)$ and $s_0(2\varepsilon, \beta^2, \alpha, d/2)$, where d_0 is as defined in Theorem 20, $N(\varepsilon)$ is as defined in the Regularity lemma, q_0 is as defined in Proposition 13, n_0 is as defined in Lemma 8 and s_0 is as defined in Lemma 11. Clearly, we may assume that the graph G_0 given in Theorem 19 does not contain a subgraph of connectivity greater than κ. By Theorem 20, we may assume that for some $c \geq c_0(\varepsilon^*)$ the graph G_0 contains a subgraph G of average degree

$$cn \geq \kappa/(1+\varepsilon^*), \tag{9}$$

where $n := |G|$. Then

$$4\kappa \geq cn \tag{10}$$

since otherwise, by Theorem 15, G would contain a subgraph whose connectivity is greater than κ. Set

$$s := 8\sqrt{(1-\lambda)\kappa}/3.$$

Apply the Regularity lemma to G to obtain a spanning subgraph G' of G and a reduced graph R. Throughout the proof, unless stated otherwise, we say that two vertices $x, y \in V(G) = V(G')$ are neighbours if they are neighbours in G'. Let L denote the size of the clusters given by the Regularity lemma and set $k := |R|$. Let μ denote the maximum density of an edge in R. Thus $\mu \geq d$ and Proposition 6 shows that

$$\delta(R) \geq (c - 2d)k/\mu =: \delta. \tag{11}$$

We will first deal with the case when $\mu \leq 9/32$.

Case 1. $\mu \leq 9/32$.

Let $X \in V(R)$ be any cluster. Choose disjoint sets N_X^1 and N_X^2 of neighbours of X in R such that $|N_X^1| = \tau k$ and $|N_X^2| = \delta - 10\tau k$. Next choose a set $\widetilde{N}_X^1$ consisting of τk vertices of R such that R contains a perfect matching between N_X^1 and $\widetilde{N}_X^1$ and such that $\widetilde{N}_X^1$ is disjoint from each of $\{X\}$, N_X^1 and N_X^2. We will fix such a perfect matching between N_X^1 and $\widetilde{N}_X^1$. By Proposition 7, all but at most $3\varepsilon L$ vertices in X have at least $dL/2$ neighbours in at least $2/3$ of the clusters in N_X^1. Let $X' \subseteq X$ be the set of these vertices. Thus $|X'| \geq (1-3\varepsilon)L$. Together with Lemma 11 and Corollary 12 this implies that X' contains an s-element subset S which is $(\alpha s, \beta)$-dense for each cluster $W \in N_X^2$ and which in addition is $(\alpha s, \beta^2)$-attached to each cluster $W \in N_X^2$. (Indeed, since for each $W \in N_X^2$ the graph $(X', W)_{G'}$ is 2ε-regular of density at least $d/2$, Lemma 11 and Corollary 12 together imply that the probability that an s-element subset S of X' chosen uniformly at random without repetitions fails to satisfy the above conditions is at most $2|N_X^2|\mathrm{e}^{-s} < 1$.) S will be the set of branch vertices of our subdivided clique. Let Z be the set of all those vertices of G that lie in some cluster belonging to N_X^2.

To find the subdivided edges of our clique, for every pair of vertices $x, y \in S$ in turn, we select a vertex $z \in Z$ which is adjacent to both x and y and was not already chosen to connect a previous pair (provided that such a vertex exists). We call a vertex $x \in S$ *bad* if, after we have considered all pairs of vertices in S in this way, there are still at least αs vertices in S which are not yet joined to x. The following claim implies that we were able to join most of the pairs of branch vertices in the above way.

Claim A. *At most αs vertices in S are bad.*

Suppose not and let S' be an αs-element subset of S consisting of bad vertices. Let $Z' \subseteq Z$ be the set of all those vertices in Z which have not been selected to join some pair of vertices in S. Then, since $\mu \le 9/32$,

$$\begin{aligned}
|Z'| > |Z| - \binom{s}{2} &\ge |N_X^2|L - \frac{s^2}{2} \\
&\ge \frac{(c-2d)kL}{\mu} - 10\tau kL - \frac{32(1-\lambda)\kappa}{9} \\
&\overset{(9)}{\ge} \frac{32(c-2d)(1-\varepsilon)n}{9} - 10\tau n - \frac{32(1-\lambda)(1+\varepsilon^*)cn}{9} \\
&\overset{(7),(8)}{\ge} 2\beta n \ge 2\beta|Z|.
\end{aligned}$$

But since S was $(\alpha s, \beta)$-dense for each cluster belonging to N_X^2, it follows that at least half of the vertices in Z' have at least $\beta S'$ neighbours in the bad set S'. Thus there is a vertex $x \in S'$ with at least $\beta|Z'|/2 > \beta^2|Z|$ neighbours in Z'. Hence there exists a cluster $W \in N_X^2$ such that x has more than $\beta^2 L$ neighbours in $W \cap Z'$. Since S was $(\alpha s, \beta^2)$-attached to W, there must be an edge joining some neighbour z of x in $W \cap Z'$ to one of the at least αs vertices in S which are not yet joined to x, y say. But this means that when we considered the pair x, y we could have selected z in order to join them, a contradiction. This proves the claim.

Now we have to show that we can find a subdivided edge for each of the at most $2\alpha s^2$ left-over pairs of vertices in S. We will join up each such left-over pair greedily by a path of length 4. This 4-path will have its midpoint in some cluster $V \in \widetilde{N}_X^1$ and its other two inner vertices in the unique cluster in N_X^1 that is matched to V. (Recall that we have fixed a perfect matching between N_X^1 and $\widetilde{N}_X^1$.) We have to show that for all the left-over pairs in turn we can find (greedily) internally disjoint such paths. Suppose that we are about to join the left-over pair $x, y \in S$. Recall that, since $S \subseteq X'$, both x and y have at least $dL/2$ neighbours in at least $2/3$ of the clusters in N_X^1. Thus they have at least $1/3$ of these clusters in common. However, we may have used up some of the neighbours of x and y before to join up previous left-over pairs. But since the number of paths constructed previously is at most $2\alpha s^2$, we have used at most $6\alpha s^2 \le 48\alpha n$ vertices for this. Thus at most

$$48\alpha n \cdot \frac{4}{dL} \overset{(8)}{<} \frac{\tau k}{3} = \frac{|N_X^1|}{3}$$

clusters in N_X^1 contain at least $dL/4$ vertices which we have already used before. This shows that there is a cluster $U \in N_X^1$ in which both x and y still have at least $dL/4$ unused neighbours. Let $V \in \widetilde{N}_X^1$ be the cluster that is matched to U. Since by construction the number of used vertices in U is exactly twice the number of used vertices in V, there must be at least $dL/2$ vertices in V which we have not used already. Together with the ε-regularity of $(U, V)_{G'}$ this implies that V contains a vertex z which is joined to both some neighbour z_1 of x in U and some neighbour z_2 of y in U such that $z_1 \neq z_2$ and such that none of z_1, z_2, z_3 has been used to join previous left-over pairs. Thus xz_1zz_2y is a 4-path as required.

Case 2. $\mu > 9/32$.

The proof of this case is an extension of that of Case 1. Let XY be an edge in R of density μ. Since μ is large, the lower bound (11) on $\delta(R)$ is now weaker and so we cannot choose all the branch vertices in a single cluster, X say, as we did in Case 1. Indeed, the number of vertices lying in a neighbouring cluster of X could be smaller than $\binom{s}{2}$. So if we put all the branch vertices into X, there may not be enough room for all the subdivided edges of our topological clique. Therefore we split our branch vertices into two sets $S_X \subseteq X$ and $S_Y \subseteq Y$ such that the density of the bipartite subgraph between S_X and S_Y is about μ. Since μ is quite large, this has the advantage that we can join many pairs of branch vertices directly by these S_X–S_Y edges and so we need less vertices in the other neighbouring clusters of X (respectively of Y) for the remaining subdivided edges. However, we now face the additional difficulty that we also have to join each vertex in S_X to all those vertices in S_Y for which there is no S_X–S_Y edge. This is the point where we use the connectivity of the graph $G_0 \supseteq G$ we started with. (Note that in Case 1 we did not make any use of it.)

Select τk-element sets N_X^1 and $\widetilde{N}_X^1$ and a $(\delta - 10\tau k)$-element set N_X^2 similarly as in Case 1. But now we additionally require that Y does not belong to any of these sets. Next choose analogous sets N_Y^1, $\widetilde{N}_Y^1$ and N_Y^2. Since $G_0 \supseteq G$ is H-free, Lemma 8 implies that the neighbourhoods of both X and Y are disjoint. Thus, we can choose all the sets $N_X^1, \widetilde{N}_X^1, N_X^2, N_Y^1, \widetilde{N}_Y^1, N_Y^2$ to be pairwise disjoint. (This is the only time we need the fact that G_0 is H-free.) The sets N_X^1 and $\widetilde{N}_X^1$ have the same purpose as in Case 1, namely to connect those left-over pairs $x, x' \in S_X$ of branch vertices by paths of length 4 which we were not able to link by paths of length 2. Each other path linking a pair of branch vertices will be routed through N_X^2 and/or N_Y^2 (see Fig. 1).

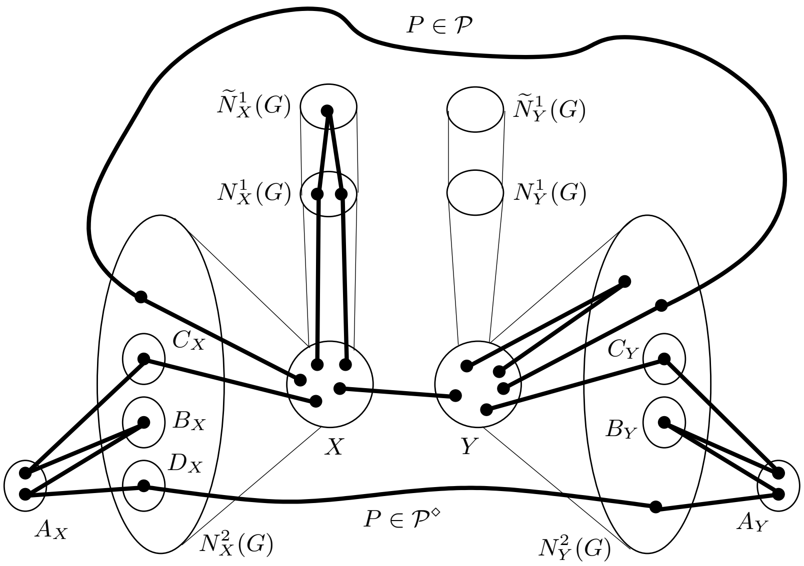

Figure 1: Five possible ways of connecting two branch vertices. The sets $N_X^1(G)$ etc. denote the subsets of $V(G)$ which correspond to the sets N_X^1 etc.

Let $1/2 \le \gamma \le 9/10$ be any number which satisfies the following two inequalities:

$$(1-\mu+10^6\varepsilon)\gamma s(1-\gamma)s \le (1-2\varepsilon^*)\kappa, \tag{12}$$

$$\binom{\gamma s}{2} + (1-\mu+10^6\varepsilon)\gamma s(1-\gamma)s + 10\alpha s^2 \le |N_X^2|L - \tau n. \tag{13}$$

We defer the technical proof of the existence of such a γ until later (Proposition 21). Inequality (12) will imply that the connectivity of G_0 is large enough to guarantee at least as many paths between the neighbourhood of S_X (inside clusters belonging to N_X^2) and the neighbourhood of S_Y as we will need to join all those pairs $x \in S_X$, $y \in S_Y$ of branch vertices for which $xy \notin G_0$. Inequality (13) will show that the neighbourhood of S_X is large enough to accommodate both an endvertex of each such path as well as a midpoint of each subdivided edge joining two branch vertices in S_X. (Similarly as in Case 1, we will join almost all pairs of branch vertices in S_X by paths of length 2.)

Set

$$s_X := \gamma s \quad \text{and} \quad s_Y := (1-\gamma)s.$$

We will now choose the set $S_X \cup S_Y =: S$ of branch vertices for our subdivided clique where $S_X \subseteq X$, $S_Y \subseteq Y$, $|S_X| = s_X$ and $|S_Y| = s_Y$. Note that, by Propositions 4 and 7, all but at most $(10^5+4)\varepsilon L$ vertices $x \in X$ satisfy the following three properties:

(i) The proportion of clusters $U \in N_X^2$ for which x has at most $d|U|/2$ neighbours in U is at most 10^{-5}.

(ii) The proportion of the clusters $U \in N_X^1$ for which x has at most $d|U|/2$ neighbours in U is at most $1/3$.

(iii) The neighbourhood of x in Y has size at least $(\mu-\varepsilon)L$.

Let X' be the set of all those at least $(1-(10^5+4)\varepsilon)L$ vertices in X. Define $Y' \subseteq Y$ similarly. Just as in Case 1, one can apply Lemma 11 and Corollary 12 to obtain an s_X-element set $S_X \subseteq X'$ which is $(\alpha s_X, \beta)$-dense for each cluster $U \in N_X^2$ and which in addition is $(\alpha s_X, \beta^2)$-attached to each cluster $U \in N_X^2$. Similarly, using Lemma 11, Corollary 12 and Proposition 13, it is easy to see that there exists an s_Y-element subset $S_Y \subseteq Y'$ which is $(\alpha s_Y, \beta)$-dense for each cluster $V \in N_Y^2$, which in addition is $(\alpha s_Y, \beta^2)$-attached to each cluster $V \in N_Y^2$ and for which the bipartite graph $(S_X, S_Y)_{G'}$ has density at least $\mu - 10^6\varepsilon$. Indeed, to ensure that the latter property is also satisfied, let $\mathcal{A} := \{N_{G'}(x) \cap Y' \,|\, x \in S_X\}$. Since (iii) implies that $|A| \ge (\mu-\varepsilon)L - |Y \setminus Y'| \ge (\mu - (10^5+5)\varepsilon)|Y'|$ for all $A \in \mathcal{A}$, Proposition 13 (with $T := Y'$ and $Q := S_Y$) tells us that the probability that there exists a vertex $x \in S_X$ which has less than $(\mu - 10^6\varepsilon)s_Y$ neighbours in S_Y is at most $1/2$. This completes the choice of the branch vertices.

As indicated earlier, we will use the connectivity of G_0 to find a set $\mathcal{P}$ of almost κ disjoint paths whose first vertex lies in a cluster belonging to N_X^2 and whose last vertex lies in a cluster belonging to N_Y^2. Most of those pairs x, y of branch vertices for which $x \in S_X$, $y \in S_Y$ and $xy \notin G_0$ will be joined by a path of the form xPy where $P \in \mathcal{P}$. However, for some such pairs x, y this will not be possible. Each of those left-over pairs x, y will be joined by an extended path of the form $xu_1 \dots u_4 P v_4 \dots v_1 y$ where $P \in \mathcal{P}$. All these extension vertices $u_1, \dots, u_4$ and $v_1, \dots, v_4$ will lie in a relatively small set I' which we set aside (before determining $\mathcal{P}$) for this purpose and which will be avoided by the paths in $\mathcal{P}$. I' will be the union of six disjoint sets A_X, B_X, C_X, A_Y, B_Y and C_Y. All vertices of the form u_1 will lie in C_X, all vertices of the form u_2 and u_4 will lie in A_X and all vertices of the form u_3 will lie in B_X. The vertices of the form v_i will satisfy analogous properties for the sets A_Y, B_Y and C_Y (see Fig. 1).

Let us first choose the set A_X. For each cluster $U \in N_X^2$ we select a neighbour $W(U)$ in R such that all these $W(U)$ are distinct for different clusters U and such that none of them lies in

$$N_X^1 \cup \widetilde{N}_X^1 \cup N_Y^1 \cup \widetilde{N}_Y^1 \cup \{X, Y\} =: J. \tag{14}$$

Let U' be the set of all those vertices in U which have at least $dL/2$ neighbours in $W(U)$. Thus, by Proposition 4, $|U'| \geq (1-\varepsilon)L$. Apply Proposition 13 (with $T := W(U)$, $q := \tau L$ and $\mathcal{A} := \{N_{G'}(x) \cap W(U) \,|\, x \in U'\}$) to obtain a τL-element subset $A_X(U)$ of $W(U)$ such that every vertex in U' has at least $d|A_X(U)|/4$ neighbours in $A_X(U)$. Let $A_X := \bigcup_{U \in N_X^2} A_X(U)$. For all $U \in N_X^2$ choose any τL-element subset $B_X(U)$ of U'. Let $B_X := \bigcup_{U \in N_X^2} B_X(U)$. Thus A_X and B_X are disjoint. (This follows from the fact that G_0 is H-free and thus R is triangle-free, but here this fact is not necessary since we could simply choose each $B_X(U)$ in $U' \setminus A_X$.) Similarly, for each cluster $V \in N_Y^2$ we choose a neighbour $W(V)$ and define V' as well as τL-element sets $A_Y(V) \subseteq W(V)$ and $B_Y(V) \subseteq V'$ such that all the sets $A_Y(V)$ and $B_Y(V)$ are disjoint from $A_X \cup B_X$. Set $A_Y := \bigcup_{V \in N_Y^2} A_Y(V)$, $B_Y := \bigcup_{V \in N_Y^2} B_Y(V)$ and let

$$I := A_X \cup B_X \cup A_Y \cup B_Y.$$

Note that I meets each cluster in at most $4\tau L$ vertices. For every cluster $U \in N_X^2$, choose a ξL-element set $C_X(U) \subseteq U' \setminus I \subseteq U$ which contains at least $d|C_X(U)|/4$ neighbours of each vertex $x \in S_X$ that has at least $dL/2$ neighbours in U. (Indeed, to see that such a set $C_X(U)$ exists, observe that each vertex x with at least $dL/2$ neighbours in U has at least $dL/3$ neighbours in $U' \setminus I$ and apply Proposition 13 with $T := U' \setminus I$, $q = \xi L$ and $\mathcal{A} := \{N_{G'}(x) \cap (U' \setminus I) \,|\, x \in S_X\}$.) Thus condition (i) and the fact that $S_X \subseteq X'$ imply the following.

(iv) For each vertex $x \in S_X$ there are at least $(1 - 10^{-5})|N_X^2|$ clusters $U \in N_X^2$ such that x has at least $d|C_X(U)|/4$ neighbours in the set $C_X(U)$.

Set $C_X := \bigcup_{U \in N_X^2} C_X(U)$. For all $V \in N_Y^2$ define $C_Y(V) \subseteq V' \setminus I$ similarly and set $C_Y := \bigcup_{V \in N_Y^2} C_Y(V)$. Put

$$I' := I \cup C_X \cup C_Y$$

and

$$\kappa' := (1 - \varepsilon^*)\kappa.$$

Note that $\kappa' \le \kappa - 20\tau n$ by (8) and (10). Moreover,

$$\begin{aligned}\kappa' &\le \frac{(1-\varepsilon^*)\kappa}{\mu} \overset{(9)}{\le} \frac{(1-(\varepsilon^*)^2)cn}{\mu} \le \frac{ckL + \varepsilon n - (\varepsilon^*)^2 cn}{\mu} \\ &\overset{(8)}{\le} \frac{(c-2d-30\tau)kL}{\mu} \overset{(11)}{\le} (\delta - 10\tau k)L - 20\tau kL = |N_X^2|L - 20\tau kL. \qquad (15)\end{aligned}$$

Let $J(G)$ be the set of all those vertices in G which lie in a cluster belonging to J (which was defined in (14)). Since

$$\left| J(G) \cup I' \cup (\bigcup_{U \in N_X^2} U \setminus U') \cup (\bigcup_{V \in N_X^2} V \setminus V') \right| \le 20\tau kL$$

and G_0 is κ-connected, Menger's theorem implies that we can choose a set $\mathcal{P}$ of κ' disjoint paths in the graph $G_0 \setminus (J(G) \cup I')$ such that each of these paths joins a vertex in $\bigcup_{U \in N_X^2} U'$ to a vertex in $\bigcup_{V \in N_Y^2} V'$ but has no other vertex in a cluster belonging to $N_X^2 \cup N_Y^2$.

Next we will choose a small set $\mathcal{P}^\diamond \subseteq \mathcal{P}$ which will be set aside to connect pairs $x \in S_X$, $y \in S_Y$ of branch vertices (with $xy \notin G_0$) for which we fail to find a path xPy with $P \in \mathcal{P}$. Each such pair x, y will be connected by a path of the form $xu_1 \dots u_4 P v_4 \dots v_1 y$ with $P \in \mathcal{P}^\diamond$. For all pairs of clusters $U \in N_X^2$, $V \in N_Y^2$, the paths in $\mathcal{P}^\diamond$ will have the property that a significant proportion of paths in $\mathcal{P}$ joins U to V whenever a significant proportion of paths in $\mathcal{P}^\diamond$ joins U to V (see (17)). Roughly speaking, this property will enable us to deduce that every reasonably large set $\mathcal{P}_x \subseteq \mathcal{P}^\diamond$ of paths will have the property that the endvertices of these paths are distributed over a large number of clusters in N_Y^2. This in turn will enable us to find the path $v_1 \dots v_4$ joining y to some $P \in \mathcal{P}_x$. (The paths $\mathcal{P}_x$ will be defined in such a way that we can join their endvertices in the clusters belonging to N_X^2 to x via a suitable path $u_1 \dots u_4$.) For each cluster $U \in N_X^2$, let $\mathcal{P}(U)$ denote the set of all the paths in $\mathcal{P}$ that start in U (and thus in $U' \setminus I'$). Put

$$\eta := \frac{1}{80N(\varepsilon)}.$$

Let $\mathcal{P}'(U)$ denote the set of all those paths in $\mathcal{P}(U)$ which end in a cluster $V \in N_Y^2$ that meets (and thus contains the endvertices of) at least ηL paths in $\mathcal{P}(U)$. Note that

$$|\mathcal{P}(U) \setminus \mathcal{P}'(U)| \le \eta L |N_Y^2| \le \eta L k \le L/80. \qquad (16)$$

It is easy to see that for all $U \in N_X^2$ with $|\mathcal{P}(U)| \ge L/40$ we can choose a set $\mathcal{P}^\diamond(U)$ consisting of $\xi|\mathcal{P}'(U)|$ paths in $\mathcal{P}'(U)$ such that each cluster $V \in N_Y^2$ satisfies

$$\frac{\text{no. of paths in } \mathcal{P}^\diamond(U) \text{ ending in } V}{\text{no. of paths in } \mathcal{P}'(U) \text{ ending in } V} \le 2\xi. \qquad (17)$$

(Indeed, for all $V \in N_Y^2$ choose approximately a ξ-proportion of the set of those paths in $\mathcal{P}'(U)$ which end in V. As ηL is a large number, this ξ-proportion is still a large number and so the rounding effects are sufficiently small.) If $|\mathcal{P}(U)| < L/40$, set $\mathcal{P}^\diamond(U) := \emptyset$. Let $D_X(U) \subseteq U$ be the subset of all endvertices of paths in $\mathcal{P}^\diamond(U)$. Set $D_X := \bigcup_{U \in N_X^2} D_X(U)$, $\mathcal{P}^\diamond := \bigcup_{U\in N_X^2} \mathcal{P}^\diamond(U)$ and $\mathcal{P}^* := \mathcal{P} \setminus \mathcal{P}^\diamond$. Thus

$$|\mathcal{P}^*| \geq \kappa' - \xi n \overset{(8),(10)}{\geq} (1-2\varepsilon^*)\kappa + \tau n. \tag{18}$$

Since $\mu > 9/32$ we have

$$|\mathcal{P}| = (1-\varepsilon^*)\kappa \overset{(10)}{\geq} \frac{ckL}{5} \overset{(11)}{\geq} \frac{\delta L}{20} \geq \frac{|N_X^2|L}{20}.$$

So on average at least 1/20 of the vertices in a cluster $U \in N_X^2$ are endvertices of paths in $\mathcal{P}$. Hence the proportion of clusters $U \in N_X^2$ which satisfy $|\mathcal{P}(U)| \geq L/40$ is at least 1/40, i.e. for at least $|N_X^2|/40$ clusters $U \in N_X^2$ the set $\mathcal{P}^\diamond(U)$ is non-empty and thus has size $\xi|\mathcal{P}'(U)|$. Together with (iv) this implies the following.

(v) For each vertex $x \in S_X$ there is a set $\mathcal{U}_x$ of at least $|N_X^2|/50$ clusters $U \in N_X^2$ such that for each $U \in \mathcal{U}_x$ the vertex x has at least $d|C_X(U)|/4$ neighbours in the set $C_X(U)$ and $|D_X(U)| = |\mathcal{P}^\diamond(U)| = \xi|\mathcal{P}'(U)| \geq \xi L/80$.

(The last inequality follows from (16).)

We will now choose the subdivided edges for all pairs x, y of branch vertices of the form $x \in S_X$, $y \in S_Y$. Clearly, we only have to consider pairs for which $xy \notin G_0$. For each such pair x, y in turn we first try to select a path $P \in \mathcal{P}^*$ whose first vertex is adjacent to x, whose last vertex is adjacent to y and such that P was not selected for a previous pair of branch vertices (if such a path P exists). We call a vertex $x \in S_X$ *useless* if after we have considered all such pairs of branch vertices there are still at least αs_Y vertices in S_Y which are not yet joined to x (neither by an edge $xy \in G_0$ nor by a path of the form xPy where $P \in \mathcal{P}^*$). The following claim implies that we were able to join most of these pairs of branch vertices in this way.

Claim B. *At most αs_X vertices in S_X are useless.*

Suppose not and let S'_X be an αs_X-element subset of S_X consisting of useless vertices. Let $\mathcal{P}'$ be the set of all those paths in $\mathcal{P}^*$ which we have not used to connect pairs x, y of branch vertices. Let Z' be the set of all those endvertices of paths in $\mathcal{P}'$ that lie in some cluster belonging to N_X^2. Recall that $d(S_X, S_Y)_{G'} \geq \mu - 10^6\varepsilon$. Together with inequalities (8), (12) and (18) this implies that $|Z'| = |\mathcal{P}'| \geq \tau n > 2\beta n$. But since S_X was $(\alpha s_X, \beta)$-dense for each cluster belonging to N_X^2, it follows that more than half of the vertices in Z' have at least $\beta|S'_X|$ neighbours in S'_X. Thus there is a vertex $x \in S'_X$ with more than $\beta|Z'|/2 > \beta^2 n$ neighbours in Z'. Let $\mathcal{P}''$ be the set of all those paths in $\mathcal{P}'$ that start in a neighbour of x in Z'. Thus $|\mathcal{P}''| > \beta^2 n$. Hence there must be a cluster $V \in N_Y^2$ which contains endvertices of more than $\beta^2 L$ paths in $\mathcal{P}''$. But since S_Y was $(\alpha s_Y, \beta^2)$-attached to each cluster belonging to N_Y^2 and thus also to V, there must be a path $P \in \mathcal{P}''$ whose endvertex in V is adjacent to one of the at least

αs_Y vertices in S_Y that are not yet joined to x. Let $y \in S_Y$ be such a vertex. Then when considering the pair x, y we could have chosen P in order to connect it, a contradiction. This proves the claim.

Thus we are left with at most $2\alpha s_X s_Y \leq \alpha s^2$ pairs $x \in S_X$, $y \in S_Y$ of branch vertices for which we have not yet found a subdivided edge. As indicated before, for each such left-over pair x, y in turn, we will now select a subdivided edge P_{xy} which is of the form $xu_1 \dots u_4 P v_4 \dots v_1 y$ where P is some path in $\mathcal{P}^\diamond$. If U denotes the cluster in N_X^2 which contains an endvertex of P, then u_1 will be a neighbour of x in $C_X(U)$, both u_2 and u_4 will lie in $A_X(U)$ and u_3 will lie in $B_X(U)$. The path $v_1 \dots v_4$ will satisfy analogous properties.

We have to prove that for each left-over pair x, y in turn we can find such a path P_{xy} so that all these paths are internally disjoint. So suppose that we are about to consider the left-over pair x, y. Note that

$$\sum_{U \in \mathcal{U}_x} |C_X(U)| = \xi |\mathcal{U}_x| L \overset{\text{(v)}}{\geq} \frac{\xi |N_X^2| L}{50} \overset{(8)}{>} 2 \cdot 10^4 \alpha s^2,$$

where $\mathcal{U}_x$ was defined in (v). Thus, for at least half of the clusters $U \in \mathcal{U}_x$ at most $|C_X(U)|/10^4$ of the vertices in $C_X(U)$ have been used to join up previous (left-over) pairs. (Recall that each $C_X(U)$ is disjoint from all the paths in $\mathcal{P} \supseteq \mathcal{P}^\diamond$.) Let $\mathcal{U}'_x$ denote the set of all these clusters. So $|\mathcal{U}'_x| \geq |N_X^2|/100$. Consider the set $\mathcal{P}_x$ of all those paths in $\mathcal{P}^\diamond$ which we have not used for previous left-over pairs and whose first point lies in some set $D_X(U)$ with $U \in \mathcal{U}'_x$, i.e. $\mathcal{P}_x$ is obtained from $\bigcup_{U \in \mathcal{U}'_x} \mathcal{P}^\diamond(U)$ by deleting all the paths which we used before. Note that for each $U \in \mathcal{U}'_x$ the number of vertices in $D_X(U)$ which we used to join previous left-over pairs is precisely the number of vertices in $C_X(U)$ which we used to join previous left-over pairs. Thus for each $U \in \mathcal{U}'_x$ this number is at most $|C_X(U)|/10^4$. Hence

$$|\mathcal{P}_x| \geq \sum_{U \in \mathcal{U}'_x} (|\mathcal{P}^\diamond(U)| - |C_X(U)|/10^4) \overset{\text{(v)}}{\geq} |\mathcal{U}'_x| \left(\frac{\xi L}{80} - \frac{\xi L}{10^4} \right) \geq \frac{|N_X^2| \xi L}{10^4}. \quad (19)$$

We will now show that there exists a cluster $V \in N_Y^2$ which contains an endvertex of some path in $\mathcal{P}_x$ and for which the set $C_Y(V) \subseteq V$ contains a neighbour of y which is still unused. This neighbour will play the role of v_1. Let $\mathcal{V}_x$ denote the subset of all those clusters in N_Y^2 that contain an endvertex of some path in $\mathcal{P}_x$. Then (17) and (19) together imply that our original set of paths $\mathcal{P}$ contains at least $|\mathcal{P}_x|/2\xi \geq |N_X^2|L/(2 \cdot 10^4)$ paths which end in a cluster belonging to $\mathcal{V}_x$ (and start in a cluster belonging to $\mathcal{U}'_x$). Thus $|\mathcal{V}_x| \geq |N_X^2|/(2 \cdot 10^4) = |N_Y^2|/(2 \cdot 10^4)$. But now the analogue of condition (iv) for vertices in S_Y shows that at least $|N_Y^2|(1/(2 \cdot 10^4) - 1/10^5) \geq |N_Y^2|/10^5$ clusters $V \in N_Y^2$ contain an endvertex of some path in $\mathcal{P}_x$ and are such that y has at least $d|C_Y(V)|/4 = \xi dL/4$ neighbours in $C_Y(V)$. But since

$$\frac{|N_Y^2|}{10^5} \cdot \frac{\xi dL}{4} \overset{(8),(11)}{\geq} \frac{\xi cdkL}{10^6 \mu} \geq \frac{\xi dcn}{10^7} \overset{(8)}{>} \alpha s^2,$$

there must be one such cluster V for which at least one of the neighbours of y in $C_Y(V)$ has not been used to connect previous left-over pairs. Let v_1 be such an unused neighbour, let P be any path in $\mathcal{P}_x$ that ends in V and let $v_5 \in V$ denote the endvertex of P. It remains to connect v_1 to v_5 via $A_Y(V)$ and $B_Y(V)$. Note that at most $2|C_Y(V)| = 2\xi L \le \tau dL/8 = d|A_Y(V)|/8$ vertices in $A_Y(V)$ have been used for previous left-over pairs. Thus, since $v_1, v_5 \in V'$ and hence they have at least $d|A_Y(V)|/4$ neighbours in $A_Y(V)$, both v_1 and v_5 have at least $d|A_Y(V)|/8 > \varepsilon L$ unused neighbours in $A_Y(V)$. Since also a large proportion of the vertices in $B_Y(V)$ is still unused, we can use the fact that the graph $(V, W(V))_{G'} \supseteq (B_Y(V), A_Y(V))_{G'}$ is ε-regular of density at least d to find a neighbour v_2 of v_1 in $A_Y(V)$, a neighbour v_4 of v_5 in $A_Y(V)$ and a vertex $v_3 \in B_Y(V)$ adjacent to both v_2 and v_4 such that all these 3 vertices are still unused. Thus we have found a path $yv_1 \dots v_5$ connecting y to the endvertex of P in V. Similarly we can find a path $xu_1 \dots u_5$ connecting x to the other endvertex u_5 of P. This shows that we may join all the left-over pairs x, y of branch vertices by a path of the form $u_1 \dots u_4 P v_4 \dots v_1$.

Having joined all the pairs x, y of branch vertices with $x \in S_X$ and $y \in S_Y$ we now have to join the all the branch vertices in S_X to each other and also all the branch vertices in S_Y. We can do this in a similar way as in Case 1. Indeed, inequality (13) shows that the clusters in N_X^2 contain at least $\binom{s_X}{2} + \tau n$ vertices which we have not used before to connect a pair x, y of branch vertices with $x \in S_X$ and $y \in S_Y$. Thus exactly as in Case 1 one can show that all but at most $2\alpha s_X^2$ pairs $x_1, x_2 \in S_X$ can be joined by a path of length two whose midpoint lies in a cluster in N_X^2. Again, to join the remaining pairs we use the clusters in N_X^1 and in $\widetilde{N}_X^1$. The pairs $y_1, y_2 \in S_Y$ are then dealt with in a similar way. This is the only point where we need the disjointness of N_X^2 and N_Y^2—it implies that there are sufficiently many unused vertices in N_Y^2 to make this final step work. □

Proposition 21 *For all $9/32 \le \mu \le 1$, there exists γ with $1/2 \le \gamma \le 9/10$ which satisfies inequalities* (12) *and* (13)*.*

As one might expect, the only case for which (12) and (13) are sharp (if we ignore the error terms) is when the maximum density μ of the edges in the reduced graph is 3/4. This would be the case for the random graph considered in the proof of Proposition 16.

Proof of Proposition 21. Note that (15) implies that

$$|N_X^2|L - \tau n \ge \frac{\kappa}{\mu}(1 - \varepsilon^*). \tag{20}$$

We will now distinguish two cases.

Case 1. $\mu \ge 7/16$.

In this case we simply set $\gamma := 1/2$. Then (12) holds since

$$\begin{aligned}(1 - \mu + 10^6\varepsilon)\gamma s(1-\gamma)s &\le \left(\frac{9}{16} + 10^6\varepsilon\right)(1-\lambda)\frac{16\kappa}{9} \\ &\le \kappa + \frac{10^6 \cdot 16\varepsilon\kappa}{9} - \lambda\kappa \overset{(7),(8)}{\le} (1 - 2\varepsilon^*)\kappa.\end{aligned}$$

Let us now show that (13) holds as well. If we multiply the left hand side of (13) with μ we obtain

$$\frac{s(s-2)\mu}{8}+\mu(1-\mu+10^6\varepsilon)\frac{s^2}{4}+10\mu\alpha s^2 \leq \frac{8}{9}(1-\lambda)\kappa(\mu(3+2\cdot 10^6\varepsilon+80\alpha)-2\mu^2).$$

Together with (20) this implies that in order to show that (13) holds, it suffices to check that $\mu a-2\mu^2\leq b$ where $a:=3+2\cdot 10^6\varepsilon+80\alpha$ and $b:=\frac{9(1-\varepsilon^*)}{8(1-\lambda)}$. But $\mu a-2\mu^2$ is maximized if $\mu=a/4$ and thus $\mu a-2\mu^2\leq b$ always holds since (7) and (8) imply that $a^2/8<b$.

Case 2. $9/32<\mu<7/16$.

In this case we put

$$\gamma:=\frac{3}{\sqrt{32}}\sqrt{\frac{1}{\mu}-1}.$$

Since $9/32<\mu<7/16$ it follows that $6/10<\gamma<9/10$. We will first prove that γ satisfies the following 'pure versions' of inequalities (12) and (13):

$$(1-\mu)(\gamma-\gamma^2)\frac{64\kappa}{9}\leq\kappa, \tag{21}$$

$$\frac{32\gamma^2\kappa}{9}+\kappa\leq\frac{\kappa}{\mu}. \tag{22}$$

Note that (22) is equivalent to

$$\mu\leq\frac{1}{\frac{32}{9}\gamma^2+1}. \tag{23}$$

But our definition of γ implies that (23) holds with equality. Therefore, to show that γ also satisfies (21), we may substitute (23) as an equality in (21) and thus it suffices to check that

$$\frac{\frac{32}{9}\gamma^2}{\frac{32}{9}\gamma^2+1}(\gamma-\gamma^2)\frac{64}{9}\leq 1,$$

i.e.

$$f(\gamma):=\gamma^4-\gamma^3+\frac{9}{64}\gamma^2+\frac{3^4}{2^{11}}\geq 0.$$

To check this, we consider the roots of the derivative of $f(\gamma)$. But the only root of $f'(\gamma)=4\gamma^3-3\gamma^2+9\gamma/32$ between $1/2$ and 1 is $3/8+\sqrt{9/128}=:\gamma_0$. Since $f(\gamma_0)>0$, $f(1/2)>0$ and $f(1)>0$, this shows that our γ satisfies (21).

It remains to show that γ also satisfies (12) and (13). But if we add $2\varepsilon^*\kappa$ to the left hand side of (12) we obtain

$$\begin{aligned}&(1-\mu+10^6\varepsilon)(\gamma-\gamma^2)(1-\lambda)\frac{64\kappa}{9}+2\varepsilon^*\kappa\\&\leq(1-\mu)(\gamma-\gamma^2)\frac{64\kappa}{9}-\lambda(1-\mu)(\gamma-\gamma^2)\frac{64\kappa}{9}+10^6\varepsilon(\gamma-\gamma^2)\frac{64\kappa}{9}+2\varepsilon^*\kappa.\end{aligned}$$

Since γ satisfies (21), the first summand is at most κ. Moreover, $1-\mu\geq 9/16$ and $\gamma-\gamma^2\geq 0.9-0.9^2$. Together with (7) and (8) this shows that the remaining sum is less than 0. Thus (12) holds. This implies that the left hand side of (13) is at most

$$(1-\lambda)\frac{32\gamma^2\kappa}{9}+\kappa-2\varepsilon^*\kappa+\frac{640\alpha\kappa}{9}\overset{(8),(22)}{\leq}\frac{\kappa}{\mu}\left(1-\frac{32\lambda\gamma^2\mu}{9}\right)$$
$$\overset{(7)}{\leq}\frac{\kappa}{\mu}(1-\varepsilon^*)\overset{(20)}{\leq}|N_X^2|L-\tau n,$$

as desired. □

As the proof of Theorem 2 is similar to that of Theorem 19, we only sketch the argument.

Proof of Theorem 2 (Sketch). By Proposition 16, it suffices to prove the upper bound. Thus, given $0<\lambda<1$, we have to show that there exists $\kappa_0=\kappa_0(\lambda)$ such that for every natural number $\kappa\geq\kappa_0$ each κ-connected graph G_0 contains a subdivision of a clique of order at least $2\sqrt{(1-\lambda)\kappa}=:s$. We start exactly as in the proof of Theorem 19. Since we are now only seeking a subdivision of a smaller clique, the calculation in Claim A shows that we can proceed as in Case 1 as long as $\mu\leq 1/2$. Thus we may assume that $\mu>1/2$. If the common neighbourhood of X and Y in R has size at most τk, we can discard it and proceed precisely as in the proof of Theorem 19 (Case 2). Otherwise we choose a τk-element set N^1_{XY} of common neighbours of X and Y and a τk-element set $\widetilde{N}^1_{XY}$ such that these sets are disjoint from each other and from $N^1_X,\widetilde{N}^1_X,N^2_X,N^1_Y,\widetilde{N}^1_Y,N^2_Y$ and such that R contains a perfect matching between N^1_{XY} and $\widetilde{N}^1_{XY}$. We set $\gamma:=1/2$ and choose the set $S_X\cup S_Y$ of branch vertices as in Case 2 of the proof of Theorem 19. (Note that when $\mu\geq 1/2$ the proof of Proposition 21 immediately shows that $\gamma=1/2$ also works in the proof of Theorem 19.) The argument implies that we may additionally assume that each branch vertex has at least $d|U|/2$ neighbours in at least $2/3$ of the clusters $U\in N^1_{XY}$. Moreover, we may clearly assume that $T:=N^2_X\cap N^2_Y$ is non-empty.

Next suppose that $|T|L\leq(1-\mu+10^6\varepsilon)s^2/4$. Thus the number of vertices lying in a cluster belonging to T is not larger than the required number of subdivided edges joining branch vertices in S_X to branch vertices in S_Y. We now join almost $|T|L$ pairs $x\in S_X$, $y\in S_Y$ of branch vertices (with $xy\notin G_0$) by a path of length two whose midpoint lies in a cluster belonging to T. (The existence of these paths follows similarly as in the proof of Claim A.) The set $\mathcal{P}$ of paths will now have size only $(1-\varepsilon^*)\kappa-|T|L$ and the paths in $\mathcal{P}$ will avoid all vertices lying in clusters belonging to T. As before (see Claim B), we can join most of the remaining pairs $x\in S_X$, $y\in S_Y$ of branch vertices by a path of the form xPy with $P\in\mathcal{P}$. As in the final part of the proof of Case 1, the sets N^1_{XY} and $\widetilde{N}^1_{XY}$ can then be used to join the small proportion of left-over pairs $x\in S_X$, $y\in S_Y$ by paths of length four. Since in total we have not used more vertices in N^2_X to join up the pairs $x\in S_X$, $y\in S_Y$ than in the proof of Case 2 in Theorem 19, all the pairs $x,x'\in S_X$ can be joined as before (and the same is true for the pairs $y,y'\in S_Y$).

Finally, suppose that $|T|L>(1-\mu+10^6\varepsilon)s^2/4$. In this case we again distribute the branch vertices evenly and proceed similarly as in the previous case except that this time we can find almost all of the subdivided edges joining pairs $x \in S_X$, $y \in S_Y$ (with $xy \notin G_0$) as paths of length two whose midpoint lies in a cluster belonging to T. Thus we do not have to use the connectivity of G_0 at all. Moreover, this time the number of all those vertices in clusters belonging to N_X^2 which we have not used up so far is at least

$$\begin{aligned} |N_X^2|L - (1-\mu+10^6\varepsilon)\frac{s^2}{4} &\overset{(15)}{\geq} (1-\varepsilon^*)\frac{\kappa}{\mu} + 2\tau n - (1-\mu+10^6\varepsilon)\frac{s^2}{4} \\ &\geq \frac{s^2}{4}\left(\frac{1}{\mu} - 1 + \mu\right) + 2\tau n \geq 2\binom{s/2}{2} + 2\tau n. \end{aligned}$$

Thus there is still enough room to join up the pairs of the form $x, x' \in S_X$ and $y, y' \in S_Y$ as in the previous case. $\square$

Roughly speaking, our aim in the proof of Theorem 3 is to find an edge XY in the reduced graph R whose density is large and which has the property that R contains many disjoint paths joining the neighbourhood of X to the neighbourhood of Y. Once we have found such an edge, we can proceed as in the proof of Theorem 2 since by Lemma 10 these paths correspond to many disjoint paths in the graph G_0 we started with. (Thus as before, the branch vertices are distributed within X and Y.) The following result of Mader [15] (see also [2]) implies that to find such an edge, it suffices to find a subgraph of R which has high minimum degree and in which every edge has large density.

Theorem 22 *In every graph G there exists an edge xy such that G contains $\delta(G)$ internally disjoint paths between x and y.*

Proof of Theorem 3 (Sketch). Let $\varrho := 1.15$ and $\sigma := 9/10$. Again, by Proposition 16, it suffices to show that for each $0 < \lambda < 1$ there exists $d^* = d^*(\lambda)$ such that for every $d_0 \geq d^*$ each graph G_0 of average degree d_0 contains a subdivision of a clique of order at least $\sqrt{2\varrho(1-\lambda)d_0} =: s$. We start by choosing constants as in (7) and (8) in the proof of Theorem 19. Similarly as there, we may assume that G_0 contains a subgraph G whose average degree is cn for some constant $c \geq c_0(\varepsilon^*)$ and such that $d_0/(1+\varepsilon^*) \leq cn \leq d_0$. (As before, n denotes the order of G.) By replacing G with a subgraph if necessary, we may assume that G contains no subgraph whose average degree is larger than cn and thus $\delta(G) \geq cn/2$. Next we apply the Regularity lemma to G. Proposition 6 implies that we obtain a reduced graph R which satisfies

$$\delta(R) \geq \left(\frac{c}{2} - 2d\right)k.$$

Put $c' := c - 2d$. Since we are now looking for a subdivision of a smaller clique, the calculation in Claim A in the proof of Theorem 19 shows that we can proceed as in Case 1 as long as $\Delta(R) \geq \varrho c' k$. (Indeed, take for X any vertex of maximum degree in R.) Thus we may assume that $\Delta(R) \leq \varrho c' k$.

Given a subgraph R' of R and a vertex $X \in V(R')$, we call

$$w_{R'}(X) := \sum_{Y \in N_{R'}(X)} \frac{e_{G'}(X,Y)}{L^2}$$

the *weight of* X *in* R'. Note that $d_{R'}(X) \geq w_{R'}(X)$. Moreover,

$$\sum_{X \in V(R)} w_R(X) L^2 = 2e(G' - V_0) \geq (c - (d + \varepsilon))n^2 - \varepsilon n^2 \geq c'(kL)^2.$$

Thus

$$\frac{\sum_{X \in V(R)} w_R(X)}{k} \geq c'k, \tag{24}$$

i.e. the average weight of the vertices in R is at least $c'k$. Let A be the set of all those vertices in R whose weight is less than $\sigma c'k$. Put $B := V(R) \setminus A$ and $b := |B|$. Let w_B be such that the average weight (in R) of the vertices in B is $w_B c'k$. Then (24) implies that $(k - b)\sigma c'k + b w_B c'k \geq c'k^2$ and hence

$$w_B \geq \frac{k}{b}(1 - \sigma) + \sigma. \tag{25}$$

Let R_1 be the graph obtained from R by deleting all those edges which have both endvertices in A. Call an edge of R_1 *light* if its density is at most $1/2$. For each $b \in B$, let ν_b be defined in such a way that $\nu_b c'k$ is the number of light edges of R_1 incident to b. Since $\Delta(R) \leq \varrho c'k$, we have

$$\sum_{b \in B} (\nu_b c'k/2 + (\varrho - \nu_b)c'k) \geq \sum_{b \in B} w_{R_1}(b) = \sum_{b \in B} w_R(b) = w_B c'kb.$$

Thus, setting

$$\nu := 2(\varrho - w_B),$$

it follows that

$$\text{no. of light edges in } R_1 \leq \sum_{b \in B} \nu_b c'k \leq \nu c'kb. \tag{26}$$

Let R_2 be the graph obtained from R_1 by deleting all light edges. Then

$$\begin{aligned} d(R_2) &\overset{(26)}{\geq} d(R_1) - 2\nu c'b \geq \frac{b \cdot w_B c'k}{k} - 2\nu c'b = bc'(5w_B - 4\varrho) \\ &\overset{(25)}{\geq} c'k\left(5(1 - \sigma) + \frac{b}{k}[5\sigma - 4\varrho]\right) \\ &\geq c'k(5 - 4\varrho) = 2c'k/5 =: 2\delta. \end{aligned}$$

(To see the last line, note that the square bracket is negative.) Finally, let R_3 be a subgraph of R_2 with minimum degree at least δ and set $\kappa_R := \delta - 1$. Apply Theorem 22 to find an edge $XY \in R_3$ such that R_3 contains a set $\mathcal{P}_R$ of κ_R disjoint paths between $N_R(X) \setminus \{Y\}$ and $N_R(Y) \setminus \{X\}$ which have no inner vertices in $N_R(X) \cup N_R(Y)$. We choose $\mathcal{P}_R$ in such a way that as many paths as possible are trivial. Since all edges in $E(R_3) \ni XY$ have at least one of their endvertices in B, we may assume that $X \in B$. Moreover, since no edge of $R_2 \supseteq R_3$ is light, the density of XY is at least $1/2$.

Similarly as in the proof of Theorem 19 (Case 2), choose disjoint τk-element sets N_X^1, $\widetilde{N}_X^1$, N_Y^1 and $\widetilde{N}_Y^1$. If $|N_R(X) \cap N_R(Y)| \geq \kappa_R$, we also choose τk-element sets N_{XY}^1 and $\widetilde{N}_{XY}^1$ which are disjoint from each other and from the above four sets and such that R contains a perfect matching between N^1_{XY} and $\widetilde{N}^1_{XY}$. Next choose a set N^2_X of neighbours of X in $R-Y$ which is disjoint from the above sets and has size $(\sigma c' - 10\tau)k$. (This is possible since $X \in B$ and so $d_R(X) \geq w_R(X) \geq \sigma c' k$.) Also, choose a set N^2_Y of $(\frac{c}{2} - 3d)k$ neighbours of Y which is disjoint from all the above sets except possibly from N^2_X. Moreover, we choose N^2_X and N^2_Y so that $\mathcal{P}_R$ contains at least $\kappa_R - 6\tau k$ paths which join N^2_X to N^2_Y and avoid each of N^1_X, $\widetilde{N}^1_X$, N^1_Y, $\widetilde{N}^1_Y$, N^1_{XY} and $\widetilde{N}^1_{XY}$. Let $\mathcal{P}'_R \subseteq \mathcal{P}_R$ denote the set of all these paths.

We now proceed as in the proof of Theorem 2 except for two changes. Firstly, the set $\mathcal{P}$ of paths is now obtained by an application of Lemma 10 to all the paths in $\mathcal{P}'_R$. Thus $|\mathcal{P}| \geq (\kappa_R - 7\tau k)L$. Secondly, we have to check that we can distribute the branch vertices of our subdivided K_s among X and Y such that N^2_X, N^2_Y and $\mathcal{P}$ are large enough to accommodate (almost) all the subdivided edges. For the latter, we distinguish two cases according to the size of $T := N^2_X \cap N^2_Y$. Again, γ will denote the proportion of branch vertices which we choose in X.

Case 1. $|T| \leq \kappa_R$

In this case, we join all pairs x, y of branch vertices with $x \in X$, $y \in Y$ and $xy \notin G_0$ by paths of the form xPy with $P \in \mathcal{P}$. (Note that if $N^2_X \cap N^2_Y \neq \emptyset$, some or even all of these paths may be trivial.) This can be done as in the proof of Theorem 19 (Case 2) if the number of all these pairs x, y is a bit smaller than $|\mathcal{P}|$, i.e. if

$$\gamma(1-\gamma)s^2(1/2 + 10^6\varepsilon) \leq (\kappa_R - 20\tau k)L. \tag{27}$$

Almost all of the pairs x, x' of branch vertices with $x, x' \in X$ will be joined by a path of length two whose midpoint lies in a cluster belonging to $N^2_X \setminus N^2_Y$ and was not used before to join some branch vertex in X to some branch vertex in Y. For this, we need that the number of all those unused vertices is a bit larger than the number of all the pairs x, x', i.e. that

$$\binom{\gamma s}{2} + \kappa_R L \leq (\sigma c' - 20\tau)kL. \tag{28}$$

The next inequality ensures that almost all pairs of branch vertices in Y can be joined in a similar way.

$$\binom{(1-\gamma)s}{2} + \kappa_R L \leq \left(\frac{c}{2} - 5d\right)kL \tag{29}$$

As before, the left-over pairs $x, x' \in X$ and $y, y' \in Y$ can be joined by using the sets N^1_X, $\widetilde{N}^1_X$ and N^1_Y, $\widetilde{N}^1_Y$ respectively. It is easy to check that (27), (28) and (29) hold if we set $\gamma := 0.78$.

Case 2. $|T| > \kappa_R$.

In this case, we join almost all of the pairs x, y of branch vertices with $x \in X$, $y \in Y$ and $xy \notin G_0$ by paths of length two whose midpoints lie in clusters belonging to T. (The left-over such pairs are then joined by paths of length four using the sets N^1_{XY} and $\widetilde{N}^1_{XY}$ as in the proof of Theorem 2.) Thus, we need that the number of all these pairs x, y is at most $(|T| - \tau k)L$. Defining t by $|T| = tc'k$, this means that

$$\gamma(1-\gamma)s^2(1/2 + 10^6\varepsilon) \le (tc' - \tau)kL. \tag{30}$$

Moreover, we will join almost all of the pairs x, x' of branch vertices with $x, x' \in X$ by paths of length two whose midpoints lie in a cluster belonging to $N_X^2 \setminus N_Y^2$. This will be possible if

$$\binom{\gamma s}{2} \le (\sigma c' - tc' - 20\tau)kL. \tag{31}$$

Finally, we join almost all of the pairs y, y' of branch vertices with $y, y' \in Y$ by paths of length two whose midpoints lie in a cluster belonging to N_Y^2 but have not been used before to join some branch vertex in X to some branch vertex in Y. Thus we need that

$$\binom{(1-\gamma)s}{2} + \gamma(1-\gamma)s^2(1/2 + 10^6\varepsilon) \le \left(\frac{c}{2} - 5d\right)kL. \tag{32}$$

As before, all the left-over pairs $x, x' \in X$ and $y, y' \in Y$ of branch vertices will be joined by using the sets N_X^1, $\widetilde{N}_X^1$ and N_Y^1, $\widetilde{N}_Y^1$. It can be easily checked that inequalities (30), (31) and (32) hold if we put $\gamma := \sqrt{(\sigma - t)/\varrho}$. $\square$

5 Concluding remarks

In this section, we briefly discuss the difficulties which arise if one tries to extend Theorem 19 to arbitrary graphs by removing the condition of H-freeness. The proof of Theorem 19 still works if the intersection of the neighbourhoods $N_R(X)$ and $N_R(Y)$ of X and Y in R is non-empty but not too large (here XY is an edge in R of maximum density). Indeed, as in the proof of Theorem 2 we can use this intersection to join a corresponding number of pairs $x \in X$, $y \in Y$ of branch vertices (with $xy \notin G_0$) by paths of length two whose midpoint belongs to a cluster in $N_R(X) \cap N_R(Y)$. The connectivity of G_0 is then only used to join the remaining such pairs.

However, the argument breaks down if $N_R(X) \cup N_R(Y)$ is too small, i.e. if the number of vertices belonging to a cluster in $N_R(X) \cup N_R(Y)$ is smaller than the required number of subdivided edges. In this case one is forced to distribute the branch vertices over more than two clusters. In fact, the following example shows that up to 9 clusters may be necessary in some cases. Suppose that G has a reduced graph R which consists of a large complete graph and whose edges all have density about 9/16. This will be the case (with high probability) if each subgraph of G corresponding to an edge of R is a bipartite random graph with edge probability 9/16 and G is empty otherwise. The connectivity of this graph is about $9n/16$ where $n := |G|$. Set $s := \frac{8}{3}\sqrt{\frac{9n}{16}}$. Then, if we distribute the branch vertices of a potential subdivision of K_s over t clusters, the number of subdivided edges one needs to find is at least about

$$t\binom{s/t}{2} + \frac{7}{16}\left(\frac{s}{t}\right)^2\binom{t}{2}$$

which is significantly larger than n unless $t \geq 9$. In this example, it is of course nevertheless easy to find a subdivision of K_s in G since the intersections of the the neighbourhoods of the clusters in R are identical (and so one can proceed as in the final case of the proof of Theorem 2). However, the example indicates that for arbitrary graphs a strategy similar to ours seems to lead to an enormous number of cases which need to be considered, as the case distinctions would not only depend on the sizes of the pairwise intersections but more generally on the sizes of the common neighbourhoods of each subset of the set of all those clusters which contain the branch vertices.

References

[1] T. Böhme, K. Kawarabayashi, J. Maharry and B. Mohar, Linear connectivity forces large complete bipartite minors, preprint 2004.

[2] B. Bollobás, *Extremal Graph Theory*, Academic Press 1978.

[3] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics 184, Springer-Verlag 1998.

[4] B. Bollobás, *Random Graphs* (2nd edition), Cambridge studies in Advanced Mathematics 73, Cambridge University Press 2001.

[5] B. Bollobás and A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, *Eur. J. Comb.* **19** (1998), 883–887.

[6] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics 173, Springer-Verlag 1997.

[7] R.A. Duke and V. Rödl, On graphs with small subgraphs of large chromatic number, *Graphs and Combinatorics* **1** (1985), 91–96.

[8] S. Gerke, Y. Kohayakawa, V. Rödl and A. Steger, in preparation.

[9] S. Janson, T. Łuczak and A. Ruciński, *Random graphs*, Wiley-Interscience 2000.

[10] H.A. Jung, Eine Verallgemeinerung des n-fachen Zusammenhangs für Graphen, *Math. Ann.* **187** (1970), 95–103.

[11] J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, *Bolyai Society Mathematical Studies 2, Combinatorics, Paul Erdős is Eighty (Vol. 2)* (D. Miklós, V. T. Sós and T. Szőnyi eds.), Budapest (1996), 295–352.

[12] J. Komlós and E. Szemerédi, Topological cliques in graphs II, *Combin. Probab. Comput.* **5** (1996), 79–90.

[13] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Annalen* **174** (1967), 265–268.

[14] W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.

[15] W. Mader, Grad und lokaler Zusammenhang in endlichen Graphen, *Math. Ann.* **205** (1973), 9–11.

[16] J.S. Myers, Graphs without large complete minors are quasi-random, *Combin. Probab. Comput.* **11** (2002), 571–585.

[17] A. Thomason, The extremal function for complete minors, *J. Combin. Theory B* **81** (2001), 318–338.

[18] A. Thomason, Two minor problems, *Combin. Probab. Comput.* **13** (2004), 413–414.

Daniela Kühn & Deryk Osthus
School of Mathematics
Birmingham University
Edgbaston
Birmingham B15 2TT
UK
E-mail addresses: {kuehn,osthus}@maths.bham.ac.uk