

CYCLES OF GIVEN LENGTH IN ORIENTED GRAPHS

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ABSTRACT. We show that for each $\ell \geq 4$ every sufficiently large oriented graph G with $\delta^+(G), \delta^-(G) \geq \lfloor |G|/3 \rfloor + 1$ contains an ℓ -cycle. This is best possible for all those $\ell \geq 4$ which are not divisible by 3. Surprisingly, for some other values of ℓ , an ℓ -cycle is forced by a much weaker minimum degree condition. We propose and discuss a conjecture regarding the precise minimum degree which forces an ℓ -cycle (with $\ell \geq 4$ divisible by 3) in an oriented graph. We also give an application of our results to pancyclicity and consider ℓ -cycles in general digraphs.

1. INTRODUCTION

1.1. **Girth.** All the directed graphs (digraphs) considered in this paper have no loops and at most two edges between each pair of vertices: at most one edge in each direction. A digraph is an *oriented graph* if it is an orientation of a simple graph. A central problem in digraph theory is the Caccetta-Häggkvist conjecture [8] (which generalized an earlier conjecture of Behzad, Chartrand and Wall [5]):

Conjecture 1. *An oriented graph on n vertices with minimum outdegree d contains a cycle of length at most $\lceil n/d \rceil$.*

Note that in Conjecture 1 it does not matter whether we consider oriented graphs or general digraphs. Chvátal and Szemerédi [9] showed that a minimum outdegree of at least d guarantees a cycle of length at most $\lceil 2n/(d+1) \rceil$. For most values of n and d , this is improved by a result of Shen [25], which guarantees a cycle of length at most $3\lceil 0.44n/d \rceil$. Chvátal and Szemerédi [9] also showed that Conjecture 1 holds if we increase the bound on the cycle length by adding a constant c . They showed that $c := 2500$ will do. Nishimura [23] refined their argument to show that one can take $c := 304$. The next result of Shen gives the best known constant.

Theorem 2 (Shen [26]). *An oriented graph on n vertices with minimum outdegree d contains a cycle of length at most $\lceil n/d \rceil + 73$.*

The special case of Conjecture 1 that has attracted most interest is when $d = \lceil n/3 \rceil$. The following bound towards this case improves an earlier one of Caccetta and Häggkvist [8].

Theorem 3 (Shen [24]). *If G is any oriented graph on n vertices with $\delta^+(G) \geq 0.354n$ then G contains a directed triangle.*

If one considers the *minimum semidegree* $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$ instead of the minimum outdegree $\delta^+(G)$, then the constant can be improved slightly. The best known value for the constant in this case is currently 0.346 [13]. See the monograph [4] or the survey [22] for further partial results on Conjecture 1.

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1.2. Cycles of given length in oriented graphs. We consider the natural and related question of which minimum semidegree forces cycles of length exactly $\ell \geq 4$ in an oriented graph. We will often refer to cycles of length ℓ as ℓ -cycles. Our main result answers this question completely when ℓ is not a multiple of 3.

Theorem 4. *Let $\ell \geq 4$. If G is an oriented graph on $n \geq 10^{10}\ell$ vertices with $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ then G contains an ℓ -cycle. Moreover for any vertex $u \in V(G)$ there is an ℓ -cycle containing u .*

The extremal example showing this to be best possible for $\ell \geq 4$, $\ell \not\equiv 0 \pmod{3}$ is given by the blow-up of a 3-cycle. More precisely, let G be the oriented graph on n vertices formed by dividing $V(G)$ into 3 vertex classes V_1, V_2, V_3 of as equal size as possible and adding all possible edges from V_i to V_{i+1} , counting modulo 3. Then this oriented graph contains no ℓ -cycle and has minimum semidegree $\lfloor n/3 \rfloor$.

Also, for all those $\ell \geq 4$ which are multiples of 3, the ‘moreover’ part is best possible for infinitely many n . To see this, consider the modification of the above example formed by deleting a vertex from the largest vertex class and adding an extra vertex u with $N^+(u) = V_2$ and $N^-(u) = V_1$. This gives an oriented graph with minimum semidegree $\lfloor (n-1)/3 \rfloor$. For $\ell \equiv 0 \pmod{3}$ it contains no ℓ -cycle through u .

Perhaps surprisingly, we can do much better than Theorem 4 for some cycle lengths (if we do not ask for a cycle through a given vertex).

Indeed, we conjecture that the correct bounds are those given by the obvious extremal example: when we seek an ℓ -cycle, the extremal example is probably the blow-up of a k -cycle, where $k \geq 3$ is the smallest integer which is not a divisor of ℓ .

Conjecture 5. *Let $\ell \geq 4$ be a positive integer and let $k \geq 3$ be minimal such that k does not divide ℓ . Then there exists an integer $n_0 = n_0(\ell)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \lfloor n/k \rfloor + 1$ contains an ℓ -cycle.*

It is easy to see that the only values of k that can appear in Conjecture 5 are of the form $k = p^s$ with $k \geq 3$, where $p \geq 2$ is a prime and s a positive integer. Theorem 4 confirms this conjecture in the case when $k = 3$. The following result implies that Conjecture 5 is approximately true when $k = 4, 5$ and ℓ is sufficiently large. It also gives weaker bounds on the minimum semidegree for large values of k .

Theorem 6. *Let $\ell \geq 4$ be a positive integer and let $k \geq 3$ be minimal such that k does not divide ℓ .*

- (i) *There exists an integer $n_0 = n_0(\ell)$ such that whenever $k \geq 150$ and G is an oriented graph on $n \geq n_0$ vertices with $\delta^+(G) \geq n/k + 150n/k^2$ then G contains an ℓ -cycle.*
- (ii) *If $k = 4$ and $\ell \geq 42$ then for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\ell, \varepsilon)$ such that every oriented graph G on $n \geq n_0$ vertices with $\delta^0(G) \geq n/k + \varepsilon n$ contains an ℓ -cycle.*
- (iii) *The analogue of (ii) holds if $k = 5$ and $\ell \geq 2550$.*

Part (i) is obtained from Theorem 2 via a simple application of the Regularity lemma for digraphs (see Section 4). It would be interesting to find a proof which does not rely on the Regularity lemma. Moreover, part (i) suggests that one might be able to replace δ^0 by δ^+ in Conjecture 5. Even replacing it in Theorem 4 would be interesting.

In view of Theorem 4 and the Caccetta-Häggkvist Conjecture one might wonder whether a minimum semidegree close to $n/3$ also forces a 3-cycle through any given vertex. However the next proposition (whose straightforward proof is given in Section 3) shows that the threshold in this case is much higher.

Proposition 7.

- (i) If G is an oriented graph on n vertices with $\delta^0(G) \geq \lceil 2n/5 \rceil$ then for any vertex $u \in V(G)$ there exists a 3-cycle containing u .
- (ii) For infinitely many n there exists an oriented graph G on n vertices with $\delta^0(G) = \lfloor 2n/5 \rfloor$ containing a vertex u which does not lie on a 3-cycle.

1.3. Pancyclicity. Building on [18], Keevash, Kühn and Osthus [15] recently gave an exact minimum semidegree bound which forces a Hamilton cycle in an oriented graph. More precisely, they showed that every sufficiently large oriented graph G with $\delta^0(G) \geq (3n-4)/8$ contains a Hamilton cycle. This is best possible and settles a problem of Thomassen. The arguments in [15] can easily be modified to show that G even contains an ℓ -cycle for every $\ell \geq n/10^{10}$ through any given vertex (see [16] for details). Together with Theorems 3 and 4 this implies that G is *pancyclic*, i.e. it contains cycles of all possible lengths.

Theorem 8. *There exists an integer n_0 such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq (3n-4)/8$ contains an ℓ -cycle for all $3 \leq \ell \leq n$. Moreover, if $4 \leq \ell \leq n$ and if u is any vertex of G then G contains an ℓ -cycle through u .*

This improves a bound of Darbinyan [10], who proved that a minimum semidegree of $\lfloor n/2 \rfloor - 1 \geq 4$ implies pancyclicity. Another degree condition which implies pancyclicity in oriented graphs which are close to being tournaments is given by Song [27]. Proposition 7 shows that we cannot have $\ell = 3$ in the ‘moreover’ part of Theorem 8.

For (general) digraphs, Thomassen [28] as well as Häggkvist and Thomassen [12] gave degree conditions which imply that every digraph with minimum semidegree $> n/2$ is pancyclic. (The complete bipartite digraph whose vertex class sizes are as equal as possible shows that the latter bound is best possible.) Alon and Gutin [1] observed that one can use Ghouila-Houri’s theorem [11] (which states that a minimum semidegree of at least $n/2$ guarantees a Hamilton cycle in a digraph) to show that every digraph G with minimum semidegree $> n/2$ is even vertex-pancyclic, i.e. for every $\ell = 2, \dots, n$ each vertex of G lies on an ℓ -cycle.

1.4. Arbitrary orientations of cycles. Recently Kelly [17] proved the following result on arbitrary orientations of Hamilton cycles in oriented graphs.

Theorem 9. *For any $\alpha > 0$ there exists $n_0 = n_0(\alpha)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq (3/8 + \alpha)n$ contains every possible orientation of a Hamilton cycle.*

In this paper we extend this further to a pancyclicity result for arbitrary orientations: if an oriented graph G on n vertices contains every possible orientation of an ℓ -cycle for all $3 \leq \ell \leq n$ we say that G is *universally pancyclic*. Our main result on arbitrary orientations says that asymptotically universal pancyclicity requires the same minimum semidegree as pancyclicity.

Theorem 10. *For all $\alpha > 0$ there exists an integer $n_0 = n_0(\alpha)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq (3/8 + \alpha)n$ is universally pancyclic.*

As with standard orientations, if we look only at short cycles then we can strengthen the minimum semidegree condition in the above result. The semidegree required will depend on the so-called cycle-type. Given an arbitrarily oriented ℓ -cycle C , the *cycle-type* $t(C)$ of C is the number of edges oriented forwards in C minus the number of edges oriented backwards in C . By traversing C in the opposite direction if necessary, we may assume that

$t(C) \geq 0$. An oriented ℓ -cycle has cycle-type ℓ . Arbitrarily oriented cycles of cycle-type 0 are precisely those for which there is a digraph homomorphism into an oriented path. Moreover, if $t(C) \geq 3$ then $t(C)$ is the *maximum* length of an oriented cycle into which there is a digraph homomorphism of C .

Proposition 11.

- Let $\ell \geq 4$ and let $\alpha > 0$. Then there exists $n_0 = n_0(\ell, \alpha)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq (1/3 + \alpha)n$ contains every orientation of an ℓ -cycle.
- Let $\alpha > 0$ and let ℓ be some positive constant. Then there exists $n_0 = n_0(\alpha, \ell)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \alpha n$ contains every cycle of length at most ℓ and cycle-type 0.

In Section 5 we will derive the universal pancyclicity result (Theorem 10) by combining the short-cycle result (Proposition 11) with a probabilistic argument applied to Theorem 9 giving all long cycles.

Conjecture 5 has a natural strengthening to incorporate arbitrarily oriented cycles.

Conjecture 12. Let C be an arbitrarily oriented cycle of length $\ell \geq 4$ and cycle-type $t(C) \geq 4$. Let $k \geq 3$ be minimal such that k does not divide $t(C)$. Then there exists an integer $n_0 = n_0(\ell, k)$ such that every oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \lfloor n/k \rfloor + 1$ contains C .

As we shall see in Section 5, Conjecture 5 would imply an approximate version of Conjecture 12.

1.5. Cycles of given length in digraphs. A straightforward application of the Regularity lemma shows that a solution to Conjecture 5 would also asymptotically solve the corresponding problem for general digraphs: Let $\delta_{di}(\ell, n)$ denote the smallest integer d so that every digraph with n vertices and minimum semidegree at least d contains an ℓ -cycle and let $\delta_{orient}(\ell, n)$ denote the smallest integer d so that every oriented graph with n vertices and minimum semidegree at least d contains an ℓ -cycle.

Proposition 13. For any $\ell \geq 3$,

$$\lim_{n \rightarrow \infty} \frac{\delta_{di}(\ell, n)}{n} = \begin{cases} 1/2 & \text{if } \ell \text{ is odd;} \\ \lim_{n \rightarrow \infty} \frac{\delta_{orient}(\ell, n)}{n} & \text{otherwise.} \end{cases}$$

It is easy to see that these limits exist. The corresponding density problem for digraphs was solved by Häggkvist and Thomassen. Let $ex_{di}(\ell, n)$ denote the largest number d so that there is digraph with n vertices and at least d edges which contains no ℓ -cycle. Häggkvist and Thomassen [12] proved that

$$(1) \quad ex_{di}(\ell, n) = \binom{n}{2} + \frac{(\ell - 2)n}{2}.$$

The case $\ell = 3$ was proved earlier by Brown and Harary [6]. A transitive tournament (i.e. an acyclic orientation of a complete graph) shows that it does not make sense to consider this density problem for oriented graphs. More general extremal digraph problems are discussed in the survey [7].

2. NOTATION

Given two vertices x and y of a digraph G , we write xy for the edge directed from x to y . The *order* $|G|$ of G is the number of its vertices. We write $N_G^+(x)$ for the outneighbourhood of a vertex x and $d^+(x) := |N_G^+(x)|$ for its outdegree. Similarly, we write $N_G^-(x)$ for the inneighbourhood of x and $d^-(x) := |N_G^-(x)|$ for its indegree. Given $X \subseteq V(G)$ we denote $|N_G^+(x) \cap X|$ by $d_X^+(x)$, and define $d_X^-(x)$ similarly. We write $N_G(x) := N_G^+(x) \cup N_G^-(x)$ for the neighbourhood of x . We use $N^+(x)$ etc. whenever this is unambiguous. Given a set A of vertices of G , we write $N_G^+(A)$ for the set of all outneighbours of vertices in A . So $N_G^+(A)$ is the union of $N_G^+(a)$ over all $a \in A$. $N_G^-(A)$ is defined similarly. The directed subgraph of G induced by A is denoted by $G[A]$ and we write $e(A)$ for the number of its edges. $G - A$ denotes the digraph obtained from G by deleting A and all edges incident to A .

When referring to paths and cycles in digraphs we always mean that they are directed without mentioning this explicitly. Given two vertices x, y of a digraph G , an x - y *path* is a directed path which joins x to y . Given two subsets A and B of vertices of G , an A - B *edge* is an edge ab where $a \in A$ and $b \in B$. We write $e(A, B)$ for the number of all these edges. A *walk* in G is a sequence $v_1 v_2 \dots v_\ell$ of (not necessarily distinct) vertices, where $v_i v_{i+1}$ is an edge for all $1 \leq i < \ell$. The length of a walk is $\ell - 1$. The walk is *closed* if $v_1 = v_\ell$. Given two vertices x, y of G , the *distance* $\text{dist}(x, y)$ from x to y is the length of the shortest x - y path. The *diameter* of G is maximum distance between any ordered pair of vertices.

3. PROOFS OF THEOREM 4 AND PROPOSITION 7

We begin with two immediate facts about oriented graphs which will prove very useful.

Fact 14. *If G is an oriented graph and $X \subseteq V(G)$ is non-empty then $e(X) \leq |X|(|X| - 1)/2$. In particular, there exists $x \in X$ with $|N^+(x) \cap X| \leq |X|/2 - 1/2$ and thus $|N^+(X) \setminus X| \geq |N^+(x) \setminus X| \geq \delta^0(G) - |X|/2 + 1/2$. \square*

Fact 15. *If G is an oriented graph on n vertices then the maximum size of an independent set is at most $n - 2\delta^0(G)$. \square*

Proof of Proposition 7. First we prove (i). By Fact 14 there exists a vertex $x \in N^+(u)$ with

$$|N^+(x) \setminus N^+(u)| \geq \delta^0(G) - |N^+(u)|/2 + 1/2.$$

Hence

$$|N^+(u)| + |N^-(u)| + |N^+(x) \setminus N^+(u)| \geq 5\delta^0(G)/2 + 1/2 > n$$

and so x must have an outneighbour in $N^-(u)$.

For (ii), pick $m \in \mathbb{N}$ and define an oriented graph G on $n := 5m - 1$ vertices as follows. Let A, B, C be disjoint vertex sets of sizes $2m - 1, 2m - 1$ and m respectively. Add all possible edges from A to B , B to C and C to A . Let $G[A]$ and $G[B]$ induce regular tournaments. So for example every vertex in A will have $m - 1$ outneighbours and $m - 1$ inneighbours in A . (It is easy to see that such oriented graphs exist.) Add a single vertex u with $N^+(u) := B$ and $N^-(u) := A$. Then $\delta^0(G) = 2m - 1 = \lfloor 2n/5 \rfloor$. By construction u is not contained in a 3-cycle. \square

We now prove Theorem 4 in a series of lemmas. Lemmas 16, 17 and 19 deal with the special cases $\ell = 4, 5, 6$. Lemmas 20 and 21 deal with the general case $\ell \geq 7$.

Lemma 16. *If G is an oriented graph on $n \geq 4$ vertices with $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ then for any vertex $x \in V(G)$, G contains a 4-cycle through x .*

Proof. Assume that there is a vertex $x \in V(G)$ for which no such cycle exists. Let X be a set of $\lfloor n/3 \rfloor + 1$ outneighbours of x and Y be a set of $\lfloor n/3 \rfloor + 1$ inneighbours. Suppose that both of the following hold.

- (i) There exists $x' \in X$ with $|N^+(x') \setminus (X \cup Y)| \geq (\lfloor n/3 \rfloor + 1)/2$.
- (ii) There exists $y' \in Y$ with $|N^-(y') \setminus (X \cup Y)| \geq (\lfloor n/3 \rfloor + 1)/2$.

Then

$$(N^+(x') \cap N^-(y')) \setminus (X \cup Y) \neq \emptyset$$

and hence the desired 4-cycle exists. So without loss of generality assume that (i) does not hold. (The case when (ii) does not hold is similar.) Let X' be the set of vertices $x' \in X$ with $d_X^-(x') > 0$. Note that Fact 15 implies that $X' \neq \emptyset$. Let $x' \in X'$ be such that $d_{X'}^+(x')$ is minimal. Since $N^+(x') \cap (X \setminus X') = \emptyset$, Fact 14 implies that

$$|N^+(x') \setminus X| = |N^+(x') \setminus X'| \geq \delta^0(G) - |X'|/2 \geq \delta^0(G) - |X|/2 \geq (\lfloor n/3 \rfloor + 1)/2.$$

Since we are assuming that (i) does not hold this means that x' has an outneighbour $y \in Y$. By definition of X' there exists an inneighbour $x'' \in X$ of x' . But then $xx''x'y$ is the required 4-cycle. \square

Lemma 17. *If G is an oriented graph on $n \geq 5$ vertices with $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ then for any vertex $x \in V(G)$, G contains a 5-cycle through x .*

Proof. As $N^-(x)$ is not independent by Fact 15 we can pick vertices $a, y \in N^-(x)$ such that $ya, ax, yx \in E(G)$. Let X be a set of $\lfloor n/3 \rfloor + 1$ outneighbours of x and Y be a set of $\lfloor n/3 \rfloor + 1$ inneighbours of y . Define $Z := X \cap Y$. Clearly, it suffices to prove the next claim.

Claim 1. *There exists at least one of the following:*

- (i) *an x - y path of length 4,*
- (ii) *an x - y path of length 3 avoiding a .*

Note that $x, y, a \notin X \cup Y$ since G is an oriented graph. So we may assume that $e(X, Y) = 0$, as otherwise (ii) is satisfied. In particular, Z is independent and $e(X, Z) = e(Z, Y) = 0$. The following claim immediately implies (i) (to see this, note that $x, y \notin N^+(x') \cap N^-(y')$).

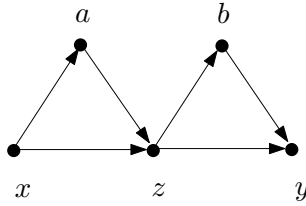
Claim 2. *Both of the following hold.*

- (a) *There exists $x' \in X$ with $|N^+(x') \setminus (X \cup Y)| \geq (\lfloor n/3 \rfloor + 1 + |Z|)/2$.*
- (b) *There exists $y' \in Y$ with $|N^-(y') \setminus (X \cup Y)| \geq (\lfloor n/3 \rfloor + 1 + |Z|)/2$.*

We will only prove (a) (the argument for (b) is similar). If $X \setminus Z = \emptyset$ then $X = Z$ and so X is independent. But $|X| = \lfloor n/3 \rfloor + 1$ which contradicts Fact 15. So assume that $X \setminus Z \neq \emptyset$ and let $x' \in X \setminus Z$ be such that $d_{X \setminus Z}^+(x')$ is minimal. Fact 14 implies that

$$d_{X \setminus Z}^+(x') > \delta^0(G) - (|X| - |Z|)/2 \geq (\lfloor n/3 \rfloor + 1 + |Z|)/2.$$

By assumption x' has no outneighbours in Y , so $d_{X \setminus Z}^+(x') = d_{X \cup Y}^+(x')$ and thus (a) holds. \square

FIGURE 1. An xy -butterfly

In order to prove the cases $\ell = 6$ and $\ell \geq 7$ of Theorem 4 we need some more notation. An xy -butterfly is an oriented graph with vertices x, y, z, a, b such that xa, xz, az, zb, zy, by are all the edges (Figure 1). The crucial fact about a butterfly is that it contains x - y paths of lengths 2, 3 and 4, and is thus a useful tool in finding cycles of prescribed length: any y - x path of length $\ell - 2$, $\ell - 3$ or $\ell - 4$ whose interior avoids the xy -butterfly yields an ℓ -cycle containing x . The following fact tells us that a large minimum semidegree guarantees the existence of a butterfly.

Fact 18. *If G is an oriented graph on n vertices with $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ then for any vertex $x \in V(G)$ there exists a vertex y such that G contains an xy -butterfly.*

Proof. By Fact 15 the outneighbourhood of x is not independent, so pick an edge az in it. Reapply Fact 15 to find an edge by in the outneighbourhood of z . Note that as $x, a \in N^-(z)$ all the vertices are distinct. \square

Lemma 19. *If G is an oriented graph on $n \geq 6$ vertices with $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ then for any vertex $x \in V(G)$, G contains a 6-cycle through x .*

Proof. Fact 18 gives us an xy -butterfly for some vertex $y \in V(G)$, with vertices a, b, z as described in the definition of an xy -butterfly. To complete the proof we may assume that each of the following holds.

- (i) There is no y - x path of length 2.
- (ii) There is no y - x path of length 3 avoiding a .
- (iii) There is no y - x path of length 4 avoiding z .

Indeed, it is easy to check that if one of these does not hold then this y - x path together with a suitable subpath of the xy -butterfly forms the required cycle.

Pick $Y \subseteq N^+(y) \setminus \{a, x\}$, $X \subseteq N^-(x) \setminus \{y\}$ such that $|Y| = \lfloor n/3 \rfloor - 1$ and $|X| = \lfloor n/3 \rfloor$. Observe that $b, z \notin Y$ and $a, z \notin X$. Moreover $X \cap Y = \emptyset$ by (i). Let $Y' := N^+(Y) \setminus Y$, $X' := N^-(X) \setminus X$. Then $X \cap Y' = \emptyset$ and $Y \cap X' = \emptyset$ by (ii). Fact 14 implies that $|Y'| \geq \lfloor n/3 \rfloor / 2 + 2$ and $|X'| \geq \lfloor n/3 \rfloor / 2 + 3/2$. By (i) and the definitions of X and Y we have $x, y \notin X, Y, X', Y'$. Altogether this shows that

$$n + |X' \cap Y'| \geq |X| + |Y| + |X'| + |Y'| + 2 \geq 3\lfloor n/3 \rfloor + 9/2 \geq (n - 2) + 9/2.$$

Hence $|X' \cap Y'| \geq 3$, and so $(X' \cap Y') \setminus \{z\} \neq \emptyset$. But this implies that there is a y - x path of length 4 avoiding z . \square

The next two lemmas deal with the case $\ell \geq 7$.

Lemma 20. *Let C be some positive integer. If G is an oriented graph on $n \geq 8 \cdot 10^9 C$ vertices with $\delta^0(G) \geq n/3 - C + 1$ then for every pair $x \neq y$ of vertices there exists an x - y path of length 3, 4 or 5.*

Proof. Let $\varepsilon := 1/10^4$ and $C' := 10C/\varepsilon$. Let X be a set of $\lceil n/3 \rceil - 2C$ outneighbours of x in $G - y$ and let Y be a set of $\lceil n/3 \rceil - 2C$ inneighbours of y in $G - x$, chosen so that $|X \setminus Y|, |Y \setminus X| \geq C$. Let $Z := X \cap Y$. If there is an X - Y edge then we have an x - y path of length 3. So suppose there is no such edge. In particular this implies that Z is independent and there are no X - Z or Z - Y edges.

Let $X' := N^+(X \setminus Z) \setminus X$ and $Y' := N^-(Y \setminus Z) \setminus Y$. Note that $X' \cap Y = \emptyset$ and $Y' \cap X = \emptyset$, as otherwise we have an X - Y edge. Moreover, we may assume that $X' \cap Y' = \emptyset$, as otherwise we have an x - y path of length 4. As no vertex in $X \setminus Z$ has an outneighbour in Z we have $X' = N^+(X \setminus Z) \setminus (X \setminus Z)$. Hence by Fact 14

$$|X'| \geq \delta^0(G) - |X \setminus Z|/2 \geq \lceil n/3 \rceil/2 + |Z|/2.$$

Similarly, $|Y'| \geq \lceil n/3 \rceil/2 + |Z|/2$. Observe that this implies

$$(2) \quad |V(G) \setminus (X \cup X' \cup Y \cup Y')| \leq 4C.$$

Note that

$$(3) \quad |X'| \leq n - |X \cup Y| - |Y'| \leq n - (2n/3 - |Z| - 4C) - (n/6 + |Z|/2) = n/6 + |Z|/2 + 4C.$$

We call a vertex $x' \in X \setminus Z$ *good* if $|N^+(x') \setminus X| \geq n/6 + |Z|/2 - C' \geq |X'| - 4C'/3$ (the last inequality follows from (3)). Suppose that at least $\varepsilon|X \setminus Z|$ vertices in $X \setminus Z$ are not good. Since $d_{X \setminus Z}^+(x') \geq \delta^0(G) - |N^+(x') \setminus (X \setminus Z)| = \delta^0(G) - |N^+(x') \setminus X|$ for every $x' \in X \setminus Z$ this implies that

$$\begin{aligned} e(X \setminus Z) &\geq \varepsilon|X \setminus Z|(\delta^0(G) - (n/6 + |Z|/2 - C')) + (1 - \varepsilon)|X \setminus Z|(\delta^0(G) - |X'|) \\ &\stackrel{(3)}{\geq} \varepsilon|X \setminus Z|(n/6 - |Z|/2 + C'/2) + (1 - \varepsilon)|X \setminus Z|(n/6 - |Z|/2 - 5C) \\ &= |X \setminus Z|(n/6 - |Z|/2 + \varepsilon C'/2 - 5C(1 - \varepsilon)) \\ &\geq |X \setminus Z|(n/6 - |Z|/2) \geq |X \setminus Z|^2/2. \end{aligned}$$

But this is a contradiction as G is an oriented graph. Thus we may assume that all but at most $\varepsilon|X \setminus Z|$ vertices in $X \setminus Z$ are good, and hence, since $|X'| \geq n/6 \geq 4C'/(3\varepsilon)$ we have

$$(4) \quad e(X \setminus Z, X') \geq (1 - \varepsilon)|X \setminus Z|(|X'| - 4C'/3) \geq (1 - 2\varepsilon)|X \setminus Z||X'|.$$

Call a vertex $x' \in X'$ *nice* if $|N^-(x') \cap (X \setminus Z)| \geq (1 - 2\sqrt{\varepsilon})|X \setminus Z|$. Then at least $(1 - 2\sqrt{\varepsilon})|X'|$ vertices in X' are nice, as otherwise

$$e(X \setminus Z, X') \leq 2\sqrt{\varepsilon}|X'|((1 - 2\sqrt{\varepsilon})|X \setminus Z|) + (1 - 2\sqrt{\varepsilon})|X'|(|X \setminus Z|) < (1 - 2\varepsilon)|X'|(|X \setminus Z|),$$

which contradicts (4). Consider a nice vertex $x' \in X' \setminus \{y\}$. Note that $N^+(x') \cap (Y \cup Y')$ is either empty or equal to $\{x\}$ (as otherwise we get an x - y path of length 4 or 5). Since x' is nice it has at most $2\sqrt{\varepsilon}|X \setminus Z|$ outneighbours in $X \setminus Z$ and so

$$(5) \quad |N^+(x') \cap X'| \stackrel{(2)}{\geq} \delta^0(G) - 2\sqrt{\varepsilon}|X \setminus Z| - 1 - 4C \geq n/3 - \sqrt{\varepsilon}n.$$

In particular, $|X'| \geq n/3 - \sqrt{\varepsilon}n$. Similarly, $|Y'| \geq n/3 - \sqrt{\varepsilon}n$. But $|X \cup Y| \geq n/3 - C$ and so

$$(6) \quad |X'| \leq n - |X \cup Y| - |Y'| \leq n/3 + 2\sqrt{\varepsilon}n.$$

Now we combine this with the fact that at least $|X'| - 1 - 2\sqrt{\varepsilon}|X'| \geq (1 - 3\sqrt{\varepsilon})|X'|$ vertices in $X' \setminus \{y\}$ are nice to obtain

$$|X'|^2/2 \geq e(X') \stackrel{(5)}{\geq} (1 - 3\sqrt{\varepsilon})|X'|(|X'| - \sqrt{\varepsilon}n) \stackrel{(6)}{\geq} (1 - 3\sqrt{\varepsilon})|X'|(|X'| - 3\sqrt{\varepsilon}n) > 2|X'|^2/3.$$

This contradiction completes the proof. \square

Lemma 21. *Suppose $\ell \geq 7$ and $n \geq 10^{10}\ell$. If G is an oriented graph on n vertices with $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ then for every vertex $x \in V(G)$, G contains an ℓ -cycle through x .*

Proof. Fact 18 gives us an xy -butterfly for some vertex $y \in V(G)$, with a, b and z as in the definition of an xy -butterfly. Greedily pick a path P of length $\ell - 7$ from y to some vertex v such that P avoids a, b, x, z (the minimum semidegree condition implies the existence of such a path).

Now apply Lemma 20 to $G - (\{a, b, z\} \cup (V(P) \setminus \{v\}))$ with $C := \ell$ (say) to find a v - x path of length 3, 4 or 5. Pick a path from x to y in the xy -butterfly of appropriate length to obtain the desired ℓ -cycle through x . \square

4. PROOFS OF THEOREM 6 AND PROPOSITION 13

The following lemma implies that if we allow ourselves a linear ‘error term’ in the degree conditions then instead of finding an ℓ -cycle, it suffices to look for a closed walk of length ℓ . We will use (i) and (ii) in the proof of Theorem 6, (iii) in the proof of Proposition 13 and (iv) in the proof of Proposition 11.

Lemma 22. *Let $\ell \geq 2$ be an integer.*

- (i) *Suppose that $c > 0$ and there exists an integer n_0 such that every oriented graph H on $n \geq n_0$ vertices with $\delta^0(H) \geq cn$ contains a closed walk W of length ℓ . Then for each $\varepsilon > 0$ there exists $n_1 = n_1(\varepsilon, \ell, n_0)$ such that if G is an oriented graph on $n \geq n_1$ vertices with $\delta^0(G) \geq (c + \varepsilon)n$ then G contains an ℓ -cycle.*
- (ii) *The analogue holds if we replace $\delta^0(H)$ by $\delta^+(H)$ and $\delta^0(G)$ by $\delta^+(G)$.*
- (iii) *The analogue of (i) holds if we consider directed graphs instead of oriented graphs.*
- (iv) *The analogue of (i) holds if we ask for a copy of some specific (not necessarily closed) walk W of length ℓ and for an orientation of a cycle which has a homomorphism into W .*

Note that (iv) is actually a strengthening of (i). The proof of Lemma 22 is a standard application of the Regularity lemma for digraphs. So we omit the details, which can be found in [16]. As mentioned in the introduction, it would be interesting to find a proof which avoids the Regularity lemma. This would probably yield a much better bound on n_1 .

Sketch of proof of Lemma 22. We only consider (i). (The arguments for the remaining parts are similar.) A directed version of the Regularity lemma was proved by Alon and Shapira [2]. Apply the degree form of this Regularity lemma to G to obtain a partition of $V(G)$ into clusters and a reduced digraph R' . (R' is sometimes also called the cluster digraph). Roughly speaking, the vertices of R' are the clusters and there is a directed edge from A to B in R' if the bipartite subdigraph of G consisting of the edges from A to B is ε' -regular and has density at least d , where $\varepsilon' \ll d \ll \varepsilon$. One can show that R' almost inherits the minimum semidegree of G , i.e. $\delta^0(R') \geq (c + \varepsilon/2)|R'|$. However, R' need not be oriented. But for every double edge of R' one can delete one of the two edges randomly (with suitable probability) in order to obtain an oriented spanning subgraph R of R' which still satisfies $\delta^0(R) \geq c|R|$ (see [18, Lemma 7] for a proof). Applying our assumption with $H := R$ gives a closed walk of length ℓ in R . Since n_1 is large compared to ℓ , this also holds for size of the clusters. So we can apply the Embedding lemma (also called Key lemma) to find an ℓ -cycle in G . For a statement and proof of the Embedding lemma, see e.g. the survey [19]. \square

Proof of Theorem 6(i). Note that Lemma 22(ii) implies that in order to prove part (i) it suffices to show that every oriented graph H with $\delta^+(H) \geq |H|/k + 149|H|/k^2$ contains a closed walk of length ℓ . Theorem 2 implies that H contains an a -cycle C for some $a \leq 1/(1/k + 149/k^2) + 74 < k$. But $a > 2$ since H is oriented and thus a divides ℓ by our definition of k . By traversing C precisely ℓ/a times we obtain the required closed walk of length ℓ in H . \square

Note that the proof actually shows the following: Let c be such that every oriented graph G with $\delta^+(G) \geq d$ has a cycle of length at most $\lceil cn/d \rceil$. Then for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon, \ell)$ such that every oriented graph G on $n \geq n_0$ vertices with $\delta^+(G) \geq cn/(k-1) + \varepsilon n$ contains an ℓ -cycle (where ℓ and k are as in Theorem 6). In particular, if we assume the Caccetta-Häggkvist conjecture, then this implies that Conjecture 5 is approximately true if we replace k by $k-1$. Similarly, the result in [9] which gives a cycle of length at most $\lceil 2n/(d+1) \rceil$ in an oriented graph of minimum outdegree at least d implies that we may take $c := 2$. It would be interesting to find improved approximate versions of Conjecture 5.

To prove parts (ii) and (iii) of Theorem 6, we will use the following lemma.

Lemma 23. *Let G be an oriented graph on n vertices.*

- (i) *If $\delta^0(G) \geq n/4$ then either the diameter of G is at most 6 or G contains a 3-cycle.*
- (ii) *If $\delta^0(G) > n/5$ then either the diameter of G is at most 50 or G contains a 3-cycle.*

Proof. We first prove (i). Consider $x \in V(G)$ and define $X_1 := N^+(x)$ and $X_{i+1} := N^+(X_i) \cup X_i$ for $i \geq 1$. If there exists an i with $\delta^+(G[X_i]) > 3|X_i|/8$ then $G[X_i]$ contains a 3-cycle by Theorem 3. So assume not. Then there exists a vertex $x_i \in X_i$ with $|N^+(x_i) \cap X_i| \leq 3|X_i|/8$. Hence

$$|X_{i+1}| \geq |X_i| + (\delta^0(G) - 3|X_i|/8) \geq 5|X_i|/8 + n/4.$$

In particular $|X_2| \geq 13n/32$ and $|X_3| \geq 65n/256 + n/4 = 129n/256 > n/2$. Similarly, for any vertex $y \neq x$ we have that $|\{v \in V(G) : \text{dist}(v, y) \leq 3\}| > n/2$, and thus there exists an x - y path of length at most 6, which completes the proof of (i).

To prove (ii), define sets X_i as before. Consider any i for which $|X_i| \leq n/2$. Similarly as before

$$\begin{aligned} |X_{i+1}| &\geq |X_i| + (\delta^0(G) - 3|X_i|/8) > |X_i| + (n/5 - 3|X_i|/8) \geq |X_i| + (n/5 - 3n/16) \\ &= |X_i| + n/80. \end{aligned}$$

Thus $|X_{25}| > n/2$. Similarly, for any vertex $y \neq x$ we have that $|\{v \in V(G) : \text{dist}(v, y) \leq 25\}| > n/2$. Thus there exists an x - y path of length at most 50. \square

Proof of Theorem 6(ii). As in the proof of (i), by Lemma 22(i) it suffices to show that every sufficiently large oriented graph H with $\delta^0(H) \geq |H|/4 + 1$ contains a closed walk of length ℓ . If H has a 3-cycle then it contains such a walk since 3 divides ℓ by definition of k . Thus we may assume that H has no 3-cycle. Fact 15 implies that the maximum size of an independent set is smaller than the neighbourhood $N_H(v)$ of any vertex v . Thus H contains some orientation of a triangle. By assumption this is not a 3-cycle, and so it must be transitive, i.e. the triangle consists of vertices x, y, z and edges xz, xy, zy .

Since $H - z$ has no 3-cycle, Lemma 23(i) implies that $H - z$ contains a y - x path P of length $t \leq 6$. This gives us 2 cycles $C_1 := yPxy$ and $C_2 := yPxyz$ of lengths $t+1$ and $t+2$ respectively. Write ℓ as $\ell = a(t+1) + r$ with $0 \leq r \leq t \leq 6$. We can wind r times around C_2 and $(a-r)$ times around C_1 to find a closed walk of length ℓ in H provided that $r \leq a$. But the latter holds as $a = \lfloor \ell/(t+1) \rfloor \geq 6$. \square

In the proof of Theorem 6(iii), we will use the following result (on undirected graphs) of Andrásfai, Erdős and Sós [3]:

Theorem 24. *Every triangle-free graph F on n vertices with minimum degree $\delta(F) > 2n/5$ is bipartite.*

Proof of Theorem 6(iii). Again, by Lemma 22(i) it suffices to show that every sufficiently large oriented graph H on n vertices with $\delta^0(H) > n/5 + 1$ contains a closed walk of length ℓ .

Let F be the underlying undirected graph of H . Since H has no double edges, we have $\delta(F) > 2n/5$. Suppose first that F contains a triangle. This cannot correspond to a 3-cycle in H , as this in turn immediately yields a closed walk of length ℓ in H . So H must contain a transitive triangle, i.e. vertices x, y, z with $xz, xy, zy \in E(H)$. We can now proceed similarly as in the proof of Theorem 6(ii): by Lemma 23(ii) we can find a y - x path P of length $t \leq 50$ in $H - z$. This gives us 2 cycles $C_1 := yPxy$ and $C_2 := yPxzy$ of lengths $t + 1$ and $t + 2$ respectively. To obtain a closed walk of length ℓ , write ℓ as $\ell = a(t + 1) + r$ with $0 \leq r \leq t \leq 50$. We can wind r times around C_2 and $(a - r)$ times around C_1 to find a closed walk of length ℓ in H provided that $r \leq a$. But the latter holds as $a = \lfloor \ell / (t + 1) \rfloor \geq 50$.

So now suppose that F does not contain a triangle. Then Theorem 24 implies that F (and thus H) is bipartite. We will now use this to find a 4-cycle in H . (This immediately yields a closed walk of length ℓ in H .) So suppose that H has no 4-cycle. Write $\delta_0 := \lceil n/5 \rceil + 1$. Denote the vertex classes of H by A and B . Let $a := |A|$ and $b := |B|$, where without loss of generality we have $b \leq n/2$. On the other hand $b \geq \delta(F) \geq 2n/5$ and so $a \leq 3n/5$. Now consider any $v \in A$. Choose a set $X_1 \subseteq N^+(v)$ and $Y_1 \subseteq N^-(v)$ with $|X_1| = |Y_1| = \delta_0$. Let $X_2 := N^+(X_1)$ and $Y_2 := N^-(Y_1)$. Note that X_2 and Y_2 are disjoint, as otherwise we would have a 4-cycle (through v) in H . The number of edges from X_1 to X_2 is at least $|X_1|\delta_0$, so by averaging there is a vertex $x \in X_2$ which receives at least $|X_1|\delta_0/|X_2|$ edges from X_1 . This in turn means that x sends at most $|X_1|(1 - \delta_0/|X_2|)$ edges to X_1 . Recall that x does not send an edge to Y_1 since otherwise $x \in X_2 \cap Y_2 = \emptyset$. So if we let $Z := B \setminus (X_1 \cup Y_1)$, then x sends at least $\delta_0 - |X_1|(1 - \delta_0/|X_2|) = \delta_0^2/|X_2|$ edges to Z . In particular, $|Z| \geq \delta_0^2/|X_2|$. On the other hand, $|Z| = b - 2\delta_0 \leq n/10$. So $|X_2| \geq \delta_0^2/(n/10) \geq 2\delta_0$. Since X_2 and Y_2 are disjoint, this implies that $|Y_2| \leq a - |X_2| \leq 3n/5 - 2\delta_0 < n/5$. On the other hand, the definition of Y_2 implies that $|Y_2| \geq \delta^0(H)$, a contradiction. \square

Proof of Proposition 13. First suppose that ℓ is even. The inequality $\delta_{di}(\ell, n) \geq \delta_{orient}(\ell, n)$ is trivial. For the upper bound on $\delta_{di}(\ell, n)$, suppose we are given a digraph H on n vertices with $\delta^0(H) \geq \delta_{orient}(\ell, n)$. If H has a double edge, it has a closed walk of length ℓ . If it has no double edge, then H has an ℓ -cycle by definition of $\delta_{orient}(\ell, n)$. So in both cases, H has a closed walk of length ℓ . So part (iii) of Lemma 22 implies that for each $\varepsilon > 0$ there is an n_0 so that for all $n \geq n_0$ we have $\delta_{di}(\ell, n) \leq \delta_{orient}(\ell, n) + \varepsilon n$, as required.

If ℓ is odd, we obtain the lower bound by considering the complete bipartite digraph with vertex class sizes as equal as possible. The upper bound follows e.g. from (1). \square

5. PROOF OF RESULTS ON ARBITRARY ORIENTATIONS

5.1. Proof of Proposition 11. For both parts of Proposition 11, the proof divides into three steps.

- (1) For a given ℓ -cycle C with cycle-type k find an appropriate walk W with prescribed orientation (which will be a cycle for $k \geq 3$) into which there is a digraph homomorphism of C .

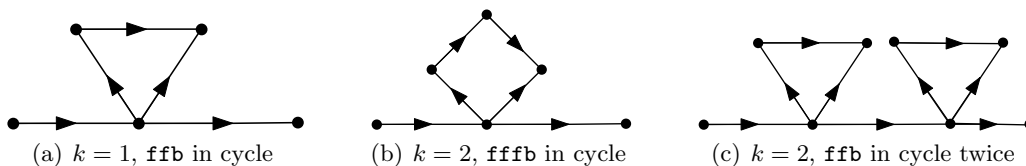


FIGURE 2. The walks needed in the cases $k = 1$ and $k = 2$.

- (2) Prove that the minimum semidegree condition in Proposition 11 guarantees a copy of W in any sufficiently large oriented graph G .
- (3) Apply Lemma 22(iv) to ‘lift’ this result to one on the cycle C itself.

Let us start with the first step. For $k = 0$ it is clear that there is a digraph homomorphism of C into a directed path of length ℓ . For $k \geq 3$ we can let W be a directed k -cycle. Suppose that $k = 1$. Then the number of edges of C oriented forwards is one larger than the number of its edges oriented backwards. So C must contain a subpath of the form \mathbf{ffb} , where we write \mathbf{f} for an edge oriented forwards and \mathbf{b} for an edge oriented backwards. But this means that there exist constants $0 \leq k_1, k_2 < \ell$ (depending on C) such that there is a digraph homomorphism of C into the oriented walk W obtained by adding a transitive triangle to the k_1 th vertex of a directed path of length k_2 (see Figure 2(a)).

Finally suppose that $k = 2$. So C contains two more edges oriented forwards than backwards. Hence C contains two subpaths of the form \mathbf{ffb} or one subpath of the form \mathbf{fffb} . In the first case we take W to be a suitable directed path of length less than ℓ with two transitive triangles attached, possibly to the same vertex (see Figure 2(c)). In the second case we let W be a suitable directed path with a 4-cycle oriented \mathbf{fffb} attached (see Figure 2(b)).

For the second step we have to show that the relevant minimum semidegree condition implies the existence of W in G . If W is a path then we only need the minimum semidegree to be at least ℓ . If W is a k -cycle then we just apply Theorem 4. So suppose that $k = 1, 2$ and consider any vertex x of G . The minimum semidegree condition $\delta^0(G) \geq (1/3 + \alpha)n$ implies each vertex $y \in N^+(x)$ has at least αn neighbours in $N^+(x)$. So G contains the transitive triangles needed in Figures 2(a) and (c). To see that we can also find the 4-cycle oriented \mathbf{fffb} , suppose that $N := G[N^+(x)]$ does not contain a directed path of length 2 (otherwise we are done). Then N must contain two distinct vertices y and y' such that y has no outneighbours in N and y' has no inneighbours in N . But this means that there is some $z \in N^+(y) \cap N^-(y')$ and then $xyzy'$ has the required orientation \mathbf{fffb} . Hence we can find any of the walks in Figure 2 greedily. An application of Lemma 22(iv) now completes the proof of Proposition 11. The argument for the case $k \geq 3$ also shows that Conjecture 5 would imply an approximate version of Conjecture 12.

5.2. Proof of universal pancyclicity result. To deduce Theorem 10 from Theorem 9 and Proposition 11 we will use the following observation which is similar to one in [20].

Lemma 25. *There exists an integer n_1 such that the following holds for all $0 < \alpha < 1$. Suppose we are given an oriented graph G on $n \geq n_1$ vertices with minimum semidegree $\delta^0(G) \geq (3/8 + \alpha - n^{-3/8})n$ where $n/2 \in \mathbb{N}$. Then there is a subset $U \subseteq V(G)$ of size $|U| = n/2 := u$ such that $\delta^0(G[U]) \geq (3/8 + \alpha - u^{-3/8})u$.*

To prove it we need a large deviation bound for the hypergeometric distribution (see e.g. [14, Theorem 2.10]).

Lemma 26. *Given $q \in \mathbb{N}$ and sets $A \subseteq T$ with $|T| \geq q$, let Q be a subset of size q of T chosen uniformly at random. Let $X := |A \cap Q|$. Then for all $0 < \varepsilon < 1$ we have*

$$\mathbb{P}[|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)] \leq 2 \exp\left(-\frac{\varepsilon^2}{3} \mathbb{E}(X)\right).$$

Proof of Lemma 25. Consider a subset U of vertices of G chosen uniformly at random from all subsets of $V(G)$ of size u . Let $\varepsilon := (1 - 2^{-3/8})u^{-3/8}$. Consider any vertex x of G and define a random variable $X^+ := |N^+(x) \cap U|$. Observe that $\varepsilon \mathbb{E}(X^+) \leq \varepsilon u = (1 - 2^{-3/8})u^{5/8}$ and hence

$$\mathbb{E}(X^+) \geq (3/8 + \alpha - n^{-3/8})u = (3/8 + \alpha - u^{-3/8})u + \varepsilon \mathbb{E}(X^+).$$

Then by Lemma 26 we have

$$\mathbb{P}[X^+ \leq (3/8 + \alpha - u^{-3/8})u] \leq \mathbb{P}[X^+ \leq (1 - \varepsilon)\mathbb{E}(X^+)] \leq 2 \exp\left(-\frac{(1 - 2^{-3/8})^2 u}{3u^{3/4} \cdot 4}\right) \leq n^{-2}.$$

The final inequality holds since we assume n , and hence u , to be sufficiently large. The same bound holds when we consider inneighbourhoods of vertices. Hence with positive probability there exists a set $U \subseteq V(G)$ with the desired minimum semidegree property. \square

We are now in a position to derive Theorem 10.

Proof of Theorem 10. Given $\alpha > 0$, set $\ell_0 := \max\{n_0(\alpha/3), n_1, (6/\alpha)^{8/3}\}$, where n_0 is the function defined in Theorem 9 and n_1 is as in Lemma 25. Let $n \gg \ell_0, 1/\alpha$ and consider an oriented graph G on n vertices with minimum semidegree $\delta^0(G) \geq (3/8 + \alpha)n$. Choose any $3 \leq \ell \leq n$ and any orientation C of an ℓ -cycle. We have to show that G contains a copy of C . This is clear if $\ell \leq \ell_0$, since $n \gg \ell_0, 1/\alpha$ and thus an application of Proposition 11 gives us C immediately.

So we may assume that $\ell > \ell_0$. Let k be an integer such that $2^k \ell \leq n < 2^{k+1} \ell$. A straightforward application of Lemma 26 implies the existence of a subgraph G' of G on $n' := 2^k \ell$ vertices with $\delta^0(G') \geq (3/8 + \alpha/2)n'$. Apply Lemma 25 k times to obtain a subgraph G'' of G' on ℓ vertices with $\delta^0(G'') \geq (3/8 + \alpha/2 - \ell^{-3/8})\ell \geq (3/8 + \alpha/3)\ell$. Since $\ell > n_0(\alpha/3)$ we can now apply Theorem 9 to obtain a Hamilton cycle oriented as C in G'' and hence the desired orientation of an ℓ -cycle in G . \square

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