OPTIMAL PATH AND CYCLE DECOMPOSITIONS OF DENSE QUASIRANDOM GRAPHS

STEFAN GLOCK, DANIELA KÜHN AND DERYK OSTHUS

ABSTRACT. Motivated by longstanding conjectures regarding decompositions of graphs into paths and cycles, we prove the following optimal decomposition results for random graphs. Let $0 < p < 1$ be constant and let $G \sim G_{n,p}$. Let $odd(G)$ be the number of odd degree vertices in G. Then a.a.s. the following hold:
  (i) G can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles and a matching of size $odd(G)/2$.
  (ii) G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.
  (iii) G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests.
Each of these bounds is best possible. We actually derive (i)–(iii) from 'quasi-random' versions of our results. In that context, we also determine the edge chromatic number of a given dense quasirandom graph of even order. For all these results, our main tool is a result on Hamilton decompositions of robust expanders by Kühn and Osthus.

1. INTRODUCTION

There are several longstanding and beautiful conjectures on decompositions of graphs into cycles and/or paths. In this paper, we consider four of the most well-known in the setting of dense quasirandom and random graphs: the Erdős-Gallai conjecture, the Gallai conjecture on path decompositions, the linear arboricity conjecture as well as the overfull subgraph conjecture.

1.1. **Decompositions of random graphs.** A classical result of Lovász [23] on decompositions of graphs states that the edges of any graph on n vertices can be decomposed into at most $\lfloor n/2 \rfloor$ cycles and paths. Erdős and Gallai [9, 10] made the related conjecture that the edges of every graph G on n vertices can be decomposed into $\mathcal{O}(n)$ cycles and edges. Conlon, Fox and Sudakov [6] recently showed that $\mathcal{O}(n \log \log n)$ cycles and edges suffice and that the conjecture holds for graphs with linear minimum degree. They also proved that the conjecture holds a.a.s. for the binomial random graph $G \sim G_{n,p}$. Korándi, Krivelevich and Sudakov [18] carried out a more systematic study of the problem for $G_{n,p}$: for a large range of p, a.a.s. $G_{n,p}$ can be decomposed into $n/4 + np/2 + o(n)$ cycles and edges, which is asymptotically best possible. They also asked for improved error terms. For constant p, we will give an exact formula.

A further related conjecture of Gallai (see [23]) states that every connected graph on n vertices can be decomposed into $\lceil n/2 \rceil$ paths. The result of Lovász mentioned above implies that for every (not necessarily connected) graph $n - 1$ paths suffice. This has been improved to $\lfloor 2n/3 \rfloor$ paths [8, 28]. Here we determine the number of paths in an optimal path decomposition of $G_{n,p}$ for constant p. In particular this implies that Gallai's conjecture holds (with room to spare) for almost all graphs.

Next, recall that an edge colouring of a graph is a partition of its edge set into matchings. A matching can be viewed as a forest whose connected components are edges. As a relaxation of this, a *linear forest* is a forest whose components are paths, and the least possible number of linear forests needed to partition the edge set of a graph G is called the *linear arboricity* of G, denoted by $la(G)$. Clearly, in order to cover all edges at any vertex of maximum degree, we need at least $\lceil \Delta(G)/2 \rceil$ linear forests. However, for some graphs (e.g. complete graphs on an odd number of vertices) we need at least $\lceil (\Delta(G)+1)/2 \rceil$ linear forests. The following conjecture is known as the linear arboricity conjecture and can be viewed as an analogue to Vizing's theorem.

Conjecture 1.1 (Akiyama, Exoo, Harary [1])**.** *For every graph G, $la(G) \leq \lceil (\Delta(G)+1)/2 \rceil$.*

This is equivalent to the statement that for all d-regular graphs G, $la(G) = \lceil (d+1)/2 \rceil$. Alon [2] proved an approximate version of the conjecture for sufficiently large values of $\Delta(G)$. Using his approach, McDiarmid and Reed [24] confirmed the conjecture for random regular graphs with fixed degree. We will show that, for a large range of p, a.a.s. the random graph $G \sim G_{n,p}$ can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests. Moreover, we use the recent confirmation [7] of the so-called 'Hamilton decomposition conjecture' to deduce that the linear arboricity conjecture holds for large and sufficiently dense regular graphs (see Corollary 6.4).

The following theorem summarises our optimal decomposition results for dense random graphs. We denote by $odd(G)$ the number of odd degree vertices in a graph G.

Theorem 1.2. *Let $0 < p < 1$ be constant and let $G \sim G_{n,p}$. Then a.a.s. the following hold:*

    (i) *G can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles and a matching of size $odd(G)/2$.*
    (ii) *G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.*
    (iii) *G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests, i.e. $la(G) = \lceil \Delta(G)/2 \rceil$.*

Clearly, each of the given bounds is best possible. Moreover, as observed e.g. in [18] for a large range of p, a.a.s. $odd(G_{n,p}) = (1 + o(1))n/2$. This means that for fixed $p < 1/2$, the size of an optimal path decomposition of $G_{n,p}$ is determined by the number of odd degree vertices, whereas for $p > 1/2$, the maximum degree is the crucial parameter.

A related result of Gao, Pérez-Giménez and Sato [11] determines the arboricity and spanning tree packing number of $G_{n,p}$. Optimal results on packing Hamilton cycles in $G_{n,p}$ which together cover essentially the whole range of p were proven in [17, 19].

One can extend Theorem 1.2(iii) to the range $\frac{\log^{117} n}{n} \leq p = o(1)$ by applying a recent result in [14] on covering $G_{n,p}$ by Hamilton cycles (see Corollary 6.2). It would be interesting to obtain corresponding exact results also for (i) and (ii). In particular we believe that the following should hold.

Conjecture 1.3. *Suppose $p = o(1)$ and $\frac{pn}{\log n} \to \infty$. Then a.a.s. $G \sim G_{n,p}$ can be decomposed into $odd(G)/2$ paths.*

By tracking the number of cycles in the decomposition constructed in [18] and by splitting every such cycle into two paths, one immediately obtains an approximate version of Conjecture 1.3. Note that this argument does not yield an approximate version of Theorem 1.2(ii) in the case when p is constant.

1.2. **Dense quasirandom graphs.** As mentioned earlier, we will deduce Theorem 1.2 from quasirandom versions of these results. As our notion of quasirandomness, we will consider the following one-sided version of ε-regularity. Let $0 < \varepsilon, p < 1$. A graph G on n vertices is called *lower-(p, ε)-regular* if we have $e_G(S, T) \geq (p - \varepsilon)|S||T|$ for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$.

The next theorem is a quasirandom version of Theorem 1.2(i). Similarly, we will also prove quasirandom versions of parts (ii) and (iii) (see Theorem 5.1).

Theorem 1.4. *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p, ε)-regular graph on n vertices. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$ and that G is Eulerian. Then G can be decomposed into $\Delta(G)/2$ cycles.*

This confirms the following conjecture of Hajós (see [23]) for quasirandom graphs (with room to spare): Every Eulerian graph on n vertices has a decomposition into $\lfloor n/2 \rfloor$ cycles. (It is easy to see that this conjecture implies the Erdős-Gallai conjecture.)

We will also apply our approach to edge colourings of dense quasirandom graphs. Recall that in general it is NP-complete to decide whether a graph G has chromatic index $\Delta(G)$ or $\Delta(G) + 1$ (see for example [15]). We will show that for dense quasirandom graphs of even order this decision problem can be solved in quadratic time without being trivial. For this, call a subgraph H of G *overfull* if $e(H) > \Delta(G)\lfloor |H|/2 \rfloor$. Clearly, if G contains any overfull subgraph, then $\chi'(G) = \Delta(G) + 1$. The following conjecture is known as the overfull subgraph conjecture and dates back to 1986.

Conjecture 1.5 (Chetwynd, Hilton [5])**.** *A graph G on n vertices with $\Delta(G) > n/3$ satisfies $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph.*

This conjecture implies the 1-factorization conjecture, that every regular graph of sufficiently high degree and even order can be decomposed into perfect matchings, which was recently proved for large graphs in [7]. Minimum degree conditions under which the overfull subgraph conjecture is true were first investigated in [4, 26]. (We refer to [27] for a more thorough discussion of the area.) We prove the overfull subgraph conjecture for quasirandom graphs of even order, even if the maximum degree is smaller than stated in the conjecture, as long as it is linear.

Theorem 1.6. *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p, ε)-regular graph on n vertices and n is even. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$. Then $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph. Further, there is a polynomial time algorithm which finds an optimal colouring.*

At the first glance, the overfull subgraph criterion seems not very helpful in terms of time complexity, as it involves all subgraphs of G. (On the other hand, Niessen [25] proved that in the case when $\Delta(G) \geq |G|/2$ there is a polynomial time algorithm which finds all overfull subgraphs.) Our proof of Theorem 1.6 will actually yield a simple criterion whether G is class 1 or class 2. Moreover, the proof is constructive, thus using appropriate running time statements for our tools, this yields a polynomial time algorithm which finds an optimal colouring.

The condition of n being even is essential for our proof as we will colour Hamilton cycles with two colours each. It would be interesting to obtain a similar result for graphs of odd order.

Conjecture 1.7. *For every $0 < p < 1$ there exist $\varepsilon, \eta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Whenever G is a lower-(p, ε)-regular graph on $n \geq n_0$ vertices, where n is odd, and $\Delta(G) - \delta(G) \leq \eta n$, then $\chi'(G) = \Delta(G)$ if and only if $\sum_{x \in V(G)} (\Delta(G) - d_G(x)) \geq \Delta(G)$.*

Note that the condition $\sum_{x \in V(G)} (\Delta(G) - d_G(x)) \geq \Delta(G)$ in Conjecture 1.7 is equivalent to the requirement that G itself is not overfull. Also note that the corresponding question for $G_{n,p}$ is easily solved if p does not tend to 0 or 1 too quickly: It is well-known that in this case a.a.s. $G \sim G_{n,p}$ satisfies $\chi'(G) = \Delta(G)$, which follows from the fact that a.a.s. G has a unique vertex of maximum degree.

This paper is organised as follows. In the next section, we will introduce the basic terminology. Section 3 discusses lower-ε-regularity, robust expansion and collects some probabilistic results that we will use later on. It also introduces our main tool, namely that robust expanders of linear degree have a Hamilton decomposition, which was recently proven in [20]. In Section 4 we will prove Theorem 1.4. Building on this, we will prove a quasirandom version of Theorem 1.2(ii) and (iii) in Section 5, which altogether implies Theorem 1.2. Section 6 contains two additional results concerning linear arboricity. The last section is devoted to edge colouring, i.e. the proof of Theorem 1.6.

2. Notation

All graphs considered in this paper are finite and do not contain loops. A graph with parallel edges is referred to as a multigraph. Let G be a graph, multigraph or digraph. As usual, we let $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively. We further let $|G| := |V(G)|$ and $e(G) := |E(G)|$. Given $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U, and $G - U := G[V(G) \setminus U]$. If $F \subseteq E(G)$, then let $G \setminus F$ be obtained from G by removing all edges in F, and if H is a subgraph of G, then $G \setminus H := G \setminus E(H)$. Moreover, if G is a graph and F a set of edges in the complement of G, then we let $G \cup F$ denote the graph obtained from G by adding the edges in F.

Given a graph G, the degree of a vertex v is denoted by $d_G(v)$, and the set of all neighbours of v is denoted by $N_G(v)$. For a set $S \subseteq V(G)$, we define $d_S(v) := |N_G(v) \cap S|$. For $S, T \subseteq V(G)$ disjoint, let $e_G(S, T)$ be the number of edges in G between S and T.

For a digraph G, let $N_G^+(v)$ and $N_G^-(v)$ denote the outneighbourhood and inneighbourhood of a vertex v, respectively. Furthermore, $d_G^+(v)$ is the outdegree of v and $d_G^-(v)$ the indegree. We also set $d_G(v) := d_G^+(v) + d_G^-(v)$ and then define $\Delta(G) := \max_{v \in V(G)} d_G(v)$ and $\delta(G) := \min_{v \in V(G)} d_G(v)$. A digraph G is called *Eulerian* if $d_G^+(v) = d_G^-(v)$ for every vertex v, and *r-regular* if $d_G^+(v) = d_G^-(v) = r$ for every vertex v. The *minimum semidegree* of G is defined as $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$, where $\delta^\pm(G) := \min_{v \in V(G)} d_G^\pm(v)$. For $S, T \subseteq V(G)$ disjoint, let $e_G(S, T)$ be the number of edges in G directed from S to T. Given a fixed digraph G, for a vertex v and a set $S \subseteq V(G)$, we define $d_S^\pm(v) := |N_G^\pm(v) \cap S|$. Paths and cycles in digraphs are always supposed to be directed.

If G is a multigraph and $v \in V(G)$, then $d_G(v)$ counts the number of edges v is incident with, where parallel edges are counted with multiplicity.

We denote by $G_{n,p}$ the binomial random graph on n vertices, that is, the random graph which is generated by including each of the $\binom{n}{2}$ possible edges independently

with probability p. If a random variable X has binomial distribution with parameters n and p, we write $X \sim Bin(n, p)$. We say that a property $P = P(n)$ holds *asymptotically almost surely (a.a.s.)* if the probability that P holds tends to 1 as n tends to infinity. For the sake of readability, we assume large quantities to be integers whenever this does not affect the argument.

We will also make use of the following notation: $0 < a \ll b \leq 1$. More precisely, if we claim that a statement is true whenever $0 < a \ll b \leq 1$, then this means that there exists a non-decreasing function $f: (0, 1] \to (0, 1]$ such that the statement holds for all $0 < a, b \leq 1$ satisfying $a \leq f(b)$. Intuitively, this means that for every $b > 0$, the statement is true provided that a is sufficiently small compared to b.

3. Tools and preliminary results

3.1. **Quasirandomness.** Here we collect some basic properties of lower-(p, ε)-regular graphs and digraphs. Let $0 < \varepsilon, p < 1$. A (di-)graph G on n vertices is called *lower-(p, ε)-regular* if we have

$$e_G(S, T) \geq (p - \varepsilon)|S||T|$$

for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$. Note that p cannot be viewed as the density of G, as every spanning supergraph of a lower-(p, ε)-regular graph is also lower-(p, ε)-regular.

The following proposition says that we can modify a lower-ε-regular graph slightly such that it is still lower-ε'-regular. Its proof is straightforward using the definition of lower-ε-regularity.

Proposition 3.1. *Let $0 < 1/n_0 \ll \varepsilon, p < 1$. Let G be a lower-(p, ε)-regular (di-)graph on $n \geq n_0$ vertices. Then the following hold:*

(i) *If G' is obtained from G by adding a new vertex w and arbitrary edges at w, then G' is lower-$(p, 2\varepsilon)$-regular.*

(ii) *Let H be a graph on $V(G)$ such that $\Delta(H) \leq \eta n$. Let $\varepsilon' := \max\{2\varepsilon, 2\sqrt{\eta}\}$. Then $G \setminus H$ is lower-(p, ε')-regular.*

(iii) *If $U \subseteq V(G)$ has size at least βn, then $G[U]$ is lower-$(p, \varepsilon/\beta)$-regular.*

Proposition 3.2. *Let $0 < 1/n_0 \ll \varepsilon, \eta \ll p < 1$. Let G be a lower-(p, ε)-regular digraph on $n \geq n_0$ vertices. Assume that G is Eulerian and $\Delta(G) - \delta(G) \leq \eta n$. Then $\delta^0(G) \geq pn/2$.*

Proof. Consider a partition of $V(G)$ into sets S, T such that $|S| = \varepsilon n$ and $|T| = (1 - \varepsilon)n$. We have $e_G(S, T) \geq (p - \varepsilon)|S||T|$ and $e_G(S, T) \leq |S|\Delta(G)/2$. Thus, $\Delta(G) \geq 2(p - \varepsilon)(1 - \varepsilon)n$, implying $\delta^0(G) = \delta(G)/2 \geq (p - \varepsilon)(1 - \varepsilon)n - \eta n/2 \geq pn/2$. □

The next lemma states that in a lower-(p, ε)-regular digraph with linear minimum semidegree, we can always find a Hamilton cycle. This is a folklore observation for ε-regularity, but in fact, carries over to lower-ε-regular graphs and digraphs. It can be seen either by using the fact that every lower-ε-regular digraph is a robust outexpander (see Subsection 3.3) and then applying a result from [22], or directly by considering a longest directed path and using the lower-ε-regularity to find a cycle on the same vertex set, which must then be a Hamilton cycle. Using the latter approach, it is moreover easy to see that a Hamilton cycle can be found in polynomial time: Start with any path and extend it in both directions until all inneighbours of the startvertex and all outneighbours of the endvertex lie on the path. Then use the

lower-ε-regularity to find a cycle on the same vertex set. If this is not a Hamilton cycle, modify the cycle into a path on more vertices and proceed as above.

Lemma 3.3. *Let* $0 < 1/n_0 \ll \varepsilon \ll \alpha, p < 1$.

(i) *Let* G *be a lower-*(p, ε)*-regular digraph on* $n \geq n_0$ *vertices such that* $\delta^0(G) \geq \alpha n$. *Then* G *contains a Hamilton cycle.*

(ii) *Let* G *be a lower-*(p, ε)*-regular graph on* $n \geq n_0$ *vertices such that* $\delta(G) \geq \alpha n$. *Then* G *contains a Hamilton cycle.*

(iii) *Let* G *be a lower-*(p, ε)*-regular graph on* $n \geq n_0$ *vertices such that* $\delta(G) \geq \alpha n$. *Then for all distinct* $a, b \in V(G)$, *there exists a Hamilton path joining* a *and* b.

Moreover, the Hamilton cycles in (i)–(ii) and the Hamilton path in (iii) can be found in polynomial time.

Clearly, (i) immediately implies (ii). To deduce (iii) from (i), contract $\{a, b\}$ into a single vertex and orient the edges at $\{a, b\}$ from $\{a, b\}$ towards $N_G(a) \setminus \{b\}$ and from $N_G(b) \setminus \{a\}$ towards $\{a, b\}$. Replace all other edges by two edges which are oriented in opposite directions. By (i) and Proposition 3.1, the graph thus obtained contains a Hamilton cycle, which in turn induces a Hamilton path from a to b in G.

3.2. **Probabilistic preliminaries.** In this subsection, the following Chernoff-Hoeffding type bound is often used, sometimes without stating the explicit calculation.

Lemma 3.4 (see [16]). *Suppose that* X *is the sum of finitely many independent indicator variables. Then, for all* $0 < \varepsilon < 1$,

(i) $\mathbb{P}(X \leq (1 - \varepsilon)\mathbb{E}X) \leq e^{-\frac{\varepsilon^2}{3}\mathbb{E}X}$,

(ii) $\mathbb{P}(X \geq (1 + \varepsilon)\mathbb{E}X) \leq e^{-\frac{\varepsilon^2}{3}\mathbb{E}X}$.

In particular, the above bounds hold if X *has binomial distribution.*

The next lemma collects some well-known properties of random graphs.

Lemma 3.5. *Let* $0 < \varepsilon, p < 1$ *be constant. The following holds a.a.s. for the random graph* $G \sim G_{n,p}$:

(i) $\Delta(G) - \delta(G) \leq 4\sqrt{n \log n}$,

(ii) G *is lower-*(p, ε)*-regular,*

(iii) G *has a unique vertex of maximum degree.*

Indeed, using Lemma 3.4, it is easy to see that a.a.s. $|d_G(x) - np| \leq 2\sqrt{n \log n}$ for all $x \in V(G)$, which implies (i). Similarly, using Lemma 3.4, it is also straightforward to show (ii). For (iii), we refer to Theorem 3.15 in [3].

The following result is a variant of the observation that a random partition of a graph splits all vertex degrees evenly. Given a (di-)graph G and a matching M in the complete graph on $V(G)$, we say that a set $S \subseteq V(G)$ *separates* M if there exists an edge e of M such that e has precisely one endvertex in S.

Lemma 3.6. *Let* $0 < 1/n_0 \ll \alpha < 1$. *Suppose that* G *is a digraph on* $n \geq n_0$ *vertices with* $\delta^0(G) \geq \alpha n$. *Let* M *be any matching in the complete graph on* $V(G)$. *Then there exists a set of vertices* S *for which the following hold:*

(i) $n/3 \leq |S| \leq 2n/3$.

(ii) *Every* $x \in V(G)$ *satisfies* $d_S^+(x), d_S^-(x), d_{\overline{S}}^+(x), d_{\overline{S}}^-(x) \geq \alpha n/6$, *where* $\overline{S} := V(G) \setminus S$.

(iii) S *does not separate* M.

Proof. Write $M = \{x_1y_1, \ldots, x_my_m\}$. By arbitrarily enlarging M if necessary, we may assume that M covers all but at most one vertex, i.e. $m = \lfloor n/2 \rfloor$. Let $X := \{x_1, \ldots, x_m\}$ and $Y := \{y_1, \ldots, y_m\}$. Let S be a random vertex set obtained by including each $\{x_i, y_i\}$ independently with probability $1/2$, for all $i \in [m]$. We now show that S satisfies (i)–(iii) simultaneously with positive probability. S automatically satisfies (iii). Let $S_X := S \cap X$. Then $|S_X| \sim Bin(m, 1/2)$. Hence by Lemma 3.4,

$$\mathbb{P}(||S_X| - m/2| > m/8) \leq 2\mathrm{e}^{-(1/4)^2 m/6}.$$

Since $|S| = 2|S_X|$, this implies that a.a.s. $n/3 \leq |S| \leq 2n/3$.

Consider any vertex $v \in V(G)$. Assume that $d_X^+(v) \geq (d_G^+(v) - 1)/2$. Otherwise, $d_Y^+(v) \geq (d_G^+(v) - 1)/2$ and we can proceed analogously. We have $d_{S_X}^+(v) \sim Bin(d_X^+(v), 1/2)$. Thus,

$$\mathbb{P}(|d_{S_X}^+(v) - d_X^+(v)/2| > d_X^+(v)/8) \leq 2\mathrm{e}^{-(1/4)^2 d_X^+(v)/6} \leq 2\mathrm{e}^{-\alpha n/300} \leq 1/n^2.$$

Note that $3d_X^+(v)/8 \leq d_{S_X}^+(v) \leq 5d_X^+(v)/8$ and our assumption that $d_X^+(v) \geq (d_G^+(v) - 1)/2$ together imply that $d_S^+(v) \geq d_{S_X}^+(v) \geq d_G^+(v)/6$ and $d_S^+(v) \leq d_{S_X}^+(v) + d_Y^+(v) \leq 5d_X^+(v)/8 + (d_G^+(v) - d_X^+(v)) \leq 5d_G^+(v)/6$. We conclude

$$\mathbb{P}(|d_S^+(v) - d_G^+(v)/2| > d_G^+(v)/3) \leq 1/n^2.$$

Taking a union bound, we see that a.a.s. $d_G^+(v)/6 \leq d_S^+(v) \leq 5d_G^+(v)/6$ for all vertices $v \in V(G)$. The same applies to $d_S^-(v)$. $\qquad\square$

In order to obtain our optimal decomposition of $G_{n,p}$ into cycles and edges, we want to find a perfect matching on the odd degree vertices, which is achieved by the following lemma.

Lemma 3.7. *Let $0 < p < 1$ be constant and $G \sim G_{n,p}$. Then a.a.s. $G[D]$ has a perfect matching, where D is the set of odd degree vertices in G.*

Proof. Let $\alpha := p \cdot p_{min}/4$, where $p_{min} := \min\{p, 1 - p\}$. Choose a constant ε such that $0 < \varepsilon \ll \alpha, p$. First, we prove that a.a.s. $\delta(G[D]) \geq \alpha n$.

For any vertex v, let $d_G^{odd}(v)$ denote the number of neighbours of v which have odd degree in G. Fix a vertex v and choose any vertex w distinct from v. Let $U := V(G) \setminus \{w\}$ and let Ω contain all possible outcomes of $E(G[U])$. For $\tilde{E} \in \Omega$ and any vertex $x \in U$, let $d_{\tilde{E}}(x)$ denote the degree of x in the graph $(U, \tilde{E})$. Let $\Omega_1 := \{\tilde{E} \in \Omega : d_{\tilde{E}}(v) \geq np/2\}$ and $\Omega_2 := \Omega \setminus \Omega_1$. We want to show that $\mathbb{P}(d_G^{odd}(v) < \alpha n) \leq 1/n^2$. Fix some $\tilde{E} \in \Omega_1$ and let $d := d_{\tilde{E}}(v)$. Let $v_1, \ldots, v_d$ be the neighbours of v in $(U, \tilde{E})$. For every v_i, let X_i be the random indicator variable such that $X_i = 1$ if $d_{\tilde{E}}(v_i) + e_G(\{v_i\}, \{w\})$ is odd and $X_i = 0$ otherwise. Let $X := X_1 + \cdots + X_d$ and observe that X_i depends only on the presence of the edge v_iw. Hence, $X_1, \ldots, X_d$ are independent. Observe that $\mathbb{E}(X \mid E(G[U]) = \tilde{E}) \geq p_{min}d \geq 2\alpha n$. Therefore, by using Lemma 3.4,

$$\mathbb{P}(d_G^{odd}(v) < \alpha n \mid E(G[U]) = \tilde{E}) \leq \mathbb{P}(X < \alpha n \mid E(G[U]) = \tilde{E}) \leq \mathrm{e}^{-(1/2)^2 \alpha n/3}.$$

Moreover, since $d_{G-w}(v) \sim Bin(n - 2, p)$, we get

$$\mathbb{P}(E(G[U]) \in \Omega_2) = \mathbb{P}(d_{G-w}(v) < np/2) \leq \mathbb{P}(d_{G-w}(v) < 2(n-2)p/3) \leq \mathrm{e}^{-(1/3)^2(n-2)p/3}.$$

Therefore, we can conclude that

$$\mathbb{P}(d_G^{odd}(v) < \alpha n) \leq \mathrm{e}^{-\alpha n/12} + \mathrm{e}^{-np/28} \leq 1/n^2,$$

and a union bound yields that a.a.s. $d_G^{odd}(v) \geq \alpha n$ for all $v \in V(G)$.

Further, Lemma 3.5 implies that a.a.s. G is lower-(p, ε)-regular. So we may assume that G is lower-(p, ε)-regular and that $\delta(G[D]) \geq \alpha n$. Since $|D| \geq \alpha n$, Proposition 3.1(iii) now implies that $G[D]$ is lower-$(p, \varepsilon/\alpha)$-regular. Thus, $G[D]$ has a perfect matching by Lemma 3.3(ii). □

3.3. **Robust expansion.** Robust expansion is a natural relaxation of the concept of quasirandomness or ε-regularity. Roughly speaking, a graph is a robust out-expander if every vertex set S which is not too small and not too large has an outneighbourhood that is at least a little larger than S even if we delete a small number of vertices or edges. More precisely, let G be a digraph on n vertices and let $0 < \nu < 1$. For a set $S \subseteq V(G)$, the ν-robust outneighbourhood $RN_{\nu,G}^+(S)$ of S is the set of all those vertices $x \in V(G)$ which have at least νn inneighbours in S. If $\nu \leq \tau < 1$ and $|RN_{\nu,G}^+(S)| \geq |S| + \nu n$ for all $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1 - \tau)n$, then G is called a *robust (ν, τ)-outexpander*. The main result of [20] states that every regular robust outexpander of linear degree has a Hamilton decomposition, that is, a decomposition of its edges into edge-disjoint Hamilton cycles.

Theorem 3.8 (Kühn, Osthus [20]). *For all $\alpha > 0$ there exists $\tau > 0$ such that for every $\nu > 0$ there is an integer $n_0 = n_0(\alpha, \tau, \nu)$ for which the following holds. If G is a robust (ν, τ)-outexpander on $n \geq n_0$ vertices and G is an r-regular digraph, where $r \geq \alpha n$, then G has a Hamilton decomposition.*

Note that the function $\tau = \tau(\alpha)$ in Theorem 3.8 can be taken to be non-decreasing. Let G be a graph on n vertices and let $0 < \nu < 1$. For a set $S \subseteq V(G)$, the ν-robust neighbourhood $RN_{\nu,G}(S)$ of S is the set of all those vertices $x \in V(G)$ which have at least νn neighbours in S. If $\nu \leq \tau < 1$ and $|RN_{\nu,G}(S)| \geq |S| + \nu n$ for all $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1 - \tau)n$, then G is called a *robust (ν, τ)-expander*. The following undirected version of Theorem 3.8 is deduced from Theorem 3.8 in [21].

Theorem 3.9 (Kühn, Osthus [21]). *For all $\alpha > 0$ there exists $\tau > 0$ such that for every $\nu > 0$ there is an integer $n_0 = n_0(\alpha, \tau, \nu)$ for which the following holds. If G is a robust (ν, τ)-expander on $n \geq n_0$ vertices and G is an r-regular graph where $r \geq \alpha n$ and r is even, then G has a Hamilton decomposition. Moreover, this decomposition can be found in time polynomial in n.*

It is easy to see that lower-ε-regularity implies robust expansion, so we can apply the above results to lower-ε-regular graphs.

Proposition 3.10. *Let $0 < 1/n_0 \ll \varepsilon \ll \tau, p < 1$ and suppose that G is a lower-(p, ε)-regular (di-)graph on $n \geq n_0$ vertices. Then G is also a robust (ε, τ)-(out)expander.*

Our next aim is to prove that an Eulerian lower-ε-regular graph can be oriented in such a way that the digraph thus obtained is Eulerian and lower-ε-regular. To this end, we need the following lemma from [20] involving subgraphs of robust out-expanders, which can be proven by means of the Max-Flow-Min-Cut theorem.

Lemma 3.11. *Suppose that $0 < 1/n_0 \ll \gamma, \xi \ll \nu \leq \tau \ll \alpha < 1$. Let G be a digraph on $n \geq n_0$ vertices with $\delta^0(G) \geq \alpha n$ which is a robust (ν, τ)-outexpander. For every vertex $x \in V(G)$, let $n_x^+, n_x^- \in \mathbb{N}$ be such that $(1 - \gamma)\xi n \leq n_x^+, n_x^- \leq (1 + \gamma)\xi n$ and such that $\sum_{x \in V(G)} n_x^+ = \sum_{x \in V(G)} n_x^-$. Then G contains a spanning subdigraph G' which satisfies $d_{G'}^+(x) = n_x^+$ and $d_{G'}^-(x) = n_x^-$ for every $x \in V(G)$.*

Lemma 3.12. *Let $0 < 1/n_0 \ll \varepsilon \ll p, \alpha < 1$. Suppose that G is an Eulerian graph on $n \geq n_0$ vertices. Assume further that G is lower-(p, ε)-regular and that $\delta(G) \geq \alpha n$. Then there exists an orientation G' of G such that G' is Eulerian and lower-$(p/4, \varepsilon)$-regular.*

Proof. Choose new constants γ, ξ, τ such that $0 < 1/n_0 \ll \gamma, \xi \ll \varepsilon \ll \tau \ll p, \alpha < 1$. Observe that there exists a decomposition of the edges of G into two digraphs G_1, G_2 such that the following holds for $i = 1, 2$.

(a) $|d_{G_i}^{\pm}(x) - d_G(x)/4| \leq \gamma \xi n/2$ for all $x \in V(G)$.
(b) G_i is lower-$(p/4, \varepsilon)$-regular.

This can be seen as follows. If $e = xy \in E(G)$, orient e from x to y with probability $1/2$ and from y to x otherwise and include e in G_1 with probability $1/2$ and in G_2 otherwise. Do this independently for each edge and then use Lemma 3.4.

In particular, we have $|d_{G_1}^{+}(x) - d_{G_1}^{-}(x)| \leq \gamma \xi n$ for all $x \in V(G)$ and $\delta^0(G_2) \geq \alpha n/5$. Moreover, (b) and Proposition 3.10 together imply that G_2 is a robust (ε, τ)-outexpander.

Now, for every vertex x, let $n_x^+ := \xi n$ and $n_x^- := d_{G_1}^{+}(x) - d_{G_1}^{-}(x) + \xi n$. Thus, $n_x^+, n_x^- = (1 \pm \gamma)\xi n$ and $\sum_{x \in V(G)} n_x^+ = \sum_{x \in V(G)} n_x^-$. Apply Lemma 3.11 (with ε playing the role of ν) to G_2 to find a spanning subdigraph G_2' of G_2 such that $d_{G_2'}^{+}(x) = n_x^+$ and $d_{G_2'}^{-}(x) = n_x^-$ for all $x \in V(G)$. Then, $G_1 \cup G_2'$ is Eulerian. Let H be the spanning subgraph of G containing all those edges which are not contained in $G_1 \cup G_2'$. Thus, H is Eulerian and can be decomposed into cycles. Orient each cycle consistently to obtain an Eulerian orientation H' of H. Then, $G' := G_1 \cup G_2' \cup H'$ is the desired orientation of G. Indeed, G' is lower-$(p/4, \varepsilon)$-regular since G_1 is. $\square$

4. Cycle decompositions

We will now prove Theorem 1.4. In fact, we prove a directed version (Theorem 4.1) which also gives additional control on the decomposition. We then derive the undirected version (and thus Theorem 1.4) from this directed version. The additional control as well as the directed version will be of use when considering linear forests in Section 5.

In order to state Theorem 4.1, we need one more definition. Let G be a (di-)graph and let $M = \{x_1 y_1, \ldots, x_m y_m\}$ be a matching in the complete graph on $V(G)$ such that $d_G(x_i) \leq d_G(y_i)$ for all $i \in \{1, \ldots, m\}$. We then say that a sub-(di-)graph F of G is *consistent with* M, if for all $i \in \{1, \ldots, m\}$, $x_i \in V(F)$ implies $y_i \in V(F)$. Note that F can be consistent with M and separate M at the same time, as F might contain y_i but not x_i.

Theorem 4.1. *Let $0 < 1/n_0 \ll \eta, \varepsilon \ll p < 1$ be such that $\varepsilon^2 \leq \eta$. Suppose G is a lower-(p, ε)-regular digraph on $n \geq n_0$ vertices. Moreover, assume that G is Eulerian and that $\Delta(G) - \delta(G) \leq \eta n$. Let M be any matching in the complete graph on $V(G)$. Then G can be decomposed into $\Delta(G)/2$ cycles which are consistent with M.*

Clearly, if an Eulerian graph G has a decomposition into $\Delta(G)/2$ cycles, then all vertices of maximum degree must be contained in every cycle of the decomposition. Our proof strategy for Theorem 4.1 is thus to inductively take out cycles that contain all vertices of maximum degree. If we can make sure that the minimum degree decreases at a significantly slower rate than the maximum degree, we will

eventually obtain a dense regular graph. We will ensure that this graph is still a robust outexpander so that we can apply Theorem 3.8 to decompose it into Hamilton cycles.

We will deduce the undirected version of Theorem 4.1 by means of Lemma 3.12. (Note that our proof of Theorem 4.1 also works directly for the undirected case, by using Theorem 3.9 instead of Theorem 3.8.)

Proof of Theorem 4.1. Let $\tau := \tau(p/3)$ be as defined in Theorem 3.8. We may assume $0 < 1/n_0 \ll \eta, \varepsilon \ll \tau, p < 1$. By Proposition 3.2, $\delta^0(G) \geq pn/2$.

Apply Lemma 3.6 to obtain a partition S_1, S_2 of $V(G)$ with $|S_1|, |S_2| \geq n/3$ such that

$$(1) \qquad d_{S_1}^+(x), d_{S_1}^-(x), d_{S_2}^+(x), d_{S_2}^-(x) \geq pn/12$$

for all $x \in V(G)$ and such that neither S_1 nor S_2 separates M.

We will inductively construct spanning subgraphs $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_k$ of G, sets of vertices $Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_{k-1}$ and edge-disjoint cycles $C_1, \ldots, C_{k-1}$ such that the following holds:

(P1) Z_i is the set of vertices of maximum degree in G_i,
(P2) C_i is a cycle in G_i that contains all vertices of Z_i,
(P3) if $i \geq 2$, then G_i is obtained from G_{i-1} by removing the edges of C_{i-1},
(P4) $\Delta(G_i) = \Delta(G) - 2(i-1)$,
(P5) if $i \geq 2$ and $d_{G_i}(v) < d_{G_{i-1}}(v) < \Delta(G_{i-1})$, then $v \in S_1$ if i is even and $v \in S_2$ if i is odd,
(P6) G_k is regular,
(P7) $k \leq 2\eta n$.

Let $G_1 := G$. Assume that for some $1 \leq i \leq 2\eta n$, we have already defined $G_1 \supseteq \cdots \supseteq G_i$, $Z_1 \subseteq \cdots \subseteq Z_{i-1}$ and $C_1, \ldots, C_{i-1}$ such that (P1)–(P5) are satisfied. If G_i is regular, then stop. Otherwise, let Z_i be the set of vertices of maximum degree in G_i. Since G_{i-1} is Eulerian, we know that $Z_i \supseteq Z_{i-1}$.

We claim that $G_i[S_1 \cup Z_i]$ and $G_i[S_2 \cup Z_i]$ are Hamiltonian. Since $|S_1 \cup Z_i| \geq n/3$, Proposition 3.1(iii) implies that $G[S_1 \cup Z_i]$ is lower-$(p, 3\varepsilon)$-regular. Let H be the spanning subgraph of $G[S_1 \cup Z_i]$ that contains the edges from $C_1, \ldots, C_{i-1}$. Thus, $G[S_1 \cup Z_i] \setminus H = G_i[S_1 \cup Z_i]$. As $\Delta(H) \leq 2(i-1) \leq 4\eta n \leq 12\eta|S_1 \cup Z_i|$, Proposition 3.1(ii) implies that $G_i[S_1 \cup Z_i]$ is lower-$(p, 2\sqrt{12\eta})$-regular. Moreover, $\delta^0(G_i[S_1 \cup Z_i]) \geq pn/12 - \Delta(H) \geq p|S_1 \cup Z_i|/13$ by (1). Thus, by Lemma 3.3(i), $G_i[S_1 \cup Z_i]$ has a Hamilton cycle. The same applies to S_2.

Now, if i is odd, then let C_i be a Hamilton cycle in $G_i[S_1 \cup Z_i]$, and if i is even, then let C_i be a Hamilton cycle in $G_i[S_2 \cup Z_i]$. Set $G_{i+1} := G_i \setminus E(C_i)$.

By construction, (P1)–(P5) hold. Let us now show that the procedure will terminate after $k - 1 \leq 2\eta n - 1$ steps with an r-regular digraph G_k for some $r \in \mathbb{N}$. Indeed, first note that (P3) and (P5) together imply that

$$(2) \qquad \delta(G_{i+1}) \geq \delta(G) - (i+1).$$

Thus, together with (P4), it follows that

$$0 \leq \Delta(G_{i+1}) - \delta(G_{i+1}) \leq \Delta(G) - 2i - \delta(G) + i + 1 \leq \eta n - i + 1$$

and so $i + 1 \leq 2\eta n$.

(2) and (P7) imply that $\delta(G_k) \geq \delta(G) - k \geq pn - 2\eta n$. Thus, $r \geq (p/2 - \eta)n \geq pn/3$. Since $E(G \setminus G_k) = E(C_1) \cup \cdots \cup E(C_{k-1})$, we have $\Delta(G \setminus G_k) \leq 2(k-1) \leq 4\eta n$. Thus, G_k is lower-$(p, 4\sqrt{\eta})$-regular by Proposition 3.1(ii) and therefore a robust $(4\sqrt{\eta}, \tau)$-outexpander by Proposition 3.10. Finally, we can apply Theorem 3.8 to

decompose G_k into Hamilton cycles. Let $\mathcal{C}$ be the set consisting of all these Hamilton cycles together with the cycles $C_1, \ldots, C_{k-1}$. We will show that $\mathcal{C}$ is the required cycle decomposition of G.

Let x be a vertex of maximum degree in G. Thus, $x \in Z_i$ for all $i < k$ and so x is contained in every cycle in $\mathcal{C}$. Hence, $|\mathcal{C}| = d_G^+(x) = \Delta(G)/2$, as desired. By the choice of S_1 and S_2, all cycles in $\mathcal{C}$ will be consistent with M. To see this, let $xy \in M$ and assume that $d_G(x) \leq d_G(y)$. Suppose that there is a cycle in $\mathcal{C}$ which contains x but not y. Let $i^* := \min\{i \; : \; C_i \text{ contains } x \text{ but not } y\}$. Since $V(C_{i^*}) = Z_{i^*} \cup S_j$ for some $j \in \{1, 2\}$, and since neither S_1 nor S_2 separates M, it follows that $x \in Z_{i^*}, y \notin Z_{i^*}$. In particular, x has higher degree in G_{i^*} than y, which is a contradiction, since by definition of i^*, any C_j with $j < i^*$ which contains x must also contain y. □

Together with Lemma 3.12, we obtain the following undirected version of Theorem 4.1, which immediately implies Theorem 1.4.

Corollary 4.2. *Let $0 < 1/n_0 \ll \eta, \varepsilon \ll p < 1$. Suppose G is a lower-(p, ε)-regular graph on $n \geq n_0$ vertices. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$ and that G is Eulerian. Let M be any matching in the complete graph on $V(G)$. Then G can be decomposed into $\Delta(G)/2$ cycles which are consistent with M.*

5. Path decompositions

In this section, we will prove our quasirandom result concerning path and linear forest decompositions and then easily deduce Theorem 1.2. As mentioned earlier, any path decomposition of a graph G must contain at least $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths, and $la(G) \geq \lceil \Delta(G)/2 \rceil$. However, there are some simple cases in which these bounds cannot be attained. For example, suppose G is r-regular and r is even. Then it is impossible to decompose G into $r/2$ paths or $r/2$ linear forests. This motivates the additional assumption in the following theorem that G has a unique vertex of maximum degree. This additional assumption is stronger than what we would really need to prove Theorem 5.1, but still enables us to deduce Theorem 1.2(ii)–(iii).

Theorem 5.1. *Let $0 < 1/n_0 \ll \eta, \varepsilon \ll p < 1$. Suppose G is a lower-(p, ε)-regular graph on $n \geq n_0$ vertices such that $\Delta(G) - \delta(G) \leq \eta n$. Then the following hold.*

(i) *G can be decomposed into $\max\{odd(G)/2, \lceil (\Delta(G) + 1)/2 \rceil\}$ paths. If G has a unique vertex of maximum degree, then G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.*

(ii) *G can be decomposed into $\lceil (\Delta(G) + 1)/2 \rceil$ linear forests. If G has a unique vertex of maximum degree, then G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests.*

To prove Theorem 5.1, we will construct a suitable Eulerian auxiliary graph G' to which we can apply Theorem 4.1 or Corollary 4.2. The resulting cycle decomposition of G' will be transformed into a path decomposition of G.

Proof. Let $D = \{a_1, b_1, a_2, b_2, \ldots, a_s, b_s\}$ be the set of odd degree vertices in G. If $a_i b_i \notin E(G)$, we still refer to the pair $\{a_i, b_i\}$ as the edge $a_i b_i$, thereby meaning the respective edge in the complement of G.

Case 1: $odd(G) \geq \Delta(G)$

Let G' be the graph obtained from G in the following way. Add a new vertex w and choose $\lceil \Delta(G)/2 \rceil$ of the pairs $\{a_i, b_i\}$. Connect w to all the vertices in these pairs. For the remaining pairs, let E^* be the set of edges $a_i b_i \in E(G)$ and let

E° be the set of edges $a_i b_i$ which are in the complement of G. Remove $E^\star$ from the graph and add E°. Hence, G' is Eulerian, $\Delta(G') - \delta(G') \le \Delta(G) - \delta(G) + 2$, and by Proposition 3.1(i) and (ii), G' is lower-$(p, 4\varepsilon)$-regular. We thus can apply Corollary 4.2 to G' to find a decomposition of G' into $\Delta(G')/2$ cycles which are consistent with $E^\star$. Since $d_{G'}(w) = 2\lceil \Delta(G)/2 \rceil = \Delta(G')$, w is contained in each of these cycles. This naturally gives a decomposition $\mathcal{P}$ of $(G \setminus E^\star) \cup E^\circ$ into $\lceil \Delta(G)/2 \rceil$ paths.

We now use $\mathcal{P}$ to prove (i). Remove the edges of E° one by one from the paths in $\mathcal{P}$. Every time when an edge is removed, the number of paths increases by one. Finally, for every edge in $E^\star$, simply take this edge as a path of length one. Thus, we have a decomposition of G into $\lceil \Delta(G)/2 \rceil + |E^\circ| + |E^\star| = odd(G)/2$ paths.

We next use $\mathcal{P}$ to prove (ii). When we are looking for linear forests, we can remove the edges of E° from the paths in $\mathcal{P}$ and thus obtain a collection $\mathcal{F}$ of $\lceil \Delta(G)/2 \rceil$ linear forests in $G - E^\star$. We now distribute the edges of $E^\star$ among these linear forests. To this end, consider any $a_i b_i \in E^\star$. We may assume that $d_G(a_i) \le d_G(b_i)$. Since $a_i b_i$ was deleted from G, $d_{G'}(b_i) < \Delta(G)$. Moreover, b_i is not a leaf in any of the linear forests in $\mathcal{F}$. Therefore, there is a linear forest $F \in \mathcal{F}$ which does not contain b_i. Since the cycle decomposition of G' was chosen to be consistent with $E^\star$, F does not contain a_i either. Hence, we can add $a_i b_i$ to F to obtain a new linear forest. We proceed analogously with the other edges in $E^\star$ to obtain a decomposition of G into $\lceil \Delta(G)/2 \rceil$ linear forests.

Case 2: $odd(G) < \Delta(G)$

Let $E^\star$ contain all edges $a_i b_i \in E(G)$ and let E° contain all edges $a_i b_i$ which are in the complement of G. Let $G' := (G \setminus E^\star) \cup E^\circ$. Hence, G' is Eulerian and lower-$(p, 2\varepsilon)$-regular by Proposition 3.1(ii). By Lemma 3.12, there exists an orientation G^{orient} of G' which is Eulerian and lower-$(p/4, 2\varepsilon)$-regular. Let G'' be the digraph obtained from G^{orient} in the following way. Add a new vertex w and connect it to all vertices of D by one edge as follows. If $a_i b_i \in E(G^{orient})$, remove it and add $a_i w$ and $w b_i$. If $b_i a_i \in E(G^{orient})$, remove it and add $b_i w$ and $w a_i$. Else, add the edges $a_i b_i$, $b_i w$ and $w a_i$. Note that the graph obtained from $G'' - w$ by ignoring the orientation of its edges is precisely G.

Case 2.1: G has a unique vertex of maximum degree and $\Delta(G)$ is even.

Let $k := (\Delta(G) - odd(G))/2$. Since G has a unique vertex of maximum degree, we can choose k distinct vertices not contained in D and not of maximum degree and connect w to each of them by two edges which are oriented in opposite directions. The digraph G''' obtained from G'' in this way is Eulerian and $d_{G'''}(w) = \Delta(G''') = \Delta(G)$. Note that Proposition 3.1 implies that G''' is lower-$(p/4, 8\varepsilon)$-regular. Decompose G''' into $\Delta(G)/2$ cycles by means of Theorem 4.1. Recall that the graph obtained from $G'' - w = G''' - w$ by ignoring the orientation of its edges is precisely G. Together with the fact that $d_{G'''}(w) = \Delta(G''')$, this implies that the cycle decomposition of G''' yields a decomposition of G into $\Delta(G)/2$ paths (and thus also a decomposition into $\Delta(G)/2$ linear forests).

Case 2.2: G does not have a unique vertex of maximum degree or $\Delta(G)$ is odd.

Let $k' := \lceil (\Delta(G) + 1 - odd(G))/2 \rceil$. We pick k' distinct vertices not contained in D and connect w to each of them by two edges which are oriented in opposite directions. Let G''' be the digraph obtained from G'' in this way. Then $d_{G'''}(w) = odd(G) + 2k' = \Delta(G''') = 2\lceil (\Delta(G) + 1)/2 \rceil$. So we can argue similarly as in Case 2.1 to obtain a decomposition of G into $\lceil (\Delta(G) + 1)/2 \rceil$ paths (and thus also a decomposition into $\lceil (\Delta(G) + 1)/2 \rceil$ linear forests). $\qquad\square$

Proof of Theorem 1.2. Let $G \sim G_{n,p}$. Let D denote the set of odd degree vertices in G. Recall that a.a.s. G has the properties stated in Lemma 3.5. Moreover, Lemma 3.7 implies that a.a.s. $G[D]$ has a perfect matching M.

To prove (i), we can apply Theorem 1.4 to $G - M$. Since $\Delta(G - M) = 2\lfloor \Delta(G)/2 \rfloor$, $G - M$ can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles.

Parts (ii) and (iii) follow immediately from Theorem 5.1. □

6. LINEAR ARBORICITY

Recall that Theorem 1.2(iii) verifies the linear arboricity conjecture for $G_{n,p}$ when p is constant. In this section we additionally verify this conjecture for the following situations: (i) for $G_{n,p}$ where $p \to 0$ but p is not too small, and (ii) for large and sufficiently dense regular graphs. We obtain these results as corollaries of (i) an optimal Hamilton cover result in [14], and (ii) the recent proof of the Hamilton decomposition conjecture in [7].

Theorem 6.1 (Hefetz, Kühn, Lapinskas, Osthus [14])**.** *Let* $\frac{\log^{117} n}{n} \leq p \leq 1 - n^{-1/8}$ *and* $G \sim G_{n,p}$. *Then a.a.s. the edges of* G *can be covered by* $\lceil \Delta(G)/2 \rceil$ *Hamilton cycles.*

Corollary 6.2. *Let* $\frac{\log^{117} n}{n} \leq p = o(1)$ *and* $G \sim G_{n,p}$. *Then a.a.s.* G *can be decomposed into* $\lceil \Delta(G)/2 \rceil$ *linear forests.*

Proof. We note that for the given range of p, Lemma 3.5(iii) still holds, which again can be seen by applying Theorem 3.15 from [3]. Let G' be obtained from G by adding a new vertex w and adding each potential edge at w independently with probability p. Hence, $G' \sim G_{n+1,p'}$, where (as a function of the order $n + 1$ of the graph) $p'(n + 1) = p(n)$. By Theorem 6.1, a.a.s. the edges of G' can be covered by $\lceil \Delta(G')/2 \rceil$ Hamilton cycles. Clearly, this Hamilton cycle cover of G' yields a decomposition of G into $\lceil \Delta(G')/2 \rceil$ linear forests. It remains to check that a.a.s. $\Delta(G') = \Delta(G)$. A.a.s. G has a unique vertex v_{max} of maximum degree. Since $p = o(1)$, it follows that a.a.s. $d_{G'}(v_{max}) = d_G(v_{max})$. Moreover, a.a.s. G' has a unique vertex of maximum degree. The probability that w is this very vertex is $1/(n + 1) = o(1)$. Hence, a.a.s. $\Delta(G') = d_{G'}(v_{max}) = \Delta(G)$. □

With more work, the above argument can also be extended to the case when p is constant, which would give an alternative proof of Theorem 1.2(iii).

We now use the result of [7] mentioned earlier to deduce that the linear arboricity conjecture holds for large and sufficiently dense regular graphs.

Theorem 6.3 (Csaba, Kühn, Lo, Osthus, Treglown [7])**.** *There exists an* $n_0 \in \mathbb{N}$ *such that whenever* $n \geq n_0$, $d \geq \lfloor n/2 \rfloor$ *and* G *is a* d-regular graph on n vertices, *then* G *has a decomposition into Hamilton cycles and at most one perfect matching.*

Corollary 6.4. *For sufficiently large* n, *every* d-regular graph G on n vertices with $d \geq \lfloor (n - 1)/2 \rfloor$ *has a decomposition into* $\lceil (d + 1)/2 \rceil$ *linear forests.*

Proof. In our argument below we will use the fact that any graph H with minimum degree at least r has a matching that covers at least r vertices.
Case 1: d is odd.

Then n must be even. Clearly, the complement $\overline{G}$ of G is $(n - 1 - d)$-regular. Hence, there is a matching M in $\overline{G}$ that covers $n - (d + 1)$ vertices. Let G' be the graph obtained from G by adding the edges of M and joining a new vertex w to the

$d+1$ vertices not covered by M. Hence, G' is $(d+1)$-regular and $d+1 \geq \lfloor |G'|/2 \rfloor$. By Theorem 6.3 and because $d+1$ is even, G' has a decomposition into $(d+1)/2$ Hamilton cycles. By removing w and the edges of M from each of these Hamilton cycles, we obtain a decomposition of G into $(d+1)/2$ linear forests.

Case 2: d is even and n is even.

Let M be a matching in $\overline{G}$ that covers $n-(d+2)$ vertices. Since $\overline{G} \setminus M$ has minimum degree at least $n-(d+2)$, $\overline{G} \setminus M$ contains another matching M' that covers $n-(d+2)$ vertices. Let G' be the graph obtained from G by adding the edges of M and M' and two new vertices w, w' such that w is joined to the $d+2$ vertices in G not covered by M and w' is joined to the $d+2$ vertices in G not covered by M'. Thus, G' is $(d+2)$-regular and $d+2 \geq \lfloor |G'|/2 \rfloor$. By Theorem 6.3 and because $d+2$ is even, G' has a decomposition into $(d+2)/2$ Hamilton cycles. By removing w, w' and the edges of $M \cup M'$ from each of these Hamilton cycles, we obtain a decomposition of G into $(d+2)/2 = \lceil (d+1)/2 \rceil$ linear forests.

Case 3: d is even and n is odd.

Let M be a matching in $\overline{G}$ that covers $n-(d+1)$ vertices. Let G' be the graph obtained from G by adding the edges of M and joining a new vertex w to the $d+1$ vertices not covered by M. Then G' is $(d+1)$-regular and $d+1 \geq \lfloor (|G'|-1)/2 \rfloor$. By Case 1, G' can be decomposed into $(d+2)/2 = \lceil (d+1)/2 \rceil$ linear forests. As G is a subgraph of G', G can be decomposed into the same number of linear forests. $\quad\square$

7. EDGE COLOURINGS

In this section we build on Theorem 3.9 to prove the overfull subgraph conjecture for dense quasirandom graphs of even order. We need the following well-known result on multigraphic degree sequences.

Theorem 7.1 (Hakimi [13]). *Let $0 \leq d_n \leq \cdots \leq d_1$ be integers. Then there exists a multigraph G on vertices $x_1, \ldots, x_n$ such that $d_G(x_i) = d_i$ for all i if and only if $\sum_{i=1}^{n} d_i$ is even and $\sum_{i>1} d_i \geq d_1$.*

Though it is not explicitly stated in [13], the inductive proof yields a polynomial time algorithm which finds an appropriate multigraph if it exists. Our strategy in proving Theorem 1.6 is similar to that of Theorem 4.1, yet different in detail. Roughly speaking, instead of inductively removing cycles, we aim to remove paths in order to make our graph regular and then apply Theorem 3.9 to decompose the regular remainder into Hamilton cycles. We can then simply colour each path with two colours and, since our graph has even order, each Hamilton cycle with two colours. An auxiliary multigraph obtained from Theorem 7.1 will tell us which vertices to choose as the endvertices of the paths to be removed. In order to actually find these paths, we observe that lower-(p, ε)-regular graphs contain 'spanning linkages' for arbitrary pairs of vertices.

Lemma 7.2. *Let $0 < 1/n_0 \ll \varepsilon \ll \alpha, p < 1$. Let G be a lower-(p, ε)-regular graph on $n \geq n_0$ vertices such that $\delta(G) \geq \alpha n$. Moreover, let $M = \{a_1 b_1, \ldots, a_m b_m\}$ be a matching in the complete graph on $V(G)$ of size at most $\alpha n/5$. Then there exist vertex-disjoint paths $P_1, \ldots, P_m$ in G such that $\bigcup V(P_i) = V(G)$ and P_i joins a_i to b_i, and these paths can be found in polynomial time.*

Proof. Our paths $P_1, \ldots, P_{m-1}$ will have length at most 3 and P_m will contain all the remaining vertices. Suppose that for some $1 \leq i \leq m$, we have already defined vertex-disjoint paths $P_1, \ldots, P_{i-1}$ in G of length at most 3 and such that P_j joins a_j to

b_j and is disjoint from $\{a_i, b_i, \ldots, a_m, b_m\}$. Let $X_i := (V(M) \setminus \{a_i, b_i\}) \cup \bigcup_{j=1}^{i-1} V(P_j)$. Let $G_i := G - X_i$. Since $|X_i| \leq 2m + 2(i-1) \leq 4m \leq 4\alpha n/5$, we have $\delta(G_i) \geq \alpha n/5$. As $|G_i| \geq n/5$, G_i is lower-$(p, 5\varepsilon)$-regular by Proposition 3.1(iii). If $i < m$, we seek a path of length at most 3. Suppose there is no path of length at most two in G_i joining a_i to b_i. Then $N_{G_i}(a_i)$ and $N_{G_i}(b_i)$ are disjoint and both have size at least $\alpha n/5 \geq 5\varepsilon n$. Since G_i is lower-$(p, 5\varepsilon)$-regular, there exists an edge in G_i between $N_{G_i}(a_i)$ and $N_{G_i}(b_i)$ which gives us a path P_i of length 3 joining a_i and b_i. Finally, if $i = m$, then Lemma 3.3(iii) tells us that we can find a Hamilton path P_m in G_m from a_m to b_m (in polynomial time). $\qquad\square$

For a graph G and $x \in V(G)$, let $\mathrm{def}_G(x) := \Delta(G) - d_G(x)$ denote the *deficiency of x in G*.

Proof of Theorem 1.6. Choose constants ε, η and $n_0 \in \mathbb{N}$ such that $0 < 1/n_0 \ll \varepsilon \leq \eta \ll p$. Let G be any lower-(p, ε)-regular graph on $n \geq n_0$ vertices with $\Delta(G) - \delta(G) \leq \eta n$, where n is even. Let $\alpha := p/2$. It is then easy to see that $\delta(G) \geq \alpha n$.

If G contains any overfull subgraph, then $\chi'(G) > \Delta(G)$. So let us assume that G contains no overfull subgraph. Label the vertices of G such that $d_G(x_1) \leq \cdots \leq d_G(x_n)$ and let $\mathrm{def}_i := \mathrm{def}_G(x_i)$.

We claim that $\mathrm{def}_1 \leq \sum_{i>1} \mathrm{def}_i$. Suppose not, and let $H := G - x_1$. Then $e(H) = e(G) - d_G(x_1)$ and so

$$2e(H) = \sum_{i=1}^{n} d_G(x_i) - 2d_G(x_1) = \sum_{i>1} d_G(x_i) - d_G(x_1)$$
$$= \sum_{i>1} (\Delta(G) - \mathrm{def}_i) - \Delta(G) + \mathrm{def}_1 > (n-2)\Delta(G) = (|H| - 1)\Delta(G).$$

Since $|H|$ is odd, H is overfull, a contradiction. Hence, we have $\mathrm{def}_1 \leq \sum_{i>1} \mathrm{def}_i$. Moreover, since n is even, the sum of all deficiencies is even. So by Theorem 7.1, there exists a multigraph A on $V(G)$ such that $d_A(x_i) = \mathrm{def}_i$. We will use this auxiliary multigraph A to find a dense spanning regular subgraph G_k of G. Observe that $\Delta(A) = \mathrm{def}_1 = \Delta(G) - \delta(G) \leq \eta n$. Thus, $\chi'(A) \leq 2\eta n$. Hence, we can (greedily) partition $E(A)$ into $k \leq 6\eta n/\alpha$ matchings $M_1, \ldots, M_k$ of size at most $\alpha n/6$. We will now inductively take out linear forests from G by applying Lemma 7.2 with $M_1, \ldots, M_k$. More precisely, we define spanning subgraphs $G_0, \ldots, G_k$ of G and edge-disjoint linear forests $F_1, \ldots, F_k$ such that

(a) $G_0 := G$ and $G_i = G_{i-1} \setminus F_i$ for $i \geq 1$,
(b) F_i is a spanning linear forest in G_{i-1} whose leaves are precisely the vertices in M_i.

Let $G_0 := G$ and suppose that for some $1 \leq i \leq k$ we have already defined $G_0, \ldots, G_{i-1}$ and $F_1, \ldots, F_{i-1}$. Since $\Delta(F_1 \cup \cdots \cup F_{i-1}) \leq 2(i-1) \leq 12\eta n/\alpha$, it follows that $\delta(G_{i-1}) \geq (\alpha - 12\eta/\alpha)n \geq 5\alpha n/6$. Moreover, let $\varepsilon' := 2\sqrt{12\eta/\alpha}$. Then, by Proposition 3.1(ii), G_{i-1} is lower-(p, ε')-regular. Hence, since M_i has size at most $\alpha n/6$, we can apply Lemma 7.2 to G_{i-1} and M_i to obtain a spanning linear forest F_i in G_{i-1} whose leaves are precisely the vertices in M_i. Set $G_i := G_{i-1} \setminus F_i$.

We claim that G_k is regular. Consider any vertex $x \in V(G)$. Then, $d_{G_k}(x) = d_G(x) - \sum_{i=1}^{k} d_{F_i}(x)$. For every $1 \leq i \leq k$, $d_{F_i}(x) = 1$ if x is an endvertex of some edge of M_i, and $d_{F_i}(x) = 2$ otherwise. Since $M_1, \ldots, M_k$ partition $E(A)$, we conclude that

$\sum_{i=1}^{k} d_{F_i}(x) = 2k - d_A(x) = 2k - \mathrm{def}_G(x)$. Thus, $d_{G_k}(x) = d_G(x) - 2k + \mathrm{def}_G(x) = \Delta(G) - 2k$.

Let $r := \Delta(G) - 2k$. We have shown that G_k is r-regular. Moreover, $r \geq 5\alpha n/6$ and G_k is lower-(p, ε')-regular. If r is even, we can decompose G_k into Hamilton cycles by means of Theorem 3.9 and Proposition 3.10. If r is odd, we can use Lemma 3.3(ii) to find a perfect matching M^* in G_k. Let $G_k^* := G_k \setminus M^*$, then G_k^* is even-regular and we can proceed analogously.

In order to obtain an optimal edge colouring of G, we colour each F_i with two colours, each Hamilton cycle with two colours, and if r is odd, M^* with one additional colour. Since x_n is incident with an edge of every colour, we use $\Delta(G)$ colours.

Let us finally check that this yields a polynomial time algorithm. Given G, first check if $\mathrm{def}_1 \leq \sum_{i>1} \mathrm{def}_i$. If not, then we know that $\chi'(G) = \Delta(G) + 1$ and can use any polynomial time algorithm that attains Vizing's bound (see e.g. [12]). If the inequality holds, then we can construct a $\Delta(G)$-colouring by proceeding as above. Since Theorem 7.1, Lemma 7.2, Lemma 3.3(ii) and Theorem 3.9 give appropriate running time statements, this can be achieved in time polynomial in n. □

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Stefan Glock, Daniela Kühn, Deryk Osthus
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

E-mail addresses: {sxg426,d.kuhn,d.osthus}@bham.ac.uk