On well-quasi-ordering infinite trees – Nash-Williams's theorem revisited

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Abstract

Nash-Williams proved that the infinite trees are well-quasi-ordered (indeed, better-quasi-ordered) under the topological minor relation. We combine ideas of several authors into a more accessible and essentially self-contained short proof.

1 Introduction and terminology

A fundamental result of Nash-Williams [5] states that the infinite trees are well-quasi-ordered under the topological minor relation. To prove this, he introduced the stronger concept of better-quasi-ordered sets, and showed that the infinite trees are even better-quasi-ordered. In this paper we give an essentially self-contained proof of this theorem. In general, the proof follows the lines of the original one. Nash-Williams's definition of a better-quasi-ordering is purely combinatorial; however, we use an equivalent topological concept, which is due to Simpson [8]. We remark that Laver [2] generalized Nash-Williams's result to a certain class of order theoretic trees. Thomas [9] extended Nash-Williams's result by proving that every class of infinite graphs with linked tree decompositions of bounded width is well-quasi-ordered under the minor relation.

We write $[n]$ for the set $\{1, \dots, n\}$. We denote by $\mathcal{C}$ the class of all cardinals, and by $\mathcal{O}$ that of all ordinals. We denote the domain of a function f by $\mathrm{D}f$.

For an infinite set $X \subseteq \mathbb{N}$ we define $X^{(\omega)}$ to be the set of all infinite subsets of X. We often identify an element $s \in X^{(\omega)}$ with the strictly ascending sequence whose elements are those of s; and conversely. Thus, if we write $s = (s_1, s_2, \dots)$ for an element of $X^{(\omega)}$, we mean that $s_1 < s_2 < \dots$. The *Ellentuck topology* on $X^{(\omega)}$ is defined by taking as basic open neighbourhoods of an element $s \in X^{(\omega)}$ all sets of the form $\{t \in s^{(\omega)} \mid u \subseteq t\}$, where u is a finite initial segment of s. Thus the Ellentuck topology is a refinement of the Tychonov (product) topology. Given a function $f : X^{(\omega)} \to D$, where D is some topological space, we say that f is *Ellentuck-continuous*, if f is continuous when we impose the Ellentuck topology on $X^{(\omega)}$. In particular, if D is discrete, then f is Ellentuck-continuous if and only if for every $s \in X^{(\omega)}$ there exists a finite initial segment u of s such that $f(s) = f(t)$ for all infinite subsequences t of s beginning with u.

We will repeatedly make use of the following theorem of Ellentuck, which says that Ellentuck-open sets are Ramsey (for a proof see e.g. [1, §20]). Apart from this, our presentation is self-contained.

Theorem 1 *Let $X \in \mathbb{N}^{(\omega)}$. For every Ellentuck-open set $A \subseteq X^{(\omega)}$ there exists $B \in X^{(\omega)}$ such that either $B^{(\omega)} \subseteq A$ or $B^{(\omega)} \cap A = \emptyset$.*

A reflexive and transitive relation is called a *quasi-ordering.* A quasi-ordered set $Q, \leq$ is *well-quasi-ordered* (*wqo*), if for every infinite sequence $q_1, q_2, \dots$ in Q there are indices $i < j$ such that $q_i \leq q_j$. In what follows Q will always denote a quasi-ordered set, and we also view Q as a discrete topological space. Q is *better-quasi-ordered* (*bqo*) if for every $X \in \mathbb{N}^{(\omega)}$ and for every Ellentuck-continuous function $f : X^{(\omega)} \to Q$ there exists an $s \in X^{(\omega)}$ such that $f(s) \leq f(s \backslash \{\min s\})$. We remark that a result of Mathias [3] implies that one obtains an equivalent definition by replacing Ellentuck-continuity by Tychonov-continuity, or by requiring Borel measurability. A *Q-array* is an Ellentuck-continuous function $f : X^{(\omega)} \to Q$, for some $X \in \mathbb{N}^{(\omega)}$. If there is no $s \in X^{(\omega)}$ such that $f(s) \leq f(s \backslash \{\min s\})$, then f is a *bad* Q-array. Thus Q is bqo if and only if there is no bad Q-array.

All trees considered in this paper will have a root. For two trees T and U with roots t and u, respectively, we call an injective mapping $\varphi : V(T) \to V(U)$ an *embedding* of T into U, if φ can be extended to an isomorphism between a subdivision of T and the smallest subtree U' of U containing all vertices in $\varphi(V(T))$, and furthermore, the path between $\varphi(t)$ and u in U contains no vertex of U' other than $\varphi(t)$. We say that T is a *rooted topological minor* of U, abbreviated by $T \preccurlyeq U$, if there is an embedding of T into U. This defines a quasi-ordering on the class of all trees.

Given two vertices x and y of a tree T, we say that x is *above* y if y lies on the path from x to the root of T. If x and y are adjacent and x is above y, we call y the *predecessor* of x and x the *successor* of y. The *branch above* x, abbreviated by $\mathrm{br}(x)$, is the subtree of T spanned by all vertices above x (including x itself). For the root of $\mathrm{br}(x)$ we choose x.

2 Better-quasi-ordering infinite trees

Lemma 1 *Every bqo set Q is wqo.*

Proof. Let $q_1, q_2, \dots$ be any infinite sequence in Q. Define a function $f : \mathbb{N}^{(\omega)} \to Q$ by $f(s) := q_{\min s}$. Then f is Ellentuck-continuous, and thus a Q-array. Hence, since Q is bqo, there exists an $s \in \mathbb{N}^{(\omega)}$ such that $f(s) \leq f(s \backslash \{\min s\})$. But this means that $q_{s_1} \leq q_{s_2}$, where $s = (s_1, s_2, \dots)$. Thus Q is wqo. □

If Q is a quasi-ordered set, then we may quasi-order the elements of the power set of Q by saying that $A \leq B$ if for all $a \in A$ there exists $b \in B$ such that $a \leq b$ in Q. We denote the power set of Q with this quasi-ordering by $\mathcal{S}(Q)$. The following lemma implies that if Q is bqo then so is $\mathcal{S}(Q)$.

Lemma 2 *If f is a bad $\mathcal{S}(Q)$-array, then there exists a bad Q-array g such that $Dg = Df$ and $g(s) \in f(s)$ for all $s \in Dg$.*

Proof. Let $s \in \mathrm{D}f$. Since $f(s) \not\leq f(s \backslash \{\min s\})$ there exists an $x_s \in f(s)$ such that $x_s \not\leq y$ for all $y \in f(s\backslash\{\min s\})$. We can choose x_s such that it depends only on the pair $f(s), f(s\backslash\{\min s\})$ and not on s itself, i.e. if $f(s) = f(t)$ and $f(s\backslash\{\min s\}) = f(t\backslash\{\min t\})$, then $x_s = x_t$. We now define a function $g : \mathrm{D}f \to Q$ by setting $g(s) := x_s$. Then the Ellentuck-continuity of f and the fact that x_s depends only on the pair $f(s), f(s\backslash\{\min s\})$ imply that g is Ellentuck-continuous, and thus a Q-array. It is also bad, since $g(s) \leq g(s\backslash\{\min s\})$ would contradict the choice of x_s. □

Given two quasi-ordered sets Q and Q', we define a quasi-ordering on $Q \times Q'$ by saying that $(q_1, q_1') \leq (q_2, q_2')$ if $q_1 \leq q_2$ and $q_1' \leq q_2'$.

Lemma 3 *If $f = (f_1, f_2)$ is a bad $\mathcal{C} \times Q$-array, then there exists a bad Q-array g such that $Dg \subseteq Df$ and $g(s) = f_2(s)$ for all $s \in Dg$.*

Proof. Let $A := \{s \in \mathrm{D}f \mid f_1(s) \leq f_1(s\backslash\{\min s\})\}$. Then the Ellentuck-continuity of f implies that A is Ellentuck-open. Hence by Theorem 1, there exists a $B \in \mathrm{D}f$ such that either $B^{(\omega)} \subseteq A$ or $B^{(\omega)} \cap A = \emptyset$. But the latter cannot hold, since then for $s = (s_1, s_2, \dots) \in B^{(\omega)}$ we would have $f_1(s_1, s_2, \dots) > f_1(s_2, s_3, \dots) > f_1(s_3, s_4, \dots) > \dots$, contradicting the fact that $\mathcal{C}$ is well-ordered. Thus $B^{(\omega)} \subseteq A$, and so $g : B^{(\omega)} \to Q$ defined by $g(s) := f_2(s)$ must be a bad Q-array, as required. □

Let $\mathrm{Seq}(Q)$ be the set of all transfinite sequences with elements in Q. For a transfinite sequence $F : \alpha \to Q$ we define $\mathrm{length}(F)$ to be α. If $\beta < \alpha$, we write $F|_\beta$ for the restriction of F to β. Given $F, G \in \mathrm{Seq}(Q)$, we call a mapping $\varphi : \mathrm{length}(F) \to \mathrm{length}(G)$ an *embedding* of F into G if φ is strictly increasing and $F(\alpha) \leq G(\varphi(\alpha))$ for all $\alpha < \mathrm{length}(F)$. We impose a quasi-ordering on $\mathrm{Seq}(Q)$ by saying that $F \leq G$ if there exists an embedding from F into G.

The following lemma implies that if a set Q is bqo then so is $\mathrm{Seq}(Q)$, which is also a result due to Nash-Williams [4]. In the proof we present here, we closely follow Prömel and Voigt [6].

Lemma 4 *If f is a bad Seq(Q)-array, then there exists a bad Q-array g such that $Dg \subseteq Df$ and $g(s) \in f(s)$ for all $s \in Dg$.*

Proof. For sequences $F, G \in \mathrm{Seq}(Q)$ we write $F \leq^* G$ if F is an initial segment of G, and $F <^* G$ if F is a proper initial segment of G. If h and h' are $\mathrm{Seq}(Q)$-arrays, we write $h \leq^* h'$ if $\mathrm{D}h \subseteq \mathrm{D}h'$ and $h(s) \leq^* h'(s)$ for all $s \in \mathrm{D}h$. Furthermore, we write $h <^* h'$ if $h \leq^* h'$ and there exists an $s \in \mathrm{D}h$ such that $h(s) <^* h'(s)$. We will first prove the following claim.

There exists a minimal bad Seq(Q)-array h such that $h \leq^* f$. $\quad (*)$

We may assume that f itself is not minimal. Put $f_0 := f$ and $X_0^{(\omega)} := \mathrm{D}f_0$. For a $\mathrm{Seq}(Q)$-array g and $s \in \mathrm{D}g$ we define

$$k_{g,s} := \min\{k \mid k \in s \text{ such that } g(s) = g(t) \text{ for all } t \in s^{(\omega)} \text{ with } s \cap [k] = t \cap [k]\}.$$

Thus $k_{g,s}$ is the smallest integer $k \in s$ such that g is constant on the set of all $t \in s^{(\omega)}$ that begin with the initial segment $s \cap [k]$ of s. (Note that $k_{g,s}$ exists, since g is Ellentuck-continuous.) We now choose a bad $\mathrm{Seq}(Q)$-array $f_1' <^* f_0$ such that

$$\min\{k_{f_1',s} \mid s \in \mathrm{D}f_1' \text{ with } f_1'(s) <^* f_0(s)\} \;=:\; k_1$$

is minimal. Choose an element $s_1 \in \mathrm{D}f_1'$ such that $f_1'(s_1) <^* f_0(s_1)$ and $k_{f_1',s_1} = k_1$. Define a function $f_1 : (s_1 \cup (X_0 \cap [k_1]))^{(\omega)} \to \mathrm{Seq}(Q)$ by

$$f_1(s) := \begin{cases} f_1'(s) & \text{if } s \in s_1^{(\omega)}; \\ f_0(s) & \text{otherwise.} \end{cases}$$

It is easily checked that f_1 is a bad $\mathrm{Seq}(Q)$-array and $f_1 <^* f_0$. If f_1 is not minimal, we continue in this fashion to construct f_2', f_2, s_2 and k_2. Thus we may assume that we have constructed infinite sequences $f_1', f_2', \ldots$ and $f_1, f_2, \ldots$ and $s_1, s_2, \ldots$ and $k_1, k_2, \ldots$. Then $k_{i+1} \geq k_i$ for all $i \geq 1$, since f_{i+1}' was a candidate for the choice of f_i'. Moreover, the sequence (k_i) is unbounded. Indeed, suppose that there is an i such that $k_i = k_j$ for all $j \geq i$. Then there exists an infinite sequence $i < j_1 < j_2 < \ldots$ such that $s_{j_1} \cap [k_i] = s_{j_\ell} \cap [k_i]$ for all $\ell \geq 1$. This yields $s_{j_{\ell+1}} \subseteq s_{j_\ell}$ for all $\ell \geq 1$. Hence the definition of $k_{f_{j_\ell}', s_{j_\ell}}$ implies that

$$f_{j_\ell}(s_{j_\ell}) \;=\; f_{j_\ell}'(s_{j_\ell}) \;=\; f_{j_\ell}'(s_{j_{\ell+1}}) \;=\; f_{j_\ell}(s_{j_{\ell+1}}).$$

By the choice of $s_{j_{\ell+1}}$ it follows that

$$f_{j_{\ell+1}}(s_{j_{\ell+1}}) \;=\; f_{j_{\ell+1}}'(s_{j_{\ell+1}}) \;<^*\; f_{j_\ell}(s_{j_{\ell+1}}) \;=\; f_{j_\ell}(s_{j_\ell}).$$

Thus $\mathrm{length}(f_{j_1}(s_{j_1})), \mathrm{length}(f_{j_2}(s_{j_2})), \ldots$ is an infinite strictly descending chain of ordinals, a contradiction.

Let $X := \bigcap_{i \geq 1} X_i$, where $X_i^{(\omega)} := \mathrm{D}f_i$. Since X contains every k_i, the unboundedness of the sequence (k_i) implies that X is infinite. Also, note that for all $s \in X^{(\omega)}$ there exists an integer $i = i(s)$ such that $f_i(s) = f_j(s)$ for all $j \geq i$. (Otherwise there would be an infinite strictly descending chain of ordinals, since $f_{j+1}(s) \leq^* f_j(s)$.) Define a function $h' : X^{(\omega)} \to \mathrm{Seq}(Q)$ by putting $h'(s) := f_{i(s)}(s)$.

We will now find an Ellentuck-continuous restriction of h' that will do for h in $(*)$. Let A be the set of all $s \in X^{(\omega)}$ such that h' is Ellentuck-continuous in s. Thus A is Ellentuck-open. By Theorem 1 there exists a $B \in X^{(\omega)}$ such that either $B^{(\omega)} \subseteq A$ or $B^{(\omega)} \cap A = \emptyset$. Suppose first that the latter holds, and let $t_1 \in B^{(\omega)}$. Since $f_{i(t_1)}$ is Ellentuck-continuous, there is a basic Ellentuck-neighbourhood N_1 of t_1 on which $f_{i(t_1)}$ is constant. Since h' is not Ellentuck-continuous in t_1, there exists an $t_2 \in N_1$ such that $h'(t_2) \neq h'(t_1)$, and thus from the definition of h' it follows that $h'(t_2) <^* f_{i(t_1)}(t_2) = f_{i(t_1)}(t_1) = h'(t_1)$. But t_2 is a subsequence of t_1 (since it lies in a basic Ellentuck-neighbourhood of t_1), and so $t_2 \in B^{(\omega)}$. Continuing in this fashion we obtain an infinite sequence $t_1, t_2, \ldots$ such that $h'(t_1) >^* h'(t_2) >^* \ldots$, i.e. $\mathrm{length}(h'(t_1))$, $\mathrm{length}(h'(t_2)), \ldots$ is an infinite strictly descending chain of ordinals, a contradiction. Thus $B^{(\omega)} \subseteq A$, and hence the restriction h of h' on $B^{(\omega)}$ is Ellentuck-continuous.

The definition of h' implies that h is a bad $\mathrm{Seq}(Q)$-array and $h \leq^* f$ (in fact, $h \leq^* f_i$ for all $i \geq 0$). Suppose that h is not minimal, and let φ be a bad $\mathrm{Seq}(Q)$-array such that $\varphi <^* h$. Let

$$k := \min\{k_{\varphi,s} \mid s \in \mathrm{D}\varphi \text{ and } \varphi(s) <^* h(s)\}.$$

Since the sequence (k_i) is unbounded, there is an i with $k_i > k$, contradicting the fact that φ was a candidate for the choice of f_i'. This shows that h is also minimal, and thus h is as required in $(*)$.

We now use $(*)$ to complete the proof of the lemma. For all $s \in \mathrm{D}h$ define

$$\psi(s) := \sup\{\alpha \in \mathcal{O} \mid h(s)|_\alpha \leq h(s\backslash\{\min s\})\}.$$

Then $\psi(s) < \mathrm{length}(h(s))$, since h is a bad $\mathrm{Seq}(Q)$-array; and the Ellentuck-continuity of h implies that of ψ. Moreover, it is straightforward to show that

$$h(s)|_{\psi(s)} \leq h(s\backslash\{\min s\}),$$

but

$$h(s)|_{\psi(s)+1} \not\leq h(s\backslash\{\min s\}).$$

Let

$$C := \{s \in \mathrm{D}h \mid h(s)|_{\psi(s)} \leq h(s\backslash\{\min s\})|_{\psi(s\backslash\{\min s\})}\}.$$

Since both h and ψ are Ellentuck-continuous, C is Ellentuck-open. Thus by Theorem 1 there exists an $D \in \mathrm{D}h$ such that either $D^{(\omega)} \subseteq C$ or $D^{(\omega)} \cap C = \emptyset$. If the latter holds, then $\chi : D^{(\omega)} \to \mathrm{Seq}(Q)$ defined by $\chi(s) := h(s)|_{\psi(s)}$ would be a bad $\mathrm{Seq}(Q)$-array with $\chi <^* h$, contradicting the choice of h.

Thus $D^{(\omega)} \subseteq C$. We now define $g : D^{(\omega)} \to Q$ by putting $g(s) := h(s)(\psi(s))$, the value of $h(s)$ at $\psi(s)$. Then g is Ellentuck-continuous, since h and ψ are. Moreover, $\mathrm{D}g \subseteq \mathrm{D}f$ and $g(s) \in f(s)$ for all $s \in \mathrm{D}g$. If there were an $s \in \mathrm{D}g$ with $g(s) \leq g(s\backslash\{\min s\})$, then we could define an embedding of $h(s)|_{\psi(s)+1}$ into $h(s\backslash\{\min s\})$ by first embedding $h(s)|_{\psi(s)}$ into $h(s\backslash\{\min s\})|_{\psi(s\backslash\{\min s\})}$ (this is possible since $s \in D^{(\omega)} \subseteq C$), and secondly, by sending $h(s)(\psi(s)) = g(s)$ to $h(s\backslash\{\min s\})(\psi(s\backslash\{\min s\})) = g(s\backslash\{\min s\})$. This contradicts the definition of ψ. Thus g is a bad $\mathrm{Seq}(Q)$-array as required. □

If Q is a quasi-ordered set, we may quasi-order the elements of the power set of Q by saying that $A \leq B$ if there is an injective function $f : A \to B$ such that $a \leq f(a)$ in Q for all $a \in A$. Let $\mathcal{S}^\sharp(Q)$ denote the power set of Q with this quasi-ordering. Lemma 4 implies the following assertion.

Corollary 5 *If f is a bad $\mathcal{S}^\sharp(Q)$-array, then there exists a bad Q-array g such that $Dg \subseteq Df$ and $g(s) \in f(s)$ for all $s \in Dg$.*

An example of Rado [7] shows that there are wqo sets Q such that $\mathcal{S}(Q)$ (and thus also $\mathcal{S}^\sharp(Q)$ and $\mathrm{Seq}(Q)$) are not wqo. This lack of closure properties under certain infinite operations is the reason why the stronger concept of bqo was introduced.

Denote the class of all trees by $\mathcal{R}$, and recall that the elements of $\mathcal{R}$ are quasi-ordered by the rooted topological minor relation. Let $\mathcal{R}_0$ be the subclass containing all trees T with the property that there is no infinite sequence $x_1, x_2, \dots$ of vertices in T such that x_{i+1} is above x_i and $\mathrm{br}(x_i) \not\preccurlyeq \mathrm{br}(x_{i+1})$ for all $i \geq 1$. Given a tree T, let $\mathrm{S}(T)$ be the set of all its vertices x for which $T \not\preccurlyeq \mathrm{br}(x)$. If $x \in \mathrm{S}(T)$, we call $\mathrm{br}(x)$ a *strict branch* of T. For a vertex $x \in T$ we denote the set of its successors by $\mathrm{succ}(x)$, and let

$$\Gamma(x) \ := \ (\,|\{\mathrm{succ}(x) \backslash \mathrm{S}(T)\}|\,,\, \{\mathrm{br}(y) \,|\, y \in \mathrm{succ}(x) \cap \mathrm{S}(T)\}\,).$$

We view $\Gamma(x)$ as an element of the quasi-ordered set $\mathcal{C} \times \mathcal{S}^\sharp(\mathcal{R})$.

Lemma 6 *Suppose that T and U are trees such that for every vertex $x \in T$ there exists a vertex $y \in U$ with $\Gamma(x) \leq \Gamma(y)$. Then $T \preccurlyeq U$.*

Proof. For $n = 0, 1, \dots$, let W_n denote the set of all vertices of T which have distance at most n from the root of T. We shall inductively define an embedding φ of T into U such that, at stage n, we have defined φ on a set $V_n \subseteq V(T)$ satisfying the following conditions:

(i) $W_n \subseteq V_n$, and if $x \in V_n$ then the predecessor of x in T lies in V_n. If $x \in V_n \backslash W_n$, then $V(\mathrm{br}(x)) \subseteq V_n$.

(ii) Suppose that $x \in W_{n+1} \backslash V_n$, and let z be the predecessor of x. Then $x \notin \mathrm{S}(T)$ and there exists a vertex $v_x^n \in \mathrm{succ}(\varphi(z)) \backslash \mathrm{S}(U)$ such that no vertex of $\mathrm{br}(v_x^n)$ lies in $\varphi(V_n)$. Furthermore, the vertices v_x^n are distinct for distinct $x \in W_{n+1} \backslash V_n$.

Let x_0 be the root of T. Then by the assumptions of the lemma, there is a vertex $y_0 \in U$ such that $\Gamma(x_0) \leq \Gamma(y_0)$. Thus for all $x \in \mathrm{succ}(x_0)$, there is a vertex $v_x^0 \in \mathrm{succ}(y_0)$ such that, firstly, the vertices v_x^0 are distinct for distinct x, secondly, if $x \notin \mathrm{S}(T)$, then $v_x^0 \notin \mathrm{S}(U)$, and thirdly, if $x \in \mathrm{S}(T)$, then $\mathrm{br}(x) \preccurlyeq \mathrm{br}(v_x^0)$. Put $\varphi(x_0) := y_0$, and extend φ by embedding $\mathrm{br}(x)$ into $\mathrm{br}(v_x^0)$ for all $x \in \mathrm{succ}(x_0) \cap \mathrm{S}(T)$. Setting

$$V_0 \ := \ \{x_0\} \cup \bigcup \{V(\mathrm{br}(x)) \ | \ x \in \mathrm{succ}(x_0) \cap \mathrm{S}(T)\}$$

starts the induction. Suppose that $n > 0$ and conditions (i) and (ii) hold for $n-1$. If $W_n \subseteq V_{n-1}$, then $V_{n-1} = V(T)$ by (i), and we are done. Thus let us assume that $W_n \not\subseteq V_{n-1}$, and let x be any vertex in $W_n \backslash V_{n-1}$. By the assumption of the lemma there is a vertex $y \in U$ such that $\Gamma(x) \leq \Gamma(y)$. Let v_x^{n-1} be as in condition (ii). Then $U \preccurlyeq \mathrm{br}(v_x^{n-1})$, since $v_x^{n-1} \notin \mathrm{S}(U)$. Let y' be the image of y in $\mathrm{br}(v_x^{n-1})$ under this embedding. The fact that $\Gamma(x) \leq \Gamma(y)$ now implies that for all $a \in \mathrm{succ}(x)$ there exists a vertex $v_a^n \in \mathrm{succ}(y')$ satisfying the following three conditions. Firstly, the v_a^n are distinct for distinct a. Secondly, if $a \notin \mathrm{S}(T)$ then $v_a^n \notin \mathrm{S}(U)$, and thirdly, if $a \in \mathrm{S}(T)$, then $\mathrm{br}(a) \preccurlyeq \mathrm{br}(v_a^n)$. Put $\varphi(x) := y'$ and extend φ further by embedding $\mathrm{br}(a)$ into $\mathrm{br}(v_a^n)$ for all $a \in \mathrm{succ}(x) \cap \mathrm{S}(T)$. Proceed similarly for every $x \in W_n \backslash V_{n-1}$. Then, setting

$$V_n \ := \ V_{n-1} \cup W_n \cup \bigcup \{V(\mathrm{br}(a)) \,|\, a \in \mathrm{succ}(x) \cap \mathrm{S}(T) \text{ for some } x \in W_n \backslash V_{n-1}\}$$

completes the induction step. $\square$

Lemma 7 *If f is a bad $\mathcal{R}_0$-array, then there exists a bad $\mathcal{R}_0$-array g such that $Dg \subseteq Df$ and $g(s)$ is a strict branch of $f(s)$ for all $s \in Dg$.*

Proof. For a tree $T \in \mathcal{R}_0$ we define $\Sigma(T) := \{\Gamma(x) \mid x \in T\}$ and think of it as an element of the quasi-ordered set $\mathcal{S}(\mathcal{C} \times \mathcal{S}^\sharp(\mathcal{R}_0))$. Lemma 6 implies that for all $T, U \in \mathcal{R}_0$,

$$\Sigma(T) \leq \Sigma(U) \;\Rightarrow\; T \preccurlyeq U.$$

Hence $\Sigma \circ f$ is a bad $\mathcal{S}(\mathcal{C} \times \mathcal{S}^\sharp(\mathcal{R}_0))$-array. By Lemma 2, there is a bad $\mathcal{C} \times \mathcal{S}^\sharp(\mathcal{R}_0)$-array φ such that $\mathrm{D}\varphi = \mathrm{D}\Sigma \circ f = \mathrm{D}f$ and $\varphi(s) \in \Sigma \circ f(s)$ for all $s \in \mathrm{D}\varphi$. Now Lemma 3 implies that there is a bad $\mathcal{S}^\sharp(\mathcal{R}_0)$-array ψ such that $\mathrm{D}\psi \subseteq \mathrm{D}\varphi$ and $\psi(s) = \varphi_2(s)$ for all $s \in \mathrm{D}\psi$. Finally, by Corollary 5, there is a bad $\mathcal{R}_0$-array g such that $\mathrm{D}g \subseteq \mathrm{D}\psi$ and $g(s) \in \psi(s)$ for all $s \in \mathrm{D}g$. Clearly, $\mathrm{D}g \subseteq \mathrm{D}f$. Furthermore, for all $s \in \mathrm{D}g$, $g(s)$ is an element of the second component of an element of $\Sigma \circ f(s)$, and thus a strict branch of $f(s)$, as required. □

If h and h' are $\mathcal{R}_0$-arrays, we write $h \leq' h'$ if $\mathrm{D}h \subseteq \mathrm{D}h'$, and if $h(s)$ is a branch of $h'(s)$ for all $s \in \mathrm{D}h$. Furthermore, we write $h <' h'$ if $h \leq' h'$ and there exists an $s \in \mathrm{D}h$ such that $h(s)$ is a strict branch of $h'(s)$.

Lemma 8 *If f is a bad $\mathcal{R}_0$-array, then there exists a minimal bad $\mathcal{R}_0$-array h such that $h \leq' f$.*

We omit the proof, since it is an easy modification of the proof of assertion $(*)$ in the proof of Lemma 4. Indeed, the only difference is the following. In Lemma 4 we repeatedly made use of the fact that we could not have an infinite sequence $F_1, F_2, \ldots$ in $\mathrm{Seq}(Q)$ such that F_{i+1} is a proper initial segment of F_i for all $i \geq 1$, since $\mathrm{length}(F_1), \mathrm{length}(F_2), \ldots$ would then have been an infinite strictly descending chain of ordinals. In the proof of Lemma 8 an infinite sequence $F_1, F_2, \ldots$ in $\mathcal{R}_0$ such that F_{i+1} is a strict branch of F_i for all $i \geq 1$ would contradict the definition of $\mathcal{R}_0$.

Lemmas 7 and 8 immediately imply the following result.

Corollary 9 *$\mathcal{R}_0$ is bqo.*

Given a tree T, let $\mathrm{F}(T) := \{x \in T \mid \mathrm{br}(x) \in \mathcal{R}_0\}$ and $I(T) := V(T) \backslash F(T)$. For a vertex $x \in T$ define

$$\Delta(x) \;:=\; (\,|\mathrm{succ}(x) \cap \mathrm{I}(T)|\,,\; \{\mathrm{br}(z) \,|\, z \in \mathrm{succ}(x) \cap \mathrm{F}(T)\}\,).$$

We view $\Delta(x)$ as an element of $\mathcal{C} \times \mathcal{S}^\sharp(\mathcal{R}_0)$.

Lemma 10 *Suppose that T is a tree and $x_0, y_0 \in I(T)$ are such that*

$$\forall x \in br(x_0) \cap I(T) \;\; \forall y \in br(y_0) \cap I(T) \;\; \exists z \in br(y) : \;\; \Delta(x) \leq \Delta(z).$$

Then $br(x_0) \preccurlyeq br(y_0)$.

Proof. The proof is very similar to that of Lemma 6. For $n = 0, 1, \ldots$, let W_n denote the set of all vertices of $\mathrm{br}(x_0)$ which have distance at most n from x_0. We shall inductively define an embedding φ of $\mathrm{br}(x_0)$ into $\mathrm{br}(y_0)$ such that, at stage n, we have defined φ on a set $V_n \subseteq V(\mathrm{br}(x_0))$ satisfying the following conditions:

(i) $W_n \subseteq V_n$, and if $x \in V_n$ then the predecessor of x in $\mathrm{br}(x_0)$ lies in V_n. If $x \in V_n \backslash W_n$, then $V(\mathrm{br}(x)) \subseteq V_n$.

(ii) Suppose that $x \in W_{n+1} \backslash V_n$, and let y be the predecessor of x. Then $x \in \mathrm{I}(T)$, and there exists a vertex $v_x^n \in \mathrm{succ}(\varphi(y)) \cap \mathrm{I}(U)$ such that no vertex of $\mathrm{br}(v_x^n)$ lies in $\varphi(V_n)$. Furthermore, the vertices v_x^n are distinct for distinct $x \in W_{n+1} \backslash V_n$.

By the assumptions of the lemma, there is a vertex $z_0 \in \mathrm{br}(y_0)$ such that $\Delta(x_0) \leq \Delta(z_0)$. Thus for all $x \in \mathrm{succ}(x_0)$ there is a vertex $v_x^0 \in \mathrm{succ}(z_0)$ such that, firstly, the vertices v_x^0 are distinct for distinct x, secondly, if $x \in \mathrm{I}(T)$ then $v_x^0 \in \mathrm{I}(T)$, and thirdly, if $x \in \mathrm{F}(T)$, then $\mathrm{br}(x) \preccurlyeq \mathrm{br}(v_x^0)$. Put $\varphi(x_0) := z_0$, and extend φ by embedding $\mathrm{br}(x)$ into $\mathrm{br}(v_x^0)$ for all $x \in \mathrm{succ}(x_0) \cap \mathrm{F}(T)$. Setting

$$V_0 \;:=\; \{x_0\} \cup \bigcup \{V(\mathrm{br}(x)) \mid x \in \mathrm{succ}(x_0) \cap \mathrm{F}(T)\}$$

starts the induction. Suppose that $n > 0$ and conditions (i) and (ii) hold for $n-1$. If $W_n \subseteq V_{n-1}$, then $V_{n-1} = V(\mathrm{br}(x_0))$ by (i), and we are done. Thus we may assume that $W_n \not\subseteq V_{n-1}$. Let x be any vertex in $W_n \backslash V_{n-1}$, and let v_x^{n-1} be as in condition (ii). Then by the assumption of the lemma there is a vertex $z \in \mathrm{br}(v_x^{n-1})$ such that $\Delta(x) \leq \Delta(z)$. Thus for all $a \in \mathrm{succ}(x)$ there exists a vertex $v_a^n \in \mathrm{succ}(z)$ such that, firstly, the v_a^n are distinct for distinct a, secondly, if $a \in \mathrm{I}(T)$ then $v_a^n \in \mathrm{I}(T)$, and thirdly, if $a \in \mathrm{F}(T)$, then $\mathrm{br}(a) \preccurlyeq \mathrm{br}(v_a^n)$. Put $\varphi(x) := z$, and extend φ further by embedding $\mathrm{br}(a)$ into $\mathrm{br}(v_a^n)$ for all $a \in \mathrm{succ}(x) \cap \mathrm{F}(T)$. Proceed similarly for every $x \in W_n \backslash V_{n-1}$. Then, setting

$$V_n \;:=\; V_{n-1} \cup W_n \cup \bigcup \{V(\mathrm{br}(a)) \,|\, a \in \mathrm{succ}(x) \cap \mathrm{F}(T) \text{ for some } x \in W_n \backslash V_{n-1}\}$$

completes the induction step. □

Theorem 11 *The infinite trees are bqo under the rooted topological minor relation.*

Proof. By Corollary 9 it suffices to show that every tree lies in $\mathcal{R}_0$. Suppose not, and let T be a tree that does not lie in $\mathcal{R}_0$. Let x_0 be the root of T. Since $T \notin \mathcal{R}_0$, there is a vertex $y_1 \in \mathrm{br}(x_0) \cap \mathrm{I}(T)$ such that $\mathrm{br}(x_0) \not\preccurlyeq \mathrm{br}(y_1)$. Then Lemma 10 implies that there exist vertices $z_1 \in \mathrm{br}(x_0) \cap \mathrm{I}(T)$ and $x_1 \in \mathrm{br}(y_1) \cap \mathrm{I}(T)$ such that $\Delta(z_1) \not\leq \Delta(z)$ for all $z \in \mathrm{br}(x_1)$. Since $x_1 \in \mathrm{I}(T)$, there is a vertex $y_2 \in \mathrm{br}(x_1) \cap \mathrm{I}(T)$ such that $\mathrm{br}(x_1) \not\preccurlyeq \mathrm{br}(y_2)$. Again, Lemma 10 implies that there exist vertices $z_2 \in \mathrm{br}(x_1) \cap \mathrm{I}(T)$ and $x_2 \in \mathrm{br}(y_2) \cap \mathrm{I}(T)$ such that $\Delta(z_2) \not\leq \Delta(z)$ for all $z \in \mathrm{br}(x_2)$. Continuing in this fashion, we obtain an infinite sequence $z_1, z_2, \ldots$ such that $\Delta(z_i) \not\leq \Delta(z_j)$ in $\mathcal{C} \times \mathcal{S}^\sharp(\mathcal{R}_0)$ for all $1 \leq i < j$. But since $\mathcal{R}_0$ is bqo by Corollary 9, $\mathcal{C} \times \mathcal{S}^\sharp(\mathcal{R}_0)$ is bqo by Lemma 3 and Corollary 5, and thus it is wqo by Lemma 1, a contradiction. □

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