EMBEDDING LARGE SUBGRAPHS INTO DENSE GRAPHS

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ABSTRACT. What conditions ensure that a graph G contains some given spanning subgraph H? The most famous examples of results of this kind are probably Dirac's theorem on Hamilton cycles and Tutte's theorem on perfect matchings. Perfect matchings are generalized by perfect F-packings, where instead of covering all the vertices of G by disjoint edges, we want to cover G by disjoint copies of a (small) graph F. It is unlikely that there is a characterization of all graphs G which contain a perfect F-packing, so as in the case of Dirac's theorem it makes sense to study conditions on the minimum degree of G which guarantee a perfect F-packing.

The Regularity lemma of Szemerédi and the Blow-up lemma of Komlós, Sárközy and Szemerédi have proved to be powerful tools in attacking such problems and quite recently, several long-standing problems and conjectures in the area have been solved using these. In this survey, we give an outline of recent progress (with our main emphasis on F-packings, Hamiltonicity problems and tree embeddings) and describe some of the methods involved.

1. INTRODUCTION, OVERVIEW AND BASIC NOTATION

In this survey, we study the question of when a graph G contains some given large or spanning graph H as a subgraph. Many important problems can be phrased in this way: one example is Dirac's theorem, which states that every graph G on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. Another example is Tutte's theorem on perfect matchings which gives a characterization of all those graphs which contain a perfect matching (so H corresponds to a perfect matching in this case). A result which gives a complete characterization of all those graphs G which contain H (as in the case of Tutte's theorem) is of course much more desirable than a sufficient condition (as in the case of Dirac's theorem). However, for most H that we consider, it is unlikely that such a characterization exists as the corresponding decision problems are usually NP-complete. So it is natural to seek simple sufficient conditions. Here we will focus mostly on degree conditions. This means that G will usually be a dense graph and that we have to restrict H to be rather sparse in order to get interesting results. We will survey the following topics:

- a generalization of the matching problem, which is called the *F-packing* or *F-tiling* problem (here the aim is to cover the vertices of G with disjoint copies of a fixed graph F instead of disjoint edges);
- Hamilton cycles (and generalizations) in graphs, directed graphs and hypergraphs;

- large subtrees of graphs;
- arbitrary subgraphs H of bounded degree;
- Ramsey numbers of sparse graphs.

A large part of the progress in the above areas is due to the Regularity lemma of Szemerédi [125] and the Blow-up lemma of Komlós, Sárközy and Szemerédi [83]. Roughly speaking, the former states that one can decompose an arbitrary large dense graph into a bounded number of random-like graphs. The latter is a powerful tool for embedding spanning subgraphs H into such random-like graphs. In the final section we give a formal statement of these results and describe in detail an application to a special case of the F-packing problem. We hope that readers who are unfamiliar with these tools will find this a useful guide to how they can be applied.

There are related surveys in the area by Komlós and Simonovits [89] (some minor updates were added later in [88]) and by Komlós [80]. However, much has happened since these were written and the emphasis is different in each case. So we hope that the current survey will be a useful complement and update to these. In particular, as the title indicates, our focus is mainly on embedding large subgraphs and we will ignore other aspects of regularity/quasi-randomness. There is also a recent survey on F-packings (and so-called F-decompositions) by Yuster [130], which is written from a computational perspective.

2. Packing small subgraphs in graphs

2.1. F-packings in graphs of large minimum degree.

Given two graphs F and G, an F-*packing in G* is a collection of vertex-disjoint copies of F in G. (Alternatively, this is often called an F-tiling.) F-packings are natural generalizations of graph matchings (which correspond to the case when F consists of a single edge). An F-packing in G is called *perfect* if it covers all vertices of G. In this case, we also say that G contains an F-*factor* or a *perfect F-matching*. If F has a component which contains at least 3 vertices then the question whether G has a perfect F-packing is difficult from both a structural and algorithmic point of view: Tutte's theorem characterizes those graphs which have a perfect F-packing if F is an edge but for other connected graphs F no such characterization is known. Moreover, Hell and Kirkpatrick [61] showed that the decision problem of whether a graph G has a perfect F-packing is NP-complete if and only if F has a component which contains at least 3 vertices. So as mentioned earlier, this means that it makes sense to search for degree conditions which ensure the existence of a perfect F-packing. The fundamental result in the area is the Hajnal-Szemerédi theorem:

Theorem 1. (Hajnal and Szemerédi [55]) *Every graph whose order n is divisible by r and whose minimum degree is at least $(1 - 1/r)n$ contains a perfect K_r-packing.*

The minimum degree condition is easily seen to be best possible. (The case when $r = 3$ was proved earlier by Corrádi and Hajnal [30].) The result is often phrased in terms of colourings: any graph G whose order is divisible by k and with

$\Delta(G) \leq k - 1$ has an equitable k-colouring, i.e. a colouring with colour classes of equal size. (So $k := n/r$ here.) Theorem 1 raises the question of what minimum degree condition forces a perfect F-packing for arbitrary graphs F. The following result gives a general bound.

Theorem 2. (Komlós, Sárközy and Szemerédi [86]) *For every graph F there exists a constant $C = C(F)$ such that every graph G whose order n is divisible by $|F|$ and whose minimum degree is at least $(1 - 1/\chi(F))n + C$ contains a perfect F-packing.*

This confirmed a conjecture of Alon and Yuster [9], who had obtained the above result with an additional error term of εn in the minimum degree condition. As observed in [9], there are graphs F for which the above constant C cannot be omitted completely (e.g. $F = K_{s,s}$ where $s \geq 3$ and s is odd). Thus one might think that this settles the question of which minimum degree guarantees a perfect F-packing. However, we shall see that this is *not* the case. There are graphs F for which the bound on the minimum degree can be improved significantly: we can often replace $\chi(F)$ by a smaller parameter. For a detailed statement of this, we define the *critical chromatic number* $\chi_{cr}(F)$ of a graph F as

$$\chi_{cr}(F) := (\chi(F) - 1)\frac{|F|}{|F| - \sigma(F)},$$

where $\sigma(F)$ denotes the minimum size of the smallest colour class in an optimal colouring of F. (We say that a colouring of F is *optimal* if it uses exactly $\chi(F)$ colours.) So for instance a k-cycle C_k with k odd has $\chi_{cr}(C_k) = 2 + 2/(k-1)$. Note that $\chi_{cr}(F)$ always satisfies $\chi(F) - 1 < \chi_{cr}(F) \leq \chi(F)$ and equals $\chi(F)$ if and only if for every optimal colouring of F all the colour classes have equal size. The critical chromatic number was introduced by Komlós [81]. He (and independently Alon and Fischer [8]) observed that for *any* graph F it can be used to give a lower bound on the minimum degree that guarantees a perfect F-packing.

Proposition 3. *For every graph F and every integer n that is divisible by $|F|$ there exists a graph G of order n and minimum degree $\lceil (1 - 1/\chi_{cr}(F))n \rceil - 1$ which does not contain a perfect F-packing.*

Given a graph F, the graph G in the proposition is constructed as follows: write $k := \chi(F)$ and let $\ell \in \mathbb{N}$ be arbitrary. G is a complete k-partite graph with vertex classes $V_1, \ldots, V_k$, where $|V_1| = \sigma(F)\ell - 1$, $n = \ell|F|$ and the sizes of $V_2, \ldots, V_k$ are as equal as possible. Then any perfect F-packing would consist of ℓ copies of F. On the other hand, each such copy would contain at least $\sigma(F)$ vertices in V_1, which is impossible.

Komlós also showed that the critical chromatic number is the parameter which governs the existence of *almost* perfect packings in graphs of large minimum degree. (More generally, he also determined the minimum degree which ensures that a given fraction of vertices is covered.)

Theorem 4. (Komlós [81]) *For every graph F and every $\gamma > 0$ there exists an integer $n_0 = n_0(\gamma, F)$ such that every graph G of order $n \geq n_0$ and minimum*

degree at least $(1 - 1/\chi_{cr}(F))n$ contains an F-packing which covers all but at most γn vertices of G.

By making V_1 slightly smaller in the previous example, it is easy to see that the minimum degree bound in Theorem 4 is also best possible. Confirming a conjecture of Komlós [81], Shokoufandeh and Zhao [121, 122] subsequently proved that the number of uncovered vertices can be reduced to a constant depending only on F.

We [96] proved that for any graph F, either its critical chromatic number or its chromatic number is the relevant parameter which governs the existence of perfect packings in graphs of large minimum degree. The classification depends on a parameter which we call the *highest common factor* of F.

This is defined as follows for non-bipartite graphs F. Given an optimal colouring c of F, let $x_1 \leq x_2 \leq \cdots \leq x_\ell$ denote the sizes of the colour classes of c. Put $\mathcal{D}(c) := \{x_{i+1} - x_i \,|\, i = 1, \ldots, \ell - 1\}$. Let $\mathcal{D}(F)$ denote the union of all the sets $\mathcal{D}(c)$ taken over all optimal colourings c. We denote by $\mathrm{hcf}(F)$ the highest common factor of all integers in $\mathcal{D}(F)$. If $\mathcal{D}(F) = \{0\}$ we set $\mathrm{hcf}(F) := \infty$. Note that if all the optimal colourings of F have the property that all colour classes have equal size, then $\mathcal{D}(F) = \{0\}$ and so $\mathrm{hcf}(F) \neq 1$ in this case. In particular, if $\chi_{cr}(F) = \chi(F)$, then $\mathrm{hcf}(F) \neq 1$. So for example, odd cycles of length at least 5 have $\mathrm{hcf} = 1$ whereas complete graphs have $\mathrm{hcf} \neq 1$.

The definition can be extended to bipartite graphs F. For connected bipartite graphs, we always have $\mathrm{hcf}(F) \neq 1$, but for disconnected bipartite graphs the definition also takes into account the relative sizes of the components of F (see [96]).

We proved that in Theorem 2 one can replace the chromatic number by the critical chromatic number if $\mathrm{hcf}(F) = 1$. (A much simpler proof of a weaker result can be found in [93].)

Theorem 5. (Kühn and Osthus [96]) *Suppose that F is a graph with $\mathrm{hcf}(F) = 1$. Then there exists a constant $C = C(F)$ such that every graph G whose order n is divisible by $|F|$ and whose minimum degree is at least $(1 - 1/\chi_{cr}(F))n + C$ contains a perfect F-packing.*

Note that Proposition 3 shows that the result is best possible up to the value of the constant C. A simple modification of the examples in [8, 81] shows that there are graphs F for which the constant C cannot be omitted entirely. Moreover, it turns out that Theorem 2 is already best possible up to the value of the constant C if $\mathrm{hcf}(F) \neq 1$. To see this, for simplicity assume that $k := \chi(F) \geq 3$ and $n = k\ell|F|$ for some $\ell \in \mathbb{N}$ and let G be a complete k-partite graph with vertex classes $V_1, \ldots, V_k$, where $|V_1| := \ell|F| - 1$, $|V_2| := \ell|F| + 1$ and $|V_i| = \ell|F|$ for $i \geq 3$. Consider any F-packing $F_1, \ldots, F_t$ in G. Let G_i be the graph obtained from G by removing $F_1, \ldots, F_i$. So $G = G_0$. If $t := \mathrm{hcf}(F) \neq 1$, then the vertex classes V_i^1 of G_1 still have property that $|V_1^1| - |V_k^1| \not\equiv 0$ modulo t. More generally, this property is preserved for all G_i, so the original F-packing cannot cover all the vertices in $V_1 \cup V_k$.

One can now combine Theorems 2 and 5 (and the corresponding lower bounds which are discussed in detail in [96]) to obtain a complete answer to the question of which minimum degree forces a perfect F-packing (up to an additive constant). For this, let

$$\chi^*(F) := \begin{cases} \chi_{cr}(F) & \text{if } \mathrm{hcf}(F) = 1; \\ \chi(F) & \text{otherwise.} \end{cases}$$

Also let $\delta(F, n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|F|$ and with $\delta(G) \geq k$ contains a perfect F-packing.

Theorem 6. (Kühn and Osthus [96]**)** *For every graph F there exists a constant $C = C(F)$ such that*
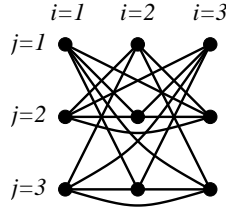
$$\left(1 - \frac{1}{\chi^*(F)}\right) n - 1 \leq \delta(F, n) \leq \left(1 - \frac{1}{\chi^*(F)}\right) n + C.$$

The constant C appearing in Theorems 5 and 6 is rather large since it is related to the number of partition classes (clusters) obtained by the Regularity lemma. It would be interesting to know whether one can take e.g. $C = |F|$ (this holds for large n in Theorem 2). Another open problem is to characterize all those graphs F for which $\delta(F, n) = \lceil (1 - 1/\chi^*(F))n \rceil$. This is known to be the case for complete graphs by Theorem 1 and all graphs with at most 4 vertices (see Kawarabayashi [71] for a proof of the case when F is a K_4 minus an edge and a discussion of the other cases). If n is large, this is also known to hold for cycles (this follows from Theorem 32 below) and for the case when F is a complete graph minus an edge [29] (the latter was conjectured in [71]).

2.2. **Ore-type degree conditions.** Recently, a simple proof (based on an inductive argument) of the Hajnal-Szemerédi theorem was found by Kierstead and Kostochka [78]. Using similar methods, they subsequently strengthened this to an Ore-type condition [79]:

Theorem 7. (Kierstead and Kostochka [79]**)** *Let G be a graph whose order n is divisible by r. If $d(x) + d(y) \geq 2(1 - 1/r)n - 1$ for all pairs $x \neq y$ of nonadjacent vertices, then G has a perfect K_r-packing.*

Equivalently, if a graph G whose order is divisible by k satisfies $d(x) + d(y) \leq 2k - 1$ for every edge xy, then G has an equitable k-colouring. (So $k := n/r$.) Recently, together with Treglown [99], we proved an Ore-type analogue of Theorem 6 (but with a linear error term εn instead of the additive constant C). The result in this case turns out to be genuinely different: again, there are some graphs F for which the degree condition depends on $\chi(F)$ and some for which it depends on $\chi_{cr}(F)$. However, there are also graphs F for which it depends on a parameter which lies strictly between $\chi_{cr}(F)$ and $\chi(F)$. This parameter in turn depends on how many additional colours are necessary to extend colourings of neighbourhoods of certain vertices of F to a colouring of F. It is an open question whether the linear error term in [99] can be reduced to a constant one.

FIGURE 1. The graph $\Gamma_3 = \Gamma_{3,1}$ in Conjecture 8

2.3. r-**partite versions.** Also, it is natural to consider r-partite versions of the Hajnal-Szemerédi theorem. For this, given an r-partite graph G, let $\delta'(G)$ denote the minimum over all vertex classes W of G and all vertices $x \notin W$ of the number of neighbours of x in W. The obvious question is what value of $\delta'(G)$ ensures that G has a perfect K_r-packing. The following (surprisingly difficult) conjecture is implicit in [101]. Fischer [39] originally made a stronger conjecture which did not include the 'exceptional' graph $\Gamma_{r,n}$ defined below.

Conjecture 8. *Suppose that $r \geq 2$ and that G is an r-partite graph with vertex classes of size n. If $\delta'(G) \geq (1 - 1/r)n$, then G has a perfect K_r-packing unless both r and n are odd and $G = \Gamma_{r,n}$.*

To define the graph $\Gamma_{r,n}$, we first construct a graph Γ_r: its vertices are labelled g_{ij} with $1 \leq i, j \leq r$. We have an edge between g_{ij} and $g_{i'j'}$ if $i \neq i'$, $j \neq j'$ and $j \leq r - 2$ or $j' \leq r - 2$. We also have an edge if $i \neq i'$ and we have either $j = j' = r - 1$ or $j = j' = r$ (see Fig. 1). $\Gamma_{r,n}$ is then obtained from Γ_r by replacing each vertex with an independent set of size n/r and replacing each edge with a complete bipartite graph.

To see that $\Gamma_{r,n}$ has no perfect K_r-packing when both r and n are odd, let W_ℓ denote the set of vertices of $\Gamma_{r,n}$ which correspond to a vertex of Γ_r with $j = \ell$. Note that every copy of K_r which covers a vertex in $W_1 \cup \cdots \cup W_{r-2}$ has to contain at least 2 vertices in W_{r-1} or at least 2 vertices in W_r. So in order to cover all vertices in $W_1 \cup \cdots \cup W_{r-2}$ we can only use copies of K_r which contain exactly 2 vertices in W_{r-1} or exactly 2 vertices in W_r. But since $|W_{r-1}| = |W_r| = n$ is odd this means that it is impossible to cover all vertices of $\Gamma_{r,n}$ with vertex-disjoint copies of K_r. (Note that the argument uses only that n is odd, but we cannot have that n is odd and r is even.)

A much simpler example which works for all r and n but which gives a weaker bound when r and n are odd is obtained as follows: choose a set A which has less than $(1 - 1/r)n$ vertices in each vertex class and include all edges which have at least one endpoint in A. For large n, the case $r = 3$ of Conjecture 8 was solved by Magyar and Martin [101] and the case $r = 4$ by Martin and Szemerédi [102], both using the Regularity lemma (the case $r = 2$ is elementary). Johansson [67] had earlier proved an approximate version of the case $r = 3$. Csaba and Mydlarz [33] proved a result which implies that Conjecture 8 holds approximately when r is large (and n large compared to r). Generalizations to packings of arbitrary graphs

were considered in [63, 103, 132]. A variant of the problem (where one considers usual minimum degree $\delta(G)$) was considered by Johansson, Johansson and Markström [68]. They solved the case $r = 3$ and gave bounds for the case $r > 3$. This problem is related to bounding the so-called 'strong chromatic number'.

2.4. **Hypergraphs.** (Perfect) F-packings have also been investigated for the case when F is a uniform hypergraph. Unsurprisingly, the hypergraph problem turns out to be much more difficult than the graph problem. There are two natural notions of (minimum) degree of the 'dense' hypergraph G. Firstly, one can consider the vertex degree. Secondly, given an r-uniform hypergraph G and an $(r-1)$-tuple W of vertices in G, the degree of W is defined to be the number of hyperedges which contain W. This notion of degree is called *collective degree* or *co-degree*. In contrast to the graph case, even the minimum collective degree which ensures a perfect matching (i.e. when F consists of a single edge) is not easy to determine. Rödl, Ruciński and Szemerédi [118] gave a precise solution to this problem, the answer turns out to be close to $n/2$. This improved bounds of [94, 115]. An r-partite version (which is best possible for infinitely many values of n) was proved by Aharoni, Georgakopoulos and Sprüssel [3]. The minimum vertex degree which forces the existence of a perfect matching is unknown. It is natural to make the following conjecture (a related r-partite version is conjectured in [3]).

Conjecture 9. *For all integers r and all $\varepsilon > 0$ there is an integer $n_0 = n_0(r, \varepsilon)$ so that the following holds for all $n \geq n_0$ which are divisible by r: if G is an r-uniform hypergraph on n vertices whose minimum vertex degree is at least*

$$(1 - (1 - 1/r)^{r-1} + \varepsilon)\binom{n}{r-1},$$

then G has a perfect matching.

The following construction gives a corresponding lower bound: let V be a set of n vertices and let $A \subseteq V$ be a set of less than n/r vertices and include as hyperedges all r-tuples with at least one vertex in A. The case $r = 3$ of the conjecture was proved recently by Han, Person and Schacht [56].

A hypergraph analogue of Theorem 6 currently seems out of reach. So far, the only hypergraph F (apart from the single edge) for which the approximate minimum collective degree which forces a perfect F-packing has been determined is the 3-uniform hypergraph with 4 vertices and 2 edges [95]. Pikhurko [113] gave bounds on the minimum collective degree which forces the complete 3-uniform hypergraph on 4 vertices. In the same paper, he also shows that if $\ell \geq r/2$ and G is an r-uniform hypergraph where every ℓ-tuple of vertices is contained in at least $(1/2 + o(1))\binom{n}{r-\ell}$ hyperedges, then G has a perfect matching, which is best possible up to the $o(1)$-term. This result is rather surprising in view of the fact that Conjecture 9 (which corresponds to the case when $\ell = 1$) has a rather different form. Further results on this question are also proved in [56].

3. Trees

One of the earliest applications of the Blow-up lemma was the solution by Komlós, Sárközy and Szemerédi [82] of a conjecture of Bollobás on the existence of given bounded degree spanning trees. The authors later relaxed the condition of bounded degree to obtain the following result.

Theorem 10. (Komlós, Sárközy and Szemerédi [87]) *For any $\gamma > 0$ there exist constants $c > 0$ and n_0 with the following properties. If $n \geq n_0$, T is a tree of order n with $\Delta(T) \leq cn/\log n$, and G is a graph of order n with $\delta(G) \geq (1/2+\gamma)n$, then T is a subgraph of G.*

The condition $\Delta(T) \leq cn/\log n$ is best possible up to the value of c. (The example given in [87] to show this is a random graph G with edge probability 0.9 and a tree of depth 2 whose root has degree close to $\log n$.)

It is an easy exercise to see that every graph of minimum degree at least k contains any tree with k edges. The following classical conjecture would imply that we can replace the minimum degree condition by one on the average degree.

Conjecture 11. (Erdős and Sós [37]) *Every graph of average degree greater than $k - 1$ contains any tree with k edges.*

This is trivially true for stars. (On the other hand, stars also show that the bound is best possible in general.) It is also trivial if one assumes an extra factor of 2 in the average degree. It has been proved for some special classes of trees, most notably those of diameter at most 4 [105]. The conjecture is also true for 'locally sparse' graphs – see Sudakov and Vondrak [124] for a discussion of this.

The following result proves (for large n) a related conjecture of Loebl. An approximate version was proved earlier by Ajtai, Komlós and Szemerédi [5].

Theorem 12. (Zhao [131]) *There is an integer n_0 so that every graph G on $n \geq n_0$ vertices which has at least $n/2$ vertices of degree at least $n/2$ contains all trees with at most $n/2$ edges.*

This would be generalized by the following conjecture.

Conjecture 13. (Komlós and Sós) *Every graph G on n vertices which has at least $n/2$ vertices of degree at least k contains all trees with k edges.*

Again, the conjecture is trivially true (and best possible) for stars. Piguet and Stein [111] proved an approximate version for the case when k is linear in n and n is large. Cooley [26] as well as Hladký and Piguet [62] proved an exact version for this case. All of these proofs are based on the Regularity lemma. As with Conjecture 11, there are several results on special cases which are not based on the Regularity lemma. For instance, Piguet and Stein proved it for trees of diameter at most 5 [112].

4. Hamilton cycles

4.1. **Classical results for graphs and digraphs.** As mentioned in the introduction, the decision problem of whether a graph has a Hamilton cycle is NP-complete, so it makes sense to ask for degree conditions which ensure that a graph has a Hamilton cycle. One such result is the classical theorem of Dirac.

Theorem 14. (Dirac [36]) *Every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle.*

For an analogue in directed graphs it is natural to consider the *minimum semidegree* $\delta^0(G)$ of a digraph G, which is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$. (Here a directed graph may have two edges between a pair of vertices, but in this case their directions must be opposite.) The corresponding result is a theorem of Ghouila-Houri [45].

Theorem 15. (Ghouila-Houri [45]) *Every digraph on n vertices with minimum semidegree at least $n/2$ contains a Hamilton cycle.*

In fact, Ghouila-Houri proved the stronger result that every strongly connected digraph of order n where every vertex has total degree at least n has a Hamilton cycle. (When referring to paths and cycles in directed graphs we always mean that these are directed, without mentioning this explicitly.) All of the above degree conditions are best possible. Theorems 14 and 15 were generalized to a degree condition on pairs of vertices for graphs as well as digraphs:

Theorem 16. (Ore [110]) *Suppose that G is a graph with $n \geq 3$ vertices such that every pair $x \neq y$ of nonadjacent vertices satisfies $d(x) + d(y) \geq n$. Then G has a Hamilton cycle.*

Theorem 17. (Woodall [129]) *Let G be a strongly connected digraph on $n \geq 2$ vertices. If $d^+(x) + d^-(y) \geq n$ for every pair $x \neq y$ of vertices for which there is no edge from x to y, then G has a Hamilton cycle.*

There are many generalizations of these results. The survey [46] gives an overview for undirected graphs and the monograph [10] gives a discussion of directed versions. Below, we describe some recent progress on degree conditions for Hamilton cycles, much of which is based on the Regularity lemma.

4.2. **Hamilton cycles in oriented graphs.** Thomassen [127] raised the natural question of determining the minimum semidegree that forces a Hamilton cycle in an *oriented graph* (i.e. in a directed graph that can be obtained from a simple undirected graph by orienting its edges). Thomassen initially believed that the correct minimum semidegree bound should be $n/3$ (this bound is obtained by considering a 'blow-up' of an oriented triangle). However, Häggkvist [52] later gave the following construction which gives a lower bound of $\lceil (3n-4)/8 \rceil - 1$ (see Fig. 2). For n of the form $n = 4m+3$ where m is odd, we construct G on n vertices as follows. Partition the vertices into 4 parts A, B, C, D, with $|A| = |C| = m$, $|B| = m + 1$ and $|D| = m + 2$. Each of A and C spans a regular tournament, B and D are joined by a bipartite tournament (i.e. an orientation of the complete
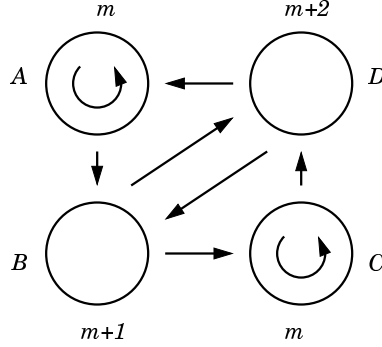
FIGURE 2. An extremal example for Theorem 18

bipartite graph) which is as regular as possible. We also add all edges from A to B, from B to C, from C to D and from D to A. Since every path which joins two vertices in D has to pass through B, it follows that every cycle contains at least as many vertices from B as it contains from D. As $|D| > |B|$ this means that one cannot cover all the vertices of G by disjoint cycles. This construction can be extended to arbitrary n (see [74]). The following result exactly matches this bound and improves earlier ones of several authors, e.g. [52, 54, 76].

Theorem 18. (Keevash, Kühn and Osthus [74]**)** *There exists an integer n_0 so that any oriented graph G on $n \geq n_0$ vertices with minimum semidegree $\delta^0(G) \geq \frac{3n-4}{8}$ contains a Hamilton cycle.*

The proof of this result is based on some ideas in [76]. Häggkvist [52] also made the following conjecture which is closely related to Theorem 18. Given an oriented graph G, let $\delta(G)$ denote the minimum degree of G (i.e. the minimum number of edges incident to a vertex) and set $\delta^*(G) := \delta(G) + \delta^+(G) + \delta^-(G)$.

Conjecture 19. (Häggkvist [52]**)** *Every oriented graph G on n vertices with $\delta^*(G) > (3n - 3)/2$ contains a Hamilton cycle.*

(Note that this conjecture does not quite imply Theorem 18 as it results in a marginally greater minimum semidegree condition.) In [76], Conjecture 19 was verified approximately, i.e. if $\delta^*(G) \geq (3/2 + o(1))n$, then G has a Hamilton cycle (note this implies an approximate version of Theorem 18). The same methods also yield an approximate version of Theorem 17 for oriented graphs.

Theorem 20. (Kelly, Kühn and Osthus [76]**)** *For every $\alpha > 0$ there exists an integer $n_0 = n_0(\alpha)$ such that every oriented graph G of order $n \geq n_0$ with $d^+(x) + d^-(y) \geq (3/4 + \alpha)n$ whenever G does not contain an edge from x to y contains a Hamilton cycle.*

The above construction of Häggkvist shows that the bound is best possible up to the term αn. It would be interesting to obtain an exact version of this result.

Note that Theorem 18 implies that every sufficiently large regular tournament on n vertices contains at least $n/8$ edge-disjoint Hamilton cycles. (To verify this,

note that in a regular tournament, all in- and outdegrees are equal to $(n-1)/2$. We can then greedily remove Hamilton cycles as long as the degrees satisfy the condition in Theorem 18.) It is the best bound so far towards the following conjecture of Kelly (see e.g. [10]).

Conjecture 21. (Kelly) *Every regular tournament on n vertices can be partitioned into $(n-1)/2$ edge-disjoint Hamilton cycles.*

A result of Frieze and Krivelevich [43] states that every dense ε-regular digraph contains a collection of edge-disjoint Hamilton cycles which covers almost all of its edges. This implies that the same holds for almost every tournament. Together with a lower bound by McKay [104] on the number of regular tournaments, it is easy to see that the above result in [43] also implies that almost every regular tournament contains a collection of edge-disjoint Hamilton cycles which covers almost all of its edges. Thomassen made the following conjecture which replaces the assumption of regularity by high connectivity.

Conjecture 22. (Thomassen [128]) *For every $k \geq 2$ there is an integer $f(k)$ so that every strongly $f(k)$-connected tournament has k edge-disjoint Hamilton cycles.*

The following conjecture of Jackson is also closely related to Theorem 18 – it would imply a much better degree condition for regular oriented graphs.

Conjecture 23. (Jackson [66]) *For $d > 2$, every d-regular oriented graph G on $n \leq 4d + 1$ vertices is Hamiltonian.*

The disjoint union of two regular tournaments on $n/2$ vertices shows that this would be best possible. An undirected analogue of Conjecture 23 was proved by Jackson [65]. It is easy to see that every tournament on n vertices with minimum semidegree at least $n/4$ has a Hamilton cycle. In fact, for tournaments T of large order n with minimum semidegree at least $n/4 + \varepsilon n$, Bollobás and Häggkvist [18] proved the stronger result that (for fixed k) T even contains the kth power of a Hamilton cycle. It would be interesting to find corresponding degree conditions which ensure this for arbitrary digraphs and for oriented graphs.

4.3. **Degree sequences forcing Hamilton cycles in directed graphs.** For undirected graphs, Dirac's theorem is generalized by Chvátal's theorem [22] that characterizes all those degree sequences which ensure the existence of a Hamilton cycle in a graph: suppose that the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. This condition on the degree sequence is best possible in the sense that for any degree sequence violating this condition there is a corresponding graph with no Hamilton cycle. Nash-Williams [109] raised the question of a digraph analogue of Chvátal's theorem quite soon after the latter was proved: for a digraph G it is natural to consider both its outdegree sequence $d_1^+, \ldots, d_n^+$ and its indegree sequence $d_1^-, \ldots, d_n^-$. Throughout, we take the convention that $d_1^+ \leq \cdots \leq d_n^+$ and $d_1^- \leq \cdots \leq d_n^-$ without mentioning this explicitly. Note that the terms d_i^+ and d_i^- do not necessarily correspond to the degree of the same vertex of G.

Conjecture 24 (Nash-Williams [109])**.** *Suppose that G is a strongly connected digraph on $n \geq 3$ vertices such that for all $i < n/2$*

(i) $d_i^+ \geq i + 1$ *or* $d_{n-i}^- \geq n - i$,

(ii) $d_i^- \geq i + 1$ *or* $d_{n-i}^+ \geq n - i$.

Then G contains a Hamilton cycle.

It is even an open problem whether the conditions imply the existence of a cycle through any pair of given vertices (see [12]). It is easy to see that one cannot omit the condition that G is strongly connected. The following example (which is a straightforward generalization of the corresponding undirected example) shows that the degree condition in Conjecture 24 would be best possible in the sense that for all $n \geq 3$ and all $k < n/2$ there is a non-Hamiltonian strongly connected digraph G on n vertices which satisfies the degree conditions except that $d_k^+, d_k^- \geq k + 1$ are replaced by $d_k^+, d_k^- \geq k$ in the kth pair of conditions. To see this, take an independent set I of size $k < n/2$ and a complete digraph K of order $n - k$. Pick a set X of k vertices of K and add all possible edges (in both directions) between I and X. The digraph G thus obtained is strongly connected, not Hamiltonian and

$$\underbrace{k, \ldots, k}_{k \text{ times}}, \underbrace{n - 1 - k, \ldots, n - 1 - k}_{n - 2k \text{ times}}, \underbrace{n - 1, \ldots, n - 1}_{k \text{ times}}$$

is both the out- and indegree sequence of G. In contrast to the undirected case there exist examples with a similar degree sequence to the above but whose structure is quite different (see [98]). In [98], the following approximate version of Conjecture 24 for large digraphs was proved.

Theorem 25 (Kühn, Osthus and Treglown [98])**.** *For every $\alpha > 0$ there exists an integer $n_0 = n_0(\alpha)$ such that the following holds. Suppose G is a digraph on $n \geq n_0$ vertices such that for all $i < n/2$*

- $d_i^+ \geq i + \alpha n$ *or* $d_{n-i-\alpha n}^- \geq n - i$,
- $d_i^- \geq i + \alpha n$ *or* $d_{n-i-\alpha n}^+ \geq n - i$.

Then G contains a Hamilton cycle.

Theorem 25 was derived from a result in [74] on the existence of a Hamilton cycle in an oriented graph satisfying a 'robust' expansion property.

The following weakening of Conjecture 24 was posed earlier by Nash-Williams [108]. It would yield a digraph analogue of Pósa's theorem which states that a graph G on $n \geq 3$ vertices has a Hamilton cycle if its degree sequence $d_1 \leq \cdots \leq d_n$ satisfies $d_i \geq i + 1$ for all $i < (n - 1)/2$ and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd [114]. Note that Pósa's theorem is much stronger than Dirac's theorem but is a special case of Chvátal's theorem.

Conjecture 26 (Nash-Williams [108])**.** *Let G be a digraph on $n \geq 3$ vertices such that $d_i^+, d_i^- \geq i + 1$ for all $i < (n - 1)/2$ and such that additionally $d_{\lceil n/2 \rceil}^+, d_{\lceil n/2 \rceil}^- \geq \lceil n/2 \rceil$ when n is odd. Then G contains a Hamilton cycle.*

The previous example shows the degree condition would be best possible in the same sense as described there. The assumption of strong connectivity is not

necessary in Conjecture 26, as it follows from the degree conditions. Theorem 25 immediately implies an approximate version of Conjecture 26.

It turns out that the conditions of Theorem 25 even guarantee the digraph G to be *pancyclic*, i.e. G contains a cycle of length t for all $t = 2, \ldots, n$. Thomassen [126] as well as Häggkvist and Thomassen [53] gave degree conditions which imply that every digraph with minimum semidegree $> n/2$ is pancyclic. The latter bound can also be deduced directly from Theorem 15. The complete bipartite digraph whose vertex class sizes are as equal as possible shows that the bound is best possible. For oriented graphs the minimum semidegree threshold which guarantees pancyclicity turns out to be $(3n - 4)/8$ (see [77]).
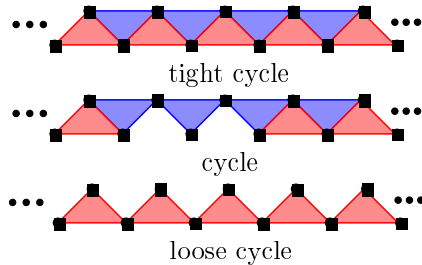
4.4. **Powers of Hamilton cycles in graphs.** The following result is a common extension (for large n) of Dirac's theorem and the Hajnal-Szemerédi theorem. It was originally conjectured (for all n) by Seymour.

Theorem 27. (Komlós, Sárközy and Szemerédi [85]) *For every $k \geq 1$ there is an integer n_0 so that every graph G on $n \geq n_0$ vertices and with $\delta(G) \geq \frac{k}{k+1}n$ contains the kth power of a Hamilton cycle.*

Complete $(k + 1)$-partite graphs whose vertex classes have almost (but not exactly) equal size show that the minimum degree bound is best possible. Previous to this a large number of partial results had been proved (see e.g. [100] for a history of the problem). Very recently, Levitt, Sarközy and Szemerédi [100] gave a proof of the case $k = 2$ which avoids the use of the Regularity lemma, resulting in a much better bound on n_0. Their proof is based on a technique introduced by Rödl, Ruciński and Szemerédi [117] for hypergraphs. The idea of this method (as applied in [100]) is first to find an 'absorbing' path P^2: roughly, P^2 is the second power of a path P which, given any vertex x, has the property that x can be inserted into P so that $P \cup x$ still induces the second power of a path. The proof of the existence of P^2 is heavily based on probabilistic arguments. Then one finds the second power Q^2 of a path which is almost spanning in $G - P^2$. One can achieve this by repeated applications of the Erdős-Stone theorem. One then connects up Q^2 and P^2 into the second power of a cycle and finally uses the absorbing property of P^2 to incorporate the vertices left over so far.

4.5. **Hamilton cycles in hypergraphs.** It is natural to ask whether one can generalize Dirac's theorem to uniform hypergraphs. There are several possible notions of a hypergraph cycle. One generalization of the definition of a cycle in a graph is the following one. An r-uniform hypergraph C is a *cycle of order n* if there a exists a cyclic ordering $v_1, \ldots, v_n$ of its n vertices such that every consecutive pair $v_i v_{i+1}$ lies in a hyperedge of C and such that every hyperedge of C consists of consecutive vertices. Thus the cyclic ordering of the vertices of C induces a cyclic ordering of its hyperedges. A cycle is *tight* if every r consecutive vertices form a hyperedge. A cycle of order n is *loose* if all pairs of consecutive edges (except possibly one pair) have exactly one vertex in common. (So every tight cycle contains a spanning loose cycle but a cycle might not necessarily contain a spanning loose cycle.) There is also the even more general notion of a *Berge-cycle*,

which consists of a sequence of vertices where each pair of consecutive vertices is contained in a common hyperedge.

tight cycle

cycle

loose cycle

A *Hamilton cycle* of a uniform hypergraph G is a subhypergraph of G which is a cycle containing all its vertices. Theorem 28 gives an analogue of Dirac's theorem for tight hypergraph cycles, while Theorem 29 gives an analogue for 3-uniform (loose) cycles.

Theorem 28. (Rödl, Ruciński and Szemerédi [117]) *For all $r \in \mathbb{N}$ and $\alpha > 0$ there is an integer $n_0 = n_0(r, \alpha)$ such that every r-uniform hypergraph G with $n \geq n_0$ vertices and minimum degree at least $n/2 + \alpha n$ contains a tight Hamilton cycle.*

Theorem 29. (Han and Schacht [57]; Keevash, Kühn, Mycroft and Os-thus [73]) *For all $r \in \mathbb{N}$ and $\alpha > 0$ there is an integer $n_0 = n_0(\alpha)$ such that every r-uniform hypergraph G with $n \geq n_0$ vertices and minimum degree at least $n/(2r - 2) + \alpha n$ contains a loose Hamilton cycle.*

Both results are best possible up to the error term αn. In fact, if the minimum degree is less than $\lceil n/(2r-2) \rceil$, then we cannot even guarantee *any* Hamilton cycle in an r-uniform hypergraph. The case $r = 3$ of Theorems 28 and 29 was proved earlier in [116] and [95] respectively. The result in [57] also covers the notion of an r-uniform ℓ-cycle for $\ell < r/2$ (here we ask for consecutive edges to intersect in precisely ℓ vertices). Hamiltonian Berge-cycles were considered by Bermond et al. [11].

5. Bounded degree spanning subgraphs

Bollobás and Eldridge [17] as well as Catlin [21] made the following very general conjecture on embedding graphs. If true, this conjecture would be a far-reaching generalization of the Hajnal-Szemerédi theorem (Theorem 1).

Conjecture 30 (Bollobás and Eldridge [17], Catlin [21])**.** *If G is a graph on n vertices with $\delta(G) \geq \frac{\Delta n - 1}{\Delta + 1}$, then G contains any graph H on n vertices with maximum degree at most Δ.*

The conjecture has been proved for graphs H of maximum degree at most 2 [4, 7] and for large graphs of maximum degree at most 3 [34]. Recently, Csaba [31] proved it for bipartite graphs H of arbitrary maximum degree Δ, provided the order of H is sufficiently large compared to Δ. In many applications of the Blow-up lemma, the graph H is embedded into G by splitting H up into several suitable

parts and applying the Blow-up lemma to each of these parts (see e.g. the example in Section 7). It is not clear how to achieve this for H as in Conjecture 30, as H may be an 'expander'. So the proofs in [31, 34] rely on a variant of the Blow-up lemma which is suitable for embedding such 'expander graphs'. Also, Kaul, Kostochka and Yu [70] showed (without using the Regularity lemma) that the conjecture holds if we increase the minimum degree condition to $\frac{\Delta n + 2n/5 - 1}{\Delta + 1}$.

Theorem 2 suggests that one might replace Δ in Conjecture 30 with $\chi(H) - 1$, resulting in a smaller minimum degree bound for some graphs H. This is far from being true in general (e.g. let H be a 3-regular bipartite expander and let G be the union of two cliques which have equal size and are almost disjoint). However, Bollobás and Komlós conjectured that this does turn out to be true if we restrict our attention to a certain class of 'non-expanding' graphs. This conjecture was recently confirmed in [15]. The bipartite case was proved earlier by Abbasi [1].

Theorem 31. (Böttcher, Schacht and Taraz [15]) *For every $\gamma > 0$ and all integers $r \geq 2$ and Δ, there exist $\beta > 0$ and n_0 with the following property. Every graph G of order $n \geq n_0$ and minimum degree at least $(1 - 1/r + \gamma)n$ contains every r-chromatic graph H of order n, maximum degree at most Δ and bandwidth at most βn as a subgraph.*

Here the *bandwidth* of a graph H is the smallest integer b for which there exists an enumeration $v_1, \ldots, v_{|H|}$ of the vertices of H such that every edge $v_i v_j$ of H satisfies $|i - j| \leq b$. Note that kth powers of cycles have bandwidth $2k$, so Theorem 31 implies an approximate version of Theorem 27. (Actually, this is only the case if n is a multiple of $k + 1$, as otherwise the kth power of a Hamilton cycle fails to be $(k + 1)$-colourable. But [15] contains a more general result which allows for a small number of vertices of colour $k + 2$.) A further class of graphs having small bandwidth and bounded degree are planar graphs with bounded degree [13]. (See [92, 97] for further results on embedding planar graphs in graphs of large minimum degree.) Note that the discussion in Section 2 implies that the minimum degree bound in Theorem 31 is approximately best possible for certain graphs H but not for all graphs. Abbasi [2] showed that there are graphs H for which the linear error term γn in Theorem 31 is necessary. One might think that one could reduce the error term to a constant for graphs of bounded bandwidth. However, this turns out to be incorrect. (We grateful to Peter Allen for pointing this out to us.)

Alternatively, one can try to replace the bandwidth assumption in Theorem 31 with a less restrictive parameter. For instance, Csaba [32] gave a minimum degree condition on G which guarantees a copy of a 'well-separated' graph H in G. Here a graph with n vertices is *α-separable* if there is a set S of vertices of size at most αn so that all components of $H - S$ have size at most αn. It is easy to see that every graph with n vertices and bandwidth at most βn is $\sqrt{\beta}$-separable. (Moreover large trees are α-separable for $\alpha \to 0$ but need not have small bandwidth, so considering separability is less restrictive than bandwidth.)

Here is another common generalization of Dirac's theorem and the triangle case of Theorem 1 (i.e. the Corrádi-Hajnal theorem). It proves a conjecture by El-Zahar (actually, El-Zahar made the conjecture for all values of n, this is still open).

Theorem 32. (Abbasi [1]) *There exists an integer n_0 so that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices and $n_1, \dots, n_k \geq 3$ are so that*

$$\sum_{i=1}^{k} n_i = n \qquad \text{and} \qquad \delta(G) \geq \sum_{i=1}^{k} \lceil n_i/2 \rceil.$$

Then G has k vertex-disjoint cycles whose lengths are $n_1, \dots, n_k$.

Note that $\sum_{i=1}^{k} \lceil n_i/2 \rceil = \sum_{i=1}^{k} (1 - 1/\chi_{cr}(C_i)) n_i$, where C_i denotes a cycle of length n_i. This suggests the following more general question (which was raised by Komlós [81]): Given $t \in \mathbb{N}$, does there exists an $n_0 = n_0(t)$ such that whenever $H_1, \dots, H_k$ are graphs which each have at most t vertices and which together have $n \geq n_0$ vertices and whenever G is a graph on n vertices with minimum degree at least $\sum_i (1 - 1/\chi_{cr}(H_i))|H_i|$, then there is a set of vertex-disjoint copies of $H_1, \dots, H_k$ in G? In this form, the question has a negative answer by (the lower bound in) Theorem 6, but it would be interesting to find a common generalization of Theorems 6 and 32.

It is also natural to ask corresponding questions for oriented and directed graphs. As in the case of Hamilton cycles, the questions appear much harder than in the undirected case and again much less is known. Keevash and Sudakov [75] recently obtained the following result which can be viewed as an oriented version of the $\Delta = 2$ case of Conjecture 30.

Theorem 33. (Keevash and Sudakov [75]) *There exist constants c, C and an integer n_0 so that whenever G is an oriented graph on $n \geq n_0$ vertices with minimum semidegree at least $(1/2 - c)n$ and whenever $n_1, \dots, n_t$ are so that $\sum_{i=1}^{t} n_i \leq n - C$, then G contains disjoint cycles of length $n_1, \dots, n_t$.*

In the case of triangles (i.e. when all the $n_i = 3$), they show that one can choose $C = 3$ (one cannot take $C = 0$). [75] also contains a discussion of related open questions for tournaments and directed graphs. Similar questions were also raised earlier by Song [123]. For instance, given t, what is the smallest integer $f(t)$ so that all but a finite number of $f(t)$-connected tournaments T satisfy the following: Let n be the number of vertices of T and let $\sum_{i=1}^{t} n_i = n$. Then T contains disjoint cycles of length $n_1, \dots, n_t$.

6. RAMSEY THEORY

The Regularity lemma can often be used to show that the Ramsey numbers of sparse graphs H are small. (The *Ramsey number* $R(H)$ of H is the smallest $N \in \mathbb{N}$ such that for every 2-colouring of the complete graph on N vertices one can find a monochromatic copy of H.) In fact, the first result which demonstrated the use of the Regularity lemma in extremal graph theory was the following result of Chvátal, Rödl, Szemerédi and Trotter [23], which states that graphs of bounded degree have linear Ramsey numbers:

Theorem 34. (Chvátal, Rödl, Szemerédi and Trotter [23]**)** *For all $\Delta \in \mathbb{N}$ there is a constant $C = C(\Delta)$ so that every graph H with maximum degree $\Delta(H) \leq \Delta$ and n vertices satisfies $R(H) \leq Cn$.*

The constant C arising from the original proof (based on the Regularity lemma) is quite large. The bound was improved in a series of papers. Recently, Fox and Sudakov [40] showed that $R(H) \leq 2^{4\chi(H)\Delta}n$ (the bipartite case was also proved independently by Conlon [24]). For bipartite graphs, a construction from [49] shows that this bound is best possible apart from the value of the absolute constant $4 \cdot 2$ appearing in the exponent.

Theorem 34 was recently generalized to hypergraphs [27, 28, 106, 64] using hypergraph versions of the Regularity lemma. Subsequently, Conlon, Fox and Sudakov [25] gave a shorter proof which gives a better constant and does not rely on the Regularity lemma.

One of the most famous conjectures in Ramsey theory is the Burr-Erdős conjecture on d-degenerate graphs, which generalizes Theorem 34. Here a graph G is *d-degenerate* if every subgraph has a vertex of degree at most d. In other words, G has no 'dense' subgraphs.

Conjecture 35. (Burr and Erdős [19]**)** *For every d there is a constant $C = C(d)$ so that every d-degenerate graph H on n vertices satisfies $R(H) \leq Cn$.*

It has been proved in many special cases (see e.g. the introduction of [41] for a recent overview). Also, Kostochka and Sudakov [91] proved that it is 'approximately' true:

Theorem 36. (Kostochka and Sudakov [91]**)** *For every d there is a constant $C = C(d)$ so that every d-degenerate graph H on n vertices satisfies $R(H) \leq 2^{C(\log n)^{2d/(2d+1)}}n$.*

The exponent '$2d/(2d + 1)$' of the logarithm was improved to '$1/2$' in [41]. All the results in [24, 40, 41, 91] rely on variants of the same probabilistic argument, which was first applied to special cases of Conjecture 35 in [90]. To give an idea of this beautiful argument, we use a simple version to give a proof of the following density result (which is implicit in several of the above papers): it implies that bipartite graphs H whose maximum degree is logarithmic in their order have polynomial Ramsey numbers. (The logarithms in the statement and the proof are binary.)

Theorem 37. *Suppose that $H = (A', B', E')$ is a bipartite graph on $n \geq 2$ vertices and $\Delta(H) \leq \log n$. Suppose that $m \geq n^8$. Then every bipartite graph $G = (A, B, E)$ with $|A| = |B| = m$ and at least $m^2/8$ edges contains a copy of H. In particular, $R(H) \leq 2n^8$.*

An immediate corollary is that the Ramsey number of a d-dimensional cube Q_d is polynomial in its number $n = 2^d$ of vertices (this fact was first observed in [120] based on an argument similar to that in [90]). The best current bound of $R(Q_d) \leq d2^{2d+5}$ is given in [40]. Burr and Erdős [19] conjectured that the bound should actually be linear in $n = 2^d$.

Proof. Write $\Delta := \log n$. Let $b_1, \ldots, b_s$ be a sequence of $s := 2\Delta$ not necessarily distinct vertices of B, chosen uniformly and independently at random and write $S := \{b_1, \ldots, b_s\}$. Let $N(S)$ denote the set of common neighbours of vertices in S. Clearly, $S \subseteq N(a)$ for every $a \in N(S)$. So Jensen's inequality implies that

$$\mathbb{E}(|N(S)|) = \sum_{a \in A} \mathbb{P}(a \in N(S)) = \sum_{a \in A} \left(\frac{|N(a)|}{m} \right)^s = \frac{\sum_{a \in A}(d(a))^s}{m^s}$$

$$\geq \frac{m \left(\frac{\sum_{a \in A} d(a)}{m} \right)^s}{m^s} \geq \frac{m \left((m^2/8)/m \right)^s}{m^s} = \frac{m}{8^s} \geq \frac{n^8}{n^6} = n^2.$$

We say that a subset $W \subseteq A$ is *bad* if it has size Δ and its common neighbourhood $N(W)$ satisfies $|N(W)| < n$. Now let Z denote the number of bad subsets W of $N(S)$. Note that the probability that a given set $W \subseteq A$ lies in $N(S)$ equals $(|N(W)|/m)^s$ (since the probability that it lies in the neighbourhood of a fixed vertex $b \in B$ is $|N(W)|/m$). So

$$\mathbb{E}Z = \sum_{W \, \text{bad}} \mathbb{P}(W \subseteq N(S)) \leq \binom{m}{\Delta} \left(\frac{n}{m} \right)^s \leq m^\Delta \left(\frac{n}{m} \right)^s = \left(\frac{n^2}{m} \right)^\Delta \leq (1/2)^\Delta < 1.$$

So $\mathbb{E}(|N(S)| - Z) \geq n^2 - 1 \geq n$ and hence there is a choice of S with $|N(S)| - Z \geq n$. By definition, we can delete a vertex from every bad W contained in $N(S)$ to obtain a set $T \subseteq N(S)$ with $|T| \geq n$ so that every subset $W \subseteq T$ with $|W| = \Delta$ satisfies $|N(W)| \geq n$. Clearly we can now embed H: first embed A' arbitrarily into T and then embed the vertices of B' one by one into B, using the property that T has no bad subset.

The bound on $R(H)$ can be derived as follows: consider any 2-colouring of the complete graph on $2n^8$ vertices. Partition its vertices arbitrarily into two sets A and B of size n^8 and then apply the main statement to the subgraph of G induced by the colour class having the most edges between A and B. $\square$

Note that the proof immediately shows that the bound on the maximum degree of H can be relaxed: all we need is the property that every subgraph of H has a vertex $b \in B'$ of low degree. In the proof of (the bipartite case) of Theorem 36, this was exploited as follows: roughly speaking one carries out the above argument twice (of course with different parameters than the above). The first time we consider a random subset $S \subseteq B$ and the second time we consider a smaller random subset $S' \subseteq T$.

For some types of sparse graphs H, one can give even more precise estimates for $R(H)$ than the ones which follow from the above results. For instance, Theorem 12 has an immediate application to the Ramsey number of trees.

Corollary 38. *There is an integer n_0 so that if T_n is a tree on $n \geq n_0$ vertices then $R(T_n) \leq 2n - 2$.*

Indeed, to derive Corollary 38 from Theorem 12, consider a 2-colouring of a complete graph K_{2n-2} on $2n - 2$ vertices, yielding a red graph G_r and a blue graph G_b. Order the vertices x_i according to their degree (in ascending order) in

G_r. If x_{n-1} has degree at least $n-1$ in G_r, then we can apply Theorem 12 to find a red copy of T in G_r. If not, we can apply it to find a blue copy of T in G_b. For even n, the bound is best possible (let T be a star and let G_b and G_r be regular of the same degree) and proves a conjecture of Burr and Erdős [20]. For odd n, they conjectured that the answer is $2n-3$. Similarly, the Komlós-Sós conjecture (Conjecture 13) would imply that $R(T_n, T_m) \leq n+m-2$, where T_n and T_m are trees on n and m vertices respectively. Of course, Corollary 38 is not best possible for every tree. For instance, in the case when the tree is a path, Gerencsér and Gyarfas [44] showed that $R(P_n, P_n) = \lfloor (3n-2)/2 \rfloor$. Further recent results on Ramsey numbers of paths and cycles (many of which rely on the Regularity lemma) can be found e.g. in [51, 38]. Hypergraph versions (i.e. Ramsey numbers of tight cycles, loose cycles and Berge-cycles) were considered e.g. in [58, 59, 50].

7. A sample application of the Regularity and Blow-up lemma

In order to illustrate the details of the Regularity method for those not familiar with it, we now prove Theorem 2 for the case when $H := C_4$ and when we replace the constant C in the minimum degree condition with a linear error term.

Theorem 39. *For every $0 < \eta < 1/2$ there exists an integer n_0 such that every graph G whose order $n \geq n_0$ is divisible by 4 and whose minimum degree is at least $n/2 + \eta n$ contains a perfect C_4-packing.*

(Note that Theorem 39 also follows from Theorems 31 and 32.) We start with the formal definition of ε-regularity. The *density* of a bipartite graph $G = (A, B)$ with vertex classes A and B is

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

We also write $d(A, B)$ if this is unambiguous. Given $\varepsilon > 0$, we say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A, B) - d(X, Y)| < \varepsilon$. Given $d \in [0, 1)$, we say that G is *(ε, d)-superregular* if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) > d$ and, furthermore, if $d_G(a) > d|B|$ for all $a \in A$ and $d_G(b) > d|A|$ for all $b \in B$. Moreover, we will denote the neighbourhood of a vertex x in a graph G by $N_G(x)$. Given disjoint sets A and B of vertices of G, we write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all the edges of G between A and B.

Szemerédi's Regularity lemma [125] states that one can partition the vertices of every large graph into a bounded number 'clusters' so that most of the pairs of clusters induce ε-regular bipartite graphs. Proofs are also included in [16] and [35]. Algorithmic proofs of the Regularity lemma were given in [6, 42]. There are also several versions for hypergraphs (in fact, all the results in Section 4.5 are based on some hypergraph version of the Regularity lemma). The first so-called 'strong' versions for r-uniform hypergraphs were proved in [48] and [107, 119].

Lemma 40 (Szemerédi [125])**.** *For all $\varepsilon > 0$ and all integers k_0 there is an $N = N(\varepsilon, k_0)$ such that for every graph G on $n \geq N$ vertices there exists a partition of $V(G)$ into $V_0, V_1, \ldots, V_k$ such that the following holds:*

- *$k_0 \leq k \leq N$ and $|V_0| \leq \varepsilon n$,*
- *$|V_1| = \cdots = |V_k| =: m$,*
- *for all but εk^2 pairs $1 \leq i < j \leq k$ the graph $(V_i, V_j)_G$ is ε-regular.*

Unfortunately, the constant N appearing in the lemma is very large, Gowers [47] showed that it has at least a tower-type dependency on ε. We will use the following degree form of Szemerédi's Regularity lemma which can be easily derived from Lemma 40.

Lemma 41 (Degree form of the Regularity lemma)**.** *For all $\varepsilon > 0$ and all integers k_0 there is an $N = N(\varepsilon, k_0)$ such that for every number $d \in [0, 1)$ and for every graph G on $n \geq N$ vertices there exist a partition of $V(G)$ into $V_0, V_1, \ldots, V_k$ and a spanning subgraph G' of G such that the following holds:*

- *$k_0 \leq k \leq N$ and $|V_0| \leq \varepsilon n$,*
- *$|V_1| = \cdots = |V_k| =: m$,*
- *$d_{G'}(x) > d_G(x) - (d + \varepsilon)n$ for all vertices $x \in G$,*
- *for all $i \geq 1$ the graph $G'[V_i]$ is empty,*
- *for all $1 \leq i < j \leq k$ the graph $(V_i, V_j)_{G'}$ is ε-regular and has density either 0 or $> d$.*

The sets V_i ($i \geq 1$) are called *clusters*, V_0 is called the *exceptional set* and G' is called the *pure graph*.

Sketch of proof of Lemma 41 To obtain a partition as in Lemma 41, apply Lemma 40 with parameters d, ε', k_0' satisfying $1/k_0', \varepsilon' \ll \varepsilon, d, 1/k_0$ to obtain clusters $V_1', \ldots, V_{k'}'$ and an exceptional set V_0'. (Here $a \ll b < 1$ means that there is an increasing function f such that all the calculations in the argument work as long as $a \leq f(b)$.) Let $m' := |V_1'| = \cdots = |V_{k'}'|$. Now delete all edges between pairs of clusters which are not ε'-regular and move any vertices into V_0' which were incident to at least $\varepsilon n/10$ (say) of these deleted edges. Secondly, delete all (remaining) edges between pairs of clusters whose density is at most $d + \varepsilon'$. Consider such a pair (V_i', V_j') of clusters. For every vertex $x \in V_i'$ which has more than $(d + 2\varepsilon')m'$ neighbours in V_j' mark all but $(d + 2\varepsilon')m'$ edges between x and V_j'. Do the same for the vertices in V_j' and more generally for all pairs of clusters of density at most $d + \varepsilon'$. It is easy to check that in total this yields at most $\varepsilon'n^2$ marked edges. Move all vertices into V_0' which are incident to at least $\varepsilon n/10$ of the marked edges. Thirdly, delete any edges within the clusters. Finally, we need to make sure that the clusters have equal size again (as we may have lost this property during the deletion process). This can be done by splitting up the clusters into smaller subclusters (which contain almost all the vertices and have equal size) and moving a small number of further vertices into V_0'. A straightforward calculation shows that the new exceptional set V_0 has size at most εn as required. □

The *reduced graph* R is the graph whose vertices are $1, \ldots, k$ and in which i is joined to j whenever the bipartite subgraph $(V_i, V_j)_{G'}$ of G' induced by V_i and V_j is ε-regular and has density $> d$. Thus ij is an edge of R if and only if G' has an edge between V_i and V_j. Roughly speaking, the following result states that R almost 'inherits' the minimum degree of G.

Proposition 42. *If $0 < 2\varepsilon \le d \le c/2$ and $\delta(G) \ge cn$ then $\delta(R) \ge (c - 2d)|R|$.*

Proof. Consider any vertex i of R and pick $x \in V_i$. Then every neighbour of x in G' lies in $V_0 \cup \bigcup_{j \in N_R(i)} V_j$. Thus $(c - (d + \varepsilon))n \le d_{G'}(x) \le d_R(i)m + \varepsilon n$ and so $d_R(i) \ge (c - 2d)n/m \ge (c - 2d)|R|$ as required. $\qquad\square$

The proof of Proposition 42 is a point where it is important that R was defined using the graph G' obtained from Lemma 41 and not using the partition given by Lemma 40.

In our proof of Theorem 39 the reduced graph R will contain a Hamilton path P. Recall that every edge ij of $P \subseteq R$ corresponds to the ε-regular bipartite subgraph $(V_i, V_j)_{G'}$ of G' having density $> d$. The next result shows that by removing a small number of vertices from each cluster (which will be added to the exceptional set V_0) we can guarantee that the edges of P even correspond to superregular pairs.

Proposition 43. *Suppose that $4\varepsilon < d \le 1$ and that P is a Hamilton path in R. Then every cluster V_i contains a subcluster $V_i' \subseteq V_i$ of size $m - 2\varepsilon m$ such that $(V_i', V_j')_{G'}$ is $(2\varepsilon, d - 3\varepsilon)$-superregular for every edge $ij \in P$.*

Proof. We may assume that $P = 1 \ldots k$. Given any $i < k$, the definition of regularity implies that there are at most εm vertices $x \in V_i$ such that $|N_{G'}(x) \cap V_{i+1}| \le (d - \varepsilon)m$. Similarly, for each $i > 1$ there are at most εm vertices $x \in V_i$ such that $|N_{G'}(x) \cap V_{i-1}| \le (d - \varepsilon)m$. Let V_i' be a subset of size $m - 2\varepsilon m$ of V_i which contains none of the above vertices (for all $i = 1, \ldots, k$). Then $V_1', \ldots, V_k'$ are as required. $\qquad\square$

Of course, in Proposition 43 it is not important that P is a Hamilton path. One can prove an analogue whenever P is a subgraph of R of bounded maximum degree. We will also use the following special case of the Blow-up lemma of Komlós, Sárközy and Szemerédi [83]. It implies that dense superregular pairs behave like complete bipartite graphs with respect to containing bounded degree graphs as subgraphs, i.e. if the superregular pair has vertex classes V_i and V_j then any bounded degree bipartite graph on these vertex classes is a subgraph of this superregular pair. An algorithmic version of the Blow-up lemma was proved by the same authors in [84]. A hypergraph version was recently proved by Keevash [72].

Lemma 44 (Blow-up lemma, bipartite case)**.** *Given $d > 0$ and $\Delta \in \mathbb{N}$, there is a positive constant $\varepsilon_0 = \varepsilon_0(d, \Delta)$ such that the following holds for every $\varepsilon < \varepsilon_0$. Given $m \in \mathbb{N}$, let G^* be an (ε, d)-superregular bipartite graph with vertex classes of size m. Then G^* contains a copy of every subgraph H of $K_{m,m}$ with $\Delta(H) \le \Delta$.*

Proof of Theorem 39 We choose further positive constants ε and d as well as $n_0 \in \mathbb{N}$ such that

$$1/n_0 \ll \varepsilon \ll d \ll \eta < 1/2.$$

(In order to simplify the exposition we will not determine these constants explicitly.) We start by applying the degree form of the Regularity lemma (Lemma 41) with parameters ε, d and $k_0 := 1/\varepsilon$ to G to obtain clusters $V_1, \ldots, V_k$, an exceptional set V_0, a pure graph G' and a reduced graph R. Thus $k := |R|$ and

$$(1) \qquad\qquad \delta(R) \geq (1/2 + \eta - 2d)k \geq (1 + \eta)k/2$$

by Proposition 42. So R contains a Hamilton path P (this follows e.g. from Dirac's theorem). By relabelling if necessary we may assume that $P = 1 \ldots k$. Apply Proposition 43 to obtain subclusters $V_i' \subseteq V_i$ of size $m - 2\varepsilon m =: m'$ such that for every edge $i(i+1) \in P$ the bipartite subgraph $(V_i', V_{i+1}')_{G'}$ of G' induced by V_i' and V_{i+1}' is $(2\varepsilon, d/2)$-superregular. Note that the definition of ε-regularity implies that $(V_i', V_j')_{G'}$ is still 2ε-regular of density at least $d - \varepsilon \geq d/2$ whenever ij is an edge of R. We add all those vertices of G that are not contained in some V_i' to the exceptional set V_0. Moreover, if k is odd then we also add all the vertices in V_k' to V_0. We still denote the reduced graph by R, its number of vertices by k and the exceptional set by V_0. Thus

$$|V_0| \leq \varepsilon n + 2\varepsilon n + m \leq 4\varepsilon n.$$

Let M denote the perfect matching in P. So M consists of the edges $12, 34, \ldots, (k-1)k$. The Blow-up lemma would imply that for every odd i the bipartite graph $(V_i', V_{i+1}')_{G'}$ contains a perfect C_4-packing, provided that 2 divides m'. So we have already proved that G contains a C_4-packing covering *almost* all of its vertices (this can also be easily proved without the Regularity lemma). In order to obtain a perfect C_4-packing, we have to incorporate the exceptional vertices.

To make it simpler to deal with divisibility issues later on, for every odd i we will now choose a set X_i of 7 vertices of G which we can put in any of V_i' and V_{i+1}' without destroying the superregularity of $(V_i', V_{i+1}')_{G'}$. More precisely, (1) implies that the vertices i and $i+1$ of R have a common neighbour, j say. Recall that both $(V_i', V_j')_{G'}$ and $(V_{i+1}', V_j')_{G'}$ are 2ε-regular and have density at least $d/2$. So almost all vertices in V_j' have at least $(d/2 - 2\varepsilon)m'$ neighbours in both V_i' and V_{i+1}'. Let $X_i \subseteq V_j'$ be a set of 7 such vertices. Clearly, we may choose the sets X_i disjoint for distinct odd i. Remove all the vertices in $X_1 \cup X_3 \cup \cdots \cup X_{k-1} =: X$ from the clusters they belong to. By removing at most $|X|k \leq 7k^2$ further vertices and adding them to the exceptional set we may assume that the subclusters $V_i'' \subseteq V_i'$ thus obtained satisfy $|V_1''| = \cdots = |V_k''| =: m''$. (The vertices in X are not added to V_0.) Note that we now have

$$|V_0| \leq 4\varepsilon n + 7k^2 \leq 5\varepsilon n.$$

Consider any vertex $x \in V_0$. Call an odd i *good for x* if x has at least $\eta^2 m''$ neighbours in both V_i'' and V_{i+1}'' (in the graph G'). Then the number g_x of good indices satisfies

$$(1/2+\eta/2)n \leq d_{G'}(x) - |V_0| - |X| \leq 2g_x m'' + (k/2 - g_x)(1+\eta^2)m'' \leq 2g_x m'' + (1+\eta^2)n/2,$$

which shows that $g_x \geq \eta k/8 = \eta|M|/4$. Since $|V_0|/(\sqrt{\varepsilon}m'') \leq \eta|M|/4$, this implies that we can assign each $x \in V_0$ to an odd index i which is good for x in such a way that to each odd i we assign at most $\sqrt{\varepsilon}m''$ exceptional vertices. Now consider any matching edge $i(i+1) \in M$. Add each exceptional vertex assigned to i to V_i' or V_{i+1}' so that the sizes of the sets $V_i^* \supseteq V_i''$ and $V_{i+1}^* \supseteq V_{i+1}''$ obtained in this way differ by at most 1. It is easy to check that the bipartite subgraph $(V_i^*, V_{i+1}^*)_{G'}$ of G' is still $(2\sqrt{\varepsilon}, d/8)$-superregular.

Since the vertices in X_i can be added to any of V_i^* and V_{i+1}^* without destroying the superregularity of $(V_i^*, V_{i+1}^*)_{G'}$, we could now apply the Blow-up lemma to find a C_4-packing of $G'[V_i^* \cup V_{i+1}^* \cup X_i]$ which covers all but at most 3 vertices (and so altogether these packings would form a C_4-packing of G covering all but at most $3k$ vertices of G). To ensure the existence of a perfect C_4-packing, we need to make $|V_i^* \cup V_{i+1}^* \cup X_i|$ divisible by 4 for every odd i. We will do this for every $i = 1, 3, \ldots, k-1$ in turn by shifting the remainders mod 4 along the path P. More precisely, suppose that $|V_1^* \cup V_2^* \cup X_1| \equiv a \mod 4$ where $0 \leq a < 4$. Choose a disjoint copies of C_4, each having 1 vertex in V_2^*, 2 vertices in V_3^* and 1 vertex in V_4^*. Remove the vertices in these copies from the clusters they belong to and still denote the subclusters thus obtained by V_i^*. (Each such copy of C_4 can be found greedily using that both $(V_2^*, V_3^*)_{G'}$ and $(V_3^*, V_4^*)_{G'}$ are still $2\sqrt{\varepsilon}$-regular and have density at least $d/8$. Indeed, to find the first copy, pick any vertex $x \in V_2^*$ having at least $(d/8 - 2\sqrt{\varepsilon})|V_3^*|$ neighbours in V_3^*. The regularity of $(V_2^*, V_3^*)_{G'}$ implies that almost all vertices in V_2^* can play the role of x. The regularity of $(V_3^*, V_4^*)_{G'}$ now implies that its bipartite subgraph induced by the neighbourhood of x in V_3^* and by V_4^* has density at least $d/8 - 2\sqrt{\varepsilon}$. So there are many vertices $y \in V_4^*$ which have at least 2 neighbours in $N_{G'}(x) \cap V_3^*$. Then x and y together with 2 such neighbours form a copy of C_4.) Now $|V_1^* \cup V_2^* \cup X_1|$ is divisible by 4. Similarly, by removing at most 3 further copies of C_4, each having 1 vertex in V_4^*, 2 vertices in V_5^* and 1 vertex in V_6^* we can achieve that $|V_3^* \cup V_4^* \cup X_3|$ is divisible by 4. Since $n = |G|$ is divisible by 4 we can continue in this way to achieve that $|V_i^* \cup V_{i+1}^* \cup X_i|$ divisible by 4 for every odd i.

Recall that before we took out all these copies of C_4, for every odd i the sizes of V_i^* and V_{i+1}^* differed by at most 1. Thus now these sizes differ (crudely) by at most 7. But every vertex $x \in X_i$ can be added to both V_i^* and V_{i+1}^* without destroying the superregularity. Add the vertices from X_i to V_i^* and V_{i+1}^* in such a way that the sets $V_i^\diamond \supseteq V_i^*$ and $V_{i+1}^\diamond \supseteq V_{i+1}^*$ thus obtained have equal size. (This size must be even since $|V_i^* \cup V_{i+1}^* \cup X_i|$ is divisible by 4.) It is easy to check that $(V_i^\diamond, V_{i+1}^\diamond)_{G'}$ is still $(3\sqrt{\varepsilon}, d/9)$-superregular. Thus we can apply the Blow-up lemma (Lemma 44) to obtain a perfect C_4-packing in $(V_i^\diamond, V_{i+1}^\diamond)_{G'}$. The union of all these packings (over all odd i) together with the C_4's we have chosen before form a perfect C_4-packing of G. □

8. Acknowledgment

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