

# RAINBOW STRUCTURES IN LOCALLY BOUNDED COLOURINGS OF GRAPHS

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ABSTRACT. We study approximate decompositions of edge-coloured quasirandom graphs into rainbow spanning structures: an edge-colouring of a graph is *locally  $\ell$ -bounded* if every vertex is incident to at most  $\ell$  edges of each colour, and is (*globally*)  *$g$ -bounded* if every colour appears at most  $g$  times. Our results imply the existence of:

- (i) approximate decompositions of properly edge-coloured  $K_n$  into rainbow almost-spanning cycles.
- (ii) approximate decompositions of edge-coloured  $K_n$  into rainbow Hamilton cycles, provided that the colouring is  $(1 - o(1))\frac{n}{2}$ -bounded and locally  $o(\frac{n}{\log^4 n})$ -bounded.
- (iii) an approximate decomposition into full transversals of any  $n \times n$  array, provided each symbol appears  $(1 - o(1))n$  times in total and only  $o(\frac{n}{\log^2 n})$  times in each row or column.

Apart from the logarithmic factors, these bounds are essentially best possible. We also prove analogues for rainbow  $F$ -factors, where  $F$  is any fixed graph. Both (i) and (ii) imply approximate versions of the Brualdi-Hollingsworth conjecture on decompositions into rainbow spanning trees.

## 1. INTRODUCTION AND OUR RESULTS

**1.1. Transversals and rainbow colourings.** For  $n \in \mathbb{N}$ , let us write  $[n] := \{1, \dots, n\}$ . A *Latin square* is an  $n \times n$  array filled with symbols from  $[n]$ , so that each symbol appears exactly once in each row and each column. A *partial transversal* of a Latin square is a subset of its entries, each in a distinct row and column, and having distinct symbols. A partial transversal of size  $n$  is a *full transversal*.

The study of Latin squares goes back to Euler, who was, in particular, interested in finding Latin squares decomposable into full transversals. It is however not obvious whether *any* Latin square should have a large transversal. Ryser [42], Stein [44] and Brualdi [11] conjectured that any given Latin square has a partial transversal of size  $n - 1$  (it need not have a full one if  $n$  is even). The current record towards this problem is due to Hatami and Shor [25], who, correcting a mistake in an earlier work of Shor [43], proved that there always exists a partial transversal of size  $n - O(\log^2 n)$ .

Clearly, each symbol appears in a Latin square exactly  $n$  times. A more general conjecture was made by Stein [44], who suggested that any  $n \times n$  array filled with symbols from  $[n]$ , each appearing exactly  $n$  times, has a partial transversal of size  $n - 1$ . The best known positive result in this direction is due to Aharoni, Berger, Kotlar and Ziv [1], who, using a topological approach, showed that any such array has a partial transversal of size at least  $2n/3$ . On the other hand, Pokrovskiy and Sudakov [38] recently disproved Stein's conjecture: in fact, they showed that there are such arrays with largest transversal of size  $n - \Omega(\log n)$ .

Each  $n \times n$  array filled with symbols may be viewed as a colouring of a complete bipartite graph  $K_{n,n}$ : an edge  $ij$  corresponds to the entry of the array in the  $i$ -th row and  $j$ -th column, and each symbol stands for a colour. In this way, a Latin square corresponds to a properly edge-coloured  $K_{n,n}$ , and a partial transversal is a *rainbow matching* in  $K_{n,n}$ , that is, a collection of disjoint edges having pairwise distinct colours. Thus, the conjecture of Stein deals with (globally)  $n$ -bounded colourings of  $K_{n,n}$ , where we say that an edge-colouring of a graph is (*globally*)  *$g$ -bounded* if each colour appears at most  $g$  times in the colouring. An edge-colouring is *locally  $\ell$ -bounded* if each colour appears at most  $\ell$  times at any given vertex. Note that locally 1-bounded colourings are simply proper colourings.

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Studying rainbow substructures in graphs has a long history. One source of inspiration is Ramsey theory, in particular, the canonical version of Ramsey's theorem due to Erdős and Rado [18]. A general problem is to find conditions on the colourings and graphs which would allow to find certain rainbow substructures. This topic has received considerable attention recently, with probabilistic tools and techniques from extremal graph theory allowing for major progress on longstanding problems. In this context, natural (rainbow) structures to seek include matchings, Hamilton cycles, spanning trees and triangle factors (see e.g. [2, 3, 15, 16, 21, 23, 32, 37, 39]). It is easy to see that results on edge-coloured  $K_n$  also imply results on patterns in symmetric  $n \times n$  arrays.

**1.2. (Almost) spanning rainbow structures in complete graphs.** Andersen [5] conjectured that every properly edge-coloured  $K_n$  contains a rainbow path of length  $n - 2$  (which would be best possible by a construction of Maamoun and Meyniel [30]). Despite considerable research, even the existence of an almost spanning path or cycle was a major open question until recently. Alon, Pokrovskiy and Sudakov [3] were able to settle this by showing that any properly edge-coloured  $K_n$  contains a rainbow cycle of length  $n - O(n^{3/4})$  (the error term was subsequently improved in [8]). A corollary of our second main theorem (Theorem 1.5) states that we can arrive at a stronger conclusion (i.e. we obtain many edge-disjoint almost-spanning rainbow cycles) under much weaker assumptions (though with a larger error term). Note that, similarly to the case of Latin squares, any proper edge-colouring of  $K_n$  is  $n/2$ -bounded.

**Corollary 1.1.** *Any  $(1 + o(1))n/2$ -bounded, locally  $o(n)$ -bounded edge-colouring of  $K_n$  contains  $(1 - o(1))n/2$  edge-disjoint rainbow cycles of length  $(1 - o(1))n$ .*

As noted above, even for proper colourings, the corollary is best possible up to the value of the final error term, i.e. we cannot guarantee a Hamilton cycle. Moreover, a slight modification of the construction of Pokrovskiy and Sudakov in [38], shows that there are locally  $o(n)$ -bounded,  $(n - 1)/2$ -bounded edge-colourings of  $K_n$  with no rainbow cycle longer than  $n - \Omega(\log n)$ . For a more detailed discussion, see Section 5.

It is, however, more desirable to have spanning (rather than almost-spanning) structures. Which conditions guarantee the existence of a rainbow Hamilton cycle? Albert, Frieze and Reed [2] showed that there exists  $\mu > 0$ , such that in any  $\mu n$ -bounded edge-colouring of  $K_n$  there is a rainbow Hamilton cycle. Their result was greatly extended by Böttcher, Kohayakawa, and Procacci [9], who showed that any  $n/(51\Delta^2)$ -bounded edge-colouring of  $K_n$  contains a rainbow copy of  $H$  for any  $n$ -vertex graph  $H$  with maximum degree at most  $\Delta$ .

Note that these requirements are quite strong compared to the trivial (global)  $(n - 1)/2$ -boundedness condition which is the limit of what one could hope for. If we impose a global bound of  $(1 - o(1))n/2$  on the sizes of each colour class, then it turns out that we can still guarantee rainbow spanning structures, provided some moderate local boundedness conditions hold. The following is a corollary of our third and fourth main theorems (see Theorems 1.6 and 1.7). For given graphs  $F$  and  $G$ , we say that  $L \subseteq G$  is an  $F$ -factor if  $L$  consists of vertex-disjoint copies of  $F$  covering all vertices of  $G$ .

**Corollary 1.2.** *For any  $\varepsilon > 0$ , there exist  $\eta > 0$  and  $n_0$  such that for all  $n \geq n_0$ , any  $(1 - \varepsilon)\frac{n}{2}$ -bounded, locally  $\frac{\eta n}{\log^4 n}$ -bounded edge-colouring of  $K_n$  contains a rainbow Hamilton cycle and a rainbow triangle-factor (assuming that  $n$  is divisible by 3 in the latter case).*

In particular, any proper,  $(1 - o(1))n/2$ -bounded edge-colouring of  $K_n$  contains a rainbow Hamilton cycle. Bipartite versions of this, where one of the aims is to find rainbow perfect matchings in  $(1 - o(1))n$ -bounded edge-colourings of  $K_{n,n}$ , have been intensively studied, see e.g. [24, 36].

Corollary 1.2 is best possible in the following sense: as mentioned above, a proper (and thus  $n/2$ -bounded) edge-colouring of  $K_n$  does not guarantee a rainbow Hamilton cycle. In fact, this condition does not even ensure the existence of  $n$  different colours required for a Hamilton cycle.

**1.3. (Approximate) decompositions of complete graphs into rainbow structures.** As already mentioned, Euler was interested in finding Latin squares that are decomposable into full transversals. This corresponds to finding decompositions of properly edge-coloured complete bipartite graphs  $K_{n,n}$  into perfect rainbow matchings. More generally, we say that a graph  $G$  has a *decomposition* into graphs  $H_1, \dots, H_k$  if  $E(G) = \bigcup_{i=1}^k E(H_i)$  and the edge sets of the  $H_i$  are pairwise disjoint. The existence of various decompositions of  $K_n$  is a classical topic in design theory, related

to Room squares [45], Howell designs [41] and Kotzig factorizations [13]. In the setting of these questions, however, one is allowed to construct both the colouring and the decomposition. But, once again, it is natural to ask what one can say for arbitrary colourings with certain restrictions.

The most studied case is that of decompositions into trees. The following conjecture was raised, with some variations, by Brualdi and Hollingsworth [10], Kaneko, Kano, and Suzuki [28] and Constantine [14]: prove that every properly coloured complete graph is (almost) decomposable into (possibly isomorphic) rainbow spanning trees. Recently Pokrovskiy and Sudakov [39] as well as Balogh, Liu and Montgomery [7] independently showed that in a properly edge-coloured  $K_n$  one can find a collection of linearly many edge-disjoint rainbow spanning trees.

Our results actually work in the setting of *approximate decompositions*. We say that a collection of edge-disjoint subgraphs  $L_1, \dots, L_t$  of  $G$  is an  $\varepsilon$ -*decomposition* of  $G$ , if they contain all but at most an  $\varepsilon$ -proportion of the edges of  $G$ . The following result is a special case of Theorem 1.7.

**Corollary 1.3.** *For any  $\varepsilon > 0$ , there exist  $\eta > 0$  and  $n_0$  such that for all  $n \geq n_0$ , any  $(1 - \varepsilon)\frac{n}{2}$ -bounded, locally  $\frac{\eta n}{\log^4 n}$ -bounded edge-colouring of  $K_n$  has an  $\varepsilon$ -decomposition into rainbow Hamilton cycles.*

Note that this corollary implies an approximate version of the three conjectures on decompositions into spanning rainbow trees mentioned above. Indeed, for proper edge-colourings of  $K_n$  with an additional mild restriction on the size of each colour class ( $(1 - \varepsilon)n/2$  instead of  $n/2$ ), rainbow Hamilton cycles with one edge removed give us an approximate decomposition into isomorphic spanning paths. Similarly, Corollary 1.1 also implies an approximate version of the above conjectures as it gives (without any restriction on the sizes of the colour classes) an approximate decomposition into almost-spanning paths.

**1.4. Rainbow spanning structures and decompositions in quasirandom graphs.** Our results actually hold not only for colourings of  $K_n$ , but in the much more general setting of quasirandom graphs (and thus for example with high probability for dense random graphs). One of our main proof ingredients is a recent powerful result of Glock and Joos [23], who proved a rainbow blow-up lemma which allows to find rainbow copies of spanning subgraphs in a suitably quasirandom graph  $G$ , provided that the colouring is  $o(n)$ -bounded (see Theorem 2.10). As a consequence, they proved a rainbow bandwidth theorem under the same condition on the colouring. Note however that their blow-up lemma does not directly apply in our setting, as the restriction on the colouring is much stronger than in our case. We nevertheless can use it in our proofs since we apply it in a small random subgraph, on which the colouring has the necessary boundedness condition.

To formulate our results, we need the definition of a quasirandom graph. This will require some preparation. For  $a, b, c \in \mathbb{R}$  we write  $a = b \pm c$  if  $b - c \leq a \leq b + c$ . We define  $\binom{X}{k} := \{A \subseteq X : |A| = k\}$ . For a vertex  $v$  in a graph  $G$ , let  $d_G(v)$  denote its degree and  $N_G(v)$  its set of neighbours. The maximum and minimum degrees of  $G$  are denoted by  $\Delta(G)$  and  $\delta(G)$ , respectively. For  $u, v \in V(G)$ , we put  $N_G(u, v) := N_G(u) \cap N_G(v)$  and  $d_G(u, v) = |N_G(u, v)|$ . The latter function we call the *codegree* of  $u$  and  $v$ . We will sometimes omit the subscript  $G$  when the graph is clear from the context.

We say that an  $n$ -vertex graph  $G$  is  $(\varepsilon, d)$ -*quasirandom* if  $d(v) = (d \pm \varepsilon)n$  for each  $v \in V(G)$  and

$$\left| \left\{ uv \in \binom{V(G)}{2} : d(u, v) \neq (d^2 \pm \varepsilon)n \right\} \right| \leq \varepsilon n^2. \quad (1.1)$$

Note that this is weaker than the standard notion of  $(\varepsilon, d)$ -quasirandomness, where the set of exceptional vertex pairs having the “wrong” codegree is required to be empty (on the other hand, our notion is very close to the classical notion of  $\varepsilon$ -superregularity). Our first theorem guarantees the existence of an approximate decomposition into almost-spanning  $F$ -factors. For graphs  $F, G$  and  $0 \leq \alpha \leq 1$ , we say that  $L$  is an  $\alpha$ -*spanning  $F$ -factor* in  $G$ , if  $L$  is a subgraph of  $G$ , consisting of vertex-disjoint copies of  $F$  and containing all but at most an  $\alpha$ -proportion of the vertices of  $G$ . We define an  $\alpha$ -*spanning cycle* in  $G$  analogously.

**Theorem 1.4.** *For given  $\alpha, d_0 > 0$  and  $f, h \in \mathbb{N}$ , there exist  $\eta > 0$  and  $n_0$  such that the following holds for all  $n \geq n_0$  and  $d \geq d_0$ . Suppose that  $G$  is an  $n$ -vertex  $(\eta, d)$ -quasirandom graph and  $F$  is an  $f$ -vertex  $h$ -edge graph. If  $\phi$  is a  $(1 + \eta)\frac{fdn}{2h}$ -bounded, locally  $\eta n$ -bounded edge-colouring of  $G$ , then  $G$  contains an  $\alpha$ -decomposition into rainbow  $\alpha$ -spanning  $F$ -factors.*

Note that the  $(1 + o(1))\frac{fdn}{2h}$ -boundedness of the colouring cannot be replaced by a weaker condition even for a single  $o(1)$ -spanning  $F$ -factor, since we are only guaranteed roughly  $|E(G)|/((1 + o(1))\frac{fdn}{2h}) = (1 - o(1))\frac{hn}{f}$  distinct colours in such a colouring. On the other hand, an  $o(1)$ -spanning  $F$ -factor also contains  $(1 - o(1))\frac{hn}{f}$  edges of distinct colours. In the case when  $F$  is an edge (i.e. when we are looking for an almost perfect rainbow matching), a much stronger conclusion holds: we can in fact drop the quasirandomness condition and consider much sparser graphs (see Section 5).

The next theorem guarantees the existence of an approximate decomposition into almost-spanning rainbow cycles.

**Theorem 1.5.** *For given  $\alpha, d_0 > 0$ , there exist  $\eta > 0$  and  $n_0$  such that the following holds for all  $n \geq n_0$  and  $d \geq d_0$ . Suppose that  $G$  is an  $n$ -vertex  $(\eta, d)$ -quasirandom graph. If  $\phi$  is a  $\frac{1}{2}(1 + \eta)dn$ -bounded, locally  $\eta n$ -bounded edge-colouring of  $G$ , then  $G$  contains an  $\alpha$ -decomposition into rainbow  $\alpha$ -spanning cycles.*

For the same reasons as in Theorem 1.4, the  $\frac{1}{2}(1 + \eta)dn$ -boundedness condition cannot be replaced by a significantly weaker one.

If we slightly strengthen both the local and the global boundedness condition, we can obtain spanning structures, as guaranteed by the next two theorems below. The first theorem guarantees the existence of an approximate decomposition into rainbow  $F$ -factors. Let us denote  $a(F) := \max\{\Delta(F), a'(F), a''(F)\}$ , where  $a'(F)$  is the maximum of the expression  $d(u) + d(v) - 2$  over all edges  $uv \in E(F)$ , and  $a''(F)$  is the maximum of the expression  $d(u) + d(v) + d(w) - 4$  over all paths  $uvw$  in  $F$ . Note that  $a(F) \leq \max\{\Delta(F), 3\Delta(F) - 4\}$ .

**Theorem 1.6.** *For given  $\alpha, d_0 > 0$  and  $a, f, h \in \mathbb{N}$ , there exist  $\eta > 0$  and  $n_0$  such that the following holds for all  $n \geq n_0$  which are divisible by  $f$  and all  $d \geq d_0$ . Suppose that  $F$  is an  $f$ -vertex  $h$ -edge graph with  $a(F) \leq a$ . Suppose that  $G$  is an  $n$ -vertex  $(\eta, d)$ -quasirandom graph. If  $\phi$  is a  $(1 - \alpha)\frac{fdn}{2h}$ -bounded, locally  $\frac{\eta n}{\log^{2a} n}$ -bounded edge-colouring of  $G$ , then  $G$  has an  $\alpha$ -decomposition into rainbow  $F$ -factors.*

In a similar setting, we can also obtain an approximate decomposition into rainbow spanning cycles.

**Theorem 1.7.** *For given  $\alpha, d_0 > 0$ , there exist  $\eta > 0$  and  $n_0$  such that the following holds for all  $n \geq n_0$  and  $d \geq d_0$ . Suppose that  $G$  is an  $n$ -vertex  $(\eta, d)$ -quasirandom graph. If  $\phi$  is a  $\frac{1}{2}(1 - \alpha)dn$ -bounded, locally  $\frac{\eta n}{\log^4 n}$ -bounded edge-colouring of  $G$ , then  $G$  contains an  $\alpha$ -decomposition into rainbow Hamilton cycles.*

We will discuss multipartite analogues of our results in Section 5. (Recall that the bipartite case is of particular interest, as such results can be translated into the setting of arrays.) There are numerous open problems that arise from the above results: in particular, it is natural to seek decompositions into more general rainbow structures such as regular spanning graphs of bounded degree. It would also be very desirable to obtain improved error terms or even exact results.

The remainder of this paper is organized as follows. In Section 2 we collect the necessary definitions and auxiliary results, some of which are new and may be of independent interest (in particular, we prove a result on matchings in not necessarily regular hypergraphs). In Section 3, we prove Theorems 1.4 and 1.5. In Section 4 we prove Theorems 1.6 and 1.7. In Section 5, we add some concluding remarks. In the appendix we prove the rainbow counting lemma, which plays an important role in the proofs.

## 2. PRELIMINARIES

In this section, we introduce and derive several key tools that we will need later on: in particular, we state the rainbow blow-up lemma from [23] and derive a result on random matchings in (not necessarily regular) hypergraphs as well as two probabilistic partition results.

**2.1. Notation.** In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever  $1/n \ll a \ll b \leq 1$  (where  $n \in \mathbb{N}$  is typically the order of a graph), then this

means that there are non-decreasing functions  $f^* : (0, 1] \rightarrow (0, 1]$  and  $g^* : (0, 1] \rightarrow (0, 1]$  such that the result holds for all  $0 < a, b \leq 1$  and all  $n \in \mathbb{N}$  with  $a \leq f^*(b)$  and  $1/n \leq g^*(a)$ . We will not calculate these functions explicitly.

The auxiliary hierarchy constants used in this paper will be denoted by the Greek letters from  $\alpha$  to  $\eta$  (reserved throughout for this purpose). In what follows,  $n$  is the number of vertices in a graph or a part of a multipartite graph;  $d$  stands for the density of a graph. We use  $i, j, k$ , along with possible primes and subscripts, to index objects. We use letters  $u, v, w$  to denote vertices and  $e$  to denote graph edges. Colours are usually denoted by  $c$  and the colouring itself by  $\phi$ , while capital  $C$  (with possible subscripts) stands for various constants. We reserve other capital Latin letters except  $N$  for different sets or graphs. In the case of graphs or sets, having a prime in the notation means that later in the proof/statement we refine this object by removing some exceptional elements (note that primes do not have this meaning for the indexing variables). Of course, a double prime will then mean that we remove the exceptional elements in two stages. Calligraphic letters will stand for collections of sets, such as partitions or hypergraphs.

All graphs considered in this paper are simple. However, we allow our hypergraphs to have multiple edges. We use standard notations  $V(\cdot)$  and  $E(\cdot)$  for vertex and edge sets of graphs and hypergraphs. The number of edges in a graph  $G$  is denoted by  $e(G)$ . For a vertex set  $U$  and an edge set  $E$ , we denote by  $G \setminus U$  the graph we obtain from  $G$  by deleting all vertices in  $U$  and  $G - E$  denotes the graph we obtain from  $G$  by deleting all edges in  $E$ . For a set  $U \subseteq V(G)$  and  $u, v \in V(G)$ , we put

$$d_{G,U}(u) := |N_G(u) \cap U| \quad \text{and} \quad d_{G,U}(u, v) := |N_G(u, v) \cap U|.$$

For a graph  $G$  and two disjoint sets  $U, V \subseteq V(G)$ , let  $G[U, V]$  denote the graph with vertex set  $U \cup V$  and edge set  $\{uv \in E(G) : u \in U, v \in V\}$ . More generally, given disjoint sets  $U_1, \dots, U_k \subseteq V(G)$ , we define the  $k$ -partite subgraph  $G[U_1, \dots, U_k]$  of  $G$  in a similar way. We denote by  $P_k$  a path with  $k$  edges.

Since in this paper we deal with edge-colourings only, we simply refer to them as *colourings*. For shorthand, we call a colouring  $\phi : E(G) \rightarrow [m]$  of  $G$  in  $m$  colours an  $m$ -colouring of  $G$ . We denote by  $G(\phi, c)$  the spanning subgraph of  $G$  that contains all its edges of colour  $c$  in  $\phi$ . More generally, for a set  $I \subseteq [m]$ , we put  $G(\phi, I) = \bigcup_{c \in I} G(\phi, c)$ . An  $m$ -colouring  $\phi$  is  $g$ -bounded if and only if  $e(G(\phi, c)) \leq g$  for each  $c \in [m]$  and is *locally  $\ell$ -bounded* if and only if  $\Delta(G(\phi, c)) \leq \ell$  for each  $c \in [m]$ . We say that  $\phi$  is  $(g, \ell)$ -bounded if it is  $g$ -bounded and locally  $\ell$ -bounded.

**2.2. Probabilistic tools.** In this section, we collect the large deviation results we need.

**Lemma 2.1** (Chernoff-Hoeffding's inequality, see [27]). *Suppose that  $X_1, \dots, X_N$  are independent random variables taking values 0 or 1. Let  $X = \sum_{i \in [N]} X_i$ . Then*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2e^{-\frac{t^2}{2(\mathbb{E}[X] + t/3)}}.$$

In particular, if  $t \geq 7\mathbb{E}[X]$ , then  $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2e^{-t}$ .

We shall need two large deviation results for martingales.

**Theorem 2.2** (Azuma's inequality [6]). *Suppose that  $\lambda > 0$  and let  $X_0, \dots, X_N$  be a martingale such that  $|X_i - X_{i-1}| \leq \vartheta_i$  for all  $i \in [N]$ . Then*

$$\mathbb{P}[|X_N - X_0| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{2\sum_{i \in [N]} \vartheta_i^2}}.$$

**Theorem 2.3** ([12], Theorems 6.1 and 6.5). *Suppose that  $\lambda > 0$  and let  $X_0, \dots, X_N$  be a martingale such that  $|X_i - X_{i-1}| \leq \vartheta$  and  $\text{Var}[X_i | X_0, \dots, X_{i-1}] \leq \sigma_i^2$  for all  $i \in [N]$ . Then*

$$\mathbb{P}[|X_N - X_0| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{2\sum_{i \in [N]} \sigma_i^2 + \lambda\vartheta}}.$$

**2.3. Regularity.** In this part, we discuss the relation between quasirandomness and superregularity, as well as collect some tools to deal with "exceptional" pairs of vertices that have high codegree.

We say that a bipartite graph  $G$  with parts  $U, V$  is  $(\varepsilon, d)$ -regular if for all sets  $X \subseteq U, Y \subseteq V$  with  $|X| \geq \varepsilon|U|, |Y| \geq \varepsilon|V|$  we have

$$\left| \frac{e(G[X, Y])}{|X||Y|} - d \right| \leq \varepsilon.$$

If  $G$  is  $(\varepsilon, d)$ -regular and  $d_G(u) = (d \pm \varepsilon)|V|$  for all  $u \in U, d_G(v) = (d \pm \varepsilon)|U|$  for all  $v \in V$ , then we say that  $G$  is  $(\varepsilon, d)$ -superregular. We remark that, although the notions of  $\varepsilon$ -superregularity and  $(\varepsilon, d)$ -quasirandomness imply very similar properties of graphs, it is much handier to use the first one for bipartite graphs and the second one for general graphs. The following observation follows directly from the definitions, so we omit the proof.

**Proposition 2.4.** *Suppose  $1/n \ll \varepsilon \ll \delta \ll d \leq 1$ .*

- (i) *If  $G'$  is an  $(\varepsilon, d)$ -regular bipartite graph with vertex partition  $V_1, V_2$  with  $|V_1|, |V_2| \geq n$  and  $E \subseteq E(G')$  is a set of edges with  $|E| \leq \varepsilon n^2$ , then  $G' - E$  is  $(\delta, d)$ -regular.*
- (ii) *Suppose  $G'$  is an  $(\varepsilon, d)$ -quasirandom  $n$ -vertex graph,  $E \subseteq E(G')$  is a set of edges with  $|E| \leq \varepsilon n^2$  and  $V \subseteq V(G')$  is a set of vertices with  $|V| \leq \varepsilon n$ . Then  $(G' \setminus V) - E$  contains a  $(\delta, d)$ -quasirandom subgraph  $G$  on at least  $(1 - \delta)n$  vertices.*

In quasirandom graphs defined as in (1.1) there are exceptional pairs of vertices that have “incorrect” codegree. Similarly, in locally bounded colourings some pairs of vertices have large “monochromatic codegree”. To deal with such exceptional pairs of vertices we introduce *irregularity graphs*.

For an  $n$ -vertex graph  $G$ , we define the *irregularity graph*  $\text{Ir}_G(\varepsilon, d)$  to be the graph on  $V(G)$  and whose edge set is as defined in (1.1), i.e.  $uv \in E(\text{Ir}_G(\varepsilon, d))$  if and only if  $d_G(u, v) \neq (d^2 \pm \varepsilon)n$ . Similarly, for a partition  $\mathcal{V}$  of  $V(G)$  into subsets  $V_1, \dots, V_r$ , we let  $\text{Ir}_{G, \mathcal{V}}(\varepsilon, d)$  be the graph on  $V(G)$  with the edge set

$$\left\{ uv \in \binom{V(G)}{2} : u \in V_j, v \in V_{j'}, d_G(u, v) \neq (d^2 \pm \varepsilon)|V_{j''}| \text{ for some } j'' \in [r] \setminus \{j, j'\} \right\}.$$

(Here we allow  $j = j'$ .)

**Theorem 2.5.** [17] *Suppose  $0 < 1/n \ll \varepsilon \ll \alpha, d \leq 1$ . Suppose that  $G$  is a bipartite graph with a vertex partition  $\mathcal{V} = (U, V)$  such that  $n = |U| \leq |V|$ . If  $e(\text{Ir}_{G, \mathcal{V}}(\varepsilon, d)) \leq \varepsilon n^2$  and  $d(u) = (d \pm \varepsilon)|V|$  for all but at most  $\varepsilon n$  vertices  $u \in U$ , then  $G$  is  $(\varepsilon^{1/6}, d)$ -regular.*

The following lemma is an easy consequence of Theorem 2.5 and the definition of  $\varepsilon$ -superregularity. Thus we omit the proof.

**Lemma 2.6.** *Suppose  $0 < 1/n \ll \varepsilon \ll 1/r, \alpha, d \leq 1$ .*

- (i) *Suppose that  $G$  is an  $n$ -vertex,  $(\varepsilon, d)$ -quasirandom graph. Then  $\Delta(\text{Ir}_G(\varepsilon^{1/10}, d)) \leq \varepsilon^{1/10}n$ .*
- (ii) *Suppose that  $\mathcal{V} = (V_1, \dots, V_r)$  is a partition of  $G$  such that  $n \leq |V_i| \leq \alpha^{-1}n$  for each  $i \in [r]$  and  $G[V_i, V_j]$  is  $(\varepsilon, d)$ -superregular for all  $i \neq j \in [r]$ . Then  $\Delta(\text{Ir}_{G, \mathcal{V}}(\varepsilon^{1/10}, d)) \leq \varepsilon^{1/10}n$ .*

For  $u, v \in V(G)$  and a colouring  $\phi$  of  $G$ , let  $C_G^\phi(u, v) := \{w \in N_G(u, v) : \phi(uw) = \phi(vw)\}$  and let  $c_G^\phi(u, v)$  be its size, that is, the *monochromatic codegree* of  $u, v$ .

For a given colouring  $\phi$  of  $G$ , we define the *colour-irregularity graph*  $\text{Ir}_G^\phi(\ell)$  to be the graph on vertex set  $V(G)$  and edge set  $\{uv \in \binom{V(G)}{2} : c_G^\phi(u, v) \geq \ell\}$ . In words, we include a pair  $uv$  in the edge set if there are at least  $\ell$  choices of  $w \in N_G(u, v)$  such that  $\phi(uw) = \phi(vw)$ .

**Lemma 2.7.** *Let  $\ell, n \in \mathbb{N}$ . If  $\phi$  is a locally  $\ell$ -bounded colouring of an  $n$ -vertex graph  $G$ , then we have  $\Delta(\text{Ir}_G^\phi(\sqrt{\ell n})) \leq \sqrt{\ell n}$ .*

*Proof.* Suppose that for some vertex  $v$  there is a set  $U$  of more than  $\sqrt{\ell n}$  vertices  $u$  such that  $c_G^\phi(u, v) \geq \sqrt{\ell n}$ . For each  $u \in U$ , consider the set  $C_G^\phi(u, v) \subseteq N_G(v)$ , which is of size at least  $\sqrt{\ell n}$ . In total, we have more than  $\sqrt{\ell n}$  such sets of size  $\sqrt{\ell n}$ , and thus there exists a vertex  $w \in N_G(v)$  which belongs to  $C_G^\phi(u, v)$  for more than  $\ell n/d_G(v) \geq \ell$  vertices  $u \in U$ . Take some  $\ell + 1$  of these vertices, say,  $u_1, \dots, u_{\ell+1}$ . We have  $\phi(u_i w) = \phi(u_j w)$  for all  $i, j \in [\ell + 1]$ , which contradicts the assumption that  $\phi$  is locally  $\ell$ -bounded.  $\square$

**2.4. Counting rainbow subgraphs.** In the proof of Theorems 1.4 and 1.6, we deal with rainbow  $F$ -factors. The proofs of these theorems rely on a hypergraph-matching result in the spirit of the Rödl nibble and the Pippenger-Spencer theorem (Theorem 2.11 below). To make the transition from hypergraphs to coloured graphs, roughly speaking, we associate a hyperedge with each rainbow copy of  $F$ . We will need to ensure that the degree and codegree conditions hold for the auxiliary hypergraph in order for the nibble machinery to work. Therefore, we need certain results that will allow us to estimate the number of rainbow copies of  $F$  in a quasirandom (or superregular) graph  $G$ .

For given graphs  $F, G$ , a subgraph  $H$  of  $G$  and a colouring  $\phi$  of  $G$ , we denote by  $R_G^\phi(F, H)$  the collection of  $\phi$ -rainbow subgraphs  $\bar{F}$  of  $G$  that are isomorphic to  $F$  and contain  $H$  as an induced subgraph. Normally,  $\phi$  is obvious from the context, so we often omit it from the notation.

For a vertex partition  $\mathcal{X} = \{X_1, \dots, X_{r'}\}$  of  $F$  and a collection  $\mathcal{V} = \{V_1, \dots, V_r\}$  of disjoint subsets of  $V(G)$ , we say that an embedding  $\psi$  of  $F$  into  $G$  or a copy  $\psi(F)$  of  $F$  in  $G$  *respects*  $(\mathcal{X}, \mathcal{V})$ , if there exists an injective map  $\pi : [r'] \rightarrow [r]$  such that  $\psi(X_i) \subseteq V_{\pi(i)}$  for each  $i \in [r']$ . By abuse of notation, we also use  $V(F)$  to denote the partition of the vertex set of  $F$  into singletons.

For a subgraph  $H \subseteq G$  we denote by  $R_{G, \mathcal{X}, \mathcal{V}}(F, H)$  the collection of  $\phi$ -rainbow copies  $\bar{F}$  of  $F$  in  $G$  that respect  $(\mathcal{X}, \mathcal{V})$  and that contain  $H$  as an induced subgraph. Put  $r_{G, \mathcal{X}, \mathcal{V}}(F, H) := |R_{G, \mathcal{X}, \mathcal{V}}(F, H)|$ . If  $H$  is the order-zero graph, then we omit  $H$  from the expression. If  $H$  is a vertex  $v \in V(G)$  or an edge  $uv \in E(G)$ , then we write  $R_{G, \mathcal{X}, \mathcal{V}}(F, v)$  or  $R_{G, \mathcal{X}, \mathcal{V}}(F, uv)$ , respectively. Note that  $R_{G, \mathcal{X}, \mathcal{V}}(F, uv)$  and  $R_{G, \mathcal{X}, \mathcal{V}}(F, \{u, v\})$  are different since the former does not count the rainbow copies of  $F$  containing  $u, v$  but not the edge  $uv$ , while the latter counts only those.

For a given graph  $F$  with a vertex partition  $\mathcal{X} = \{X_1, \dots, X_r\}$  of  $V(F)$  into independent sets, let  $\text{Aut}_{\mathcal{X}}(F)$  denote the set of automorphisms  $\pi$  of  $F$  such that  $\{X_1, \dots, X_r\} = \{\pi(X_1), \dots, \pi(X_r)\}$ . We have  $\text{Aut}(F) = \text{Aut}_{V(F)}(F)$ , where  $\text{Aut}(F)$  is the set of all automorphisms of  $F$ .

The following two lemmas are easy corollaries of the ‘rainbow counting lemma’ given in the appendix. Their deduction is also deferred to the appendix. Roughly speaking, the proof relies on the fact that the global and local boundedness of the colouring  $\phi$  together imply that the number of non-rainbow copies of  $F$  in  $G$  containing a specific vertex or a specific edge is negligible, and so the number of *rainbow* copies of  $F$  in  $G$  is roughly the same as the total number of copies of  $F$  in  $G$ .

**Lemma 2.8.** *Let  $0 < 1/n \ll \zeta \ll \varepsilon \ll d, 1/r, 1/C, 1/f, 1/h \leq 1$ . Take a graph  $F$  with  $h$  edges and a vertex partition  $\mathcal{X} = \{X_1, \dots, X_r\}$  of  $V(F)$  into independent sets, where  $|X_i| = f$ . Take a graph  $G$  with a vertex partition  $\mathcal{V} = \{V_1, \dots, V_r\}$  into independent sets. Suppose that  $\phi$  is a  $(Cn, \zeta n)$ -bounded colouring of  $G$ . Fix  $j', j'' \in [r]$  and an edge  $vw \in E(G)$  with  $v \in V_{j'}$  and  $w \in V_{j''}$ . Suppose that the following conditions hold.*

- (A1)<sub>2.8</sub> *For each  $i \in [r]$ , we have  $|V_i| = (1 \pm \zeta)n$ .*
- (A2)<sub>2.8</sub> *For all  $i \neq j \in [r]$ , the bipartite graph  $G[V_i, V_j]$  is  $(\zeta, d)$ -superregular.*
- (A3)<sub>2.8</sub> *Either  $d_{G, V_i}(v, w) = (d^2 \pm \zeta)|V_i|$  and  $vw \notin \text{Ir}_G^\phi(\zeta n)$  for all  $i \in [r] \setminus \{j', j''\}$ , or  $F$  is triangle-free.*

*Then for any vertex  $u \in V(G)$ , we have*

$$r_{G, \mathcal{X}, \mathcal{V}}(F, u) = (1 \pm \varepsilon) \frac{r! f d^h n^{f r - 1}}{|\text{Aut}_{\mathcal{X}}(F)|} \quad \text{and} \quad r_{G, \mathcal{X}, \mathcal{V}}(F, vw) = (1 \pm \varepsilon) \frac{r! h d^{h-1} n^{f r - 2}}{\binom{r}{2} |\text{Aut}_{\mathcal{X}}(F)|}.$$

**Lemma 2.9.** *Let  $0 < 1/n \ll \zeta \ll \varepsilon \ll d, 1/C, 1/f, 1/h \leq 1$ . Take a graph  $F$  with  $f$  vertices and  $h$  edges and an  $n$ -vertex graph  $G$  which is  $(\zeta, d)$ -quasirandom. Suppose that  $\phi$  is a  $(Cn, \zeta n)$ -bounded colouring of  $G$ . Fix  $vw \in E(G)$ . Suppose that the following holds.*

- (A1)<sub>2.9</sub> *Either  $vw \notin \text{Ir}_G(\zeta, d) \cup \text{Ir}_G^\phi(\zeta n)$  or  $F$  is triangle-free.*

*Then for any vertex  $u \in V(G)$ , we have*

$$r_G(F, u) = (1 \pm \varepsilon/3) \frac{f d^h n^{f-1}}{|\text{Aut}(F)|} \quad \text{and} \quad r_G(F, vw) = (1 \pm \varepsilon/3) \frac{2 h d^{h-1} n^{f-2}}{|\text{Aut}(F)|}.$$

Note that in our applications of these lemmas, (A3)<sub>2.8</sub> and (A1)<sub>2.9</sub> will be satisfied for *all* edges, and thus the conclusion will hold for all edges as well.

**2.5. A rainbow blow-up lemma.** The following statement is an easy consequence of the rainbow blow-up lemma proved by Glock and Joos [23], which is our main tool to turn almost-spanning structures into spanning ones. Note however that the boundedness condition on  $\phi$  is much more restrictive than in our results.

**Theorem 2.10.** *Let  $0 < 1/n \ll \delta_2 \ll \gamma, 1/r, d, 1/\Delta \leq 1$ . Suppose that  $H$  is a graph with vertex partition  $\{X_0, X_1, \dots, X_r\}$  and  $G$  is a graph with vertex partition  $\{V_0, V_1, \dots, V_r\}$ . Let  $\phi$  be a  $\delta_2 n$ -bounded colouring of  $G$ . Suppose that the following conditions hold.*

- (A1)<sub>2.10</sub> *For each  $i \in [r] \cup \{0\}$ ,  $X_i$  is an independent set of  $H$  and  $\Delta(H) \leq \Delta$ . Moreover, no two vertices of  $X_0$  have a common neighbour.*
- (A2)<sub>2.10</sub>  *$\psi' : X_0 \rightarrow V_0$  is an injective map and  $|X_0| \leq \delta_2 n$ .*
- (A3)<sub>2.10</sub> *For each  $i \in [r]$ , we have  $|X_i| \leq |V_i|$  and  $|V_i| = (1 \pm \delta_2)n$ .*
- (A4)<sub>2.10</sub> *For all  $i \neq j \in [r]$ , the graph  $G[V_i, V_j]$  is  $(\delta_2, d)$ -superregular.*
- (A5)<sub>2.10</sub> *For all  $x \in X_0$  and  $i \in [r]$ , if  $N_H(x) \cap X_i \neq \emptyset$ , then we have  $d_{G, V_i}(\psi'(x)) \geq \frac{\gamma d}{2} |V_i|$ .*

*Then there is an embedding  $\psi$  of  $H$  into  $G$  which extends  $\psi'$  such that  $\psi(X_i) \subseteq V_i$  for each  $i \in [r]$  and  $\psi(H)$  is a rainbow subgraph of  $G$ . Moreover, if  $|X_i| \leq (1 - \sqrt{\delta_2})n$  for all  $i \in [r]$ , then the prefix “super” in (A4)<sub>2.10</sub> may be omitted.*

**2.6. Matchings in hypergraphs.** This section starts with a classical result due to Pippenger and Spencer on matchings in hypergraphs. We then prove a “defect” version of this (see Lemma 2.13). We conclude the section with Lemma 2.14, which is a translation of results on almost-spanning matchings in hypergraphs to results on approximate decompositions into rainbow almost-spanning factors. Lemma 2.14 is an essential step in the proofs of our theorems, allowing to obtain an approximate rainbow structure, which we then complete using the rainbow blow-up lemma.

Recall that we allow hypergraphs to have multiple edges. For a hypergraph  $\mathcal{H}$  and  $u, v \in V(\mathcal{H})$ , we let  $d_{\mathcal{H}}(v) := |\{H \in E(\mathcal{H}) : v \in H\}|$  and  $d_{\mathcal{H}}(uv) := |\{H \in E(\mathcal{H}) : \{u, v\} \subseteq H\}|$ . We let

$$\Delta(\mathcal{H}) := \max_{v \in V(\mathcal{H})} d_{\mathcal{H}}(v) \quad \text{and} \quad \Delta_2(\mathcal{H}) := \max_{u \neq v \in V(\mathcal{H})} d_{\mathcal{H}}(uv)$$

be the maximum degree and codegree of  $\mathcal{H}$ , respectively. A *matching* in a hypergraph is a collection of disjoint edges. It is *perfect* if it covers all the vertices of the hypergraph. If all sets in a matching have size  $r$ , then we call it an *r-matching*.

**Theorem 2.11.** [35] *Let  $0 < 1/n \ll \varepsilon \ll \delta, 1/r < 1$ . If  $\mathcal{H}$  is an  $n$ -vertex  $r$ -uniform hypergraph satisfying  $\delta(\mathcal{H}) \geq (1 - \varepsilon)\Delta(\mathcal{H})$  and  $\Delta_2(\mathcal{H}) \leq \varepsilon\Delta(\mathcal{H})$ , then  $E(\mathcal{H})$  can be partitioned into  $(1 + \delta)\Delta(\mathcal{H})$  matchings.*

Applying this theorem, we can prove a variation in which the hypergraph is allowed to have vertices of smaller degree, but the matchings are only required to cover the vertices of “correct” degree. We will need the following classical result on resolvable block designs due to Ray-Chaudhuri and Wilson, formulated in terms of matchings of  $r$ -sets.

**Theorem 2.12** ([40]). *For any  $r \in \mathbb{N}$  there exists  $b'_0 \in \mathbb{N}$ , such that the following holds for any  $b' \geq b'_0$ . For any  $\rho \leq 1$  there exists an  $r$ -uniform regular hypergraph  $\mathcal{A}$  on vertex set  $X$  of size  $b := r(r-1)b' + r$ , such that its degree is  $\lfloor \rho g \rfloor$ , where  $g := rb' + 1 = (b-1)/(r-1)$ , and its maximum codegree is 1. Moreover,  $\mathcal{A}$  is decomposable into  $\lfloor \rho g \rfloor$  perfect  $r$ -matchings.*

Note that if we take  $\rho = 1$  in the theorem above, then codegree of any two vertices in  $X$  is 1, that is, any pair is contained in exactly one edge of a matching. We now state our “defect” version of Theorem 2.11.

**Lemma 2.13.** *Let  $0 < 1/n \ll \varepsilon \ll \delta, 1/r < 1$ . Suppose that  $\mathcal{H}$  is an  $r$ -uniform hypergraph satisfying  $\Delta_2(\mathcal{H}) \leq \varepsilon\Delta(\mathcal{H})$ . Put*

$$U := \{u \in V(\mathcal{H}) : d_{\mathcal{H}}(u) < (1 - \varepsilon)\Delta(\mathcal{H})\} \quad \text{and} \quad V' := V(\mathcal{H}) \setminus U.$$

*Suppose  $V \subseteq V'$  with  $|V| = n$ .*

- (i) *There exist at least  $(1 - \delta)\Delta(\mathcal{H})$  edge-disjoint matchings of  $\mathcal{H}$  such that each matching covers at least  $(1 - \delta)n$  vertices of  $V$  and each vertex  $v$  of  $V$  belongs to at least  $(1 - \delta)\Delta(\mathcal{H})$  of the matchings.*



(ii) *There exists a randomized algorithm which always returns a matching  $\mathcal{M}$  of  $\mathcal{H}$  covering at least  $(1 - \delta)n$  vertices of  $V$  such that for each  $v \in V$  we have*

$$\mathbb{P}[v \in V(\mathcal{M})] \geq 1 - \delta.$$

*Proof.* Note that  $\Delta_2(\mathcal{H}) \geq 1$  implies  $\Delta(\mathcal{H}) \geq \varepsilon^{-1}$ . Before we can apply Theorem 2.11, we have to preprocess our hypergraph and make it nearly regular, without increasing the codegree too much. We shall do this in two stages.

The first stage is the following process. We iteratively obtain a sequence of hypergraphs  $\mathcal{H} =: \mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \dots$  on the same vertex set, until we have that  $|U_i| \leq \varepsilon n$  at some step, where

$$U_i := \{u \in V(\mathcal{H}) : d_{\mathcal{H}_i}(u) < (1 - \varepsilon)\Delta(\mathcal{H})\}. \quad (2.1)$$

We additionally require that throughout our process the following hold for each  $i$ :

$$\begin{aligned} \Delta(\mathcal{H}_i) &\leq (1 + \varepsilon^{1/3})\Delta(\mathcal{H}), \quad \Delta_2(\mathcal{H}_i) \leq \Delta_2(\mathcal{H}) + 2i \quad \text{and} \\ \delta(\mathcal{H}_i) &\geq \min\{(1 - \varepsilon)\Delta(\mathcal{H}), \delta(\mathcal{H}) + \varepsilon^{-1/2}i\}. \end{aligned} \quad (2.2)$$

Note that  $\mathcal{H}_0$  satisfies (2.2). Suppose that we have constructed  $\mathcal{H}_i$  and assume that  $|U_i| > \varepsilon n$ . Then we find two (not necessarily disjoint) sets  $U_i^1$  and  $U_i^2$  of the same size  $r(r-1)b' + r$  for some integer  $b'$ , such that  $U_i^1 \cup U_i^2 = U_i$ . We apply Theorem 2.12 to  $U_i^j$ ,  $j \in [2]$ , and find an  $r$ -uniform regular hypergraph  $\mathcal{A}_i^j$  on  $U_i^j$  with degree  $\varepsilon^{-1/2}$  and maximum codegree 1. Then we put  $\mathcal{H}_{i+1} := \mathcal{H}_i \cup \mathcal{A}_i^1 \cup \mathcal{A}_i^2$ , and repeat the procedure until  $|U_i| \leq \varepsilon n$  for some  $i$ , say  $i = k$ . Note that we may well be adding the same edge multiple times and, should this be the case, keep multiple copies of it.

Let us verify the validity of (2.2). Clearly, the minimum degree increases by at least  $\varepsilon^{-1/2}$  at each step, while the codegree of any two vertices increases by at most 2. We also have

$$\Delta(\mathcal{H}_{i+1}) \leq \max\{\Delta(\mathcal{H}_i), (1 - \varepsilon)\Delta(\mathcal{H}) + 2\varepsilon^{-1/2}\} \leq (1 + \varepsilon^{1/3})\Delta(\mathcal{H}).$$

Recall that  $|U_k| \leq \varepsilon n$  and  $k$  is the smallest index for which it holds. Due to the minimum degree condition in (2.2), we have  $k \leq \varepsilon^{1/2}\Delta(\mathcal{H})$ , and thus

$$\Delta_2(\mathcal{H}_k) \stackrel{(2.2)}{\leq} 3\varepsilon^{1/2}\Delta(\mathcal{H}). \quad (2.3)$$

This concludes the first stage of modification.

The goal of the second stage is to fix the degrees in the small exceptional set  $U_k$ . Put  $t := \sum_{u \in U_k} (\Delta(\mathcal{H}) - d_{\mathcal{H}_k}(u))$ . Note that  $t \leq \varepsilon n \Delta(\mathcal{H})$ . Consider a family  $\mathcal{W}$  of disjoint  $(r-1)$ -sets on  $V(\mathcal{H}) \setminus U_k$ , such that

$$|\mathcal{W}| = \left\lfloor \frac{|V(\mathcal{H}) \setminus U_k|}{r-1} \right\rfloor > \frac{n}{r}.$$

Consider an  $r$ -uniform hypergraph  $\mathcal{B}$  on  $V(\mathcal{H})$ , such that each edge of  $\mathcal{B}$  has the form  $\{u\} \cup W$ , where  $u \in U_k$ ,  $W \in \mathcal{W}$ , and, moreover, each  $u$  is contained in exactly  $\Delta(\mathcal{H}) - d_{\mathcal{H}_k}(u)$  edges of  $\mathcal{B}$  and each  $W$  is contained in at most  $\lceil \frac{t}{|\mathcal{W}|} \rceil \leq r\varepsilon\Delta(\mathcal{H})$  edges. Note that  $\Delta_2(\mathcal{B}) \leq r\varepsilon\Delta(\mathcal{H})$ , as well as that for any  $v \in V(\mathcal{H}) \setminus U_k$  we have  $d_{\mathcal{B}}(v) \leq r\varepsilon\Delta(\mathcal{H})$ . Consider the hypergraph  $\mathcal{G} := \mathcal{H}_k \cup \mathcal{B}$ . Then, clearly,  $\delta(\mathcal{G}) \geq (1 - \varepsilon)\Delta(\mathcal{H})$ , but also

$$\Delta(\mathcal{G}) \leq \Delta(\mathcal{H}_k) + r\varepsilon\Delta(\mathcal{H}) \stackrel{(2.2)}{\leq} (1 + \varepsilon^{1/4})\Delta(\mathcal{H}),$$

$$\Delta_2(\mathcal{G}) \leq \Delta_2(\mathcal{H}_k) + r\varepsilon\Delta(\mathcal{H}) \stackrel{(2.3)}{\leq} \varepsilon^{1/4}\Delta(\mathcal{H}).$$

Since  $\varepsilon \ll \delta$ , we are now in a position to apply Theorem 2.11, and decompose  $\mathcal{G}$  into a family  $\mathcal{F}''$  of  $(1 + \delta^5)\Delta(\mathcal{G})$  matchings. At least  $(1 - \delta^2)\Delta(\mathcal{H})$  of these matchings must cover at least  $(1 - \delta^2)n$  vertices of  $V$ .

Let  $\mathcal{F}'$  denote the family of these almost-spanning matchings and let  $\mathcal{F} := \{\mathcal{M} \cap E(\mathcal{H}) : \mathcal{M} \in \mathcal{F}'\}$ . We claim that the collection  $\mathcal{F}$  satisfies the assertion of the first part of the lemma, moreover, the algorithm which chooses one of the matchings from  $\mathcal{F}$  uniformly at random and returns its intersection with  $\mathcal{H}$  satisfies the assertion of the second part of the lemma.

To see this, first note that any matching  $\mathcal{M} \in \mathcal{F}'$  covers at least  $(1 - \delta^2)n$  vertices of  $V$ , and, since any edge in  $\mathcal{G} - \mathcal{H}$  either entirely lies in  $U$  or intersects  $U_k$ , the matching  $\mathcal{M} \cap E(\mathcal{H})$  covers at

least  $(1 - \delta^2)n - r|U_k| \geq (1 - \delta)n$  vertices of  $V$ . Second, for each  $v \in V(\mathcal{H})$ , the vertex  $v$  belongs to  $d_{\mathcal{H}}(v) \pm |\mathcal{F}'' \setminus \mathcal{F}'| \geq (1 - \delta)\Delta(\mathcal{H})$  matchings from  $\mathcal{F}'$  that cover  $v$  by an edge from  $\mathcal{H}$ . This proves (i). To prove (ii), note that  $|\mathcal{F}| = |\mathcal{F}'| = (1 \pm \delta^2)\Delta(\mathcal{H})$ , hence for a randomly chosen  $\mathcal{M} \in \mathcal{F}$ , for each  $v \in V$ , we have

$$\mathbb{P}[v \in V(\mathcal{M})] = \frac{d_{\mathcal{H}}(v) \pm |\mathcal{F}'' \setminus \mathcal{F}'|}{|\mathcal{F}|} = \frac{d_{\mathcal{H}}(v)}{|\mathcal{F}|} \pm 3\delta^2 \geq 1 - \delta.$$

□

For an edge-coloured graph  $G$  and a given family  $\mathcal{F}$  of rainbow subgraphs of  $G$ , we denote by  $\mathcal{F}(v_1, v_2; c_1, c_2)$  the subfamily of all those graphs from  $\mathcal{F}$  which contain the vertices  $v_1, v_2$  and edges of colours  $c_1, c_2$ . We define  $\mathcal{F}(v_1; c_1)$ ,  $\mathcal{F}(v_1, v_2)$ ,  $\mathcal{F}(c_1, c_2)$ ,  $\mathcal{F}(v_1)$  and  $\mathcal{F}(c_1)$  in a similar way. For  $uw, u'w' \in E(G)$  we denote by  $\mathcal{F}(uw)$  the subfamily of graphs from  $\mathcal{F}$  that contain the edge  $uw$ , and define  $\mathcal{F}(uw, u'w')$  similarly. The next lemma is the key to the proof of Theorem 1.4 and is also very important for the proofs of the other theorems from Section 1.4.

**Lemma 2.14.** *Let  $0 < 1/n \ll \varepsilon \ll \delta, 1/f \leq 1$ . Suppose that  $F$  is a graph on  $f \geq 3$  vertices and with  $h \geq 1$  edges. Suppose that  $G = (V, E)$  is an  $n$ -vertex graph and  $\phi$  is an  $m$ -colouring of  $G$ . Consider a family  $\mathcal{F}$  of rainbow copies of  $F$  in  $G$  that satisfies the following requirements.*

(A1)<sub>2.14</sub> *For any  $v, v_1, v_2 \in V$  and  $c_1, c_2 \in [m]$  we have*

$$\max\{|\mathcal{F}(v_1, v_2)|, |\mathcal{F}(v_1; c_1)|, |\mathcal{F}(c_1, c_2)|\} \leq \varepsilon|\mathcal{F}(v)|.$$

(A2)<sub>2.14</sub> *For any  $c \in [m]$  and  $v \in V$  we have  $|\mathcal{F}(v)| \geq (1 - \varepsilon)|\mathcal{F}(c)|$ .*

(A3)<sub>2.14</sub> *For all  $v \in V$  and  $uw \in E$  we have*

$$(1 \pm \varepsilon) \frac{|\mathcal{F}(v)|}{|\mathcal{F}(uw)|} = \frac{f|E|}{h|V|} =: t.$$

(A4)<sub>2.14</sub> *For any  $uw \in E$  we have  $|\mathcal{F}(uw)| \geq 10\varepsilon^{-1} \log n$ .*

(A5)<sub>2.14</sub> *For any  $uw, u'w', u''w'' \in E$  we have  $\varepsilon|\mathcal{F}(uw)| \geq |\mathcal{F}(u'w', u''w'')|$ .*

*Then there exists a randomized algorithm which always returns  $(1 - \delta)t$  edge-disjoint rainbow  $\delta$ -spanning  $F$ -factors  $\mathcal{M}_1, \dots, \mathcal{M}_{(1-\delta)t}$  of  $G$ , such that each  $\mathcal{M}_i$  consists of copies of  $F$  from  $\mathcal{F}$  and for all  $v \in V$  and  $i \in [(1 - \delta)t]$  we have*

$$\mathbb{P}[v \in V(\mathcal{M}_i)] \geq 1 - \delta.$$

Clearly, the union of all the  $\mathcal{M}_i$  covers all but at most a  $2\delta$ -proportion of edges of  $G$ .

*Proof.* The idea is to apply Lemma 2.13 (ii) to a suitable auxiliary (multi-) hypergraph  $\mathcal{H}$ . However, the choice of  $\mathcal{H}$  is not straightforward, since Lemma 2.13 (ii) gives only a single random matching while we need an almost-decomposition. We can resolve this by turning both the edges and the vertices of  $G$  into vertices of  $\mathcal{H}$ . However, this gives rise to the issue that the potential degrees of vertices and edges in the corresponding auxiliary hypergraph are very different. This in turn can be overcome by the following random splitting process.

Consider a random partition  $\mathcal{F}_1, \dots, \mathcal{F}_t$  of  $\mathcal{F}$  into  $t$  parts, where for all  $\bar{F} \in \mathcal{F}$  and  $i \in [t]$  we have  $\bar{F} \in \mathcal{F}_i$  with probability  $1/t$  independently of all other graphs in  $\mathcal{F}$ . Using Lemma 2.1 combined with the fact that the expected value of  $\mathcal{F}(v) \cap \mathcal{F}_i$  is sufficiently large (it is at least  $9\varepsilon^{-1} \log n$  by (A4)<sub>2.14</sub> and (A3)<sub>2.14</sub>) for each  $v \in V(G)$ , we can conclude that for any  $uw \in E$ ,  $v_1, v_2 \in V$ ,  $c, c_1, c_2 \in [m]$  and  $i, i' \in [t]$  we have

$$(1 \pm 3\varepsilon^{1/2})|\mathcal{F}(uw)| \stackrel{(A3)_{2.14}}{=} |\mathcal{F}(v) \cap \mathcal{F}_i| \stackrel{(A2)_{2.14}}{\geq} (1 - 5\varepsilon^{1/2})|\mathcal{F}(c) \cap \mathcal{F}_{i'}|, \quad (2.4)$$

$$3\varepsilon^{1/2}|\mathcal{F}(v) \cap \mathcal{F}_i| \stackrel{(A1)_{2.14}}{\geq} \max\{|\mathcal{F}(v_1, v_2) \cap \mathcal{F}_{i'}|, |\mathcal{F}(v_1; c_1) \cap \mathcal{F}_{i'}|, |\mathcal{F}(c_1, c_2) \cap \mathcal{F}_{i'}|\}. \quad (2.5)$$

Consider the hypergraph  $\mathcal{H}$  defined by

$$\begin{aligned} V(\mathcal{H}) &:= E \cup (V \times [t]) \cup ([m] \times [t]) && \text{and} \\ E(\mathcal{H}) &:= \{E(\bar{F}) \cup (V(\bar{F}) \times \{i\}) \cup (\phi(E(\bar{F})) \times \{i\}) : \bar{F} \in \mathcal{F}_i, i \in [t]\}. \end{aligned} \quad (2.6)$$

Thus each edge of  $\mathcal{H}$  corresponds to some  $\bar{F} \in \mathcal{F}$ . Condition (2.4) guarantees that the vertices from  $E \cup (V \times [t])$  have roughly the same degree, while the vertices from  $[m] \times [t]$  have degree that is at most  $(1 - 8\varepsilon^{1/2})^{-1}$  times the degrees of any vertex from  $E \cup (V \times [t])$ , but may be significantly smaller. Moreover, we have  $\Delta_2(\mathcal{H}) \leq 3\varepsilon^{1/2}\Delta(\mathcal{H})$  due to (2.5) and (A5)<sub>2.14</sub> (note here that vertices of  $\mathcal{H}$  that have different indices  $i, i' \in [t]$  have zero codegree and that  $|\mathcal{F}(uw)| \leq |\mathcal{F}(u, w)|$  for any  $uw \in E$ ). Therefore, we can apply Lemma 2.13 (ii) with  $8\varepsilon^{1/2}, \delta^3$  and  $V \times [t]$  playing the roles of  $\varepsilon, \delta$  and  $V$ , respectively and obtain an algorithm producing a random matching  $\mathcal{M}$  of  $\mathcal{H}$ , covering at least a  $(1 - \delta^3)$ -proportion of vertices from  $V \times [t]$  and such that each vertex of  $V \times [t]$  is contained in  $\mathcal{M}$  with probability at least  $1 - \delta^3$ . In particular, this implies that

$$\mathcal{M} \text{ covers all but at most a } \delta\text{-proportion of } V \times \{i\} \text{ for at least } (1 - \delta)t \text{ values of } i. \quad (2.7)$$

Let  $\mathcal{M}'_i$  be the collection of all those  $\bar{F} \in \mathcal{F}_i$  which correspond to some edge of  $\mathcal{M}$ . Then  $\mathcal{M}'_i$  forms a rainbow  $c_i$ -spanning  $F$ -factor in  $G$  for some  $c_i > 0$ , moreover, these factors are edge-disjoint for different  $i_1, i_2 \in [t]$ . Furthermore, by Lemma 2.13 for each  $v \in V$  we have  $\mathbb{P}[v \in V(\mathcal{M}'_i)] \geq 1 - \delta^3$ . However,  $\mathcal{M}'_i$  does not necessarily form a  $\delta$ -spanning  $F$ -factor. This can be fixed easily. Since for each  $i \in [t]$  the matching  $\mathcal{M}$  covers each vertex from  $V \times \{i\}$  with probability  $1 - \delta^3$ , for each  $i$  with probability at least  $1 - \delta/2$  the factor  $\mathcal{M}'_i$  is  $\delta$ -spanning, and thus with probability at least  $1 - \delta$  it is both  $\delta$ -spanning and covers a given vertex  $v$ . Moreover, the factors  $\mathcal{M}'_i$  are  $\delta$ -spanning for at least  $(1 - \delta)t$  values of  $i$  (cf. (2.7)). Let the algorithm return the factors  $\mathcal{M}_i$ ,  $i \leq (1 - \delta)t$ , where  $\mathcal{M}_i := \mathcal{M}'_i$  if  $\mathcal{M}'_i$  is  $\delta$ -spanning, and otherwise  $\mathcal{M}_i := \mathcal{M}'_j$  for some  $j > (1 - \delta)t$ , where  $\mathcal{M}'_j$  is a  $\delta$ -spanning factor not yet used to substitute for  $\mathcal{M}'_{i'}$  with  $i' < i$ . Note that for any given  $v \in V$  and any  $i \in [(1 - \delta)t]$ , we have  $\mathbb{P}[v \in V(\mathcal{M}_i)] \geq \mathbb{P}[v \in V(\mathcal{M}'_i) \wedge \mathcal{M}_i = \mathcal{M}'_i] \geq 1 - \delta$  as required.  $\square$

**2.7. Partitions.** To have better control over the colours and vertices used when constructing the decompositions, we need to split vertices and colours into groups. The results in this section will be needed to ensure that the relevant properties of the original graph are inherited by its subgraphs induced by suitable random partitions.

We say that  $\mathcal{V}$  is a *partition of a set  $V$  chosen at random with probability distribution  $(p_1, \dots, p_r)$* , if  $p_1, \dots, p_r$  are nonnegative real numbers satisfying  $\sum_{i \in [r]} p_i \leq 1$  and  $\mathcal{V}$  is a random variable such that for each  $v \in V$ , we have  $\mathbb{P}[v \in V_i] = p_i$  independently at random.

**Lemma 2.15.** *Let  $0 < 1/n \ll \eta \ll \zeta \ll \delta, d, 1/C \leq 1$ . Let  $G'$  be a  $(\eta, d)$ -quasirandom  $n$ -vertex graph. Suppose that  $\phi$  is a  $(Cn, \eta n)$ -bounded  $m$ -colouring of  $G'$ . Then there exists a  $(\zeta, d)$ -quasirandom spanning subgraph  $G$  of  $G'$ , such that for any  $uv \in E(G)$  we have*

$$d_G(u, v) = (d^2 \pm \zeta)n \text{ and } c_G^\phi(u, v) \leq \zeta n. \quad (2.8)$$

Moreover, the following holds. For a random partition  $\mathcal{V}$  of  $V(G)$  with probability distribution  $(p_1, \dots, p_r)$ , where  $p_i \geq n^{-1/2}$  for each  $i \in [r]$ , with probability at least 0.9 we have:

- (A1)<sub>2.15</sub>  $|V_i| = (1 \pm \zeta)p_i n$ .
- (A2)<sub>2.15</sub> For all  $i \neq j \in [r]$ , the bipartite graph  $G[V_i, V_j]$  is  $(\zeta, d)$ -superregular.
- (A3)<sub>2.15</sub> For all  $vw \in E(G)$  and  $i \in [r]$ , we have  $|C_G^\phi(v, w) \cap V_i| \leq \zeta|V_i|$  and  $d_{G, V_i}(v, w) = (d^2 \pm \zeta)|V_i|$ .
- (A4)<sub>2.15</sub> For all  $i \in [r]$ , the graph  $G[V_i]$  is  $(\zeta, d)$ -quasirandom.

Note that  $G$  contains all but at most a  $2\zeta/d$ -fraction of the edges of  $G'$ .

*Proof.* Let  $\text{Ir}$  be the graph with vertex set  $V(G')$  and edge set  $E(\text{Ir}_{G'}(\eta^{1/10}, d)) \cup E(\text{Ir}_{G'}^\phi(\eta^{1/2}n))$ , that is, every edge of  $\text{Ir}$  corresponds to a pair of vertices which either has “wrong” codegree or “wrong” monochromatic codegree. We first show that  $\text{Ir}$  has small maximum degree. Since  $G'$  is a  $(\eta, d)$ -quasirandom graph and  $\phi$  is locally  $\eta n$ -bounded, by Lemmas 2.6 and 2.7 we have

$$\Delta(\text{Ir}) \leq \Delta(\text{Ir}_{G'}(\eta^{1/10}, d)) + \Delta(\text{Ir}_{G'}^\phi(\eta^{1/2}n)) \leq \eta^{1/10}n + \eta^{1/2}n \leq \zeta^{10}n. \quad (2.9)$$

Consider the graph  $G := G' - E(\text{Ir})$ . For each  $uv \notin \text{Ir}$  (and so in particular for each  $uv \in E(G)$ ), we have

$$d_G(u, v) = (d^2 \pm \zeta^2)n \text{ and } c_G^\phi(u, v) \leq \zeta^2 n. \quad (2.10)$$

Clearly,  $G$  is  $(\zeta^2, d)$ -quasirandom, and (2.10) implies (2.8). Now, a standard application of Chernoff bounds (Lemma 2.1) implies that (A1)<sub>2.15</sub> and (A3)<sub>2.15</sub> hold with probability 0.99. For the same

reasons,  $d_{G, V_i}(v) = (1 \pm \zeta/2)p_i d_G(v)$  for all  $v \in V(G), i \in [r]$  with probability 0.99 (this implies the “super” part of the superregularity from (A2)2.15). Note that  $p_i \geq n^{-1/2}$  implies that  $r \leq n^{1/2}$ .

Finally, by (2.9) and Lemma 2.1, with probability 0.99 for all  $i, j \in [r]$  the maximum degree of a vertex from  $V_i$  in  $\text{Ir}[V_i, V_j]$  is at most  $\zeta^7 p_j n$ , and thus the number of edges of  $\text{Ir}$  that connect vertices of  $V_i$  is at most  $\zeta^7 p_i n |V_i| \leq \zeta^6 |V_i|^2$  for all  $i \in [r]$ . Applying Theorem 2.5, we obtain that  $G[V_i, V_j]$  is  $(\zeta, d)$ -regular. One can similarly bound the number of irregular pairs in each  $G[V_i]$ , and, combined with the bounds for the degrees and codegrees obtained above, it follows that (A4)2.15 holds with probability at least 0.99. Overall, all these events hold simultaneously with probability at least 0.9.  $\square$

**Lemma 2.16.** *Let  $n, r \in \mathbb{N}$  and  $0 < 1/n \ll \zeta \ll 1/C < 1$ . Assume that  $\ell \geq \frac{n}{\log^3 n}$ . Suppose that  $G$  is an  $n$ -vertex graph with at most  $Cn$  edges and  $\Delta(G) \leq \ell$ , and  $\mathcal{V} = (V_1, \dots, V_r)$  is a partition of  $V(G)$  chosen at random with probability distribution  $(p_1, \dots, p_r)$  with  $p_i \geq \log^{-2} n$ . Let  $p := \min_{i \in [r]} \{p_i\}$ .*

*Then with probability at least  $1 - 2r^2 e^{-\frac{\zeta^3 p n}{\ell}}$  we have for all  $i \neq j \in [r]$*

$$e(G[V_i, V_j]) = 2p_i p_j e(G) \pm \zeta p_i p_j n \quad \text{and} \quad e(G[V_i]) = p_i^2 e(G) \pm \zeta p_i^2 n.$$

*Proof.* Let  $v_1, \dots, v_n$  be the vertices of  $G$  in the decreasing degree order. Put  $t := 2Cn^{1/3}$ . Since  $e(G) \leq Cn$ , we have  $d(v_k) \leq n^{2/3}$  for  $k > t$ . Fix  $i, j \in [r]$ . We now count edges with the first (in the ordering) vertex in  $V_i$  and the second in  $V_j$ . We denote this quantity by  $\bar{e}(G[V_i, V_j])$ . Consider a martingale  $X_0, \dots, X_n$ , where

$$X_k := \mathbb{E}[\bar{e}(G[V_i, V_j]) \mid V_i \cap \{v_1, \dots, v_k\}, V_j \cap \{v_1, \dots, v_k\}].$$

We aim to apply Theorem 2.3 to this martingale. In the notation of that theorem, for  $k > t$ , we clearly have  $|X_k - X_{k-1}| \leq d(v_k) \leq n^{2/3}$ . Moreover,  $\sum_{k=t}^n \sigma_k^2 \leq \sum_{k=t}^n d^2(v_k) \leq n^{2/3} \sum_{k=t}^n d(v_k) \leq 2Cn^{5/3}$ .

We now suppose that  $k \leq t$ . Without loss of generality, we assume that there are no edges in  $G$  between vertices  $v_k, v_{k'}$  for  $k, k' \leq t$ . Indeed, this accounts for at most  $N := 4C^2 n^{2/3}$  edges, which is negligible, and we will take care of this later. Take  $k \leq t$  and fix  $V_i \cap \{v_1, \dots, v_{k-1}\}, V_j \cap \{v_1, \dots, v_{k-1}\}$ . Then  $X_k - X_{k-1}$  is the following random variable:

$$X_k - X_{k-1} = \begin{cases} p_j(1 - p_i)d(v_k), & \text{if } v_k \in V_i \\ -p_i p_j d(v_k) & \text{otherwise.} \end{cases}$$

From this formula we can easily conclude that, first,  $|X_k - X_{k-1}| \leq p_j d(v_k) \leq p_j \ell$ , and, second,  $\text{Var}[X_k \mid X_{k-1}, \dots, X_1] = \mathbb{E}[(X_k - X_{k-1})^2 \mid X_{k-1}, \dots, X_1] \leq p_i p_j^2 d^2(v_k) =: \sigma_k^2$ . Thus,  $\sum_{k=1}^t \sigma_k^2 \leq 2Cp_i p_j^2 n \ell$ . Altogether, with  $\vartheta$  defined as in Theorem 2.3, we have

$$\sum_{k=1}^n \sigma_k^2 \leq 2Cp_i p_j^2 n \ell + 2Cn^{5/3} \leq 3Cp_i p_j^2 \ell n \quad \text{and} \quad \vartheta \leq \max\{p_j \ell, n^{2/3}\} \leq p_j \ell.$$

(This is the only place where we make use of the lower bound on  $\ell$ .) Substituting into Theorem 2.3, we obtain

$$\mathbb{P}\left[|X_k - p_i p_j e(G)| \leq \frac{\zeta}{3} p_i p_j n\right] \leq 2 \exp\left(-\frac{(\frac{\zeta}{3} p_i p_j n)^2}{6Cp_i p_j^2 \ell n + \frac{\zeta}{3} p_i p_j n \cdot p_j \ell}\right) \leq 2e^{-\frac{\zeta^3 p_i n}{\ell}}. \quad (2.11)$$

Note that  $N \leq \frac{\zeta}{6} p_i p_j n$ , and thus (2.11), with  $\zeta/3$  replaced by  $\zeta/2$  on the left hand side, also holds in the situation when we may have edges between  $v_k, v_{k'}$  for  $k, k' \leq t$ . The fact that  $e(G[V_i]) = \bar{e}(G[V_i, V_i])$  and  $e(G[V_i, V_j]) = \bar{e}(G[V_i, V_j]) + \bar{e}(G[V_j, V_i])$  if  $i \neq j$ , together with a union bound over all possible choices of  $i, j \in [r]$ , implies the result.  $\square$

The next lemma allows us to extend the counting results of Lemmas 2.8 and 2.9 to the case when the graph is sparse.

**Lemma 2.17.** *Let  $n, r \in \mathbb{N}$  and  $0 < 1/n \ll \varepsilon \ll 1/f, 1/C < 1$ . Assume that  $\ell \geq n^{2/3}$ . Suppose that  $G$  is an  $n$ -vertex graph and  $\phi$  is a  $(Cn, \ell)$ -bounded  $m$ -colouring of  $G$ . Fix a  $k$ -vertex subset  $U$  of  $V(G)$ . Suppose that  $\mathcal{I} = (I_1, \dots, I_r)$  is a partition of  $[m]$  chosen at random with probability distribution  $(p_1, \dots, p_r)$ , where  $p_i \geq \log^{-2} n$ . Suppose that  $\mathcal{F}$  is a collection of  $f$ -vertex  $h$ -edge*

rainbow subgraphs of  $G$  such that  $U$  is an independent set of each  $R \in \mathcal{F}$ . Assume that, for some  $a \geq 1$ , the set  $U$  has at most  $a$  edges incident to it in each  $R \in \mathcal{F}$ . For  $j \in [r]$ , with probability at least  $1 - 2 \exp\left(-\frac{\varepsilon^4 p_j^{2a-1} n}{\ell}\right)$  the number of graphs  $R$  in  $\mathcal{F}$  which are subgraphs of  $G(\phi, I_j)$  is  $p_j^h |\mathcal{F}| \pm \varepsilon p_j^h n^{f-k}$ .

*Proof.* The proof of this lemma is similar to that of Lemma 2.16. Fix  $j \in [r]$  and let  $L$  be the random variable equal to the number of graphs  $R \in \mathcal{F}$  such that the colour of every edge of  $R$  belongs to  $I_j$ . As  $R$  contains  $h$  edges whose colours are all different, we have  $\mathbb{E}[L] = p_j^h |\mathcal{F}|$ . Order the colours in  $[m]$  by the number of graphs  $R \in \mathcal{F}$  that contain that colour, from the larger value to the smaller. Put  $t := hf!n^{1/2}$ . The number of  $R \in \mathcal{F}$  that contain some edge  $e$  of colour  $i \leq t$ , where  $e$  is not adjacent to one of the vertices of  $U$ , is at most  $t \cdot Cn \cdot f!n^{f-k-2} \leq n^{f-k-1/3}$ . We assume for the moment that there are no such  $R$ , and will deal with them later.

For each  $i \in [m]$ , we let  $X_i = \mathbb{E}[L \mid I_j \cap [i]]$ . Then  $X_0, X_1, \dots, X_m$  is an exposure martingale. Let  $C_i$  be the number of  $R \in \mathcal{F}$  which contain an edge of colour  $i$ . Let  $C_i(j)$  be the number of  $R \in \mathcal{F}$  that are coloured with colours from  $I_j$  and which contain an edge of color  $i$ . It is easy to see that for  $i \geq t$  we have  $C_i \leq n^{f-k-1/2}$ . This implies that  $|X_i - X_{i-1}| \leq n^{f-k-1/2}$  for  $i \geq t$ , moreover, in the notation of Theorem 2.3,  $\sum_{i=t}^m \sigma_i^2 \leq \sum_{i=t}^m C_i^2 \leq C_t \cdot h |\mathcal{F}| \leq hf!n^{2f-2k-1/2}$ .

Take  $i < t$  and fix  $I_j \cap [i-1]$ . Then the random variable  $X_i - X_{i-1}$  has the following form:

$$X_i - X_{i-1} = \begin{cases} \mathbb{E}[C_i(j) \mid i \in I_j, I_j \cap [i-1]] - \mathbb{E}[C_i(j) \mid I_j \cap [i-1]], & \text{if } i \in I_j, \\ -\mathbb{E}[C_i(j) \mid I_j \cap [i-1]] & \text{otherwise.} \end{cases}$$

Take any graph  $R$  containing an edge of colour  $i$ . By the assumption, all edges of  $R$  not ending in  $U$  have colours in  $[t, n]$ , and therefore,  $\mathbb{E}[C_i(j) \mid i \in I_j, I_j \cap [i-1]] \leq C_i \cdot p_j^{h-a}$  and  $\mathbb{E}[C_i(j) \mid I_j \cap [i-1]] = p_j \mathbb{E}[C_i(j) \mid i \in I_j, I_j \cap [i-1]]$ . From here we may conclude that  $|X_i - X_{i-1}| \leq C_i \cdot p_j^{h-a}$  and, moreover, in terms of Theorem 2.3,  $\text{Var}[X_i \mid X_{i-1}, \dots, X_1] = \mathbb{E}[|X_i - X_{i-1}|^2 \mid X_{i-1}, \dots, X_1] \leq C_i^2 p_j^{2(h-a)+1} =: \sigma_i^2$ .

Next, we have to bound  $C_i$ . Since  $U$  is an independent set of  $R$  for every  $R \in \mathcal{F}$ , for any edge  $uv$  of  $R$ , at least one of  $u, v$  lies outside  $U$ . Moreover, if  $u, v \notin U$  then by our assumption  $\phi(uv) > t$ . Hence, we obtain that  $C_i$  equals the number of  $R \in \mathcal{F}$  which contain an edge  $uv$  of colour  $i$  which is incident to  $U$ . Hence,  $C_i$  is at most  $f!n^{f-k-1}$  times the number of edges with colour  $i$  in  $G$  which are incident to  $U$ . Thus  $C_i \leq f!k\ell n^{f-k-1}$  for each  $i \in [t-1]$ . Moreover,  $\sum_{i \in [t-1]} C_i \leq h |\mathcal{F}|$ . In terms of Theorem 2.3, this implies that

$$\sum_{i \in [m]} \sigma_i^2 \leq p_j^{2(h-a)+1} \sum_{i=1}^{t-1} C_i^2 + hf!n^{2f-2k-1/2} \leq \varepsilon^{-1/2} p_j^{2(h-a)+1} \ell n^{2f-2k-1}.$$

We also have

$$\vartheta \leq \max\{n^{f-k-1/2}, p_j^{h-a} \cdot kf! \ell n^{f-k-1}\} \leq \varepsilon^{-1/2} p_j^{h-a} \ell n^{f-k-1}.$$

Substituting the right hand sides of the displayed formulas above in the inequality in Theorem 2.3, we have

$$\mathbb{P}[L \neq (1 \pm \frac{\varepsilon}{2}) p_j^h n^{f-k}] \leq 2 \exp\left(-\frac{\varepsilon^3 p_j^{2a-1} n^{2f-2k}}{\ell n^{2f-2k-1} + \varepsilon p_j^{a-1} \ell n^{2f-2k-1}}\right) \leq 2 \exp\left(-\frac{\varepsilon^4 p_j^{2a-1} n}{\ell}\right).$$

Finally, the at most  $n^{f-k-1/3}$  potential  $R \in \mathcal{F}$  that contain an edge of colour  $i \leq t$ , not incident to  $U$ , may change the value of  $L$  by at most  $\frac{\varepsilon}{2} p_j^h n^{f-k}$ .  $\square$

### 3. APPROXIMATE DECOMPOSITIONS INTO NEAR-SPANNING STRUCTURES

**3.1. Proof of Theorem 1.4.** The proof of this theorem is based on an application of Lemma 2.14. It suffices to carry out some preprocessing and to verify that the conditions on the graph and the colouring are fulfilled.

If  $h = 0$ , then there is nothing to prove, so we assume  $h \geq 1$ . If  $f \leq 2$  (and thus  $F$  is an edge), then we replace  $F$  by two disjoint edges, so we may assume that  $f \geq 3$ .

Choose  $\eta, \zeta, \varepsilon, \delta$  and  $n_0$  such that  $0 < 1/n_0 \ll \eta \ll \zeta \ll \varepsilon \ll \delta \ll \alpha, d_0, 1/f, 1/h$ . Consider  $G'$  as in the statement of the theorem. Let  $V := V(G')$  and define  $a, t \in \mathbb{N}$  by

$$a := \frac{2hd^{h-1}n^{f-2}}{|\text{Aut}(F)|} \quad \text{and} \quad t := \frac{fdn}{2h}. \quad (3.1)$$

Note that  $a \geq \eta n$  as  $f \geq 3$ .

By Lemma 2.15, there is a  $(\zeta, d)$ -quasirandom subgraph  $G$  of  $G'$  which satisfies (2.8). Lemma 2.9 then implies that, for all  $v \in V$  and  $e \in E(G)$ , we have

$$r_G(F, v) = (1 \pm \varepsilon/3)t \cdot a \quad \text{and} \quad r_G(F, e) = (1 \pm \varepsilon/3)a. \quad (3.2)$$

We claim that we can apply Lemma 2.14 with  $\mathcal{F}$  being all rainbow copies of  $F$  contained in  $G$  and  $\alpha/2$  playing the role of  $\delta$ . Equations (3.1) and (3.2) imply that conditions (A3)<sub>2.14</sub> and (A4)<sub>2.14</sub> are satisfied (since  $a \geq \eta n$ ). Condition (A2)<sub>2.14</sub> is satisfied since the number of copies of  $F$  containing colour  $i$  is at most

$$e(G(\phi, i)) \cdot \max_{e \in G} r_G(F, e) \leq (1 + \eta) \frac{fdn}{2h} \cdot (1 + \varepsilon/3)a \leq (1 + \varepsilon/2)ta.$$

To verify the codegree conditions (A1)<sub>2.14</sub>, first note that for each  $u \neq v \in V$ , there are at most  $f!n^{f-2}$  copies of  $F$  containing both  $u$  and  $v$ , so we have

$$|R_G(F, u) \cap R_G(F, v)| \leq f!n^{f-2}. \quad (3.3)$$

For  $c \neq c' \in [m]$ , we have

$$\begin{aligned} |\{\bar{F} \in R_G(F) : \{c, c'\} \subseteq \phi(E(\bar{F}))\}| &\leq \sum_{uv \in E(G(\phi, c))} (d_{G(\phi, c')}(u) + d_{G(\phi, c')}(v)) f!n^{f-3} \\ + \sum_{uv \in E(G(\phi, c'))} \sum_{u'v' \in E(G(\phi, c) - \{u, v\})} f!n^{f-4} &\leq fn \cdot 2\eta n \cdot f!n^{f-3} + (fn)^2 \cdot f!n^{f-4} \leq \eta^{1/2} n^{f-1}. \end{aligned} \quad (3.4)$$

Similarly, one can obtain that for  $c \in [m]$  and  $v \in V$  one has

$$R_G(F, v) \cap \{\bar{F} : c \in \phi(\bar{F})\} \leq \eta^{1/2} n^{f-1}. \quad (3.5)$$

Thus (A1)<sub>2.14</sub> holds. Finally, for any two edges  $e_1, e_2 \in E(G)$  we have  $r_G(F, e_1 \cup e_2) \leq h^2 n^{f-3}$  and thus (A5)<sub>2.14</sub> is satisfied.

Therefore, since  $\eta, \varepsilon \ll \alpha$  the conditions of Lemma 2.14 are satisfied, we obtain the desired  $\alpha$ -decomposition into rainbow  $\alpha$ -spanning  $F$ -factors.

**3.2. Proof of Theorem 1.5.** Let us first present a sketch of the proof.

- We start with splitting the graph into two smaller parts  $V_1, V_2$  and one larger part  $V_3$ . Then we split the colours into a smaller part  $I_1$  and a larger part  $I_2$ . We make sure that most of the vertices and colours “behave sufficiently nicely”: the graphs between the parts are  $\varepsilon$ -regular, the graphs inside the parts are quasirandom, and each colour appears roughly the “expected” number of times between and inside the parts (cf. Claim 3.1, (3.6) and (3.7)). We restrict our attention only to the colours and vertices that “behave nicely”.
- Using Theorem 1.4, we find an approximate decomposition of  $V_3$  into rainbow almost-spanning factors consisting of long cycles using only the colours from  $I_2$ .
- For each cycle in each of these almost-spanning factors, we randomly select a “special” edge and remove it. The endpoints of these edges will be used to glue the cycles together into one long cycle. Again, we restrict our attention to cycles and colours that “behave sufficiently nicely”: we discard all colours that appear “unexpectedly” many times between the endpoints of the “special” edges and the parts  $V_1, V_2$ , as well as all the cycles containing vertices of too high degree in these “bad” colours.
- Finally, we apply Lemma 2.10 using “good” colours from  $I_1$  to link up the endvertices of the removed edges via  $V_1$  and  $V_2$ . The fact that in the previous step we removed the “special” edges randomly guarantees us that we will be able to successively perform the connecting step for all the almost-spanning factors without causing the graph on  $V_1, V_2$  and the colours in  $I_1$  “deteriorate” too much during this process.

Let us now make this precise. We choose auxiliary constants according to the hierarchy

$$1/n_0 \ll \eta \ll \zeta \ll \zeta_1 \ll \varepsilon \ll \varepsilon_1 \ll 1/s \ll \delta \ll \delta_1 \ll \delta_2 \ll \gamma \ll \beta \ll \alpha, d_0.$$

Take a graph  $G''$  and an  $m$ -colouring  $\phi$  of  $G''$  satisfying the conditions of the theorem. Apply Lemma 2.15 to  $G''$  with probability distribution  $(q_1, q_2, q_3) = (\delta_1, \delta_1, 1 - \gamma)$ , to obtain a  $(\zeta, d)$ -quasirandom spanning subgraph  $G' \subseteq G''$ , which, for any  $\mathcal{V}'' := (V_1'', V_2'', V_3'')$  chosen according to the above probability distribution, satisfies properties (A1)2.15–(A4)2.15 with probability at least 0.9.

Using Lemma 2.16 for each colour  $c$  and the graph  $G'(\phi, c)$ , we conclude that with probability at least  $1 - 18e^{-\zeta^3 \delta_1 / \eta} \geq 1 - \eta$  we have

$$\begin{aligned} e(G'[V_1'', V_2''](\phi, c)) &\leq 2\delta_1^2 e(G'(\phi, c)) + \zeta \delta_1^2 n \\ &\leq 3\delta_1^2 n. \end{aligned} \quad (3.6)$$

Note that the number of colours of size at least  $3\delta_1^2 n$  in the *original* colouring  $\phi$  is at most  $e(G'')/(3\delta_1^2 n) \leq \eta^{-1/3} n$ . Using Markov's inequality, we also conclude that with probability at least 0.99 at most an  $\eta^{2/3}$ -fraction of these does not satisfy (3.6), so altogether (3.6) holds for all but  $\eta^{1/3} n$  colours. Similarly,

$$\begin{aligned} e(G'[V_3''](\phi, c)) &\leq (1 - \gamma)^2 e(G'(\phi, c)) + \zeta^2 (1 - \gamma)^2 n \\ &\leq \frac{1}{2} (1 + \zeta) (1 - \gamma)^2 d n \end{aligned} \quad (3.7)$$

holds for all but  $\eta^{1/3} n$  colours with probability 0.99.

Choose  $\mathcal{V}'' = (V_1'', V_2'', V_3'')$  which satisfies conditions (A1)2.15–(A4)2.15, as well as (3.6), (3.7) for all colours apart from a set  $EC$  of at most  $\eta^{1/4} n$  “exceptional” colours.

Let  $EV$  be the set of all those vertices  $v$  with  $d_{G'(\phi, EC)}(v) \geq \zeta n$ . Clearly,  $|EV| \leq \zeta n$ . Put  $G^* := (G'[V_1'', V_2'', V_3''] \setminus EV) - e(G'(\phi, EC))$  and  $V_j' := V_j'' \setminus EV$  for  $j \in [3]$ . By Proposition 2.4 (ii), we can find  $V_j' \subseteq V_j'' \setminus EV$  with  $|V_j'| \geq (1 - \zeta_1)|V_j''|$  such that

$$G := G^*[V_1', V_2', V_3'] \text{ satisfies (A1)2.15–(A4)2.15 with } V_j', \zeta_1 \text{ playing the roles of } V_j, \zeta, \quad (3.8)$$

as well as (3.6) and (3.7) for *all* colours present in the colouring of  $G$ . We also note that  $G$  has at least a  $(1 - \gamma^{1/2})$ -fraction of vertices and edges of  $G''$ , therefore, an approximate decomposition into almost-spanning cycles for  $G$  would be an approximate decomposition into almost-spanning cycles for the initial graph  $G''$ .

**Claim 3.1.** *A partition  $\mathcal{I} := (I_1, I_2)$  of the colours from  $[m] \setminus EC$ , chosen at random with probability distribution  $(p_1, p_2) = (\gamma, 1 - \gamma)$ , with probability at least 0.9 satisfies the following. There exist subsets  $V_i \subseteq V_i'$  for  $i \in [3]$ , such that*

(V1)3.1  $|V_i| \geq (1 - \zeta_1)|V_i'|.$

(V2)3.1 *The graph  $G[V_1, V_2](\phi, I_1)$  is  $(\varepsilon^{1/6}, \gamma d)$ -regular.*

(V3)3.1 *For each vertex  $v \in V_3$  and  $i \in [2]$  we have  $d_{G(\phi, I_1), V_i}(v) = (1 \pm \varepsilon)\gamma d|V_i|.$*

(V4)3.1 *The graph  $G[V_3](\phi, I_2)$  is  $(\varepsilon, (1 - \gamma)d)$ -quasirandom.*

*Proof.* Consider a partition  $\mathcal{I} := (I_1, I_2)$  of the colours as in the claim. Apply Lemma 2.17 with partition  $\mathcal{I}$  for each vertex  $v \in V(G)$ , where  $\varepsilon^2$  plays the role of  $\varepsilon$ ,  $U := \{v\}$  and  $\mathcal{F}$  is simply the collection of all edges in  $G$  from  $v$  to  $V_i'$ . Keeping in mind that, by (3.8),  $G$  is  $(\zeta_1, d)$ -superregular between the parts and  $(\zeta_1, d)$ -quasirandom inside the parts, we conclude that for all  $j \in [2]$  and  $i \in [3]$  we have

$$d_{G(\phi, I_j), V_i'}(v) = (1 \pm \varepsilon/2)p_j d|V_i'| \quad (3.9)$$

with probability at least  $1 - 12 \exp(-\frac{\varepsilon^8 \gamma}{\eta}) > 1 - \eta$ . Using Markov's inequality, with probability at least 0.99 the number of vertices not satisfying (3.9) is at most  $\eta^{1/2} n$ . Delete these vertices, obtaining sets  $V_i \subseteq V_i'$ ,  $i \in [3]$ . Note that they satisfy (V1)3.1, and that the condition (V3)3.1 is fulfilled as well. (Indeed,  $d_{G(\phi, I_j), V_i}(v) = (1 \pm \varepsilon/2)p_j d|V_i'| \pm \eta^{1/2} n = (1 \pm \varepsilon)p_j d|V_i|.$ )

Fix  $i_1, i_2 \in [3]$ . Since  $G$  satisfies (A2)2.15 with  $\zeta_1$  playing the role of  $\zeta$ , it follows that the total number of pairs of vertices  $u, v \in V_{i_1}'$ , for which  $d_{G, V_{i_2}'}(u, v) \neq (d^2 \pm \zeta_1)|V_{i_2}'|$  is at most  $\zeta_1 n^2$ .

Moreover, the total number of pairs  $u, v$ , which have more than  $\eta^{1/2}n$  monochromatic paths  $P_2$  with ends in  $v$  and  $u$  is at most  $\eta^{1/2}n^2$  by Lemma 2.7. Consider any pair of vertices  $u, v \in V_{i_1}$  which does not belong to either of these two sets  $EP_1$  and  $EP_2$  of “exceptional” pairs. Then we conclude that the number of rainbow (i.e. two-coloured)  $P_2$  with ends in  $u$  and  $v$  and middle vertex in  $V_{i_2}$  is  $(d^2 \pm 3\zeta_1)|V_{i_2}|$ . Here we both used that  $|V_{i_2}| \geq (1 - \zeta_1)|V'_{i_2}|$  and that all but  $\eta^{1/2}n$  copies of  $P_2$  with ends in  $v, w$  are rainbow. Apply Lemma 2.17 with  $U := \{u, v\}$ ,  $\varepsilon^2$  playing the role of  $\varepsilon$  and  $\mathcal{F}$  being a collection of rainbow  $P_2$  with ends in  $u$  and  $v$  and middle vertex in  $V_{i_2}$ . We conclude that for each  $j \in [2]$  the number of rainbow  $P_2$  which end in  $u, v$ , have their middle vertex in  $V_{i_2}$  and are coloured with colours from  $I_j$  is

$$(1 \pm \varepsilon/3)(d^2 \pm 3\zeta_1)p_j^2|V_{i_2}| = (1 \pm \varepsilon/2)d^2p_j^2|V_{i_2}| \quad (3.10)$$

with probability  $1 - 4\exp(-\frac{\varepsilon^8\gamma^3}{\eta}) > 1 - \eta$ . Using Markov’s inequality, with probability 0.99 the number of pairs  $(u, v) \notin EP_1 \cup EP_2$  violating (3.10) is at most  $\eta^{1/2}n^2$ . For any pair  $u, v \in V_{i_1}$  not belonging to the set  $EP_3$  of these “exceptional” pairs we have

$$d_{G(\phi, I_j), V_{i_2}}(u, v) = (1 \pm \varepsilon/2)d^2p_j^2|V_{i_2}| \pm \eta^{1/2}n = (1 \pm \varepsilon)d^2p_j^2|V_{i_2}|. \quad (3.11)$$

Proceed in a similar way for all choices of  $i_1, i_2 \in [3]$ . Then the union of the sets  $EP_1 \cup EP_2 \cup EP_3$  of all exceptional pairs (taken over all choices of  $i_1, i_2 \in [3]$ ) has size at most  $9\zeta_1n^2 + 9\eta^{1/2}n^2 + 9\eta^{1/2}n^2 \leq \varepsilon^2\delta_1^2n^2$ . In particular, we may conclude that all but at most an  $\varepsilon$ -proportion of pairs in  $V_3$  satisfy (3.11) with  $j = 2$  and  $i_2 = 3$ . Together with (3.9) this implies that  $G[V_3](\phi, I_2)$  is  $(\varepsilon, (1 - \gamma)d)$ -quasirandom, i.e., (V4)<sub>3,1</sub> holds. Similarly, by Theorem 2.5, property (V2)<sub>3,1</sub> is satisfied.  $\square$

After this preprocessing step, we are ready to proceed with the construction of our almost-decomposition. First, apply Theorem 1.4 to  $G[V_3](\phi, I_2)$  for  $F := C_s$  and with  $3\varepsilon, \beta$  playing the roles of  $\eta, \alpha$  (recall that  $\varepsilon \ll 1/s \ll \delta \ll \beta$ ). Indeed, to see that we can apply Theorem 1.4, first note that the colouring on  $G[V_3](\phi, I_2)$  is locally  $\varepsilon|V_3|$ -bounded since  $\eta n \leq \varepsilon|V_3|$ . Moreover, due to (3.7), it is  $\frac{1}{2}(1 + \zeta)(1 - \gamma)^2dn \leq \frac{1}{2}(1 + \varepsilon)(1 - \gamma)d|V_3|$ -bounded. As a result, we obtain a  $\beta$ -decomposition of  $G[V_3](\phi, I_2)$  into rainbow  $\beta$ -spanning  $C_s$ -factors. Denote by  $\mathcal{L}'_i$  the  $i$ -th factor from this decomposition, and let  $n_1$  be their total number. By deleting some cycles if necessary, we may assume that each factor includes the same number  $n'_2$  of copies of  $C_s$ , where  $n'_2 \geq (1 - 2\beta)\frac{n}{s}$ . That is,  $\mathcal{L}'_i := \bigcup_{j=1}^{n'_2} C_i^j$ , where  $C_i^j$  are the  $s$ -cycles forming  $\mathcal{L}'_i$ . Thus

$$|V(\mathcal{L}'_i)| \geq (1 - 2\beta)n \quad \text{for each } i \in [n_1]. \quad (3.12)$$

The union of all the  $\mathcal{L}'_i$  covers all but a  $4\beta$ -fraction of the edges of the initial graph  $G''$ .

The last step of the proof is to combine (most of) the cycles in each  $\mathcal{L}'_i$  into one large cycle using the vertices from  $V_1, V_2$  and the colours from  $I_1$ . For all  $i \in [n_1]$  and  $j \in [n'_2]$  select an edge  $e_i^j = x_i^j y_i^j$  in  $C_i^j$  independently uniformly at random. Put  $U'_i := \bigcup_{j=1}^{n'_2} \{x_i^j, y_i^j\}$ . We claim that the following two properties have non-zero probability to be satisfied simultaneously:

- A** Each vertex  $v \in V_3$  belongs to at most  $\delta n$  selected edges.
- B** For each  $i \in [n_1]$  define  $I^i$  to be the set of colours  $c \in I_1$  such that  $e(G[V_1 \cup V_2, U'_i](\phi, c)) > \delta n$ . Then  $e(G[V_1 \cup V_2, U'_i](\phi, I^i)) \leq \delta e(G[V_1 \cup V_2, U'_i](\phi, I_1))$ , as well as  $e(G(\phi, I^i)) \leq \delta n^2$ .

Let us verify this claim. For each  $i \in [n_1]$ , any given  $v \in V_3$  belongs to at most one  $C_i^j$ , and thus it belongs to the corresponding  $e_i^j$  with probability at most  $2/s$ . Using Lemma 2.1 and a union bound, the probability that the property **A** does not hold is at most  $2ne^{-\delta n} \leq e^{-\delta n/2}$ .

Since  $e(G[V_1 \cup V_2, U'_i]) \leq \frac{5\delta_1 n^2}{s}$  by (A1)<sub>2,15</sub>, the definition of  $I^i$  implies that  $|I^i| \leq \delta n$ , and so in particular  $e(G(\phi, I^i)) \leq \delta n^2$ . For any fixed  $i \in [n_1]$  and a colour  $c \in I_1$ , the expected value of  $e(G[V_1 \cup V_2, U'_i](\phi, c))$  is at most  $2n/s$ , and so by Markov’s inequality,  $c \in I^i$  with probability at most  $\frac{2}{\delta s} < \delta^3$ . Therefore, the expected number of edges in  $G[V_1 \cup V_2, U'_i]$  having colours from  $I^i$  is at most  $\delta^3 e(G[V_1 \cup V_2, U'_i](\phi, I_1))$ . Using Markov’s inequality again, with probability at least  $1 - \delta^2$  the number of such edges is at most  $\delta e(G[V_1 \cup V_2, U'_i](\phi, I_1))$ . Combining this bound for different values  $i$ , we obtain that property **B** is satisfied with probability at least  $(1 - \delta^2)^{n_1} > e^{-\delta n/3}$ . Therefore, with positive probability both **A** and **B** are satisfied. Fix a choice of edges satisfying both **A** and **B** simultaneously.



For each  $i \in [n_1]$ , define  $J_i \subseteq [n'_2]$  to be the set of indices such that  $j \in J_i$  if and only if at least a  $\delta_1$ -proportion of edges in  $G[V_1 \cup V_2, z](\phi, I_1)$  are coloured in colours from  $I^i$  for at least one  $z \in \{x_i^j, y_i^j\}$ . Due to **B** and (V3)<sub>3.1</sub>, we have

$$|J_i| \leq \delta_1 n'_2. \quad (3.13)$$

By (3.12) we have

$$\left| \bigcup_{j \in [n'_2] \setminus J_i} V(C_i^j) \right| \geq (1 - 3\beta)n. \quad (3.14)$$

By disregarding some cycles if necessary, we assume that the  $J_i$  have the same cardinality for any  $i$  and that the cycles are ordered in such a way that  $[n'_2] \setminus J_i = [n_2]$  for some  $n_2 < n'_2$ . Put  $U_i := \bigcup_{j=1}^{n_2} \{x_i^j, y_i^j\}$ . Then the following holds.

**C** For all  $i \in [n_1]$ ,  $z \in U_i$  and  $q \in [2]$  we have  $d_{G(\phi, I^i), V_q}(z) \leq \delta_1 d_{G(\phi, I_1), V_1 \cup V_2}(z) \stackrel{\text{(V3)}_{3.1}}{\leq} 5\delta_1 \gamma d |V_q|$ .

Finally, we are ready to apply Lemma 2.10. For each  $i \in [n_1]$  in turn, we define a mapping  $\psi_i$  as follows. Consider the graph  $H$  consisting of  $n_2$  vertex-disjoint copies of  $P_3$ , say,  $H := \bigcup_{j=1}^{n_2} P_3^j$ , where  $P_3^j := w_0^j w_1^j w_2^j w_3^j$  is a path of length 3. Consider the partition  $\mathcal{X} = (X_0, X_1, X_2)$  of  $V(H)$ , where  $X_q := \{w_q^j : j \in [n_2]\}$  for  $q = 1, 2$ , and  $X_0 := \{w_0^j, w_3^j : j \in [n_2]\}$ . Consider the map  $\psi'_i : X_0 \rightarrow U_i$ , defined by  $\psi'_i(w_0^j) = x_i^j$  and  $\psi'_i(w_3^j) = y_i^{j+1}$ , with indices taken modulo  $n_2$ . Assume that we have already defined  $\psi_1, \dots, \psi_{i-1}$ . Then, provided that the necessary conditions hold, Lemma 2.10 will guarantee

an embedding  $\psi_i$  of  $H$  into  $G_i := G[V_1, V_2, U_i](\phi, I_1 \setminus I^i) - \bigcup_{k=1}^{i-1} E(\psi_k(H))$ ,

extending  $\psi'_i$ , such that  $\psi_i(X_q) \subseteq V_q$  for  $q \in [2]$  and  $\psi_i(H)$  is rainbow. By (3.14), for each  $i \in [n_1]$ , the union of  $\psi_i(H)$  and all the  $C_i^j - e_i^j$  with  $j \in [n_2]$  gives us a rainbow cycle of length at least  $(1 - 3\beta)n$ . These cycles are edge-disjoint and altogether provide us with an  $\alpha$ -decomposition of  $G''$  (by (3.13) and since the  $\mathcal{L}'_i$  form a  $4\beta$ -decomposition of  $G''$ ). Thus, it only remains to verify the conditions of Lemma 2.10 in order to complete the proof of the theorem. (For this, we will make use of (3.8).) We apply Lemma 2.10 with  $G_i, \delta_1 n$  playing the roles of  $G, n$ .

- Due to (3.6) and the definition of  $I^i$  in **B**, the colouring of  $G_i$  is  $3\delta_1^2 n + \delta n \leq \delta_2 \delta_1 n$ -bounded.
- (A1)<sub>2.10</sub> is clear from the definition of  $H$ , with  $\Delta = 2$ .
- (A2)<sub>2.10</sub> is satisfied since  $|X_0| = |U_i| \leq 2n/s \leq \delta_2 \delta_1 n$ .
- (A3)<sub>2.10</sub> is satisfied in the form needed to apply the “moreover part” of Lemma 2.10 since  $|X_j| = \frac{1}{2}|U_i| < \delta_2 |V_j|$  for  $j \in [2]$  and

$$|V_j| \stackrel{\text{(V1)}_{3.1}}{=} (1 \pm \zeta_1) |V'_j| \stackrel{\text{(3.8)}}{=} (1 \pm \zeta_1)(1 \pm \zeta_1) \delta_1 n = (1 \pm \delta_2) \delta_1 n.$$

- Due to  $e(H) \leq 3n/s \leq \delta n$  and property **B**, the graph  $\bigcup_{k=1}^{i-1} \psi_k(H) \cup G(\phi, I^i)$  contains at most  $2\delta n^2$  edges. Assertion (V2)<sub>3.1</sub> together with Proposition 2.4 imply that  $G_i[V_1, V_2]$  is  $(\delta_2, \gamma d)$ -regular (note here that  $\delta_2 \ll \gamma$ ). Therefore, (A4)<sub>2.10</sub> is satisfied with the prefix “super” omitted, which is sufficient to apply the “moreover part” of Lemma 2.10.
- Finally, (A5)<sub>2.10</sub> is satisfied due to (V3)<sub>3.1</sub>, combined with **A** and **C**.

#### 4. SPANNING STRUCTURES

**4.1. Proof of Theorem 1.6. Almost-decomposition into  $F$ -factors.** Let us first give a sketch of the proof.

- We split the vertex set  $V$  of  $G$  into  $b = O(\log n)$  equal size parts  $U_i$ , and, using Lemma 2.15, we have that the pairs of parts induce superregular pairs. We ignore the edges inside each  $U_i$  since the number of these is negligible.

- Using the result on resolvable designs (Theorem 2.12), we split the collection of  $U_i$  into groups of  $f$  parts, such that each pair of parts belongs to exactly one group, and the set of all groups has a partition into layers with each layer covering all  $U_i$  exactly once. In other words, we consider a decomposition of the complete graph  $K_b$  into  $K_f$ -factors. Next, we aim to translate this into an almost-decomposition of  $G$ . Each  $K_f$  from the decomposition will correspond to an  $f$ -partite graph  $G_i^j \subseteq G$  between the corresponding parts.
- We randomly split the colours as follows. We set aside a small proportion of colours  $I_{q+1}$ , and the remaining ones we split into  $q := b/f$  groups of roughly equal size. By Lemma 2.17, in each of the colour groups and for each of the  $G_i^j$  the count of rainbow copies of  $F$  is “correct”. Thus, we can apply Lemma 2.14 and obtain an approximate decomposition into rainbow almost-spanning  $F$ -factors of each of  $G_i^j$  in each of the colour groups.
- Next we combine the rainbow almost-spanning  $F$ -factors in all the  $G_i^j$  into rainbow almost-spanning  $F$ -factors in  $G$ .
- Finally, we transform each such almost-spanning  $F$ -factor  $\mathcal{D}$  into an  $F$ -factor as follows. Let  $V_2$  be the set of vertices not covered by  $\mathcal{D}$ . Since we have little control over  $V_2$ , we also consider a set of vertices  $V_1$  (depending on  $\mathcal{D}$ ) which is the union of several  $V(G_i^j)$ , where these  $i, j$  are chosen in a way that, over all  $F$ -factors, each  $V(G_i^j)$  is used roughly the same number of times by the sets  $V_1$ . Moreover, we will have  $|V_2| \ll |V_1| \ll n$ . We discard  $\mathcal{D}[V_1]$  and then apply the rainbow blow-up lemma to obtain a rainbow spanning factor on  $V_1 \cup V_2$  in colours from  $I_{q+1}$ . Combined with  $\mathcal{D}[V \setminus (V_1 \cup V_2)]$ , this gives a rainbow  $F$ -factor in  $G$ . Altogether, these  $F$ -factors form the desired approximate decomposition.

The main challenge in the final step is to carry this out in such a way that the conditions of the rainbow blow-up lemma (Theorem 2.10) are satisfied in each successive application. In particular, we need to show that the colouring is well-bounded and that in each iteration, the vertex degrees are not affected too much. This is the main reason why we need the almost-spanning factors to be distributed randomly in the assertion of Lemma 2.14. This guarantees that, when we combine almost-spanning factors from different  $G_i^j$ , the vertices that are left out will “behave nicely” with respect to each colour in  $I_{q+1}$ , in particular, the edges of each colour from  $I_{q+1}$  will appear roughly the correct number of times. The degrees of the vertices are not affected too much since none is used too many times in some  $V_1$  and since the random choice of the approximate decompositions of  $G_i^j$  does not allow a vertex to appear too many times in some  $V_2$ .

Let us make this precise. If  $h = 0$ , then there is nothing to prove, so we assume  $h \geq 1$ . If  $f \leq 2$  (and thus  $F$  is an edge), then we replace  $F$  with two disjoint edges, so we may assume that  $f \geq 3$ . We remark that this does not change the value of  $a$  from the statement. However, this affects the divisibility conditions (if  $n$  is divisible by 2, but not by 4), but this problem is easy to fix, and we will come back to it when applying the rainbow blow-up lemma.

We choose auxiliary constants according to the hierarchy

$$0 < 1/n_0 \ll \eta \ll \zeta \ll \zeta_1 \ll \varepsilon \ll \delta \ll \delta_1 \ll \delta_2 \ll \gamma \ll \alpha, d_0 \quad \text{and} \quad \delta_2 \ll 1/f, 1/h. \quad (4.1)$$

Let  $G'$  be a graph with an  $m$ -colouring  $\phi$  as in the formulation of Theorem 1.6. Let us note that for any  $c \in [m]$

$$e(G'(\phi, c)) \leq \frac{f}{h}n. \quad (4.2)$$

Without loss of generality, assume that the number  $b' := \eta^{-1/3a} \log n$  is an integer, and define integers

$$b := f(f-1)b' + f, \quad g := (b-1)/(f-1), \quad q := b/f.$$

Apply Lemma 2.15 to  $G'$  to obtain a  $(\zeta, d)$ -quasirandom spanning subgraph  $G$  of  $G'$  such that a random partition  $\mathcal{U} := (U_1, \dots, U_b)$  of  $V(G')$  chosen with probability distribution  $(1/b, \dots, 1/b)$  satisfies the following with probability at least 0.9.

(U1) For each  $i \in [b]$ , we have  $|U_i| = (1 \pm \zeta)n/b$ .

(U2) For all  $i \neq j \in [b]$ , the bipartite graph  $G[U_i, U_j]$  is  $(\zeta, d)$ -superregular.

(U3) For all  $vw \in E(G)$  and  $i \in [b]$ , we have  $|C_G^\phi(v, w) \cap U_i| \leq \zeta|U_i|$  and  $d_{G, U_i}(v, w) = (d^2 \pm \zeta)|U_i|$ .

A Chernoff estimate also shows that the following holds with probability at least 0.9.

(U4) For each  $i, j \in [b]$  the colouring  $\phi$  of  $G[U_i, U_j]$  is locally  $(1 + 2\zeta)\eta^{\frac{n/b}{\log^{2a} n}}$ -bounded.

Next apply Lemma 2.16 to  $G(\phi, c)$  with a random partition  $\mathcal{U}$  as above and with  $\eta^{1/8}$  playing the role of  $\zeta$  to obtain that

$$e(G[U_i, U_j](\phi, c)) = \frac{2}{b^2}e(G(\phi, c)) \pm \frac{\zeta}{b^2}n \quad (4.3)$$

holds with probability at least 0.9 for every  $c \in [m]$  and  $i, j \in [b]$ . Fix one such partition  $\mathcal{U}$  satisfying (U1)–(U4) as well as (4.3).

Due to the choice of  $b$ , we can apply Theorem 2.12 with  $f, b'$  playing the roles of  $r, b'$  and  $\rho = 1$ . Let  $\mathcal{L}_1, \dots, \mathcal{L}_g$  be the perfect  $f$ -matchings on  $[b]$  thus obtained, and, for each  $i \in [g]$ , write  $\mathcal{L}_i = \{L_i^j : j \in [q]\}$ .

For each  $f$ -tuple  $L_i^j =: \{i'_1, \dots, i'_f\}$  with  $i \in [g]$  and  $j \in [q]$ , let  $G_i^j := G[U_{i'_1}, \dots, U_{i'_f}]$ . Next we apply Lemma 2.8 with  $G_i^j$  playing the role of  $G$  (note that the assertions (U1), (U2), (U3) immediately imply the conditions (A1)<sub>2.8</sub>, (A2)<sub>2.8</sub>, (A3)<sub>2.8</sub>, respectively, and (4.3) guarantees the required boundedness of the colouring, while (U4) implies the local boundedness of the colouring). We apply Lemma 2.8 to  $F$  with the “trivial” partition of  $V(F)$  into parts of size 1 and to  $G_i^j$  with its natural partition (these are the only partitions we use in what follows, so, by abuse of notation, we will not specify them in the notation). We obtain that for each  $v \in V(G_i^j)$  and  $uw \in E(G_i^j)$

$$r_{G_i^j}(F, v) \geq \frac{1}{2}d^h(n/b)^{f-1} \quad \text{and} \quad (4.4)$$

$$\frac{r_{G_i^j}(F, v)}{r_{G_i^j}(F, uw)} = (1 \pm 2\varepsilon)\frac{\binom{f}{2}dn}{bh} = (1 \pm 3\varepsilon)\frac{f|E(G_i^j)|}{h|V(G_i^j)|}. \quad (4.5)$$

The final equality holds since (U1) and (U2) imply that the average degree in  $G_i^j$  is  $(1 \pm 3\varepsilon)\frac{dn(f-1)}{b}$ .

Consider a random partition  $\mathcal{I} := (I_1, \dots, I_{q+1})$  of colours chosen with probability distribution  $(p_1, \dots, p_{q+1}) := (\frac{1-\gamma}{q}, \dots, \frac{1-\gamma}{q}, \gamma)$ . Note that the number of colour classes, excluding the last one, is equal to the number of  $f$ -tuples in each  $\mathcal{L}_i$ . For all  $i \in [g], j \in [q]$ , apply Lemma 2.17 to  $G_i^j$  with  $\emptyset, \varepsilon^2, |V(G_i^j)|, E(G_i^j)$  playing the roles of  $U, \varepsilon, n, \mathcal{F}$ . Together with (U4) this implies that with probability  $1 - o(1)$  for all  $j, r \in [q], i \in [g]$  we have

$$|E(G_i^j(\phi, I_r))| = (1 \pm \varepsilon/2)p_r|E(G_i^j)|. \quad (4.6)$$

Next, for each  $i \in [g], j \in [q]$  and  $v \in V(G_i^j)$ , we apply Lemma 2.17 to  $G_i^j$  with  $\{v\}, \varepsilon^2, \mathcal{F}$  playing the roles of  $U, \varepsilon, R_{G_i^j}(F, v)$ . Next, for each  $uw \in E(G_i^j)$  we apply the lemma with  $\{u, w\}, \varepsilon^2, \{F' - uw : F' \in R_{G_i^j}(F, uw)\}$  playing the roles of  $U, \varepsilon, \mathcal{F}$ . Finally, for any  $u'w_1, u'w_2 \in E(G_i^j)$  we apply the lemma with  $\{u', w_1, w_2\}, \varepsilon, \{F' - u'w_1 - u'w_2 - w_1w_2 : F' \in R_{G_i^j}(F, u'w_1 \cup u'w_2)\}$  playing the roles of  $U, \varepsilon, \mathcal{F}$ . We claim that with probability  $1 - n^{-1}$  for all  $i \in [g], j \in [q], r \in [q]$ , all vertices  $v \in V(G_i^j)$  and edges  $uw, u'w_1, u'w_2 \in E(G_i^j(\phi, I_r))$  we have

$$r_{G_i^j(\phi, I_r)}(F, v) \geq \frac{1}{4}d^h p_r^h (n/b)^{f-1}, \quad (4.7)$$

$$\frac{r_{G_i^j(\phi, I_r)}(F, v)}{r_{G_i^j(\phi, I_r)}(F, uw)} = (1 \pm 4\varepsilon)\frac{p_r f |E(G_i^j)|}{h|V(G_i^j)|} \stackrel{(4.6)}{=} (1 \pm 5\varepsilon)\frac{f|E(G_i^j(\phi, I_r))|}{h|V(G_i^j)|}, \quad (4.8)$$

$$r_{G_i^j(\phi, I_r)}(F, u'w_1 \cup u'w_2) \leq 2f!p_r^{h-3}(n/q)^{f-3}. \quad (4.9)$$

Indeed, to see (4.7) and (4.8), note that in (4.7) we combined (4.4) with the conclusion of Lemma 2.17, while in (4.8) we combined (4.5) and the conclusions of Lemma 2.17 (obtained from fixing  $v$  and then  $uw$ ). To see (4.9), we first use the trivial bound  $r_{G_i^j}(F, u'w_1 \cup u'w_2) \leq \frac{3}{2}f!(n/q)^{f-3}$  and then apply Lemma 2.17. Let us check that a union bound allows us to arrive to the desired conclusion. First note that the maximum number of edges incident to  $U$  in the applications of Lemma 2.17 is bounded by  $a$ . Using (U4), the probability that (4.7)–(4.9) hold for fixed  $i, j, r, v, u'w_1, u'w_2$  and

$uw$  is at least

$$1 - 2 \exp\left(-\frac{\varepsilon^9 p_r^{2a-1} n}{2\eta \frac{n}{b} \log^{-2a} n}\right) \geq 1 - \exp(-\eta^{-1/3} \log n) \geq 1 - n^{-10}.$$

Thus, taking a union bound over all possible choices of  $i, j, r$  and  $v$  as well as  $uv, u'w_1, u'w_2$ , we conclude that (4.7)–(4.9) hold for all such choices simultaneously with probability at least  $1 - n^{-1}$ .

Moreover, adapting the proof of (V2)<sub>3.1</sub> to our setting, we have

$$G[U_i, U_j](\phi, I_{q+1}) \quad \text{is } (\varepsilon^{1/6}, \gamma d)\text{-superregular for any } i \neq j \in [b] \quad (4.10)$$

with probability  $1 - n^{-1}$ . From now on, we fix a colour partition  $\mathcal{I} = (I_1, \dots, I_{q+1})$  which satisfies (4.6)–(4.10) simultaneously.

For each  $G_i^j(\phi, I_r)$  with  $r \in [q]$ , we aim to apply Lemma 2.14 to the family  $\mathcal{F} := R_{G_i^j(\phi, I_r)}(F)$  with  $5\varepsilon, \delta/2$  playing the role of  $\varepsilon, \delta$  respectively. Condition (A3)<sub>2.14</sub> is satisfied due to (4.8), and (A4)<sub>2.14</sub> and (A5)<sub>2.14</sub> are satisfied due to (4.7), (4.8) and the fact that  $r_{G_i^j(\phi, I_r)}(F, uw \cup u'w') \leq h^2 n^{f-3}$ . Due to (4.3) and the boundedness of the colouring, for each colour  $c \in I_r$  we have

$$|E(G_i^j(\phi, c))| \leq \frac{f(f-1)(1-\alpha)\frac{fdn}{2h} \pm \zeta^{1/2}n}{b^2} \leq \frac{(1-2\gamma)\binom{f}{2}fdn}{b^2 h} \leq \frac{(1-\gamma)p_r\binom{f}{2}dn}{bh} \leq \frac{f|E(G_i^j(\phi, I_r))|}{h|V(G_i^j)|},$$

where the final inequality follows from the second equality in (4.5) and (4.6). Note that this is the only place where we make full use of the (global) boundedness condition on the colouring. Thus, for any  $v \in V(G_i^j)$  and  $c \in I_r$ , the number of rainbow copies of  $F$  in  $G_i^j(\phi, I_r)$  containing an edge of colour  $c$  is at most

$$|E(G_i^j(\phi, c))| \cdot \max_{uw \in E(G_i^j)} \{r_{G_i^j(\phi, I_r)}(F, uw)\} \stackrel{(4.8)}{\leq} \frac{1}{1-5\varepsilon} r_{G_i^j(\phi, I_r)}(F, v),$$

and condition (A2)<sub>2.14</sub> is satisfied. Finally, the verification of the codegree assumptions (A1)<sub>2.14</sub> uses (4.9) and can be done as in the proof of Theorem 1.4. We present only the calculation for the codegree of two colours  $c, c' \in [m]$ . Recall that due to (U4) the colouring of  $G_i^j$  is locally  $\ell$ -bounded with  $\ell := \eta^{1/4} p_r^3 n$ . Then, for  $c \neq c' \in [m]$  and  $w \in V(G_i^j)$ , we have

$$\begin{aligned} |\{\bar{F} \in R_{G_i^j(\phi, I_r)}(F) : \{c, c'\} \subseteq \phi(E(\bar{F}))\}| &\stackrel{(4.9)}{\leq} \sum_{uv \in E(G_i^j(\phi, c))} (d_{G_i^j(\phi, c')}(u) + d_{G_i^j(\phi, c')}(v)) 2f! p_r^{h-3} \left(\frac{n}{q}\right)^{f-3} \\ + \sum_{uv \in E(G_i^j(\phi, c'))} \sum_{u'v' \in E(G_i^j(\phi, c) - \{u, v\})} 2f! \left(\frac{n}{q}\right)^{f-4} &\stackrel{(4.3)}{\leq} 2f^3 \frac{n}{b^2} \cdot 2\ell \cdot 2f! p_r^{h-3} \left(\frac{n}{q}\right)^{f-3} + 2(fn)^2 \cdot f! \left(\frac{n}{q}\right)^{f-4} \\ &\leq \eta^{1/5} p_r^h \left(\frac{n}{b}\right)^{f-1} \stackrel{(4.7)}{<} \eta^{1/6} |R_{G_i^j(\phi, I_r)}(F, w)|. \end{aligned}$$

The other calculations can be done similarly.

Thus, we conclude that, for all  $i \in [g]$ ,  $j, r \in [q]$ , there is a randomized algorithm which returns a  $\delta$ -decomposition of  $G_i^j(\phi, I_r)$  into rainbow  $\delta/2$ -spanning  $F$ -factors, such that each  $v \in V(G_i^j)$  belongs to each factor with probability at least  $1 - \delta/2$ . By (U2) and (4.6) we may assume that the number of  $\delta/2$ -spanning  $F$ -factors in the  $\delta$ -decomposition of  $G_i^j(\phi, I_r)$  is the same for all  $i, j, r$ . We denote this number by  $n_\delta$ . For each  $i \in [g]$ ,  $j, r \in [q]$ , delete a randomly chosen collection of  $\frac{\delta|V(G_i^j)|}{3f}$  copies of  $F$  from each of the  $\delta/2$ -spanning  $F$ -factors of  $G_i^j(\phi, I_r)$ . Then the proportion of vertices of  $V(G_i^j)$  covered by each factor is at least  $1 - \delta$  and at most  $1 - \delta/3$ , and each  $v \in V(G_i^j)$  belongs to each factor with probability at least  $1 - \delta$ . Moreover, the factors clearly form a  $2\delta$ -decomposition of  $G_i^j(\phi, I_r)$ . For  $k' \in [n_\delta]$ ,  $j \in [q]$  and  $i \in [g]$ , let  $\mathcal{D}_i^j(k', I_r)$  denote the resulting  $k'$ -th  $\delta$ -spanning  $F$ -factor in this  $2\delta$ -decomposition of  $G_i^j(\phi, I_r)$ . Note that the total number of edges in the  $\mathcal{D}_i^j(k', I_r)$  over all  $i \in [g]$ ,  $j, r \in [q]$  and  $k' \in [n_\delta]$  is at least

$$\sum_{i \in [g]} \sum_{j, r \in [q]} n_\delta \cdot (1 - \delta) \frac{h}{f} |V(G_i^j)| = (1 - \delta) \frac{h}{f} n \sum_{i \in [g]} \sum_{r \in [q]} n_\delta = (1 - \delta) \frac{h}{f} n \cdot gq n_\delta,$$

but, on the other hand, is at most  $\binom{n}{2}$ , and therefore

$$gqn_\delta \leq \frac{f}{h}n. \quad (4.11)$$

Summarizing, these almost-spanning  $F$ -factors satisfy the following properties.

- a)** For all  $i \in [g]$ ,  $j, r \in [q]$ ,  $k' \in [n_\delta]$  and  $v \in V(G_i^j)$ , we have  $v \in \mathcal{D}_i^j(k', I_r)$  with probability at least  $1 - \delta$ .
- b)** For all  $i \in [g]$  and  $j_1, j_2, r_1, r_2 \in [q]$  with  $(j_1, r_1) \neq (j_2, r_2)$ , the random variables  $\mathcal{D}_i^{j_1}(k', I_{r_1})$  and  $\mathcal{D}_i^{j_2}(k', I_{r_2})$  are independent.

For all  $i \in [g]$ ,  $r \in [q]$ ,  $k' \in [n_\delta]$  put

$$\mathcal{D}_i(k' + (r-1)n_\delta) := \bigcup_{j=1}^q \mathcal{D}_i^j(k', I_{r+j}), \quad (4.12)$$

where the index  $r+j$  is modulo  $q$ . It is easy to see that for each  $i \in [g]$  and  $k \in [qn_\delta]$  the family  $\mathcal{D}_i(k)$  is a rainbow  $\delta$ -spanning  $F$ -factor in  $G$ , and that  $\bigcup_{k=1}^{qn_\delta} \mathcal{D}_i(k) = \bigcup_{k'=1}^{n_\delta} \bigcup_{j,r=1}^q \mathcal{D}_i^j(k', I_r)$ . Moreover, the  $\mathcal{D}_i(k)$  are pairwise edge-disjoint. Denote by  $V'(\mathcal{D}_i(k))$  the set of vertices not covered by  $\mathcal{D}_i(k)$ . We have

$$\frac{\delta}{3}n \leq |V'(\mathcal{D}_i(k))| \leq \delta n. \quad (4.13)$$

We claim that the following properties hold with high probability.

- A** For any  $v \in V(G)$  we have  $v \in V'(\mathcal{D}_i(k))$  for at most  $2f\delta n$  choices of  $(k, i) \in [qn_\delta] \times [g]$ .
- B** For any  $c \in I_{q+1}$ ,  $i \in [g]$  and  $k \in [qn_\delta]$  the number of edges of colour  $c$  incident to  $V'(\mathcal{D}_i(k))$  is at most  $3f\delta n$ .

By Lemma 2.1 and properties **a)**, **b)**, for any  $i \in [g]$ ,  $j \in [q]$ ,  $v \in V(G_i^j)$  and  $k' \in [n_\delta]$ , the probability that there are at least  $2\delta q$  indices  $r \in [q]$  such that  $v \notin \mathcal{D}_i^j(k', I_r)$  is at most  $2e^{-\delta q/3} \leq n^{-4}$ . Now a union bound shows that with probability at least  $1 - o(1)$  for all  $i \in [g]$ ,  $j \in [q]$ ,  $v \in V(G_i^j)$  and  $k' \in [n_\delta]$  there are at most  $2\delta q$  indices  $r \in [q]$  such that  $v \notin \mathcal{D}_i^j(k', I_r)$ . Thus, using (4.11), with probability  $1 - o(1)$  every vertex  $v \in V(G)$  belongs to all but at most  $2\delta qgn_\delta \leq 2f\delta n$  of the  $\delta$ -spanning factors  $\mathcal{D}_i(k)$ . Consequently, **A** holds with high probability.

Fix  $c \in I_{q+1}$ ,  $k' \in [n_\delta]$ ,  $i \in [g]$  and  $r \in [q]$  and put  $k := k' + (r-1)n_\delta$ . Define the martingale  $X_0, \dots, X_q$ , where  $X_{j'}$  is equal to the expected number of edges of colour  $c$  incident to  $V'(\mathcal{D}_i(k))$ , given the choices of the almost-spanning factors  $\mathcal{D}_i^1(k', I_{r+1}), \dots, \mathcal{D}_i^{j'}(k', I_{r+j'})$ , with indices taken modulo  $q$  (cf. (4.12)). We have  $X_0 < 2f\delta n$  due to **a)** and (4.2). Moreover,  $|X_{j'} - X_{j'-1}| \leq 4f^2n/b$  due to (4.2), (4.3) and **b)**. Thus, using Theorem 2.2, we obtain  $\mathbb{P}[X_q \geq 3f\delta n] \leq n^{-5}$ , and with probability at least  $1 - n^{-1}$  none of these events for different  $i \in [g], k' \in [n_\delta], r \in [q], c \in I_{q+1}$  occurs. Hence we can choose the  $\mathcal{D}_i^j(k', I_r)$  such that **A** and **B** hold.

The remaining part of the proof is concerned with turning  $\mathcal{D}_i(k)$  for all  $i \in [g]$ ,  $k \in [qn_\delta]$  into a *spanning*  $F$ -factor using the rainbow blow-up lemma (Lemma 2.10). We cannot apply Lemma 2.10 to  $V'(\mathcal{D}_i(k))$  directly, so we add some random vertices to it as described below. Fix  $i \in [g]$  and  $k \in [qn_\delta]$ . Define a random subcollection  $\mathcal{C}_{i,k}$  of  $\{G_i^j : j \in [q]\}$  as follows.

- c)** Include each  $Q \in \{G_i^j : j \in [q]\}$  into  $\mathcal{C}_{i,k}$  independently at random with probability  $\delta_1$ .

Put

$$V_1(i, k) := \bigcup_{Q \in \mathcal{C}_{i,k}} V(Q), \quad V_2(i, k) := V'(\mathcal{D}_i(k)) \setminus V_1(i, k).$$

Recall that each  $Q \in \mathcal{C}_{i,k}$  is  $f$ -partite and for each  $j' \in [f]$  let  $V_1^{j'}(i, k)$  be the union over all  $Q \in \mathcal{C}_{i,k}$  of the  $j'$ -th vertex class of  $Q$ . In particular,  $\bigcup_{j'=1}^f V_1^{j'}(i, k) = V_1(i, k)$ . For every  $k \in [qn_\delta]$  and  $i \in [g]$ , put

$$W'(i, k) := G[V_1^1(i, k), \dots, V_1^f(i, k), V_2(i, k)](\phi, I_{q+1}).$$

Consider an arbitrary  $F$ -factor  $H_{i,k}$  on  $[N_{i,k}]$ , where  $N_{i,k} := |V_1(i, k)| + |V_2(i, k)|$ . Note that  $N_{i,k}$  is divisible by  $f$  since  $n$  is divisible by  $f$  and  $\mathcal{D}_i(k) \setminus V_1(i, k)$  is an  $F$ -factor on  $V(G) \setminus (V_1(i, k) \cup V_2(i, k))$ . Recall that in the case when  $F$  was an edge, we had to replace it with two disjoint edges. If  $n$  (and thus also  $N_{i,k}$ ) is not divisible by 4, let  $H_{i,k}$  be the union of an  $F$ -factor on  $[N_{i,k} - 2]$  and

the edge  $\{N_{i,k} - 1, N_{i,k}\}$ . Split  $V(H_{i,k})$  arbitrarily into  $f + 1$  independent sets  $S_0, \dots, S_f$ , where  $|S_{j'}| = |V_1^{j'}(i, k)|$  for each  $j' \in [f]$ , and where  $|S_0| = |V_2(i, k)|$ . Moreover, we require that each copy of  $F$  in  $H_{i,k}$  intersects each of  $S_0, \dots, S_f$  in at most one vertex. Using (U1) and (4.13), it is easy to see that such a partition always exists provided  $N_{i,k} \geq \delta_1 n/2$ , say (which will be satisfied by  $\mathbf{C}_i$  below).

Note that both  $V_1(i, k)$  and  $N_{i,k}$  are still random variables at this stage. For all pairs  $(i, k)$  in lexicographical order, where  $i \in [g]$  and  $k \in [qn_\delta]$ , we proceed iteratively as follows. Fix  $i \in [g]$ . Assume that we have already fixed a choice of all the  $V_1(i^*, k^*)$  for  $i^* < i$  and  $k^* \in [qn_\delta]$  and that we have constructed edge-disjoint embeddings  $\psi_{i^*, k^*} : H_{i^*, k^*} \rightarrow W'(i^*, k^*)$  for all  $i^* < i$ ,  $k^* \in [qn_\delta]$ , which satisfy  $\mathbf{C}_1, \dots, \mathbf{C}_{i-1}, \mathbf{D}_1, \dots, \mathbf{D}_{i-1}, \mathbf{E}_1, \dots, \mathbf{E}_{i-1}$  below. Define the graph

$$K_i := \bigcup_{i^*=1}^{i-1} \bigcup_{k^*=1}^{qn_\delta} \psi_{i^*, k^*}(H_{i^*, k^*}).$$

Next, we choose  $V_1(i, k)$  for all  $k \in [qn_\delta]$  simultaneously. We claim that there exists a choice of these  $V_1(i, k)$  such that the following hold.

$\mathbf{C}_i$  For each  $k \in [qn_\delta]$  we have  $|V_1(i, k)| = (1 \pm \delta_1)\delta_1 n$ .

$\mathbf{D}_i$  Each  $v \in V(G)$  belongs to at most  $2\delta_1 qn_\delta$  sets among  $V_1(i, 1), \dots, V_1(i, qn_\delta)$ .

$\mathbf{E}_i$  For all  $v \in V(G)$  and  $k \in [qn_\delta]$  we have  $d_{K_i, V_1(i, k)}(v) \leq \delta_1^{3/2} n$ .

To prove the claim, note that  $\mathbb{E}[|V_1(i, k)|] = \delta_1 n$ . Recall from (U1) that for all  $i_1, j_1, i_2, j_2$  we have  $|V(G_{i_1}^{j_1})| = (1 \pm 2\zeta)|V(G_{i_2}^{j_2})|$ . Using Lemma 2.1 and **c**), it follows that  $\mathbf{C}_i$  is satisfied with probability at least  $1 - e^{-\delta_1^3 q/3} \geq 1 - n^{-2}$  for fixed  $k$ . Taking a union bound over all  $k \in [qn_\delta]$ , we conclude that  $\mathbf{C}_i$  is satisfied with probability  $1 - o(1)$ .

Next, using **c**) and Lemma 2.1, for any  $j \in [q]$  the probability that  $G_i^j$  belongs to  $\mathcal{C}_{i, k}$  for at least  $2\delta_1 qn_\delta$  different values of  $k$  is at most  $e^{-\delta_1 qn_\delta/3}$ . Taking a union bound over all  $j$ , we conclude that  $\mathbf{D}_i$  holds with probability  $1 - o(1)$ .

Finally, using **A**, as well as  $\mathbf{D}_{i^*}$  for  $i^* < i$  and the fact that each  $H_{i^*, k^*}$  has maximum degree at most  $f$ , we conclude that  $d_{K_i}(v) \leq 2f^2 \delta n + f(i-1) \cdot 2\delta_1 qn_\delta \leq 4f^2 \delta_1 n$ . Fix  $v \in V(G)$  and  $k \in [qn_\delta]$ . Similarly to the proof of **B**, define the martingale  $X_0, \dots, X_q$ , where  $X_t := \mathbb{E}[d_{K_i, V_1(i, k)}(v) \mid \mathcal{C}_{i, k} \cap \{G_i^1, \dots, G_i^t\}]$ . We have  $X_0 \leq 4f^2 \delta_1^2 n$  and  $|X_t - X_{t-1}| \leq \vartheta_t$ , where  $\vartheta_t \leq |V(G_i^t)| \leq 2n/q$ . Thus,  $\sum_{t=1}^q \vartheta_t^2 \leq \frac{2n^2}{q}$ . Applying Theorem 2.2, we obtain that  $\mathbb{P}[X_q \geq \delta_1^{3/2} n] \leq n^{-4}$  and, taking a union bound over all choices of  $v \in V(G)$  and  $k \in [qn_\delta]$ , we conclude that  $\mathbf{E}_i$  holds with probability  $1 - o(1)$ . Fix choices of  $V_1(i, 1), \dots, V_1(i, qn_\delta)$  that satisfy properties  $\mathbf{C}_i, \mathbf{D}_i, \mathbf{E}_i$  simultaneously.

We remark that  $\mathbf{C}_i$  together with (U1) and (4.13) imply that

$$|\{j \in [q] : G_i^j \in \mathcal{C}_{i, k}\}| = (1 \pm 2\delta_1)\delta_1 q, \quad |V_1^{j'}(i, j)| = (1 \pm 2\delta_1)\delta_1 n/f \quad \text{and} \quad (4.14)$$

$$|V_2(i, k)| \leq \delta_1 |V_1^{j'}(i, k)| \quad \text{for each } j' \in [f]. \quad (4.15)$$

For each  $k \in [qn_\delta]$  in turn, we now intend to apply Lemma 2.10 using  $H_{i, k}$  with partition  $S_0, \dots, S_f$  and an arbitrary bijection  $\psi'_{i, k} : S_0 \rightarrow V_2(i, k)$  and with  $\delta_1 n/f, f, \gamma d, \delta_2, \gamma$  playing the roles of  $n, r, d, \delta_2, \gamma$  to

$$W(i, k) := W'(i, k) - K_i - \bigcup_{k^*=1}^{k-1} \psi_{i, k^*}(H_{i, k^*})$$

with partition  $V_2(i, k), V_1^1(i, k), \dots, V_1^f(i, k)$ . Provided that such an application is possible, we can extend  $\psi'_{i, k}$  to  $\psi_{i, k}$  and obtain a rainbow  $F$ -factor  $\psi_{i, k}(H_{i, k})$  in  $W(i, k)$  for each  $i, k$ . (Indeed, the graph  $\psi_{i, k}(H_{i, k}) \cup (\mathcal{D}_i(k) \setminus V_1(i, k))$  forms a rainbow  $F$ -factor in  $G'$ .) Moreover,  $\mathcal{H} := \bigcup_{i, k} (\psi_{i, k}(H_{i, k}) \cup (\mathcal{D}_i(k) \setminus V_1(i, k)))$  gives us an  $\alpha$ -decomposition of  $G'$ , as required. Indeed, using (U2), (4.10), (4.12) and the fact that  $\{\mathcal{D}_i^j(k', I_r) : k' \in [n_\delta]\}$  forms a  $2\delta$ -decomposition of  $G_i^j(\phi, I_r)$ , it is easy to see that already  $\bigcup_{i, k} \mathcal{D}_i(k)$  covers all but an  $\gamma^{1/2}$ -fraction of the edges of  $G'$ , and  $\mathcal{H}$  contains at least as many edges (as it consists of spanning rather than almost-spanning factors). Thus, to complete the proof, we only need to verify that Lemma 2.10 is applicable in each iteration step. Indeed, we have the following.

- Property **B**, combined with (4.2), (4.3) and the first part of (4.14) imply that the colouring  $\phi$  of  $W(i, k)$  is  $3f\delta n + \frac{3fn}{b^2} \cdot (2\delta_1 q)^2 \binom{f}{2} \leq \delta_2(\delta_1 n/f)$ -bounded.
  - (A1)2.10 is implied by the definitions of  $H_{i,k}$  and  $S_0$ , with  $\Delta = f$ .
  - (A2)2.10 is implied by the definition of  $\psi'_{i,k}$  and (4.15).
  - (A3)2.10 is implied by (4.14), together with the definition of  $H_{i,k}$ .
  - By (4.10), for  $j_1 \neq j_2 \in [f]$  the graph  $W'(i, k)[V_1^{j_1}(i, k), V_1^{j_2}(i, k)]$  is  $(\delta_1, \gamma d)$ -superregular and for each  $v \in V_2(i, k)$ ,  $j' \in [f]$  we have  $d_{W'(i,k), V_1^{j'}(i,k)}(v) = \gamma d |V_1^{j'}(i, k)| \pm \delta_1 |V_1^{j'}(i, k)|$ .
- Let

$$K_i^k := K_i \cup \bigcup_{k^*=1}^{k-1} \psi_{i,k^*}(H_{i,k^*}).$$

Then, due to **E**<sub>i</sub>, for each vertex  $v \in V(G)$  we have

$$d_{K_i^k, V_1(i,k)}(v) \leq d_{K_i, V_1(i,k)}(v) + fq n_\delta \leq \delta_1^{1/3} \cdot (\delta_1 n/f).$$

(Note that we bounded the contribution of  $\bigcup_{k^*=1}^{k-1} \psi_{i,k^*}(H_{i,k^*})$  by  $fq n_\delta \leq f^2 n/g \leq \delta n$ , using (4.11).) Using Proposition 2.4, we conclude that for all  $j_1, j_2, j' \in [f]$  and  $v \in V_2(i, k)$

$$W(i, k)[V_1^{j_1}(i, k), V_1^{j_2}(i, k)] \text{ is } (\delta_2, \gamma d)\text{-superregular and } d_{W(i,k), V_1^{j'}(i,k)}(v) \geq \frac{1}{2} \gamma d |V_1^{j'}(i, k)|.$$

Thus, (A4)2.10 and (A5)2.10 are satisfied.

This concludes the proof of Theorem 1.6.

**4.2. Proof of Theorem 1.7. Almost-decomposition into Hamilton cycles.** The proof of this theorem is very similar to that of Theorem 1.6. In particular, we use the same notation as in the proof of Theorem 1.6. We will let a cycle  $C_s$  with sufficiently large  $s$  play the role of  $F$ . (We remark that  $a(C_s) = 2$ .) We then merge each almost-spanning  $C_s$ -factor into a single Hamilton cycle. This introduces a final ‘‘gluing’’ step, and, in particular, changes the graphs  $H_{i,k}$  and embeddings  $\psi'_{i,k}$  we use. This part of the proof resembles the final part of the proof of Theorem 1.5. The main difference to Theorem 1.5 is that we have to include all the vertices into the cycle this time.

Let us make this precise. We use the following hierarchy of constants:

$$0 < 1/n_0 \ll \eta \ll \zeta \ll \zeta_1 \ll \varepsilon \ll \varepsilon_1 \ll \delta \ll \delta_1 \ll \delta_2 \ll 1/s \ll \delta_3 \ll \delta_4 \ll \gamma \ll \beta \ll \alpha, d_0. \quad (4.16)$$

Note that the position of  $f, h$  in the hierarchy in the proof of Theorem 1.6 is consistent with  $f = h = s$  and (4.1). We additionally assume that  $s$  is even.

We proceed until the stage just before properties **A** and **B**. In particular, for all  $i \in [g]$ ,  $k \in [qn_\delta]$  we define a  $\delta$ -spanning  $C_s$ -factor  $\mathcal{D}_i(k)$  in  $G$ . We write  $\mathcal{D}_i(k) = \bigcup_{j=1}^{n_2} C_i^j(k)$ , where  $C_i^j(k)$  are the  $s$ -cycles forming  $\mathcal{D}_i(k)$ . Moreover, by (randomly) disregarding some cycles if necessary, we assume that the number  $n_2$  is the same for all  $i, k$ . For every  $k \in [qn_\delta]$  we define the following sets of vertices:

$$V'(\mathcal{D}_i(k)) := V(G) \setminus V(\mathcal{D}_i(k)) \quad \text{and}$$

$$V''(\mathcal{D}_i(k)) := \bigcup_{j=1}^{n_2} \{x_i^j(k), y_i^j(k)\}, \quad \text{where } x_i^j(k)y_i^j(k) \text{ is an edge, randomly chosen from } C_i^j(k).$$

We modify the properties **A** and **B** accordingly. We claim that the following hold with high probability.

- A** For any  $v \in V(G)$  we have  $v \in V'(\mathcal{D}_i(k)) \cup V''(\mathcal{D}_i(k))$  for at most  $2s\delta n + 4n/s$  choices of  $(k, i) \in [qn_\delta] \times [g]$ .
- B** For all  $c \in I_{q+1}$ ,  $i \in [g]$  and  $k \in [qn_\delta]$  the number of edges of colour  $c$  incident to  $V'(\mathcal{D}_i(k)) \cup V''(\mathcal{D}_i(k))$  is at most  $3s\delta n + 4n/s$ .

In view of the proof of Theorem 1.6, we only have to verify the parts of the properties **A** and **B** involving  $V''(\mathcal{D}_i(k))$ .

Fix  $v \in V(G)$ . To verify the second part of **A**, note that for any  $i \in [g]$  the number  $\rho_v$  of sets among  $V''(\mathcal{D}_i(1)), \dots, V''(\mathcal{D}_i(qn_\delta))$  to which  $v$  belongs is a random variable, which is a sum of  $qn_\delta$  independent binary random variables with probability of success at most  $2/s$ . Since  $qn_\delta \cdot 2/s \leq 2n/g s$

due to (4.11), a standard application of Lemma 2.1 (together with a union bound over  $i \in [g]$ ) implies that the second part of **A** holds for all  $v \in V(G)$  with probability  $1 - o(1)$ .

In order to ensure that for all  $c \in I_{q+1}$ ,  $i \in [g]$ ,  $k \in [qn_\delta]$  the number of edges of colour  $c$  incident to  $V''(\mathcal{D}_i(k))$  is at most  $4n/s$ , we define a martingale  $X_0, \dots, X_{n_2}$ , where  $X_j$  is equal to the expected number of edges of colour  $c \in I_{q+1}$  incident to  $V''(\mathcal{D}_i(k))$  given the choices of random edges in  $C_i^1(k), \dots, C_i^j(k)$ . By (4.2) we have  $X_0 \leq 2n/s$  and, since the coloring  $\phi$  is locally  $\eta n / \log^4 n$ -bounded,  $|X_j - X_{j-1}| \leq 2\eta \frac{n}{\log^4 n}$ . Putting  $c_i^j(k)$  to be the number of edges of colour  $c$  incident to  $C_i^j(k)$ , we also get that  $\sum_{j=1}^{n_2} |X_j - X_{j-1}| \leq \sum_{j=1}^{n_2} c_i^j(k) \leq 2n$ . Thus, Theorem 2.2 implies that  $\mathbb{P}[|X_{n_2} - X_0| \geq 2n/s] \leq n^{-5}$ . Thus, **B** holds with high probability. Fix a choice of the  $\mathcal{D}_i(k)$  that satisfies **A** and **B** simultaneously.

Fix  $i \in [g]$  and  $k \in [qn_\delta]$ . Define a random subcollection  $\mathcal{C}_{i,k}$  of  $\{G_i^j : j \in [q]\}$  as follows.

**c)** Include each  $Q \in \{G_i^j : j \in [q]\}$  into  $\mathcal{C}_{i,k}$  independently with probability  $\delta_3$ .

(Note that the probability in this case is not the same as in the proof of Theorem 1.6.) Put

$$V_1(i, k) := \bigcup_{Q \in \mathcal{C}_{i,k}} V(Q), \quad V_2(i, k) := V'(\mathcal{D}_i(k)) \setminus V_1(i, k) \quad \text{and} \quad V_3(i, k) := V''(\mathcal{D}_i(k)) \setminus V_1(i, k).$$

Recall that each  $Q \in \mathcal{C}_{i,k}$ , and thus also  $V_1(i, k)$ , has  $s$  parts and that  $s$  is even. Let  $V_1^1(i, k)$  be the union of all parts with odd indices and let  $V_1^2(i, k) := V_1(i, k) \setminus V_1^1(i, k)$ . Put  $n_2(i, k) := |V_3(i, k)|/2$ .

Let  $W'(i, k)$  be the following graph:

$$W'(i, k) := G[V_1^1(i, k), V_1^2(i, k), V_2(i, k), V_3(i, k)](\phi, I_{q+1}).$$

Now we are in a position to define the graph  $H_{i,k}$  and the embedding function  $\psi'_{i,k}$ . Let  $H_{i,k} := \bigcup_{j=1}^{n_2(i,k)-1} P^j \cup P_{i,k}$ , where  $P^j = w_0^j w_1^j w_2^j w_3^j$  is a path of length 3, and  $P_{i,k} = w_0^{n_2(i,k)} \dots w_3^{n_2(i,k)}$  is a path on  $|V(W'(i, k))| - 4(n_2(i, k) - 1)$  vertices. We also let  $t := |V_2(i, k)|$  and choose vertices  $z_1, \dots, z_t$  on  $P_{i,k}$ , such that the vertices  $z_1, \dots, z_t, w_0^{n_2(i,k)}, w_3^{n_2(i,k)}$  have pairwise distance at least four along the path and for all  $i' \in [t-1]$  the distance between  $z_i$  and  $z_{i+1}$  is precisely four. Clearly, this condition is possible to fulfill since  $V_1(i, k)$  is much larger than  $V_2(i, k) \cup V_3(i, k)$  (note that  $V_1(i, k)$  has size roughly  $\delta_3 n$ , while  $V_2(i, k)$  and  $V_3(i, k)$  have sizes at most  $\delta n$  and  $2n/s$ , respectively). Note that  $|V(H_{i,k})| = |V(W'(i, k))|$ .

Take a partition of  $H_{i,k}$  into three parts  $\mathcal{X} = \{X_0, X_1, X_2\}$ , which satisfies the following:  $X_0 = \bigcup_{j=1}^{n_2} \{w_0^j, w_3^j\} \cup \{z_1, \dots, z_t\}$ ; for  $j \in [2]$  the set  $X_j$  is independent in  $H_{i,k}$  and has size  $|V_1^j(i, k)|$ . The final property is easy to satisfy since by (U1) the sizes of  $V_1^1(i, k)$  and  $V_1^2(i, k)$  differ by at most  $2\zeta n$ , which is much smaller than  $t$  due to (4.13) and thus we have sufficient flexibility in assigning the vertices of  $P_{i,k}$  to  $X_1$  and  $X_2$  since  $z_1, \dots, z_t$  are assigned to  $X_0$ . Note that  $|X_0| = |V_2(i, k) \cup V_3(i, k)|$ . Moreover, by relabeling if necessary, we may assume that  $\mathcal{D}_i(k) = \bigcup_{j=1}^{n_2(i,k)} C_i^j(k)$ . We aim to apply Lemma 2.10 with  $\psi'_{i,k}$  defined as follows:  $\psi'_{i,k}(w_0^j) = x_i^j(k)$  and  $\psi'_{i,k}(w_3^j) = y_i^{j+1}(k)$ , with indices taken modulo  $n_2(i, k)$ , and  $\psi'_{i,k}(z_j) = v_j$ , where  $V_2(i, k) =: \{v_1, \dots, v_t\}$ .

We slightly modify the properties **C<sub>i</sub>**, **D<sub>i</sub>**, **E<sub>i</sub>** from the proof of Theorem 1.6 (due to changing  $\delta_1$  to  $\delta_3$  in **c)**):

**C<sub>i</sub>** For each  $k \in [qn_\delta]$  we have  $|V_1(i, k)| = (1 \pm \delta_3)\delta_3 n$ .

**D<sub>i</sub>** Each  $v \in V(G)$  belongs to at most  $2\delta_3 q n_\delta$  sets among  $V_1(i, 1), \dots, V_1(i, qn_\delta)$ .

**E<sub>i</sub>** For any  $v \in V(G)$  and  $k \in [qn_\delta]$  we have  $d_{K_i, V_1(i, k)}(v) \leq \delta_3^{3/2} n$ .

Apply Lemma 2.10 to embed  $H_{i,k}$  with partition  $X_0, X_1, X_2$  and the bijection  $\psi'_{i,k} : X_0 \rightarrow V_2(i, k) \cup V_3(i, k)$  into  $W(i, k)$  with  $\delta_3 n/2, 2, \delta_4, \gamma$  playing the roles of  $n, r, \delta_2, \gamma$ . As in the proof of Theorem 1.6,  $W(i, k)$  is obtained from  $W'(i, k)$  by deleting the edges used in previous iterations.

The verification of the conditions of the blow-up lemma repeats the one done in the previous subsection.



## 5. CONCLUDING REMARKS

In this section we describe possible extensions of our results, along with applications. We also adapt a counterexample to Stein's conjecture due to Pokrovskiy and Sudakov [38] to the setting of Corollary 1.1, as mentioned in the introduction.

**5.1. Colourings of  $K_n$  with no rainbow cycles longer than  $n - \Omega(\log n)$ .** Pokrovskiy and Sudakov [38] constructed an  $n \times n$  array  $A$  where the entries are symbols from  $[n]$  such that each symbol occurs precisely  $n$  times and such that the largest partial transversal of  $A$  has size  $n - \Omega(\log n)$ . Any such array may be interpreted as a colouring of a complete directed graph  $G$  on  $[n]$  with one loop at each vertex. For any  $i, j$ , the edge  $(i, j)$  of  $G$  is coloured with the symbol in the  $i$ -th row and  $j$ -th column. In this interpretation, any rainbow directed cycle in  $G$  gives rise to a rainbow partial transversal of the same length in  $A$ . Assuming that  $A$  (and thus also the colouring of  $G$ ) is *symmetric*, we can construct a colouring of  $K_n$  by simply assigning the colour of  $(i, j)$  and  $(j, i)$  to the edge  $ij$  in  $K_n$ , and vice versa. Thus, any rainbow cycle of length  $k$  in the resulting  $n/2$ -bounded colouring of  $K_n$  gives rise to a partial transversal of size  $k$  in  $A$  (which avoids the diagonal).

As mentioned in [38], the array from [38] can be easily made symmetric. Moreover, the same argument works for arrays where each symbol appears at most  $n - 1$  times. This provides us with an example of an  $(n - 1)/2$ -bounded colouring of  $K_n$  whose longest rainbow cycle has length  $n - \Omega(\log n)$ . The array from [38] can be also adjusted so that the resulting colouring of  $K_n$  is locally  $n^{1/2+\varepsilon}$ -bounded for any fixed  $\varepsilon > 0$ , and so that the length of the longest rainbow cycle is still  $n - \Omega(\log n)$ , with the constant in the  $\Omega$ -term depending on  $\varepsilon$ . (In terms of the array, this means that no row or column contains more than  $n^{1/2+\varepsilon}$  copies of the same symbol.)

**5.2. Multipartite versions.** Our methods also extend to the multipartite setting. We state the following result without proof, as this is almost identical to that of Theorems 1.4 and 1.6. The two main differences are that we apply a result of MacNeish [31] instead of Theorem 2.12 to show that there is a resolvable design in the partite setting. Moreover, in the proof of (i) we apply Lemma 2.8 (with  $f$  as in Theorem 5.1) instead of Lemma 2.9. Similarly, to obtain the analogue of (4.4), (4.5) in the proof of (ii), we apply Lemma 2.8 rather than Lemma 2.9.

**Theorem 5.1.** *For given  $\alpha, d_0, f, h, r > 0$ , there exist  $\eta > 0$  and  $n_0$  such that the following holds for all  $n \geq n_0$  such that  $f$  divides  $n$  and  $d \geq d_0$ . Suppose that  $F$  is an  $fr$ -vertex  $h$ -edge graph with vertex partition  $\{X_1, \dots, X_r\}$  into independent sets of size  $f$ . Suppose that  $a(F) \leq a$ . Suppose that  $G$  is a  $rn$ -vertex  $r$ -partite graph with vertex partition  $\{V_1, \dots, V_r\}$  into sets of size  $(1 \pm \eta)n$  and such that  $G[V_i, V_j]$  is  $(\eta, d)$ -superregular for all  $i \neq j \in [r]$ .*

- (i) *If  $\phi$  is a  $(1 + \eta)\frac{f}{h}\binom{r}{2}dn$ -bounded, locally  $\eta n$ -bounded colouring of  $G$ , then  $G$  has an  $\alpha$ -decomposition into rainbow  $\alpha$ -spanning  $F$ -factors.*
- (ii) *If  $\phi$  is a  $(1 - \alpha)\frac{f}{h}\binom{r}{2}dn$ -bounded, locally  $\eta n \log^{-2a} n$ -bounded colouring of  $G$  and  $|V_i| = n$  for each  $i \in [r]$ , then  $G$  has an  $\alpha$ -decomposition into rainbow  $F$ -factors.*

Note that, if  $F = K_2$ , then the above theorem implies that in a properly coloured complete balanced bipartite graph on  $2n$  vertices, if no colour appears more than  $(1 - o(1))n$  times, then we can obtain a  $o(1)$ -decomposition into rainbow perfect matchings. This was first announced by Montgomery, Pokrovskiy and Sudakov [33]. In terms of arrays, this result states that any  $n \times n$ -array filled with symbols, none of which appears more than  $(1 - o(1))n$  times in total or is repeated in any row or column, can be  $o(1)$ -decomposed into full transversals. (Note that our theorem has a much weaker condition on the repetitions of symbols in rows or columns.)

**5.3. Further remarks and extensions.** We can easily deduce the following pancyclicity result from Theorem 1.7 and Theorem 2.10: For any  $\varepsilon > 0$  there exist  $\eta > 0$  and  $n_0$  such that whenever  $n \geq n_0$ , any  $(1 - \varepsilon)\frac{n}{2}$ -bounded, locally  $\frac{\eta n}{\log^4 n}$ -bounded colouring of  $K_n$  contains a rainbow cycle of any length. (Indeed, to obtain cycles of length  $k$  for  $k$  linear in  $n$ , apply Theorem 1.7 to a random subset of  $V(G)$  of size  $k$ , and for shorter cycles, apply Theorem 2.10 to  $G$ .) This (up to logarithmic factors) extends a result of Frieze and Krivelevich [20], who proved this for  $\eta n$ -bounded colourings.

The conditions of Theorem 1.4 (as well as in its bipartite analogue) may be substantially weakened if  $F$  is an edge. More precisely, we can prove the following theorem, which applies to *sparse* graphs.

**Theorem 5.2.** *For any  $\delta > 0$ , there exist  $\varepsilon > 0$  and  $n_0$  such that the following holds for all  $n \geq n_0$  and  $r \geq \varepsilon^{-1}$ . Suppose that  $G$  is an  $n$ -vertex graph satisfying  $d(v) = (1 \pm \varepsilon)r$ . If  $\phi$  is a  $(1 + \varepsilon)r$ -bounded, locally  $\varepsilon r$ -bounded colouring of  $G$ , then  $G$  contains a  $2\delta$ -decomposition into  $\delta$ -spanning rainbow matchings.*

*Proof.* We apply Lemma 2.14 with  $\mathcal{F}$  being a collection of pairs of disjoint edges of distinct colour. First of all, let us calculate the values of different subfamilies of  $\mathcal{F}$  (in the notation of Lemma 2.14). For every  $v \in V(G)$  we have

$$|\mathcal{F}(v)| = (1 \pm \varepsilon)r \cdot (1 \pm 2\varepsilon)\frac{rn}{2} = (1 \pm 4\varepsilon)\frac{r^2n}{2}.$$

Indeed, we first choose an edge adjacent to  $v$ , and then another edge in  $G$  of another colour and disjoint from the first one. The number of edges of  $G$  is  $(1 \pm \varepsilon)\frac{rn}{2}$ , and the two conditions imposed on the choice of the second edge exclude at most  $(3 + \varepsilon)r$  edges. We present the other calculations more concisely. For all edges  $uw, u'w' \in E(G)$ , vertices  $v_1, v_2 \in V(G)$  and colours  $c_1, c_2 \in [m]$  we have

$$\begin{aligned} |\mathcal{F}(uw)| &= (1 \pm 2\varepsilon)\frac{rn}{2}, \\ |\mathcal{F}(c_1)| &\leq (1 + \varepsilon)r \cdot (1 + \varepsilon)\frac{rn}{2} \leq (1 + 3\varepsilon)\frac{r^2n}{2}, \\ |\mathcal{F}(v_1, v_2)| &\leq (1 + \varepsilon)\frac{rn}{2} + ((1 + \varepsilon)r)^2 \leq 2rn, \\ |\mathcal{F}(c_1, v_1)| &\leq \varepsilon r \cdot (1 + \varepsilon)\frac{rn}{2} + ((1 + \varepsilon)r)^2 \leq 2\varepsilon r^2n, \\ |\mathcal{F}(c_1, c_2)| &\leq ((1 + \varepsilon)r)^2 \leq 2r^2, \\ |\mathcal{F}(uw, u'w')| &\leq 1. \end{aligned}$$

Using the displayed formulas and the fact that  $r > \varepsilon^{-1}$ , it is easy to see that the conditions of Lemma 2.14 are satisfied with  $8\varepsilon$  playing the role of  $\varepsilon$ .  $\square$

**Postscript.** As mentioned in Section 5.2, Montgomery, Pokrovskiy and Sudakov had earlier announced the case  $F = K_2$  of Theorem 5.1(ii) for proper (i.e. locally 1-bounded) colourings. After completing our manuscript, we learned that they independently obtained some other related results to ours. Slightly more precisely, they also obtained similar approximate decomposition results for rainbow Hamilton cycles, and in addition, they also obtained approximate decomposition results for rainbow trees, but do not consider general rainbow  $F$ -factors. For their results on rainbow spanning subgraphs, the colourings considered in [34] are always proper. On the other hand, the global boundedness condition in [34] is less restrictive than ours. They also deduce from their results a conjecture of Akbari and Alipur on transversals in generalized Latin squares. This conjecture was proved independently by Keevash and Yepremyan [29].

More recently, the Brualdi-Hollingsworth conjecture as well its strengthening by Constantine has been proved by Glock, Kühn, Montgomery and Osthus [22], i.e. every sufficiently large optimally edge-coloured complete graph has a decomposition into isomorphic rainbow spanning trees. Amongst others, the proof makes use of some of the ideas in the current paper. The related conjecture of Kaneko, Kano, and Suzuki (which allows for not necessarily optimal proper colourings) remains an interesting open problem.

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## APPENDIX A. RAINBOW COUNTING LEMMA

The following is a variation of the well-known subgraph counting lemma, which we state without proof.

**Lemma A.1.** *Suppose  $0 < 1/n \ll \zeta \ll \varepsilon \ll 1/r, d, 1/f, 1/h \leq 1$ . Suppose that  $F$  is an  $h$ -edge graph with vertex partition  $\mathcal{X} = \{X_1, \dots, X_r\}$  into independent sets with  $|X_i| = f$ , and  $G$  is a graph with vertex partition  $\mathcal{V} = \{V_{1,1}, \dots, V_{1,f}, V_{2,1}, \dots, V_{r,f}\}$ . For each  $i \in [r]$ , let  $V_i := \bigcup_{f' \in [f]} V_{i,f'}$  and let  $\mathcal{V}' := \{V_1, \dots, V_r\}$ . Suppose that a vertex  $u \in V(G)$  and an edge  $vw \in E(G)$  are given with  $v \in V_{j'}$  and  $w \in V_{j''}$ . Suppose the following hold.*

- (A1)<sub>A.1</sub> *For each  $(i, f') \in [r] \times [f]$ , we have  $|V_{i,f'}| = (1 \pm \zeta)n$ .*
- (A2)<sub>A.1</sub> *For all  $i \neq j \in [r]$  and  $f', f'' \in [f]$ , the bipartite graph  $G[V_{i,f'}, V_{j,f''}]$  is  $(\zeta, d)$ -superregular.*
- (A3)<sub>A.1</sub> *Either  $d_{G, V_{i,f'}}(v, w) = (d^2 \pm \zeta)|V_{i,f'}|$  for all  $i \in [r] \setminus \{j', j''\}$  and  $f' \in [f]$ , or  $F$  is triangle-free.*

*Then the number of copies of  $F$  in  $G$  containing  $u$ , and respecting both  $(\mathcal{X}, \mathcal{V}')$  and  $(V(F), \mathcal{V})$  is*

$$(1 \pm \varepsilon) \frac{r!(f!)^r d^h n^{fr-1}}{|\text{Aut}_{\mathcal{X}}(F)|}$$

*and the number of copies of  $F$  in  $G$  containing  $vw$ , and respecting both  $(\mathcal{X}, \mathcal{V}')$  and  $(V(F), \mathcal{V})$  is*

$$(1 \pm \varepsilon) \frac{hr!(f!)^r d^{h-1} n^{fr-2}}{\binom{r}{2} f^2 |\text{Aut}_{\mathcal{X}}(F)|}.$$

The proof of the following result is also straightforward, so again we omit it. We will combine it with Lemma A.1 to derive both Lemma 2.8 and 2.9.

**Lemma A.2.** *Let  $0 < 1/n \ll \zeta \ll \varepsilon \ll 1/f, 1/C \ll 1$ . Let  $F$  be an  $f$ -vertex graph and let  $G$  be an  $n$ -vertex graph. Suppose that  $\phi$  is a  $(Cn, \zeta n)$ -bounded colouring of  $G$ . Fix a vertex  $u \in V(G)$  and an edge  $vw \in E(G)$ . Suppose that either  $vw \notin \text{Ir}_G^\phi(\zeta n)$  or  $F$  is triangle-free. Then the number of non-rainbow copies of  $F$  in  $G$  containing  $u$  is at most  $\varepsilon n^{f-1}$  and the number of non-rainbow copies of  $F$  in  $G$  containing  $vw$  is at most  $\varepsilon n^{f-2}$ .*

*Proof of Lemma 2.8.* We fix  $u \in V(G)$ . For given  $G$  with partition  $\mathcal{V}$ , we duplicate each vertex  $x \in V(G)$  into  $x_1, \dots, x_f$  and let  $V_{i,1}, \dots, V_{i,f}$  be defined by  $V_{i,f'} := \{x_{f'} : x \in V_i\}$ . Let  $G'$  be the graph on  $\bigcup_{(i,f') \in [r] \times [f]} V_{i,f'}$  such that, for each  $x_{f'} \in V(G')$ , the neighbourhood  $N_{G'}(x_{f'})$  is exactly the set of duplicates of  $N_G(x)$ . Let  $\mathcal{W} := \{V_{1,1}, \dots, V_{r,f}\}$  and  $\mathcal{W}' := \{\bigcup_{f' \in [f]} V_{1,f'}, \dots, \bigcup_{f' \in [f]} V_{r,f'}\}$ . For each edge  $xx' \in E(G)$  and  $f' \neq f'' \in [f]$ , let  $\phi'(x_{f'} x'_{f''}) = \phi(xx')$ . It is easy to see that (A1)<sub>2.8</sub>–(A3)<sub>2.8</sub> imply (A1)<sub>A.1</sub>–(A3)<sub>A.1</sub> for  $G'$  with  $\zeta, \varepsilon/4, u_1, v_1 w_1$  playing the roles of  $\zeta, \varepsilon, u, vw$ . Thus we can apply Lemma A.1 and at the same time we apply Lemma A.2 (with  $\zeta, \varepsilon^2, u_1, v_1 w_1$  playing the roles of  $\zeta, \varepsilon, u, vw$ ) to estimate the number of non-rainbow copies of  $F$ . By subtracting the latter from the former, we conclude that the number of rainbow copies of  $F$  in  $G'$  containing  $u_1$  and respecting both  $(\mathcal{X}, \mathcal{W}')$  and  $(V(F), \mathcal{W})$  is

$$(1 \pm \varepsilon/2) \frac{r!(f!)^r d^h n^{fr-1}}{|\text{Aut}_{\mathcal{X}}(F)|}$$

and we also obtain that the number of rainbow copies of  $F$  in  $G'$  containing  $v_1 w_1$  respecting both  $(\mathcal{X}, \mathcal{W}')$  and  $(V(F), \mathcal{W})$  is

$$(1 \pm \varepsilon/2) \frac{hr!(f!)^r d^{h-1} n^{fr-2}}{\binom{r}{2} f^2 |\text{Aut}_{\mathcal{X}}(F)|}.$$

Note that each such copy of  $F$  is either degenerate (in the sense that it contains two duplicates of the same vertex) or corresponds to rainbow copy of  $F$  in  $G$ . It is easy to see that there are at most  $|V(F)|^3 f (fn)^{fr-2}$  degenerate copies of  $F$  containing  $u_1$ . Note that  $(f!)^{r-1} (f-1)!$  non-degenerate copies of  $F$  in  $G'$  containing  $u_1$  correspond to the same copy of  $F$  in  $G$  (which then contains  $u$ ). Thus

$$\begin{aligned} r_{G, \mathcal{X}, \mathcal{V}}(F, u) &= \frac{1}{(f!)^{r-1} (f-1)!} |R_{G', \mathcal{X}, \mathcal{W}'}(F, u_1) \cap R_{G', V(F), \mathcal{W}}(F, u_1)| \pm f(fr)^3 (fn)^{fr-2} \\ &= (1 \pm \varepsilon) \frac{r! f d^h n^{fr-1}}{|\text{Aut}_{\mathcal{X}}(F)|}. \end{aligned}$$

A similar argument works for  $r_{G,\mathcal{X},\mathcal{V}}(F, vw)$ .  $\square$

*Proof of Lemma 2.9.* For a given quasirandom graph  $G$ , we duplicate each  $x \in V(G)$  into  $x_1, \dots, x_f$  and let  $V_1, \dots, V_f$  be defined by  $V_i := \{x_i : x \in V\}$ . Let  $\mathcal{V} = \{V_1, \dots, V_f\}$ . Let  $H$  be the graph with vertex set  $\bigcup_{i \in [f]} V_i$  such that  $x_i y_j \in E(H)$  if and only if  $xy \in E(G)$ . For each edge  $xy \in E(G)$  and  $i \neq j \in [f]$ , let  $\phi'(x_i y_j) = \phi(xy)$ . By using Theorem 2.5 and (A1)2.9, it is easy to see that (A1)2.8–(A3)2.8 hold for  $H$  and  $\mathcal{V}$  with  $\zeta^{1/6}, f, 1, u_1, v_1 w_2$  playing the roles of  $\zeta, r, f, u, vw$ , respectively. By applying Lemma 2.8 with these parameters and with  $\varepsilon/5$  playing the role of  $\varepsilon$ , and by using Lemma A.2 to estimate the number of non-rainbow copies of  $F$ , we conclude that

$$r_{H,V(F),\mathcal{V}}(F, u_1) = (1 \pm \varepsilon/4) \frac{f! d^h n^{f-1}}{|\text{Aut}(F)|} \quad \text{and} \quad r_{H,V(F),\mathcal{V}}(F, v_1 w_2) = (1 \pm \varepsilon/4) \frac{h f! d^{h-1} n^{f-2}}{\binom{f}{2} |\text{Aut}(F)|}.$$

Again, similarly as in the proof of Lemma 2.8, the number of degenerate copies of  $F$  in  $H$  is negligible in both cases. Moreover,  $(f-1)!$  distinct non-degenerate copies of  $F$  in  $H$  containing  $u_1$  correspond to the same copy of  $F$  in  $G$ , and  $(f-2)!$  distinct non-degenerate copies of  $F$  in  $H$  containing  $v_1 w_2$  correspond to the same copy of  $F$  in  $G$ . Thus we have

$$r_G(F, u) = (1 \pm \varepsilon/3) \frac{f d^h n^{f-1}}{|\text{Aut}(F)|} \quad \text{and} \quad r_G(F, vw) = (1 \pm \varepsilon/3) \frac{2h d^{h-1} n^{f-2}}{|\text{Aut}(F)|}.$$

$\square$

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