Coarse geometry of groups and spaces - DRAFT VERSION

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Motivating Question

There are many natural examples and many interesting constructions of coarse embeddings. However, when we may wish to prove that there is no coarse embedding between spaces there are few known obstructions. To demonstrate this, consider products of real hyperbolic spaces, 3-regular trees and Euclidean spaces

$$X = \Pi_{j=1}^{k}\mathbb{H}^{l_j} \times (T_3)^m \times \mathbb{R}^n, \quad X' = \Pi_{j'=1}^{k'}\mathbb{H}^{l'_{j'}} \times (T_3)^{m'} \times \mathbb{R}^{n'}$$

It is natural to ask for which values of k, l, m, n there exists a coarse embedding from X to X'.

Let us start with four positive results:

(i) there are quasi-isometric embeddings $\mathbb{R}^n \to \mathbb{R}^m$, $\mathbb{R}^n \to (T_3)^m$, $(T_3)^n \to (T_3)^m$ and $\mathbb{H}^n \to \mathbb{H}^m$ whenever $n \leq m$.

(ii) there is a coarse embedding $\mathbb{R}^n \to \mathbb{H}^l$ whenever $n < l$,

(iii) there is a quasi-isometric embedding $\mathbb{H}^l \to (T_3)^m$ whenever $l \leq m$ [Buyalo-Dranishnikov-Schroeder]

(iv) there is a quasi-isometric embedding $\mathbb{H}^l \to \Pi_{j'=1}^{k'_{j'}}\mathbb{H}^{l_{j'}}$ whenever

$$l - 1 \leq \sum_{j'=1} k'_{j'}(l_{j'} - 1).$$

A natural question to ask is whether there are any pairs X, X' where there is a coarse embedding $X \to X'$ but there is not one obtained as a concatenation of the above maps.

Below I will try to list some of the smallest examples where there is no known example of a coarse embedding:

(i) $T \to \mathbb{R}^d$ for any d,

(ii) $\mathbb{H}^m \to \mathbb{R}^d$ and any $m \geq 2$, $d \geq 1$,

(iii) $\mathbb{R}^d \to \mathbb{H}^m$ for any $d \geq m$,

(iv) $T \times T \to \mathbb{H}^m$ for any m,

(v) $T \times \mathbb{R} \to \mathbb{H}^m$ for any m,

(vi) $\mathbb{H}^3 \to \mathbb{H}^2 \times \mathbb{R}^d$ for any d,

(vii) $\mathbb{H}^2 \to T \times \mathbb{R}^d$ for any d,

(viii) $\mathbb{H}^3 \to \mathbb{H}^2 \times T$.