UNIPOTENT DEGREES OF IMPRIMITIVE COMPLEX REFLECTION GROUPS

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Abstract. In the representation theory of finite groups of Lie type $G(q)$ the unipotent characters play a fundamental role. Their degrees, seen as polynomials in q, are only dependent on the Weyl group of $G(q)$. G. Lusztig (*Astérisque* **212** (1993) 191–203) has shown that one can define unipotent degrees for a general finite Coxeter group. In this article we construct, for the two infinite series of n-dimensional complex reflection groups that are generated by n reflections, a set of unipotent degrees, with the same combinatorial properties as the unipotent character degrees of a finite Weyl group. In particular they are related by a Fourier transform matrix to the fake degrees, and together with the appropriate eigenvalues of Frobenius they provide a representation of $\mathrm{SL}_2(\mathbb{Z})$.

1. Introduction

In this article we introduce a new type of polynomial associated to the finite n-dimensional imprimitive complex reflection groups W that are generated by n reflections, the *unipotent degrees*. If W is the Weyl group of a finite group of Lie type G, these degrees match the degrees of the unipotent characters of G. Generally, the unipotent degrees satisfy many of the formal properties of the degrees of the unipotent characters of the groups of Lie type. Their parametrization is based on e-*symbols*, combinatorial objects that generalize the notions of partitions and Lusztig's symbols for classical groups. The results proved here always include as a special case statements about the unipotent degrees of classical groups; even in this case these results are partly new (such as Theorems 3.14, 4.21 and 6.10). The language of e-symbols also allows us to establish a conjectural formula for the relative degrees of the *cyclotomic Hecke algebras* introduced in [2, 4, 1] for the complex reflection groups $G(e,p,n)$.

We now describe the problem and results in detail. Let W be a finite complex reflection group, and let K be a number field that is a splitting field for an n-dimensional reflection representation of W. For each irreducible character χ of W we have the following *fake degree*:

$$R_\chi(q) := (-1)^n \frac{P(W)}{|W|}(q-1)^n \sum_{w\in W} \frac{\chi(w)}{\overline{\overline{\det}}(1-qw)},$$

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where $P(W)$ denotes the Poincaré polynomial of W, a polynomial in $K[q]$. In the case of the symmetric group $W = S_n$, the fake degrees are the degrees of the unipotent characters of the general linear group $\mathrm{GL}_n(q)$. However, already for $W = Z_2 \wr S_n$ of type B_n, to go from the unipotent characters of the associated groups of Lie type $\mathrm{SO}_{2n+1}(q)$ and $\mathrm{Sp}_{2n}(q)$ to the fake degrees requires a non-trivial Fourier transform matrix (see for example [10, Prop. 2.9(i)]). These Fourier transform matrices, together with the diagonal matrix of the eigenvalues of Frobenius, produce in the case of a Weyl group W a representation of the group $\mathrm{PSL}_2(\mathbb{Z})$.

The results of this article can be summarized by the statement that, for each series of imprimitive n-dimensional complex reflection groups $W_n = Z_e \wr S_n$, $n, e \geq 1$, we have the following:

Theorem 1.1. *For $W_n = Z_e \wr S_n$ there exists a set $\mathrm{Uch}(W_n)$ of polynomials $\gamma \in \mathbb{Q}(\zeta)[q]$, $\zeta = \exp(2\pi \mathrm{i}/e)$, with the following properties:*

(a) *The γ all divide $q^{N^*} P(W_n)(q-1)^n$, where N^* is the number of reflection hyperplanes of W_n and $P(W_n)$ denotes the Poincaré polynomial of W_n.*
(b) *The set $\pm\mathrm{Uch}(W_n)$ is invariant under all field automorphisms of $\mathbb{Q}(\zeta)$.*
(c) *(Ennola property) For each $\gamma(q) \in \mathrm{Uch}(W_n)$, we have that $\pm\gamma(\zeta q)$ lies in $\mathrm{Uch}(W_n)$.*
(d) *(Harish-Chandra theories) The degree γ satisfies a Φ-Harish-Chandra theory, for every irreducible cyclotomic polynomial Φ over K. The specialization of the parameters occurring in the cyclotomic Hecke algebra are all integral powers of q, multiplied by an eth root of unity.*
(e) *(Fourier transform) The degrees and the fake degrees each fall naturally into families. Inside each family the γ and the R_χ are linked by a Fourier transform; in particular, they span the same subspace of the space of polynomials $\mathbb{Q}(\zeta)[q]$, and we have the relation*
$$\sum_\gamma \gamma \cdot \bar{\gamma} = \sum_\chi R_\chi^2,$$
taken family-wise.
(f) *(Eigenvalues of Frobenius) We can assign an eigenvalue of Frobenius $\mathrm{Fr}(\gamma)$ to each degree γ. The eigenvalues of Frobenius Fr are also compatible with all Φ-Harish-Chandra theories.*
(g) *($\mathrm{SL}_2(\mathbb{Z})$ representation) Denoting by T the Fourier transform matrix, and U the product of T with the diagonal matrix of eigenvalues of Frobenius, we have*
$$T^4 = U^3 = [T^2, U] = 1,$$
i.e., T and U yield a representation of $\mathrm{SL}_2(\mathbb{Z})$.
(h) *In the special case of a real reflection group (i.e., for $e = 1, 2$) the degrees $\mathrm{Uch}(W_n)$ agree with the degrees of the unipotent characters of the associated groups of Lie type.*

The degrees $\mathrm{Uch}(W_n)$ are introduced in Definition 3.8, and by Corollary 3.17, they are given by polynomials. The statement (a) follows from the hook formula 3.12 and Proposition 2.24, part (c) is Corollary 3.11. Part (d) is made precise in

Theorem 3.14. Part (e) is Theorem 4.17, (f) is given in Theorem 4.21 and (g) is Corollary 4.15.

In the special case of a real reflection group (a) and (b) are trivial, and the statements (c), (e) and (g) were proved by Lusztig, where here we have $T^2 = 1$, so that it induces a representation of $\mathrm{PSL}_2(\mathbb{Z})$. The statements (d) and (f) were previously known only for the usual Harish-Chandra theory, i.e., the case $\Phi = q-1$. The statements proved here are consistent with the conjecture given in [4, d-HV6] on derived equivalences for blocks of classical groups, and can be regarded as further evidence for the correctness of this conjecture.

The Fourier transform matrices occurring in (e) contain as a special case the Fourier matrices introduced by Lusztig for the elementary abelian group Z_2^{2m}, and it is apparent that they cannot be derived from a finite group in general.

We see also that the degrees $\mathrm{Uch}(W_n)$ satisfy (in spirit) the postulates of Lusztig on unipotent character degrees given in [13, 2.1–2.8]. The properties 2.4, 2.6 and 2.7 given there are special cases of Theorem 1.1(d).

We obtain a very similar result for the second infinite series $G(e, e, n)$ of n-dimensional reflection groups that are generated by n reflections. However, here we also get twisted variants, which correspond to groups of Lie type ${}^2D_n$ in the case $e = 2$, so that the formulation of the results is more complicated. Overall we still have analogues of all of the statements in Theorem 1.1, but with the exception of the compatibility of the eigenvalues of Frobenius and Harish-Chandra theory.

As special cases we obtain statements about the unipotent character degrees of D_n and ${}^2D_n$, as well as the finite Coxeter groups $I_2(p)$. This therefore provides a parametrization of the unipotent characters of ${}^2B_2$, G_2 and ${}^2G_2$ by symbols. Moreover, our degrees specialize in the case of $I_2(p)$ to those unipotent degrees described in [13], and our Fourier transform specializes to the *exotic Fourier transform* introduced by Lusztig in [14]. As a corollary we have that Lusztig's exotic Fourier transform actually does transform the unipotent degrees to the fake degrees (Example 6.29).

This article is structured as follows: the next section introduces the group W_n and the associated reflection data (see [5]). We give a formula for the fake degrees. Afterwards we recall the definition of the cyclotomic Hecke algebra for W_n, to formulate a conjecture on the relative degrees. In the third section we define e-symbols, hooks and cores; this then allows the introduction of unipotent degrees. We prove the hook formula and in particular Theorem 3.14, which states the validity of all Φ-Harish-Chandra theories. Section 4 examines the Fourier transform and the eigenvalues of Frobenius. This is done separately for each *family* of unipotent degrees. In the fifth section we introduce the (twisted) reflection data ${}^tI_n^e$ and strengthen the previously made conjecture on the relative degrees of cyclotomic Hecke algebras.

Finally, the last section contains the definition and proof of the properties of the unipotent degrees of ${}^tI_n^e$. The Ennola property is given in Corollary 6.7, the Harish-Chandra theories will be shown in Theorem 6.10, the existence of the Fourier transform by Theorem 6.26, and the representation of $\mathrm{SL}_2(\mathbb{Z})$ will be verified in Corollary 6.25. Finally Alvis–Curtis duality can be formally generalized to the unipotent degrees of ${}^tI_n^e$.

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2. Reflection data of type $B_n^{(e)}$

In this section we introduce the reflection group W_n and its associated reflection datum $B_n^{(e)}(q)$. The fake degrees of $B_n^{(e)}(q)$ can be described by a combinatorial formula. This part is inspired by the presentation of Lusztig for the case $e = 2$ in [10]. We then recall the definition of the generic cyclotomic Hecke algebra $\mathscr{H}$ of W_n and formulate a conjecture about the exact form of the relative degrees of $\mathscr{H}$.

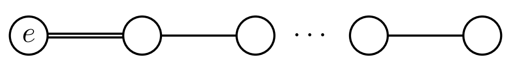
2A. **The reflection group $G(e, 1, n)$.** In this subsection let n and e be two fixed positive integers, and let $W_n = Z_e \wr S_n$ be the wreath product of the cyclic group Z_e by the symmetric group S_n. We denote by Z the kernel of the semidirect product $Z_e^n \rtimes S_n$. We identify W_n with the transitive subgroup of S_{en} generated by the n-cycle

$$s_1 = (1, n + 1, 2n + 1, \ldots, (e - 1)n + 1)$$

and the involutions

$$\begin{aligned} s_2 &= (1, 2)(n + 1, n + 2) \ldots ((e - 1)n + 1, (e - 1)n + 2),\\ \vdots \quad & \quad \vdots \\ s_n &= (n - 1, n) \ldots (en - 1, en). \end{aligned}$$

In this representation one clearly sees that the conjugacy classes of W_n are parametrized by e-tuples of partitions $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$ with $\alpha_i \vdash n_i$ and $\sum n_i = n$. The generators $s_1, \ldots, s_n$ satisfy the relations implied by the diagram

(2.1) 

and these yield a presentation of W_n on the n generators $s_1, \ldots, s_n$ (see [4, 3A]).

One obtains a faithful irreducible matrix representation of W_n over $K := \mathbb{Q}(\zeta)$, $\zeta := \exp(2\pi \mathrm{i}/e)$, by mapping s_1 to the diagonal matrix $\mathrm{diag}(\zeta, 1, \ldots, 1)$, and s_i, $i = 2, ..., n$, to the permutation matrix which swaps the $(i - 1)$th and ith basis vectors, for $V := K^n$. In particular this shows that W_n is an irreducible complex reflection group, since all generators in this representation have $n - 1$ eigenvalues equal to 1. In the notation of Shephard and Todd [16] this is $G(e, 1, n)$.

The linear characters of W_n are given by $\{\gamma_i, \varepsilon\gamma_i \mid 0 \leqslant i \leqslant e - 1\}$ with

$$(2.2) \qquad \begin{aligned} \gamma_i &: W_n \to K, \qquad s_1 \mapsto \zeta^i,\ s_k \mapsto 1 \text{ for } k \neq 1,\\ \varepsilon &: W_n \to K, \qquad s_1 \mapsto 1,\ s_k \mapsto -1 \text{ for } k \neq 1. \end{aligned}$$

We have that $\varepsilon\gamma_1 = \det_V$ is the determinant of the reflection representation of W_n. In general the irreducible complex characters χ of W_n are parametrized by e-tuples of partitions $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$, with $\alpha_i = (\alpha_{i1} \leqslant \alpha_{i2} \leqslant \cdots \leqslant \alpha_{im_i}) \vdash n_i$ and $\sum_{i=0}^{e-1} n_i = n$. Let $n_i \geq 0$ be given such that $\sum n_i = n$, and let $W_{n_0} \times \cdots \times W_{n_{e-1}}$

be the Levi subgroup of W_n corresponding to the Young subgroup $S_{n_0}\times\cdots\times S_{n_{e-1}}$ of S_n in the wreath product $Z_e \wr S_n$. An irreducible character χ_i of S_{n_i} can be considered as a character of W_{n_i} by virtue of the projection $W_{n_i}\to S_{n_i}$. For any e-tuple $(\chi_0,\dots,\chi_{e-1})$ of irreducible characters of $S_{n_0},\dots,S_{n_{e-1}}$ the induction of $\chi_0\#(\chi_1\otimes\gamma_1)\#\cdots\#(\chi_{e-1}\otimes\gamma_{e-1})$ to W_n is an irreducible character. This will be denoted by χ_α if the irreducible character χ_i of S_{n_i} is parametrized by the partition $\alpha_i\vdash n_i$:

$$\chi_{\boldsymbol{\alpha}} = \mathrm{Ind}_{W_{n_0}\times\cdots\times W_{n_{e-1}}}^{W_n}\left(\chi_0\#(\chi_1\otimes\gamma_1)\#\cdots\#(\chi_{e-1}\otimes\gamma_{e-1})\right). \tag{2.3}$$

2B. **The associated reflection datum.** A *reflection datum* is a finite-dimensional complex vector space V, together with a coset $W\phi$ of a complex reflection group W on V in the automorphism group of (V,W) (see [5]). Let $\mathbb{G}:=B_n^{(e)}(q):=(V,W_n)$ be the reflection datum associated to W_n with the trivial automorphism. Any $w\in W_n$ can be written uniquely as $w=zy$ with $z\in Z$ and $y\in S_n$. If y is an n-cycle, in the above reflection representation we obviously get $w^n=(zy)^n=\det_V(z)\cdot 1_V$, and hence $\det_V(q-zy)=q^n-\det_V(z)$. Therefore, by induction we can calculate the characteristic polynomial of any $w\in W_n$ of type $\boldsymbol{\alpha}=(\alpha_0,\dots,\alpha_{e-1})$, $\alpha_i=(\alpha_{i1},\alpha_{i2},\dots,\alpha_{im_i})$, to be

$$\det_V(q-w)=\prod_{i=0}^{e-1}\sum_{k=1}^{m_i}(q^{\alpha_{ik}}-\zeta^i). \tag{2.4}$$

Let Φ be a cyclotomic polynomial over K and ξ a zero of Φ in a suitable extension field of K, therefore a primitive dth root of unity for the appropriate d. We set $c:=\gcd(e,d)$, $d':=d/c$ and $\xi_c:=\xi^{d'}$. Then there is a Sylow Φ-torus $\mathbb{S}$ of $\mathbb{G}$ (see [5]) of order $(q-\xi)^{\lfloor n/d'\rfloor}$. The centralizer of $\mathbb{S}$ in $\mathbb{G}$ is a reflection datum of type $\mathbb{T}\times\mathbb{H}$, where $\mathbb{T}$ is a torus of order $(q^{d'}-\xi_c)^{\lfloor n/d'\rfloor}$ and $\mathbb{H}$ has type $B_m^{(e)}$ with $m=n-\lfloor n/d'\rfloor$. The Φ-split Levi subgroups thus have the structure

$$\mathbb{L}=\mathrm{GL}_{n_1}(\xi_c^{-1}q^{d'})\times\cdots\times\mathrm{GL}_{n_r}(\xi_c^{-1}q^{d'})\times B_m^{(e)}(q), \tag{2.5}$$

with Weyl group $W_{\mathbb{L}}=S_{n_1}\times\cdots\times S_{n_r}\times W_m$, where $\sum_{i=1}^r d'n_i=n-m$. It follows that the relative Weyl group $W_{\mathbb{G}}(\mathbb{L})=\mathscr{N}_W(W_{\mathbb{L}})/W_{\mathbb{L}}$ of such a Levi subgroup is of the form

$$W_{\mathbb{G}}(\mathbb{L})=Z_{ed'}\wr S_{k_1}\times\cdots\times Z_{ed'}\wr S_{k_n} \tag{2.6}$$

with $k_i:=|\{j\mid n_j=i\}|$.

2C. **Fake degrees and generic degrees.** For a finite subset S of $\mathbb{N}$ we define

$$\begin{aligned}\Delta(S,q)&:=\prod_{\substack{\lambda,\lambda'\in S\\ \lambda'<\lambda}}(q^\lambda-q^{\lambda'}),\\ \Theta(S,q)&:=\prod_{\lambda\in S}\prod_{h=1}^{\lambda}(q^h-1).\end{aligned} \tag{2.7}$$

For an irreducible character χ of W_n set

$$\delta_\chi(q) := (-1)^n \langle \chi, \mathrm{tr}_{SV}\rangle_{\mathbb{G}} = (-1)^n \frac{1}{|W_n|} \sum_{w\in W_n} \frac{\chi(w)}{\overline{\det}_V(1-qw)}, \tag{2.8}$$

so that therefore

$$R_\chi := (-1)^n \langle \chi, \mathrm{tr}_{RG}\rangle_{\mathbb{G}} = |B_n^{(e)}|_{q'}\delta_\chi = \prod_{i=1}^{n} (q^{ei}-1)\delta_\chi \tag{2.9}$$

denotes the fake degree of χ in q (see [5]). We assign to each partition $\alpha = (\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k)$ the sequence of β-numbers $(\lambda_i := \alpha_i + i - 1)$.

Proposition 2.10. *Let $\chi = \chi_{\boldsymbol{\alpha}} \in \mathrm{Irr}(W_n)$ be parametrized by the tuple of partitions $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$, $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im_i}) \vdash n_i$, and $S_i = (\lambda_{i1}, \ldots, \lambda_{im_i})$ be the corresponding sequences of β-numbers. Then we have*

$$\delta_\chi(q) = \prod_{i=0}^{e-1} \frac{\Delta(S_i, q^e)\cdot q^{in_i}}{\Theta(S_i, q^e)\cdot q^{e\binom{m_i-1}{2}+e\binom{m_i-2}{2}+\cdots}}. \tag{2.11}$$

Proof. Let $\mathbb{L}_i = (V_i, W_{n_i})$ be the reflection datum of the reflection subgroup W_{n_i} of W_n on $V_i \leqslant V$, and let $\mathbb{L} = (V, W_{n_0} \times \cdots \times W_{n_{e-1}})$ be that of the direct product $W_{n_0} \times \cdots \times W_{n_{e-1}}$. By [5] we have

$$\begin{aligned}
\delta_\chi(q) &= (-1)^n \langle \chi, \mathrm{tr}_{SV}\rangle_{\mathbb{G}} = (-1)^n \langle 1, \mathrm{tr}_{SV}\rangle_{\mathbb{G}} \cdot \langle \chi, \mathrm{tr}_{R\mathbb{G}}\rangle_{\mathbb{G}} \\
&= x^{N(\mathbb{G})} |\mathbb{G}|^{-1} \langle \chi_0 \#(\chi_1 \otimes \gamma_1)\# \cdots \#(\chi_{e-1}\otimes\gamma_{e-1}), \mathrm{Res}^{\mathbb{G}}_{\mathbb{L}} \mathrm{tr}_{R\mathbb{G}}\rangle_{\mathbb{L}} \\
&= x^{N(\mathbb{L})} |\mathbb{L}|^{-1} \prod_{i=0}^{e-1} \langle \chi_i \otimes \gamma_i, \mathrm{tr}_{R\mathbb{L}_i}\rangle_{\mathbb{L}_i} \\
&= \prod_{i=0}^{e-1} (-1)^{n_i} \langle \chi_i \otimes \gamma_i, \mathrm{tr}_{SV_i}\rangle_{\mathbb{L}_i},
\end{aligned}$$

so that we have reduced the statement to the case where $n_i = n$ for some $0 \leqslant i \leqslant e-1$ and $n_j = 0$ for $j \neq i$. Because of the known formula for $\delta_\chi(q)$ for S_n, i.e., for $e = 1$ (see for example [10, (2.2.1)]), the assertion for $n_i = n$ is equivalent to

$$\frac{1}{e^n} \sum_{u\in Z} \frac{\gamma_i(wu)}{\overline{\det}(1-qwu)} = \frac{q^{in}}{\overline{\det}(1-q^e w)} \qquad \text{for all } w \in S_n.$$

By induction it is sufficient to prove this for the case where w is an n-cycle. For such a case, we have

$$\det(1-qwu) = 1 - \gamma_1(u) q^n$$

by (2.4), and therefore

$$\frac{1}{e^n} \sum_{u\in Z} \frac{\gamma_i(wu)}{\overline{\det}(1-qwu)} = \frac{1}{e^n} \sum_{u\in Z} \frac{\gamma_i(u)}{1-\gamma_{e-1}(u) q^n} = \frac{q^{in}}{1-q^{en}}. \qquad \square$$

Now let $\mathscr{S} = (S_0, \ldots, S_{e-1})$ be a sequence of finite subsets of $\mathbb{N}$ with $|S_0| = m+1$, $|S_1| = \cdots = |S_{e-1}| = m$. We set

$$\delta_{\mathscr{S}}(q) := \prod_{i=0}^{e-1} \frac{\Delta(S_i, q^e)}{\Theta(S_i, q^e)} \cdot \frac{\prod_{i=0}^{e-1} \prod_{\lambda \in S_i} q^{i\lambda}}{q^{\binom{e(m-1)+1}{2} + \binom{e(m-2)+1}{2} + \cdots}}. \tag{2.12}$$

If $\boldsymbol{\alpha}$ is an e-tuple of partitions of n, then by adding zeroes to the α_i, we can always produce an e-tuple such that α_0 has exactly one more part than $\alpha_1, \ldots, \alpha_{e-1}$ (which all have the same number of parts), and that not every α_i has a zero as a part. If we associate an irreducible character $\chi_{\boldsymbol{\alpha}}$ of W_n with such a standardized $\boldsymbol{\alpha}$, to the sequence

$$\mathscr{S} := ((\lambda_{01}, \ldots, \lambda_{0,m+1}), (\lambda_{11}, \ldots, \lambda_{1m}), \ldots, (\lambda_{e-1,1}, \ldots, \lambda_{e-1,m})) \tag{2.13}$$

of β-numbers, our definitions obviously yield

$$\delta_{\chi_{\mathscr{S}}}(q) = \delta_{\mathscr{S}}(q). \tag{2.14}$$

We denote by $\tilde{a}(\chi_{\mathscr{S}})$ the multiplicity of the root 0 of $\delta_{\mathscr{S}}(q)$ at $q = 0$, and by $\tilde{A}(\chi_{\mathscr{S}})$ the degree in q of $R_{\mathscr{S}}(q) := (q-1)^n P(W_n) \delta_{\mathscr{S}}(q)$. From (2.12) we see

$$\begin{aligned}
\tilde{a}(\chi_{\mathscr{S}}) &= e \sum_{i=0}^{e-1} \sum_{\substack{\lambda, \lambda' \in S_i \\ \lambda' < \lambda}} \lambda' + \sum_{i=0}^{e-1} i \sum_{\lambda \in S_i} \lambda - \sum_{k=1}^{m-1} \binom{ek+1}{2}, \\
\tilde{A}(\chi_{\mathscr{S}}) &= e \sum_{i=0}^{e-1} \sum_{\substack{\lambda, \lambda' \in S_i \\ \lambda' < \lambda}} \lambda + \sum_{i=0}^{e-1} i \sum_{\lambda \in S_i} \lambda - \sum_{k=1}^{m-1} \binom{ek+1}{2} \\
&\quad - e \sum_{i=0}^{e-1} \sum_{\lambda \in S_i} \binom{\lambda+1}{2} + e \binom{n+1}{2}.
\end{aligned} \tag{2.15}$$

In addition, we note the relationship

$$\begin{aligned}
\sum_{i=0}^{e-1} \sum_{\lambda \in S_i} \lambda &= \sum_{i,j} \alpha_{ij} + \binom{m+1}{2} + (e-1)\binom{m}{2} \\
&= n + \frac{m(em - e + 2)}{2}
\end{aligned} \tag{2.16}$$

between the invariants of $\mathscr{S} = (S_0, \ldots, S_{e-1})$ and $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$.

Example 2.17. The character χ of the reflection representation of W_n on V corresponds via (2.3) for $e \geq 2$ to the tuple of partitions $\boldsymbol{\alpha} = ((n-1), (1), -, \ldots, -)$, with associated $\mathscr{S} = ((0, n), (1), (0), \ldots, (0))$. By (2.12) χ therefore has fake degree

$$R_{\mathscr{S}}(q) = \frac{q(q^{en} - 1)}{q^e - 1}$$

and $\tilde{a}(\chi) = 1$, $\tilde{A}(\chi) = en - e + 1$.

We now recall the definition of the cyclotomic Hecke algebra of the complex reflection group W_n using the diagram $B_n^{(e)}$ in (2.1) ([4, 4.1], see also [2]).

Definition 2.18. Let $\mathbf{u} = (u_0, \ldots, u_{e-1})$ be algebraically independent over $\mathbb{Q}$. We define the cyclotomic Hecke algebra $\mathscr{H}(W_n, \mathbf{u}) = \mathscr{H}(B_n^{(e)}, \mathbf{u})$ over $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ corresponding to the diagram $B_n^{(e)}$ as follows:

- $\mathscr{H}(W_n, \mathbf{u})$ is generated as a $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-algebra by the elements $\{T_i \mid 1 \leqslant i \leqslant n\}$.
- The T_i satisfy the braid relations implied by $B_n^{(e)}$, as well as

$$(T_1 - u_0)(T_1 - u_1) \ldots (T_1 - u_{e-1}) = 0,$$
$$(T_i - u_0)(T_i + 1) = 0 \quad \text{for } 2 \leqslant i \leqslant n.$$

Note that we have used a different normalization of each of the first eigenvalues of the T_i than in [4], which appears to be more suitable for our present purposes.

Finally we define for an e-tuple $\mathscr{S} = (S_0, \ldots, S_{e-1})$, constructed from $\boldsymbol{\alpha}$ as per (2.13), and for indeterminates $q, v_0, \ldots, v_{e-1}$, the rational function

$$D_{\mathscr{S}}(q;v_0, \ldots, v_{e-1}) :=$$

$$\frac{(-1)^{\binom{e}{2}\binom{m}{2}+n(e-1)}(q-1)^n \prod_{i=0}^{e-1}\prod_{j=i}^{e-1} \prod_{\substack{(\lambda,\mu)\in S_i\times S_j \\ \mu<\lambda \text{ if } i=j}} (q^\lambda v_i - q^\mu v_j) \cdot \prod_{i=0}^{e-1} v_i^n}{q^{\binom{e(m-1)+1}{2}+\binom{e(m-2)+1}{2}+\cdots} \prod_{i,j=0}^{e-1} \prod_{\lambda\in S_i} \prod_{k=1}^{\lambda} (q^k v_i - v_j) \cdot \prod_{i=0}^{e-1}\prod_{j=i+1}^{e-1} (v_i - v_j)^m} \tag{2.19}$$

The definition of $D_{\mathscr{S}}$ is invariant under simultaneous shifting of the S_i, so that $D_{\mathscr{S}}$ therefore only depends on the tuple of partitions giving rise of $\mathscr{S}$. Furthermore,

$$D_{\mathscr{S}}(q; v_0, \ldots, v_{e-1}) = D_{\mathscr{S}}(q; xv_0, \ldots, xv_{e-1})$$

for all invertible x. The $D_{\mathscr{S}}$ should be the relative degrees of the cyclotomic Hecke algebra $\mathscr{H}(B_n^{(e)}; \mathbf{u})$ with respect to a suitable quasi-symmetric basis (see [4, 1C]). To formulate this we define the specialization

$$f : \mathscr{H}(W_n, \mathbf{u}) \to KW_n, \qquad u_i \mapsto f(u_i) := \zeta^i,$$

extending linearly.

Conjecture 2.20. *$\mathscr{H}(W_n; \mathbf{u})$ has a basis $\{T_w \mid w \in W_n\}$ with the following properties:*

(1) *The linear map defined by $f(T_w) \mapsto w$ is an algebra isomorphism*

$$f(\mathscr{H}(W_n; \mathbf{u})) \xrightarrow{\sim} \mathbb{Z}[\zeta]W_n.$$

(2) *The relative degree of the character $\chi_{\boldsymbol{\alpha}}$ of the generic cyclotomic Hecke algebra $\mathscr{H}(W_n; (u_0, u_1, \ldots, u_{e-1}))$, with respect to the linear form defined by*

$$t_{\mathbf{u}} : \mathscr{H}(W_n; \mathbf{u}) \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}], \qquad t_{\mathbf{u}}(T_w) := \delta_{w,1},$$

is $D_{\mathscr{S}}(u_0; 1, u_1, \ldots, u_e - 1)$, where $\mathscr{S} = (S_0, \ldots, S_{e-1})$ denotes a sequence of β-numbers associated to $\boldsymbol{\alpha}$ as per (2.13).

For $e = 1$ the conjectured formula is

$$D_S(q) = \frac{(q-1)^n \prod_{\substack{\lambda,\mu \in S \\ \mu<\lambda}} (q^\lambda - q^\mu)}{q^{\binom{m}{2}+\binom{m-1}{2}+\cdots} \prod_{\lambda\in S}\prod_{k=1}^{\lambda}(q^k-1)},$$

and this is indeed the generic degree of $\mathscr{H}(A_{n-1};q)$ with respect to the standard basis $\{T_w \mid w \in S_n\}$ (see for example [6, 13.5] and note that here $|S| = m+1$, and the definition of the generic degree in [6] is obtained from ours by multiplication by the Poincaré polynomial of $\mathbb{G}$). For $e = 2$ we get that $D_{S_0,S_1}(q;1,-r^{-1})$, after manipulation, is equal to

$$\frac{(q-1)^n \cdot \Delta(S_0,q)\Delta(S_1,q)\cdot r^{-\binom{m}{2}} \prod_{(\lambda,\mu)\in S_0\times S_1}(q^\lambda r + q^\mu)}{(1+r)^m q^{\binom{2m-1}{2}+\binom{2m-3}{2}+\cdots} \prod_{\lambda\in S_0}\prod_{k=1}^{\lambda}(q^k-1)(q^k r+1) \prod_{\mu\in S_1}\prod_{k=1}^{\mu}(q^k-1)(q^k r^{-1}+1)}$$

and the conjecture follows from 4.6 (using 3.2.2) in [10]. Furthermore, Conjecture 2.20 holds for $n = 1$ [4, 2.4] and in the cases $(n,e) = (2,3)$, $(2,4)$, $(3,3)$.

2D. **Generic degrees for $B_n^{(e)}(q)$.** The Hecke algebra of the reflection datum $\mathbb{G}$ is obtained from $\mathscr{H}(W_n,\mathbf{u})$ by the specialization

$$u_0 \mapsto q, \qquad u_i \mapsto \zeta^i \text{ for } 1 \leqslant i \leqslant e-1$$

(see also [5]). We write $D_{\mathscr{S}}(q) := D_{\mathscr{S}}(q;1,\zeta,\ldots,\zeta^{e-1})$, for short, for the (conjectural) generic degree

$$D_{\mathscr{S}}(q) = \frac{(-1)^{\binom{e}{2}\binom{m}{2}}(q-1)^n \cdot \prod_{i=0}^{e-1}\prod_{j=i}^{e-1} \prod_{\substack{(\lambda,\mu)\in S_i\times S_j \\ \mu<\lambda \text{ if } i=j}} (q^\lambda\zeta^i - q^\mu\zeta^j)}{\tau(e)^m \cdot q^{\binom{e(m-1)+1}{2}+\binom{e(m-2)+1}{2}+\cdots} \cdot \prod_{i=0}^{e-1}\Theta(S_i,q^e)} \tag{2.21}$$

of the specialized algebra $\mathscr{H}(W_n,q) := \mathscr{H}(W_n,(q,\zeta,\ldots,\zeta^{e-1}))$, where for brevity we set

$$\tau(e) := \prod_{i=0}^{e-1}\prod_{j=i+1}^{e-1}(\zeta^i-\zeta^j) = \zeta_4^{\binom{e-1}{2}}\sqrt{e}^e, \qquad \zeta_4 := \mathrm{i}. \tag{2.22}$$

For a sequence $\mathscr{S} = (S_0,\ldots,S_{e-1})$ of β-numbers let $a(\mathscr{S})$ be the multiplicity of the zero of $D_{\mathscr{S}}(q)$ at $q = 0$. If Conjecture 2.20(2) is correct, then $a(\mathscr{S})$ agrees with the a-function of the irreducible characters $\chi_{\boldsymbol{\alpha}}$, which is parametrized by the tuple of partitions associated to $\mathscr{S}$. Furthermore, if $A(\mathscr{S})$ denotes the degree in

q of $P(W_n)D_{\mathscr{S}}(q)$, then the validity of Conjecture 2.20(2) implies that this is the A-value of $\chi_{\boldsymbol{\alpha}}$. Obviously we have

$$
\begin{aligned}
a(\mathscr{S}) &= \sum_{\{\lambda,\mu\}} \min(\lambda,\mu) - \sum_{k=1}^{m-1} \binom{ek+1}{2}, \\
A(\mathscr{S}) &= \sum_{\{\lambda,\mu\}} \max(\lambda,\mu) - \sum_{k=1}^{m-1} \binom{ek+1}{2} - e\sum_{\lambda\in\mathscr{S}} \binom{\lambda+1}{2} + e\binom{n+1}{2},
\end{aligned}
\tag{2.23}
$$

where the first sums run over all unordered pairs of entries from $\mathscr{S}$. In particular, given two β-sets $\mathscr{S}$ and $\mathscr{S}'$, whose sets of entries coincide with multiplicities, then both the a and A-values agree. (Such β-sets will be defined to lie in the same family in Section 4C.)

By (2.3) the two characters of W_n parametrized by $\mathscr{S} = (S_0, S_1, \ldots, S_{e-1})$ and

$$\bar{\mathscr{S}} := (S_0, S_{e-1}, \ldots, S_1)$$

are complex conjugates of one another, $\overline{\chi_{\mathscr{S}}} = \chi_{\bar{\mathscr{S}}}$. The following observation generalizes a result which for Coxeter groups is well known (see [6, 11.3], [11, 2.9]).

Proposition 2.24. (a) *We have $\tilde{a}(\chi_{\mathscr{S}}) \geq a(\mathscr{S})$ for all $\chi_{\mathscr{S}} \in \mathrm{Irr}(W_n)$. Equality occurs exactly for $\mathscr{S}$ of the form*

$$
\mathscr{S} = \begin{pmatrix}
\lambda_{e1} & \cdots & \lambda_{em} & \lambda_{e,m+1} \\
\lambda_{11} & \cdots & \lambda_{1m} & \\
\vdots & & & \\
\lambda_{e-1,1} & \cdots & \lambda_{e-1,m} &
\end{pmatrix}
$$

with

$$\lambda_{ij} \leqslant \lambda_{kl} \qquad \text{for } j < l \text{ or } j = l,\ i > k. \tag{2.25}$$

Such an $\mathscr{S}$ is then called special.

(b) *We have $\tilde{A}(\chi_{\mathscr{S}}) \leqslant A(\mathscr{S})$ for all $\chi_{\mathscr{S}} \in \mathrm{Irr}(W_n)$. We have equality precisely if $\bar{\mathscr{S}}$ is special, and such an $\mathscr{S}$ is called* cospecial.

(c) *For all $\chi_{\mathscr{S}} \in \mathrm{Irr}(W_n)$, we have*

$$\tilde{a}(\chi_{\mathscr{S}}) + \tilde{A}(\chi_{\mathscr{S}}) + \tilde{a}(\chi_{\bar{\mathscr{S}}}) + \tilde{A}(\chi_{\bar{\mathscr{S}}}) = a(\mathscr{S}) + A(\mathscr{S}) + a(\bar{\mathscr{S}}) + A(\bar{\mathscr{S}}).$$

(d) *We have $a(\mathscr{S}) \leqslant N^*$ and $A(\mathscr{S}) \leqslant N$. Equality occurs exactly for $\mathscr{S}$ that are equivalent to the sequence of β-numbers*

$$S_0 = (0, 1, \ldots, n), \qquad S_i = \begin{cases} (0, \ldots, n-1) & i \neq j \\ (1, \ldots, n) & i = j, \end{cases}$$

for some $1 \leqslant j \leqslant e-1$.

Proof. By (2.15) and (2.23) we have

$$\begin{aligned}\tilde{a}(\chi_{\mathscr{S}}) - a(\mathscr{S}) &= e\sum_{i=1}^{e}\sum_{\substack{\lambda,\lambda'\in S_i\\ \lambda'<\lambda}}\lambda' + \sum_{i=1}^{e} i\sum_{\lambda\in S_i}\lambda - \sum_{i\leqslant j}\sum_{\substack{\lambda\in S_i,\,\mu\in S_j\\ \mu<\lambda \text{ if } i=j}}\min(\lambda,\mu)\\ &= \sum_{i=1}^{e}\sum_{j=1}^{m}(e(m-j)+i)\lambda_{ij} - \sum_{i\leqslant j}\sum_{\substack{\lambda\in S_i,\,\mu\in S_j\\ \mu<\lambda \text{ if } i=j}}\min(\lambda,\mu).\end{aligned}$$

If $\mathscr{S}$ has the form specified in (a), the difference above obviously vanishes. An $\mathscr{S}$ not of this form is obtained from a special one by finitely many swaps of two entries, so that the greater of the two entries is moved to the position that is lower according to (2.25). The first sum in the above expression becomes strictly larger, while the second remains constant. The second part follows in an analogous way.

For part (c) we have by (2.23)

$$a(\mathscr{S}) + A(\mathscr{S}) = em\sum_{\lambda\in\mathscr{S}}\lambda - 2\sum_{k=1}^{m-1}\binom{ek+1}{2} - e\sum_{\lambda\in\mathscr{S}}\binom{\lambda+1}{2} + e\binom{n+1}{2}.$$

On the other hand, from (2.15) we get

$$\begin{aligned}\tilde{a}(\chi_{\mathscr{S}}) + \tilde{A}(\chi_{\mathscr{S}}) = \sum_{i=0}^{e-1}&(em + e\delta_{i,0} - e + 2i)\sum_{\lambda\in S_i}\lambda\\ &- 2\sum_{k=1}^{m-1}\binom{ek+1}{2} - e\sum_{\lambda\in\mathscr{S}}\binom{\lambda+1}{2} + e\binom{n+1}{2}.\end{aligned}$$

The claim follows by substituting the definition of $\bar{\mathscr{S}}$.

For the final part, we start from the tuple $(S_0,\dots,S_{e-1})$ with

$$S_0 = (0,1,\dots,n), \qquad S_i = (0,\dots,n-1) \text{ for } i>0,$$

associated to the trivial partition $\boldsymbol{\alpha} = (-,\dots,-)$. We reach all parameters for $\mathrm{Irr}(W_n)$ by raising n times, one after another, an entry of $\mathscr{S}$. If the entry to be changed is ith in order of size (in its set), then the a-value is raised by $en+1-i$. It is clear that a maximal a-value is reached, when one has raised the smallest possible entry. This leads exactly to the specified form of $\mathscr{S}$, and to the a-value $n + e\binom{n}{2}$. It is easy to see that this is equal to the number of cyclic subgroups of W_n that are generated by reflections, hence equal to the number N^* of reflection hyperplanes. □

3. Unipotent degrees

The description of the unipotent degrees is based on e-symbols, a generalization of the usual symbols for classical groups of Lie type. We define the notions of symbols, the ζ^j-hooks and -cores. The degrees $\mathrm{Uch}(B_n^{(e)})$ are defined via a combinatorial formula based on symbols of content 1. We show that the degrees satisfy the hook formula and the theorem on degrees for cyclotomic Hecke algebras, and hence the generalized Harish-Chandra theories.

3A. **e-symbols, ζ^j-hooks and (l,ζ^j)-cores.** An *e-symbol* is an ordered sequence $\mathscr{S}=(S_0,\ldots,S_{e-1})$ of e strictly increasing finite sequences of natural numbers $S_i=(\lambda_{i1},\ldots,\lambda_{im_i})$ which is also presented in the form

$$\mathscr{S}=\begin{pmatrix}\lambda_{01} & \cdots & \lambda_{0m_0}\\ \lambda_{11} & \cdots & \lambda_{1m_1}\\ \vdots & & \\ \lambda_{e-1,1} & \cdots & \lambda_{e-1,m_{e-1}}\end{pmatrix}. \tag{3.1}$$

In this form we also call the S_i the *rows* of the e-symbol. We write $|S_i|:=m_i$ for the row lengths. If the parameter e is obvious from the context we speak simply of *symbols*. We define equivalence classes of symbols via the symmetric, transitive closure of the following two operations: cyclic permutation of the S_i in $\mathscr{S}$, and simultaneous shifting of all S_i by $(\lambda_{i1},\ldots,\lambda_{i,m_i})\mapsto(0,\lambda_{i1}+1,\ldots,\lambda_{i,m_i}+1)$.

The *content* of a symbol $\mathscr{S}$ as in (3.1) is $\mathrm{ct}(\mathscr{S})=m_0+m_1+\cdots+m_{e-1}$. The content is not constant on equivalence classes, but the congruence class $\mathrm{ct}(\mathscr{S})$ mod e is. The *rank* of $\mathscr{S}$ is

$$\mathrm{rk}(\mathscr{S}):=\sum_{i,j}\lambda_{ij}-\left\lfloor\frac{(\mathrm{ct}(\mathscr{S})-1)(\mathrm{ct}(\mathscr{S})-e+1)}{2e}\right\rfloor. \tag{3.2}$$

One readily convinces oneself that the rank is an invariant of the equivalence class. Because of (2.16), in the case where $\mathscr{S}$ is associated to an e-tuple $\boldsymbol{\alpha}$ of partitions of n, we have $\mathrm{rk}(\mathscr{S})=n$.

Let $\zeta:=\exp(2\pi\mathrm{i}/e)$ be a primitive eth root of unity. A *ζ^j-hook* of an e-symbol $\mathscr{S}$ is a triple (i,λ,μ) with $\lambda\in S_i$, $\mu\notin S_{i+j}$, $\lambda>\mu$ (where the indices of the S are taken modulo e). The difference $\lambda-\mu$ is called the *length* of the ζ^j-hook. If (i,λ,μ) is a ζ^j-hook of $\mathscr{S}$, we call

$$\mathscr{S}':=(S_0,\ldots,S_i\setminus\{\lambda\},\ldots,,S_{i+j}\cup\{\mu\},\ldots,S_{e-1})$$

the symbol obtained by the removal of (i,λ,μ). Obviously the removal of ζ^i-hooks also induces a well-defined map on equivalence classes of symbols. The rank of a symbol decreases upon removal of a hook by exactly the length of it.

The *ζ^0-core* of $\mathscr{S}$ is the (uniquely determined) symbol that is obtained from $\mathscr{S}$ by removing all possible ζ^0-hooks, and therefore is

$$\begin{pmatrix}0 & \cdots & m_0-1\\ 0 & \cdots & m_1-1\\ \vdots & & \\ 0 & \cdots & m_{e-1}-1\end{pmatrix}. \tag{3.3}$$

In general, we call an e-symbol $\mathscr{S}$ an *(l,ζ^j)-core* if there are no ζ^j-hooks of length l for $\mathscr{S}$. By successive removal of ζ^j-hooks of length l from a given symbol, one obviously reaches an (l,ζ^j)-core.

Proposition 3.4. *The (l,ζ^j)-core of a symbol $\mathscr{S}$ is uniquely determined.*

Proof. Let $\mathscr{S} = (S_0, \dots, S_{e-1})$, $S_i = (\lambda_{i1}, \dots, \lambda_{im_i})$ be an e-symbol and let (l, ζ^j) be given. We associate to $\mathscr{S}$ a sequence of e abacus diagrams, each with l runners. For every abacus, let the positions of the 0th runner be 0, l, $2l$,..., for the first runner be 1, $l+1$, $2l+1$,..., and so on to the $(l-1)$th runner, with $l-1$, $2l-1$,....

A ζ^j-hook of $\mathscr{S}$ of length l now connects two positions that differ by a knight-move path across j rows and reducing the entry by l. To linearize this, we assign to an entry $\lambda \in S_i$ of $\mathscr{S}$ a bead in position $\lfloor \lambda/l \rfloor$ of the $(\lambda \bmod l)$th runner of the $(i + j\lfloor \lambda/l \rfloor)$th abacus. Now, the removal of a ζ^j-hook of length l clearly corresponds to the moving of a bead by exactly one position on a runner. The kernel is therefore obtained when all beads of all abacuses have been moved to the smallest possible positions, and is in particular uniquely determined. □

Let $\mathscr{S}$ be a symbol and let $\mathscr{S}'$ be the corresponding (l, ζ^j)-core. Then we write

$$w(\mathscr{S}, l, \zeta^j) := (\mathrm{rk}(\mathscr{S}) - \mathrm{rk}(\mathscr{S}'))/l,$$

the (l, ζ^j)-*weight* of $\mathscr{S}$. The weight therefore indicates how many (l, ζ^j)-hooks one has to remove from $\mathscr{S}$ to obtain the (l, ζ^j)-core $\mathscr{S}'$. The construction in the proof above gives us, for $\mathscr{S}$ and (l, ζ^j), an el-tuple of partitions of $w(\mathscr{S}, l, \zeta^j)$, the (l, ζ^j)-*quotient* of $\mathscr{S}$.

Let $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{e-1})$ be, as in Section 2C, an e-tuple of partitions $\alpha_i \vdash n_i$ with $\sum n_i = n$, without loss of generality with α_0 having one part more than each of the α_i for $1 \leqslant i \leqslant e-1$. With the associated β-numbers $S_i = (\alpha_{i1}, \alpha_{i2} + 1, \dots, \alpha_{im_i} + m_i - 1)$, we assign to $\boldsymbol{\alpha}$ the symbol $\mathscr{S} = (S_0, \dots, S_{e-1})$. By (3.2), we have $\mathrm{rk}(\mathscr{S}) = n$. In fact this assignment gives a bijection between the irreducible characters of W_n and the equivalence classes of symbols of rank n, with $m_0 = m_1 + 1 = m_2 + 1 = \cdots = m_{e-1} + 1$. This is a special case of the following observation.

Let $\mathscr{S}'$ be an (l, ζ^j)-core. To any el-tuple of partitions of w one can assign in a unique way an e-tuple of l-abacus diagrams, and if one interprets this as the (l, ζ^j)-quotient, a symbol $\mathscr{S}$ with $\mathrm{rk}(\mathscr{S}) = \mathrm{rk}(\mathscr{S}') + lw$. Thus we have the following.

Corollary 3.5. *The equivalence classes of symbols of rank n and content congruent to* 1 *modulo e with a given (l, ζ^j)-core $\mathscr{S}'$ (and thus of (l, ζ^j)-weight $w = (n - \mathrm{rk}(\mathscr{S}'))/l$) are in bijection with the el-tuples of partitions of w.*

3B. **Unipotent degrees of $B_n^{(e)}$.** The unipotent degrees are parametrized by classes of symbols $\mathscr{S}$ of content $\mathrm{ct}(\mathscr{S}) \equiv 1 \bmod e$.

Definition 3.6. For a symbol $\mathscr{S}$ of content $\mathrm{ct}(\mathscr{S}) \equiv 1 \bmod e$ let

$$\mathrm{def}(\mathscr{S}) := \frac{(e-1)(\mathrm{ct}(\mathscr{S}) - 1)}{2} - \sum_{i=0}^{e-1} i|S_i| \qquad \bmod e.$$

Such a symbol is called *reduced* if $\mathrm{def}(\mathscr{S}) = 0$, and not every λ_{i1} is equal to 0.

Obviously in each equivalence class of symbols of content $\mathrm{ct}(\mathscr{S}) \equiv 1 \bmod e$ there lies exactly one reduced symbol, since $\mathrm{def}(\mathscr{S})$ changes by 1 under cyclic permutation of the rows.

Example 3.7. For $e = 2$ the reduced symbols of content $2m + 1 \equiv 1 \bmod 2$ are exactly the same as Lusztig's reduced symbols of odd defect, for which the number of entries in the second row is congruent to m modulo 2 (i.e., one obtains from each class of symbols exactly one representative). The ζ^0-hooks are the standard hooks, and the ζ^1-hooks are the so-called cohooks.

We extend naturally to every $\mathscr{S} = (S_0, \ldots, S_{e-1})$ the definition given in (2.19) for the rational function $D_{\mathscr{S}}$ in the indeterminates $q, v_0, \ldots, v_{e-1}$ assigned to $\mathscr{S}$. This is invariant under shifting of the symbols, but not under cyclic permutation of the rows. If one specializes v_i to ζ^i, we see from (2.21) that $D_{\mathscr{S}}(q)$ is also now, up to sign, invariant with respect to cyclic permutation of the rows of $\mathscr{S}$.

Definition 3.8. For an e-symbol $\mathscr{S}$ of rank n and content $\mathrm{ct}(\mathscr{S}) = me + 1$ define the *degree* by

$$\gamma_{\mathscr{S}}(q) := \frac{(-1)^{\binom{e}{2}\binom{m}{2}} \prod_{i=1}^{n}(q^{ei} - 1) \cdot \prod_{i=0}^{e-1}\prod_{j=i}^{e-1} \prod_{\substack{(\lambda,\mu)\in S_i\times S_j \\ \mu<\lambda \text{ if } i=j}} (q^\lambda \zeta^i - q^\mu \zeta^j)}{\tau(e)^m \cdot q^{\binom{e(m-1)+1}{2} + \binom{e(m-2)+1}{2} + \cdots} \cdot \prod_{i=0}^{e-1} \Theta(S_i, q^e)}. \tag{3.9}$$

The multiset

$$\mathrm{Uch}(B_n^{(e)}) := \{\gamma_{\mathscr{S}} \mid \mathscr{S} \text{ is a reduced } e\text{-symbol, } \mathrm{rk}(\mathscr{S}) = n,\ \mathrm{ct}(\mathscr{S}) \equiv 1 \bmod e\}$$

is called the set of *unipotent degrees* of $B_n^{(e)}$.

Example 3.10. For the symbol $\mathscr{S} = ((0, n), (1), (0), \ldots, (0))$ for the reflection character of W_n we get

$$\gamma_{\mathscr{S}}(q) = \frac{1}{e} \frac{q(q^{n-1} - \zeta)(q^{en} - 1)}{(q - 1)(q^n - \zeta)}.$$

(Compare with Example 2.17.)

We immediately see the following (compare with [10, 3.7 and 8.11(2)] for $e = 2$).

Corollary 3.11. *For any $\gamma(q) \in \mathrm{Uch}(B_n^{(e)})$, we also have that $\pm\gamma(\zeta q)$ lies in $\mathrm{Uch}(B_n^{(e)})$.*

Proof. The symbol $\mathscr{S}'$ parametrizing $\gamma' := \pm\gamma(\zeta q)$ has an entry $\lambda \in S_{i+\lambda}$ for each entry $\lambda \in S_i$ of $\mathscr{S}$. One obtains a reduced symbol by cycling the rows. □

We let $a(\gamma_{\mathscr{S}}) = a(\mathscr{S})$ be the multiplicity of the zero of $\gamma_{\mathscr{S}}(q)$ at $q = 0$. The concept of ζ^j-hooks of symbols introduced above allows us to transform (3.9) (compare with [15, Prop. 5] for the case $e = 2$).

Proposition 3.12 (Hook formula). *We have*

$$\gamma_{\mathscr{S}}(q) = \frac{b_{\mathscr{S}} q^{a(\mathscr{S})} \prod_{i=1}^{n} (q^{ei} - 1)}{\prod_{j,l} (q^l - \zeta^j)},$$

where the product in the denominator runs over all ζ^i-hooks of $\mathscr{S}$ of length l and

$$b_{\mathscr{S}} = \frac{(-1)^{\binom{e}{2}\binom{m}{2}} \prod_{i=0}^{e-1} (-1)^{L_i} \zeta^{i(K_i+L_i)}}{\prod_{i=0}^{e-1} \prod_{j=i+1}^{e-1} (\zeta^i - \zeta^j)^{m-|S_i \cap S_j|}}$$

is a constant, with

$$K_i := |\{(j, \lambda, \mu) \mid \lambda \in S_i,\ \mu \in S_j,\ \ \mu < \lambda,\ j \geq i\}|,$$
$$L_i := |\{(j, \lambda, \mu) \mid \lambda \in S_i,\ \mu \in S_j,\ \ \mu < \lambda,\ j < i\}|,$$

3C. **Harish-Chandra theories.** Let Φ be a cyclotomic polynomial over K with $\Phi(\xi) = 0$ for a primitive dth root of unity ξ. A unipotent degree $\gamma_{\mathscr{S}} \in \mathrm{Uch}(B_n^{(e)})$ is called Φ-*cuspidal* if the full Φ-part of $|B_n^{(e)}(q)|$ divides $\gamma_{\mathscr{S}}(q)$. By the hook formula 3.12 this is equivalent to the statement that the symbol $\mathscr{S}$ possesses no (d', ξ_c)-hooks for $c = \gcd(e, d)$, $d' = d/c$ and $\xi_c = \xi^{d'}$. This definition does not depend on the choice of zero ξ of Φ. We write for short $\mathrm{Uch}_\Phi(\mathbb{G})$ for the subset of the Φ-cuspidal unipotent degrees of $\mathbb{G}$. A pair $(\mathbb{L}, \boldsymbol{\lambda})$, consisting of a Φ-split Levi subgroup $\mathbb{L}$ of $\mathbb{G}$ and a Φ-cuspidal degree $\boldsymbol{\lambda} \in \mathrm{Uch}_\Phi(\mathbb{L})$ is called a Φ-*cuspidal pair* of $\mathbb{G}$.

According to the definition of twisted reflection data, the Φ-cuspidal character degrees of $\mathrm{GL}_m(\xi_c^{-1} q^{d'})$ are in canonical bijection with the 1-cuspidal character degrees of $\mathrm{GL}_m(q)$, hence $\mathrm{GL}_m(\xi_c^{-1} q^{d'})$ only possesses a Φ-cuspidal degree for $m = 1$. By the description of the Φ-split Levi subgroups of $\mathbb{G}$ in (2.5) the Φ-cuspidal pairs of $\mathbb{G}$ have the form $(\mathbb{L}, \boldsymbol{\lambda})$, with

$$\begin{aligned} \mathbb{L} &= \mathrm{GL}_1(\xi_c^{-1} q^{d'}) \times \cdots \times \mathrm{GL}_1(\xi_c^{-1} q^{d'}) \times B_m^{(e)}(q), \\ \boldsymbol{\lambda} &= 1 \otimes \cdots \otimes 1 \otimes \boldsymbol{\lambda}' \qquad \text{where } \boldsymbol{\lambda}' \in \mathrm{Uch}_\Phi(B_m^{(e)}). \end{aligned} \tag{3.13}$$

Let $\mathscr{S}'$ be the reduced symbol of rank m that parametrizes the Φ-cuspidal degree $\boldsymbol{\lambda}'$. By the above, $\mathscr{S}'$ is then a (d', ξ_c)-core, and the symbols $\mathscr{S}$ of rank n with (d', ξ_c)-core $\mathscr{S}'$ have (d', ξ_c)-weight $w := (n - m)/d'$. The corresponding relative Weyl group, according to (2.6) therefore has the structure

$$W_{\mathbb{G}}(\mathbb{L}, \boldsymbol{\lambda}) = Z_{ed'} \wr S_w,$$

if we assume a trivial action of $W_{\mathbb{G}}(\mathbb{L})$ on $\mathrm{Uch}(\mathbb{L})$. In Corollary 3.5 we saw that the unipotent degrees $\gamma_{\mathscr{S}} \in \mathrm{Uch}(\mathbb{G})$, for which $\mathscr{S}$ has (d', ξ_c)-core $\mathscr{S}'$, are in bijection with the irreducible characters of $W_{\mathbb{G}}(\mathbb{L}, \boldsymbol{\lambda})$, via the construction of the

(d',ξ_c)-quotient. This bijection is compatible with the corresponding unipotent degrees:

Theorem 3.14. *Let $\gamma_{\mathscr{S}} \in \mathrm{Uch}(B_n^{(e)})$ be parametrized by the symbol $\mathscr{S}$. Let ξ be a primitive dth root of unity with minimal polynomial Φ over K, $c := \gcd(e,d)$, $d' := d/c$, and e' defined by $\xi^{d'} = \zeta^{e'} =: \xi_c$. Let $\mathscr{S}'$ be the (d',ξ_c)-core and $\mathscr{Q} = (Q_0,\ldots,Q_{ed'-1})$ be an ed'-symbol of the (d',ξ_c)-quotient of $\mathscr{S}$. Then we have*

$$\gamma_{\mathscr{S}}(q) = \pm\frac{|\mathbb{G}(q)|_{q'}}{|\mathbb{L}(q)|_{q'}} D_{\mathscr{Q}}(\xi_c^{-1}q^{d'}; v_0,\ldots,v_{ed'-1})\gamma_{\mathscr{S}'}(q),$$

where $\mathbb{G} = B_n^{(e)}$, $\gamma_{\mathscr{S}} \in \mathrm{Uch}_\Phi(\mathbb{L})$,

$$v_i := q^{\tilde{i}-k_i d'}\zeta^{i+e'k_i}, \quad 0 \leqslant i \leqslant ed'-1, \text{ where } \tilde{i} := \lfloor i/e \rfloor,$$

and $k_i := \max\{|Q_l| - \delta_{0,l} \mid 0 \leqslant l \leqslant ed'-1\} - |Q_i| + \delta_{0,i}$.

Proof. If $\mathscr{S}$ coincides with $\mathscr{S}'$, then the result is trivial. It remains therefore to show that

$$\frac{|\mathbb{G}'|_{q'}\gamma_{\mathscr{S}}(q)}{|\mathbb{G}|_{q'}\gamma_{\mathscr{R}}(q)} = \pm\frac{D_{\mathscr{Q}}(\xi_c^{-1}q^{d'}; v_0,\ldots,v_{ed'-1})}{D_{\mathscr{P}}(\xi_c^{-1}q^{d'}; v_0,\ldots,v_{ed'-1})} \tag{3.15}$$

where $\mathscr{R}$ is obtained from $\mathscr{S}$ by removing a (d',ξ_c)-hook, $\mathbb{G}'$ denotes the corresponding Levi subgroup, and $\mathscr{P}$ is the (d',ξ_c)-quotient of $\mathscr{R}$.

In the proof of Proposition 3.4 we associated to $\mathscr{S}$ an e-tuple of abacuses, each with d' runners. Corresponding to an entry $\lambda \in S_i$ of $\mathscr{S}$ is a bead on the $(\lambda \bmod d')$th runner of the $(i+e'\lfloor\lambda/d'\rfloor)$th abacus in position $\lfloor\lambda/d'\rfloor$. Conversely, a bead on the sth runner of the tth abacus in position u is therefore associated to an entry $d'u+s \in S_{t-e'u}$. Now let $\mathscr{T}$ be the ed'-symbol, whose ith row arises from the positions of the beads on the $\lfloor i/e\rfloor$th runner of the $(i \bmod e)$th abacus. (The ed'-tuple of partitions of $w = w(\mathscr{S},d',\xi_c)$ associated to $\mathscr{T}$ is therefore just the (d',ξ_c)-quotient of $\mathscr{S}$.) We now assume that $\mathscr{R}$ is obtained from $\mathscr{S}$ by removing the (d',ξ_c)-hook $(f,\mu,\mu-d')$ from position $\mu \in S_f$. Furthermore, let the image of $\mu \in S_f$ of $\mathscr{T}$ under the above construction be denoted by $\nu \in T_j$. Then we have

$$\begin{aligned}
&\prod_{i=0}^{e-1} \prod_{\substack{\lambda\in S_i \\ \lambda\neq\mu \text{ if } i=f}} (q^\lambda\zeta^i - q^\mu\zeta^f) \\
&= \pm \prod_{i=0}^{ed'-1} \prod_{\substack{\kappa\in T_i \\ \kappa\neq\nu \text{ if } i=j}} (q^{d'\kappa+\tilde{i}}\zeta^{i-e'\kappa} - q^{d'\nu+\tilde{j}}\zeta^{j-e'\nu}) \\
&= \pm \prod_{i=0}^{ed'-1} \prod_{\substack{\kappa\in T_i \\ \kappa\neq\nu \text{ if } i=j}} \left((\xi_c^{-1}q^{d'})^\kappa q^{\tilde{i}}\zeta^i - (\xi_c^{-1}q^{d'})^\nu q^{\tilde{j}}\zeta^j\right),
\end{aligned}$$

where we set $\tilde{i} := \lfloor i/e\rfloor$. The symbol $\mathscr{T}$ is now not in the form in which the first row has exactly one more entry than all the others. We achieve this by shifting the

ith row of $\mathscr{T}$ by $k_i := \max_l\{|T_l|-\delta_{0,l}\}-|T_i|+\delta_{0,i}$. Denote by $\mathscr{T}' = (T'_0,\dots,T'_{ed'-1})$ the symbol obtained by these shifts. Setting

$$p := \xi_c^{-1}q^{d'}, \qquad v_i := q^{\tilde{i}-k_i d'}\zeta^{i+e'k_i} \quad (0 \leqslant i \leqslant ed'-1),$$

the above expression then becomes

$$\pm \prod_{i=0}^{ed'-1} \prod_{\substack{\kappa\in T_i \\ \kappa\neq\nu \text{ if } i=j}} (p^{\kappa+k_i}v_i - p^{\nu+k_j}v_j).$$

Next, we get the factor contributed by the quotients of Θ-functions to be

$$\begin{aligned}\prod_{i=\mu-d'+1}^{\mu}(q^{ei}-1) &= \prod_{i=0}^{d'-1}(q^{e(d'\nu+\tilde{j}-i)}-1)\\ &= \prod_{i=0}^{ed'-1}(q^{d'\nu+\tilde{j}-\tilde{i}}\zeta^{j-e'\nu-i}-1)\\ &= \prod_{i=0}^{ed'-1} p^{-k_i}v_i^{-1}(p^{\nu'}v_j - p^{k_i}v_i).\end{aligned}$$

The left side of the quotient (3.15) to be examined, divided by $(q^{d'}-\xi_c)$, therefore becomes

$$\begin{aligned}&\pm\prod_{i=0}^{ed'-1}\prod_{\kappa\in T_i}{}' \frac{p^{\kappa+k_i}v_i - p^{\nu+k_j}v_j}{p^{\kappa+k_i}v_i - p^{\nu+k_j-1}v_j}\prod_{i=0}^{ed'-1}\frac{p^{k_i}v_i}{p^{\nu'}v_j - p^{k_i}v_i}\\ &\quad = \pm\prod_{i=0}^{ed'-1}\prod_{\kappa\in T'_i}{}'\frac{p^{\kappa}v_i - p^{\nu'}v_j}{p^{\kappa}v_i - p^{\nu'-1}v_j}\prod_{i=0}^{ed'-1}\frac{p^{k_i}v_i - p^{\nu'}v_j}{p^{k_i}(v_i - p^{\nu'}v_j)}\prod_{i=0}^{ed'-1}\frac{p^{k_i}v_i}{(p^{\nu'}v_j - p^{k_i}v_i)}\\ &\quad = \pm\prod_{i=0}^{ed'-1}\prod_{\kappa\in T'_i}{}'\frac{p^{\kappa}v_i - p^{\nu'}v_j}{p^{\kappa}v_i - p^{\nu'-1}v_j}\prod_{i=0}^{ed'-1}\frac{v_i}{v_i - p^{\nu'}v_j},\end{aligned}$$

where by $\prod'$ we mean the product over all $\kappa \in T_i$ with $\kappa \neq \nu$ for $i = j$, and the product over all $\kappa \in T'_i$ with $\kappa \neq \nu'$ for $i = j$ respectively. This is, by (2.19), equal to the quotient of the right side of (3.15) for the symbol $\mathscr{Q} := \mathscr{T}'$, when divided by $(q^{d'}-\xi_c)$. □

The above result was already known in special cases: for $e \in \{1,2\}$, $d = 1$, it is the classical Harish-Chandra theory for the groups of Lie type A_{n-1} and B_n (see for example [10, 3.2.1] for $e = 2$, $d = 1$). The case $e \in \{1,2\}$, d arbitrary, $w(\mathscr{S}, d', \xi_c) = 1$, was handled in [4, Bem. 2.10, 2.14, 2.19]. The important consequence of Theorem 3.14 in relation to the conjectures proposed in [4] resulting from the case $e \in \{1,2\}$, d arbitrary, is as follows:

Corollary 3.16. *Let $\mathbb{G}$ be a generic group of Lie type, of type A_n, ${}^2A_n$, B_n or C_n. Then the Φ_d-blocks of $\mathbb{G}$, assuming Conjecture 2.20(2), satisfy Conjecture (d-HV6) in* [4].

Conversely, with the help of Theorem 3.14, the truth of (d-HV6) for the groups $\mathrm{GL}_n(q)$ would establish Conjecture 2.20(2) on the relative degrees of the Hecke algebra $\mathscr{H}(B_n^{(e)};\mathbf{u})$, since all these algebras would occur in each case as the endomorphism algebras of Deligne–Lusztig induced characters for infinitely many Φ-blocks of groups $\mathrm{GL}_n(q)$. The required degrees have to also satisfy the degree formula in the theorem, with the same parameters (since these are already determined by the case of weight 1, for which the relative degrees are known from [4, Bem. 2.4]). By interpolating, this would give the desired formula. (Compare [10, 9.6] for an analogous argument.)

Furthermore, Theorem 3.14 gives the next conclusion.

Corollary 3.17. *The degrees $\gamma_{\mathscr{S}} \in \mathrm{Uch}(B_n^{(e)})$ are polynomials.*

Proof. As is easily verified using the ζ^0-core, in every equivalence class of symbols there is one, all of whose entries are either less than or equal to its rank. Therefore the possible zeroes of the denominator of $\gamma_{\mathscr{S}} \in \mathrm{Uch}(B_n^{(e)})$, according to the hook formula 3.12, come either at $q = 0$, or a primitive dth root of unity ξ with $d \mid ei$ for some $1 \leqslant i \leqslant n$. In order to show that $\gamma_{\mathscr{S}}(q)$ has no pole at $q = \xi$, we apply Theorem 3.14. Let $\mathscr{S}'$ be the (d', ξ_c)-core of $\mathscr{S}$, $c := \gcd(d, e)$, $d' = d/c$, $\mathbb{L}$ the associated Levi subgroup and Φ the minimal polynomial of ξ over K.

Because $\mathbb{L}$ is a Φ-split Levi subgroup, the quotient $|\mathbb{G}|/|\mathbb{L}|$ by definition has no $(q-\xi)$-factor. Furthermore, by the definition of Φ-cuspidal degrees, $\gamma_{\mathscr{S}'}(q)$ has no pole at $q = \xi$. Lastly, we see that, on the basis of the form of the parameters, also the generic degree $D_{\mathscr{Q}}$ of the relative Weyl group has neither a pole nor a zero at $q^{d'} - \xi_c$. So therefore $\gamma_{\mathscr{S}}(q)$ is an integer at $q = \xi$.

The integrality at $q = 0$ follows, firstly since $\mathscr{S}$ has the same a-value as the special symbol with the same entries. For special symbols, the a-value and $\tilde{a}$-value coincide by Proposition 2.24, which is non-negative, since the fake degrees are polynomials. $\square$

The definition of relative degrees (2.19) shows that the inverse of $D_{\mathscr{S}}(q;\mathbf{v})$ lies in $K[q, q^{-1}, \mathbf{v}, \mathbf{v}^{-1}]$. This allows a simple structural proof of the result of Boyce [3] for $e = 2$ for Φ-split Levi subgroups.

Corollary 3.18. *Under the hypotheses of Theorem 3.14 we have*

$$\gamma_{\mathscr{S}}(q)_{q'} \,|\, \deg\left(R_{\mathbb{L}}^{\mathbb{G}}(\gamma_{\mathscr{S}'})\right) \ \text{ in } K[q], \ \text{ where } \deg\left(R_{\mathbb{L}}^{\mathbb{G}}(\gamma_{\mathscr{S}'})\right) := \frac{|\mathbb{G}(q)|_{q'}}{|\mathbb{L}(q)|_{q'}}\gamma_{\mathscr{S}'}(q).$$

4. Fourier transform

In this section we establish a link between the unipotent degrees and the fake degrees for $B_n^{(e)}$: both span the same subspace of the vector space of polynomials in q over K, and the change of basis matrix, called the Fourier transform matrix by

Lusztig in the case $e = 2$, can be explicitly described. To every unipotent degree we assign an eth root of unity, the eigenvalue of Frobenius. The eigenvalues of Frobenius are compatible with all Φ-Harish-Chandra theories. The diagonal matrix of the eigenvalues of Frobenius and the Fourier transform matrix give a matrix representation of $\mathrm{SL}_2(\mathbb{Z})$.

4A. **Fourier transform matrices.** Let Y be an $(em+1)$-element totally ordered set, with $m > 0$. Let $\Psi = \Psi(Y)$ be the set of functions

$$\psi : Y \to \{0, \ldots, e-1\} \qquad \text{with} \qquad \sum_{y \in Y} \psi(y) \equiv m \binom{e}{2} \bmod e, \tag{4.1}$$

and Ψ_0 the subset of those $\psi \in \Psi$ with $|\psi^{-1}(i)| = m + \delta_{0,i}$. Let $^-$ denote the natural involution

$$^- : \Psi \to \Psi, \qquad \psi \mapsto \bar{\psi} \text{ where } \bar{\psi}(y) := e - \psi(y) \bmod e,$$

on Ψ. Furthermore, set $\langle\, , \rangle$ to be the symmetric pairing on Ψ, given by

$$\langle \phi, \psi \rangle := \varepsilon(\phi)\varepsilon(\psi) \prod_{y \in Y} \zeta^{-\phi(y)\psi(y)},$$

where $\zeta := \exp(2\pi \mathrm{i}/\mathrm{e})$ and

$$\begin{aligned} &\varepsilon(\psi) := (-1)^{c(\psi)}, \text{ with} \\ &c(\psi) := |\{(y, y') \in Y \times Y \mid y < y',\, \psi(y) < \psi(y')\}|. \end{aligned}$$

Lemma 4.2. *The symmetric pairing $\langle\, , \rangle : \Psi \times \Psi \to \mathbb{C}$ satisfies:*

(1) $\langle \phi, \bar{\psi} \rangle = \varepsilon(\psi)\varepsilon(\bar{\psi})\overline{\langle \phi, \psi \rangle}$,
(2) $\sum_{\nu \in \Psi} \langle \phi, \nu \rangle \langle \nu, \psi \rangle = \delta_{\phi, \bar{\psi}} \varepsilon(\psi)\varepsilon(\bar{\psi})|\Psi|$.

Proof. The first part follows directly from the definition. For the second part, we have to compute

$$\sum_{\nu \in \Psi} \langle \phi, \nu \rangle \langle \nu, \psi \rangle = \varepsilon(\phi)\varepsilon(\psi) \sum_{\nu \in \Psi} \prod_{y \in Y} \zeta^{-\nu(y)(\phi(y)+\psi(y))}.$$

If we have $\phi(y) + \psi(y) \equiv 0 \bmod e$ for all $y \in Y$, so that $\bar{\psi} = \phi$, then we get $\pm|\Psi|$ as the result. Now let $y \in Y$ be such that $\phi(y) + \psi(y) \not\equiv 0 \bmod e$. For a fixed $\nu \in \Psi$ let ν_i be the function $\nu_i(z) := \nu(z) - i + \delta_{y,z} \cdot i$. The ν_i again lie in Ψ, and are all distinct for $0 \leqslant i \leqslant e-1$. The part of the above sum taken over the ν_i, vanishes, as a sum over all powers of a non-trivial root of unity is zero. Hence the whole sum vanishes, and we get the result. $\square$

We introduce an operator $T : H \to H$ on the space of functions $H := R^{\Psi}$ from Ψ to a $\mathbb{C}$-algebra R, given by

$$T(f)(\phi) := \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\psi \in \Psi} \langle \phi, \psi \rangle f(\psi),$$

and call $T(f)$ the *Fourier transform* of f (compare with [10, 1.1]). By Lemma 4.2 and (2.22), we have

$$T^2(f)(\psi) = (-1)^{m\binom{e-1}{2}}\varepsilon(\psi)\varepsilon(\bar\psi)f(\bar\psi),$$

and the four-fold Fourier transform is just the original function. Based on Lemma 4.2, one easily proves the identity

$$\sum_{\psi\in\Psi} T(f)(\psi)\overline{T(f)(\psi)} = \sum_{\psi\in\Psi} f(\psi)\overline{f(\psi)}. \tag{4.3}$$

In addition, let g be a fixed function from Y to a $\mathbb{C}$-algebra, and γ and ∂ be defined by

$$\gamma(\psi) := \frac{1}{\tau(e)^m}\prod_{i=0}^{e-1}\prod_{j=i}^{e-1}\prod_{\substack{y,y'\colon\psi(y)=i,\,\psi(y')=j\\ y>y' \text{ if } i=j}}\left(g(y)\zeta^{\psi(y)} - g(y')\zeta^{\psi(y')}\right), \tag{4.4}$$

$$\partial(\psi) := \begin{cases} \displaystyle\prod_{i=0}^{e-1}\prod_{y\colon\psi(y)=i} g(y)^i \prod_{\substack{y'\colon\psi(y')=i\\ y>y'}}\left(g(y)^e - g(y')^e\right) & \text{if } \psi\in\Psi_0,\\ 0 & \text{otherwise.}\end{cases} \tag{4.5}$$

Proposition 4.6. *The function $\psi\mapsto(-1)^{\binom{e}{2}\binom{m}{2}}\partial(\psi)$ is the Fourier transform of $\psi\mapsto\gamma(\psi)$.*

Proof. We first interpret the quantity $\varepsilon(\psi)\gamma(\psi)$ as a Vandermonde determinant, and obtain via Laplace's formula

$$\begin{aligned}\varepsilon(\psi)\gamma(\psi) &= \frac{1}{\tau(e)^m}\prod_{\substack{y,y'\in Y\\ y>y'}}\left(g(y)\zeta^{\psi(y)} - g(y')\zeta^{\psi(y')}\right)\\ &= \frac{1}{\tau(e)^m}\sum_{\sigma\in S_{em+1}}\varepsilon(\sigma)\prod_{y\in Y}\left(g(y)\zeta^{\psi(y)}\right)^{\sigma(y)},\end{aligned}$$

where here we have identified the symmetric group S_{em+1} with the bijections from Y to $\{0,\ldots,em\}$, and the sign $\varepsilon(\sigma)$ is defined by

$$\varepsilon(\sigma) := (-1)^{c(\sigma)},\quad\text{where}\quad c(\sigma) := |\{(y,y')\in Y\times Y \mid y>y',\ \sigma(y)<\sigma(y')\}|.$$

With this, we get

$$\begin{aligned}T(\gamma)(\phi) &= \frac{(-1)^{m(e-1)}\varepsilon(\phi)}{\tau(e)^{2m}}\sum_{\psi\in\Psi}\left(\prod_{y\in Y}\zeta^{-\phi(y)\psi(y)}\right)\sum_{\sigma\in S_{em+1}}\varepsilon(\sigma)\prod_{y\in Y}\left(g(y)\zeta^{\psi(y)}\right)^{\sigma(y)}\\ &= \frac{(-1)^{m(e-1)}\varepsilon(\phi)}{\tau(e)^{2m}}\sum_{\sigma\in S_{em+1}}\varepsilon(\sigma)\left(\prod_{y\in Y}g(y)^{\sigma(y)}\right)\sum_{\psi\in\Psi}\prod_{y\in Y}\zeta^{\psi(y)(\sigma(y)-\phi(y))}.\end{aligned}$$

As in Lemma 4.2(b), the inner sum vanishes unless $\sigma(y) - \phi(y)$ is identical to 0 (modulo e) on Y. In the latter case, we have that the sets of preimages $\phi^{-1}(i)$ each have cardinality $m + \delta_{0,i}$, i.e., ϕ lies in Ψ_0, and σ induces bijections $\sigma_i : \phi^{-1}(i) \xrightarrow{\sim} \{i, e+i, \dots, (m-1)e+i\}$, $1 \leqslant i \leqslant e-1$, and $\sigma_0 : \phi^{-1}(0) \xrightarrow{\sim} \{0, e, \dots, em\}$. One defines the sign $\varepsilon(\sigma_i)$ in an analogous way to $\varepsilon(\sigma)$, so that a brief calculation shows

$$\varepsilon(\sigma) = (-1)^{\binom{e}{2}\binom{m+1}{2}} \varepsilon(\phi) \prod_{i=0}^{e-1} \varepsilon(\sigma_i).$$

Taking into account the sign arising from $\tau(e)^{2m}$ we obtain

$$T(\gamma)(\phi) = (-1)^{\binom{e}{2}\binom{m}{2}} \sum_{\sigma_0} \varepsilon(\sigma_0) \prod_{y \in \phi^{-1}(0)} g(y)^{\sigma_0(y)} \times \cdots$$
$$\times \sum_{\sigma_{e-1}} \varepsilon(\sigma_{e-1}) \prod_{y \in \phi^{-1}(e-1)} g(y)^{\sigma_{e-1}(y)}.$$

However, with the definition of ∂ this is already the desired result. □

A function $\pi : Y \to \mathbb{N}$ defines an equivalence relation $\sim_\pi$ on Ψ by

$$\phi \sim_\pi \psi \text{ if } \pi \circ \phi^{-1}(i) = \pi \circ \psi^{-1}(i) \text{ for } 0 \leqslant i \leqslant e-1.$$

The class of ψ in Ψ with respect to $\sim_\pi$ will be denoted by $[\psi]$, A $\psi \in \Psi$ is called π-*admissible* if whenever $\pi(y) = \pi(y')$ and $\psi(y) = \psi(y')$ then $y = y'$. Denote by H_π the subspace of H generated by all f such that

(4.7) $\qquad f(\phi) = f(\psi) \text{ if } [\phi] = [\psi], \qquad f(\psi) = 0 \text{ if } \psi \text{ is not } \pi\text{-admissible}$

Lemma 4.8. *The subspace H_π is invariant under the Fourier transform.*

Proof. The functions

(4.9) $$\left\{ f_{[\phi]} \mid \phi \ \pi\text{-admissible},\ f_{[\phi]}(\psi) := \delta_{[\phi],[\psi]} \right\}$$

generate H_π. Their Fourier transforms have values

$$T(f_{[\phi]})(\psi) = \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\nu \in \Psi} \langle \psi, \nu \rangle f_{[\phi]}(\nu)$$
$$= \frac{(-1)^{m(e-1)} \varepsilon(\psi)}{\tau(e)^m} \sum_{\nu \in [\phi]} \varepsilon(\nu) \prod_{y \in Y} \zeta^{-\psi(y)\nu(y)}.$$

If ψ is not π-admissible, with $y, y' \in Y$ demonstrating this, then for any $\nu \in [\phi]$, by interchanging the images of y and y' we obtain ν' also in $[\phi]$. The associated signs satisfy $\varepsilon(\nu)\varepsilon(\nu') = -1$, while the inner product for either assumes the same value. Because of this, the contributions of ν and ν' in the sum cancel out, and $T(f_{[\phi]})$ vanishes on ψ. If $[\psi] = [\psi']$, then one sees with a similar argument that (4.7) is satisfied. □

Hence T induces a Fourier transform on H_π. Let g be a function on Y as above, which factors over $\pi(Y)$ via π. Then the functions γ and ∂ introduced in (4.4) and (4.5) lie in H_π, and according to Proposition 4.6, are sent to each other via the Fourier transform.

The matrix representation $T(Y,\pi)$ of T with respect to the basis (4.9), i.e.,

$$T(Y,\pi) = \left(\frac{(-1)^{m(e-1)}}{\tau(e)^m}\sum_{\nu\in[\phi]}\varepsilon(\nu)\varepsilon(\psi)\prod_{y\in Y}\zeta^{-\nu(y)\psi(y)}\right)_{[\phi],[\psi]}, \tag{4.10}$$

will be called the *Fourier transform matrix* of (Y,π).

4B. **Eigenvalues of Frobenius.** We first recall an easy consequence of a theorem of Gauss (see for example [9, Theorem 211]):

$$\sum_{i=0}^{e-1}\zeta^{(i^2+ei)/2} = \zeta_{24}^{-3(e-1)}\sqrt{e}, \tag{4.11}$$

where $\zeta_{24} = \exp(\pi\mathrm{i}/12)$. Let Y again be an $(em+1)$-element set and Ψ be defined as above. To any $\psi \in \Psi$ we assign the *eigenvalue of Frobenius*

$$\mathrm{Fr}(\psi) := \zeta_{24}^{-2(e^2-1)m}\prod_{y\in Y}\zeta^{-(\psi(y)^2+e\psi(y))/2}. \tag{4.12}$$

Obviously $\mathrm{Fr}(\psi)$ is always a root of unity.

Lemma 4.13. *For all $\phi,\psi\in\Psi$ and γ as in (4.4) we have:*

(a) $\mathrm{Fr}(\psi)^e = 1$;
(b) $\mathrm{Fr}(\psi) = 1$ *if* $\psi\in\Psi_0$;
(c) $\mathrm{Fr}(\psi) = \mathrm{Fr}(\bar\psi)$;
(d) $\sum_{\nu,\mu\in\Psi}\mathrm{Fr}(\psi)\mathrm{Fr}(\nu)\mathrm{Fr}(\mu)\langle\psi,\nu\rangle\langle\nu,\mu\rangle\langle\mu,\phi\rangle = (-1)^{(e-1)m}\tau(e)^{3m}\delta_{\psi,\phi}$;
(e) $\sum_{\nu\in\Psi}\mathrm{Fr}(\phi)\overline{\langle\phi,\nu\rangle}\mathrm{Fr}(\nu)\gamma(\nu) = (-1)^{(e-1)m}\overline{\tau(e)}^m\gamma(\phi)$.

Proof. Part (a) follows directly from the definitions of Ψ and Fr. The second part is trivial. Part (c) follows because we have

$$(e-x)^2 + e(e-x) = e^2 - 2ex + x^2 + e^2 - ex \equiv x^2 + ex \bmod 2e,$$

for any $x\in\mathbb{Z}$. For (d), we get

$$\sum_{\nu,\mu\in\Psi}\mathrm{Fr}(\psi)\mathrm{Fr}(\nu)\mathrm{Fr}(\mu)\langle\psi,\nu\rangle\langle\nu,\mu\rangle\langle\mu,\phi\rangle$$
$$= \varepsilon(\nu)\varepsilon(\phi)\zeta_{24}^{-2(e^2-1)m}\mathrm{Fr}(\psi)\sum_{\nu\in\Psi}\mathrm{Fr}(\nu)\langle\psi,\nu\rangle\sum_{\mu\in\Psi}\prod_{y\in Y}\zeta^{-\mu(y)(\phi(y)+\nu(y)+(\mu(y)+e)/2)}.$$

Along with μ, we see that $\mu-\phi-\nu$ also lies in Ψ, and as we change the summation variable of the inner sum in this way, we obtain

$$\varepsilon(\phi)\varepsilon(\bar\phi)\zeta_{24}^{-6(e^2-1)m}\mathrm{Fr}(\psi)\sum_{\nu\in\Psi}\langle\psi,\nu\rangle\langle\nu,\bar\phi\rangle\overline{\mathrm{Fr}(\phi)}\sum_{\mu\in\Psi}\prod_{y\in Y}\zeta^{-(\mu(y)^2+e\mu(y))/2}.$$

Now, we have

$$
\begin{aligned}
\left(\sum_{i=0}^{e-1} \zeta^{-(i^2+ei)/2}\right)^{em+1} &= \sum_{\mu\in\{0,\dots,e-1\}^Y} \prod_{y\in Y} \zeta^{-(\mu(y)^2+e\mu(y))/2} \\
&= \sum_{p=0}^{e-1} \sum_{\mu\in\Psi} \prod_{y\in Y} \zeta^{-((\mu(y)+p)^2+e\mu(y)+ep)/2} \\
&= \sum_{p=0}^{e-1} \zeta^{-(p^2+ep)/2} \sum_{\mu\in\Psi} \prod_{y\in Y} \zeta^{-(\mu(y)^2+e\mu(y))/2},
\end{aligned}
$$

and hence

$$
\sum_{\mu\in\Psi} \prod_{y\in Y} \zeta^{-(\mu(y)^2+e\mu(y))/2} = \left(\sum_{i=0}^{e-1} \zeta^{-(i^2+ei)/2}\right)^{em} = \left(\zeta_{24}^{3(e-1)}\sqrt{e}\right)^{em} \tag{4.14}
$$

by (4.11), and the sum under investigation becomes

$$
\begin{aligned}
&\varepsilon(\phi)\varepsilon(\bar\phi)\mathrm{Fr}(\psi)\overline{\mathrm{Fr}(\phi)}\zeta_{24}^{-6(e^2-1)m} \left(\zeta_{24}^{3(e-1)}\sqrt{e}\right)^{em} \sum_{\nu\in\Psi} \langle\psi,\nu\rangle\langle\nu,\bar\phi\rangle \\
&\qquad = \mathrm{Fr}(\psi)\overline{\mathrm{Fr}(\phi)}\zeta_{24}^{-3(e-1)(e+2)m}\sqrt{e}^{em}\delta_{\psi,\phi}|\Psi| \\
&\qquad = \zeta_{24}^{-3(e-1)(e+2)m}\sqrt{e}^{em}\delta_{\psi,\phi}|\Psi|^{3/2},
\end{aligned}
$$

where we have used Lemma 4.2(b). The desired result now follows by (2.22).

For part (e), we initially compute as in the proof of Proposition 4.6:

$$
\begin{aligned}
&\tau(e)^m \sum_{\nu\in\Psi} \mathrm{Fr}(\phi)\overline{\langle\phi,\nu\rangle}\mathrm{Fr}(\nu)\gamma(\nu) \\
&\qquad = \varepsilon(\phi)\kappa^2 \sum_{\nu\in\Psi} \left(\prod_{y\in Y} \zeta^{\phi(y)\nu(y)-(\nu(y)^2+e\nu(y)+\phi(y)^2+e\phi(y))/2}\right) \\
&\qquad\quad \times \sum_{\sigma\in S_{em+1}} \varepsilon(\sigma) \prod_{y\in Y} \left(g(y)\zeta^{\nu(y)}\right)^{\sigma(y)},
\end{aligned}
$$

with $\kappa := \zeta_{24}^{-2(e^2-1)m}$. Now σ mod e can be interpreted as an element of Ψ_0. By summation over $\psi := \nu - \phi - \sigma$ mod e instead of over ν, we obtain

$$
\kappa^3\varepsilon(\phi) \sum_{\sigma\in S_{em+1}} \varepsilon(\sigma) \prod_{y\in Y} \left(g(y)\zeta^{\phi(y)}\right)^{\sigma(y)} \overline{\mathrm{Fr}(\sigma)} \sum_{\psi\in\Psi} \zeta^{-(\psi(y)^2+e\psi(y))/2}.
$$

The first sum now yields simply $\tau(e)^m\gamma(\phi)$, and furthermore $\mathrm{Fr}(\sigma) = 1$ by (b). The value of the last factor was determined in (4.14), and therefore we are left with

$$
\kappa^3\tau(e)^m\gamma(\phi)\zeta_{24}^{3e(e-1)m}\sqrt{e}^{em}.
$$

The expression in the lemma is thus equal to

$$\frac{(-1)^{(e-1)m}}{\overline{\tau(e)}^m}\zeta_{24}^{-6(e^2-1)m+3e(e-1)m}\sqrt{e}^{em}\gamma(\phi)=\gamma(\phi). \qquad \square$$

The eigenvalues of Frobenius can be collected into an operator

$$F : H \to H, \qquad F(f)(\phi) := \mathrm{Fr}(\phi)f(\phi).$$

For a function $\pi : Y \to \mathbb{N}$ as above, Fr is by definition constant on the associated equivalence classes $[\psi]$, and we set $\mathrm{Fr}([\psi]) := \mathrm{Fr}(\psi)$. Let $F := F(Y,\pi)$ be the matrix representing the induced operator

$$F : H_\pi \to H_\pi, \qquad F(f)([\psi]) := \mathrm{Fr}([\psi])f([\psi]),$$

with regards to the basis (4.9).

Using the two automorphisms T and F introduced above, we construct the *Shintani operator*

$$\mathrm{Sh} := F \circ T^3 \circ F = F \circ \bar{T} \circ F,$$

on H and on H_π, and denote by $\mathrm{Sh}(Y,\pi)$ the associated representing matrix with respect to the basis (4.9). One obtains by Lemmas 4.2 and 4.13(c)–(e) the following (compare with [7, VII.3]).

Corollary 4.15. *The function $\gamma \in H$ is a fixed point of* Sh. *The matrices $T := T(Y,\pi)$ and $U := T \cdot F(Y,\pi)$ satisfy*

$$T^4 = 1, \quad U^3 = 1, \quad [T^2, U] = 1.$$

In particular, T and U provide a matrix representation of $\mathrm{SL}_2(\mathbb{Z})$ *via*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto T, \qquad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mapsto U.$$

The preimages of $F(Y,\pi)$ and $\mathrm{Sh}(Y,\pi)$ *under this representation are*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto F(Y,\pi), \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \mathrm{Sh}(Y,\pi).$$

Proof. We have $F^{-1}T\gamma = F^{-1}\delta = \delta = T\gamma$, so $T^{-1}F^{-1}T\gamma = \gamma$. Using $U^3 = 1$ this yields $FTFT^2\gamma = \gamma$, so $FT^3F\gamma = \gamma$. $\square$

Because of Lemma 4.13(a), the image $\bar{G}$ of $G = \langle T, U\rangle$ in $\mathrm{PSL}_2(\mathbb{Z})$ is a $(2,3,e)$-generator group. For $e = 2$, we therefore have in particular $\bar{G} = S_3$, for $e = 3$ we have $\bar{G} = A_4$, for $e = 4$, $\bar{G}$ is S_4, and finally for $e = 5$, $\bar{G} = A_5$. As above, each $\pi : Y \to \mathbb{N}$ furnishes us with an invariant subspace H_π, so the representation of $\mathrm{SL}_2(\mathbb{Z})$ is also in general far from being irreducible.

4C. **Families.** We now return to the symbols introduced in the third section. Two symbols $\mathscr{S}$ and $\mathscr{S}'$ lie in the same *family* if the multisets of entries of $\mathscr{S}$ and $\mathscr{S}'$ coincide. This induces a decomposition of the equivalence classes of symbols into families, and hence also of the unipotent degrees $\mathrm{Uch}(B_n^{(e)})$ that are parametrized by the reduced symbols. According to Proposition 2.24, the families are in bijection with special symbols.

Let $\mathscr{S}$ be an e-symbol, $n(\lambda) := |\{i \mid \lambda \in S_i\}|$, and $N := \{\lambda \mid 0 < n(\lambda) < e\}$. Then the family containing $\mathscr{S}$ has

$$\frac{1}{e}\prod_{\lambda\in N}\binom{e}{n(\lambda)} \tag{4.16}$$

elements.

A family $\mathscr{F}$ of reduced symbols $\mathscr{S}$ of content $\mathrm{ct}(\mathscr{S}) = em+1$ can be identified in an obvious way with a set $\Psi(Y,\pi)$, as in Section 4A. Namely, let Y be an $(em+1)$-element set and $\pi : Y \to \mathbb{N}$ be chosen so that $|\pi^{-1}(n)| = |\{i \mid n \in S_i\}|$ (this obviously does not depend on the choice of symbol $\mathscr{S} \in \mathscr{F}$). We associate to a class $[\psi]$ of $\psi \in \Psi$ the symbol $\mathscr{S}$ with $S_i = \pi(\psi^{-1}(i))$. Again, this does not depend on the chosen representative $\psi \in [\psi]$. According to the definition of Ψ, the symbol $\mathscr{S}$ so obtained is reduced and lies in $\mathscr{F}$, and the above map defines a bijection between the π-admissible classes in Ψ and the elements of $\mathscr{F}$. The subset Ψ_0 corresponds precisely to the principal 1-series degrees $\mathscr{F}_0$ of $\mathscr{F}$.

Let T be the associated matrix of the Fourier transform (4.10). We extend the definition of the fake degree $R_{\mathscr{S}}$ from (2.12) to arbitrary symbols $\mathscr{S}$ via $R_{\mathscr{S}} = 0$ for $\mathscr{S} \notin \mathscr{F}_0$. We denote by $\gamma_{\mathscr{S}}$ the unipotent degree of $\mathscr{S}$, as in Section 3B.

Theorem 4.17. *Let $\mathscr{F}$ be a family. The function*

$$\mathscr{S} \mapsto R_{\mathscr{S}}(q)$$

is the Fourier transform of

$$\mathscr{S} \mapsto \gamma_{\mathscr{S}}(q).$$

In particular we have

$$\sum_{\gamma_{\mathscr{S}}\in\mathscr{F}} \gamma_{\mathscr{S}}(q)\overline{\gamma_{\mathscr{S}}(q)} = \sum_{\mathscr{S}\in\mathscr{F}_0} R_{\mathscr{S}}(q)^2.$$

Proof. The equations (3.9) and (2.12) show

$$\begin{aligned}\gamma_{\mathscr{S}}(q) &= v(\mathscr{F})\gamma(\mathscr{S}),\\ R_{\mathscr{S}}(q) &= \prod_{i=1}^{n}(q^{ei}-1)\delta_{\mathscr{S}}(q) = (-1)^{\binom{e}{2}\binom{m}{2}}v(\mathscr{F})\partial(\mathscr{S}),\end{aligned}$$

where γ and ∂ are the functions introduced in (4.4) and (4.5) with $g(y) := q^{\pi(y)}$, and

$$v(\mathscr{F}) := \frac{(-1)^{\binom{e}{2}\binom{m}{2}} \prod_{i=1}^{n}(q^{ei}-1)}{q^{\binom{e(m-1)+1}{2}+\binom{e(m-2)+1}{2}+\cdots} \prod_{i=0}^{e-1} \Theta(S_i, q^e)},$$

only depending on $\mathscr{F}$. Because g factors through π, the functions γ and ∂ lie in the space H_π, and the statement follows from Proposition 4.6. The last identity is a direct consequence of (4.3) for the restricted transformation if one observes that the $R_{\mathscr{S}}$ are polynomials in q with rational coefficients. □

In the case $e = 1$ the fake degrees are equal to the unipotent degrees, each family has one element, and the above theorem is trivial. The statement for $e = 2$ is contained in [10, Proposition 2.9(i)].

Example 4.18. For $n = 1$, $e = 4$, so that $W_n \cong Z_4$, there is a six-element family with the symbols

$$\begin{pmatrix}0&1\\1&\\0&\\0&\end{pmatrix}, \begin{pmatrix}0&1\\0&\\1&\\0&\end{pmatrix}, \begin{pmatrix}0&1\\0&\\0&\\1&\end{pmatrix}, \begin{pmatrix}0&\\0&1\\0&1\\-&\end{pmatrix}, \begin{pmatrix}-&\\0&1\\0&\\0&1\end{pmatrix}, \begin{pmatrix}0&\\-&\\0&1\\0&1\end{pmatrix}.$$

The unipotent degrees are by (3.9)

$$\frac{(1-\mathrm{i})}{4}q(q+1)(q+\mathrm{i}), \qquad \frac{1}{2}q(q^2+1),$$

$$\frac{(1+\mathrm{i})}{4}q(q+1)(q-\mathrm{i}),$$

$$\frac{(1+\mathrm{i})}{4}q(q-1)(q+\mathrm{i}), \qquad \frac{\mathrm{i}}{2}q(q-1)(q+1),$$

$$-\frac{(1-\mathrm{i})}{4}q(q-1)(q-\mathrm{i}).$$

The accompanying Fourier transform matrix (4.10) is given by

$$\frac{1}{4}\begin{pmatrix}1-\mathrm{i}&2&1+\mathrm{i}&1+\mathrm{i}&2\mathrm{i}&-1+\mathrm{i}\\2&0&2&-2&0&2\\1+\mathrm{i}&2&1-\mathrm{i}&1-\mathrm{i}&-2\mathrm{i}&-1-\mathrm{i}\\1+\mathrm{i}&-2&1-\mathrm{i}&1-\mathrm{i}&2\mathrm{i}&-1-\mathrm{i}\\2\mathrm{i}&0&-2\mathrm{i}&2\mathrm{i}&0&2\mathrm{i}\\-1+\mathrm{i}&2&-1-\mathrm{i}&-1-\mathrm{i}&2\mathrm{i}&1-\mathrm{i}\end{pmatrix},$$

and the vector of the eigenvalues of Frobenius is $(1, 1, 1, -1, -\mathrm{i}, -1)$.

Alternatively, one could proceed in the proof of [10, Theorem 4.17] as follows: for a family $\mathscr{F}$, write $S = S_0 \cap S_1 \cap \cdots \cap S_{e-1}$ for the set of those entries of a symbol in $\mathscr{F}$ that appear in all rows. Write $S_i' := S_i \setminus S$ for the complement, and $\mathscr{F}'$ for the set of symbols

$$\mathscr{F}' := \{\mathscr{S}' := (S_0', \ldots, S_{e-1}') \mid \mathscr{S} \in \mathscr{F}\}.$$

Choose Y' with $|Y'| = em'+1$, where $\mathrm{ct}(\mathscr{S}') = em'+1$, and π' as above. Using the formulae for $\gamma_{\mathscr{S}}$ and $\delta_{\mathscr{S}}$, one convinces oneself that a common factor, depending only on $\mathscr{F}$, can be factored out, so that the remaining terms are exactly the functions $\gamma(\mathscr{S}')$ and $\partial(\mathscr{S}')$. The transition between the two is then achieved by the Fourier transform matrix $T(Y', \pi')$.

Remark 4.19. If e is odd or m is even, then the set $\Psi(Y)$ additionally carries the structure of a homocyclic e-group of order e^{em}, under pointwise addition. This does not in general transfer to the associated families, since $|\mathscr{F}|$ need not be a power of e by (4.16) for $e > 2$. Because of this it does not appear possible to associate the Fourier transform to a finite abelian group as in the real case.

4D. **Harish-Chandra theories for eigenvalues of Frobenius.** Using the identification of families of unipotent degrees with appropriate sets Ψ that was carried out in the last section, we have assigned an eigenvalue of Frobenius

$$\mathrm{Fr}(\gamma_{\mathscr{S}}) = \mathrm{Fr}(\mathscr{S}) = \zeta_{24}^{-2(e^2-1)m} \prod_{i=0}^{e-1} \zeta^{-(i^2+ei)m_i/2}$$

to every $\gamma_{\mathscr{S}} \in \mathrm{Uch}(B_n^{(e)})$ with $\mathscr{S} = (S_0, \ldots, S_{e-1})$, $|S_i| = m_i$. (It is clear that this assignment is independent of the chosen identification.) Lusztig has shown that the unipotent characters γ of groups of Lie type can also be assigned a root of unity $\mathrm{Fr}(\gamma)$, so that the eigenvalues of the Frobenius map on the γ-isotypical part of each l-adic cohomology group of a Deligne–Lusztig variety is equal to $\mathrm{Fr}(\gamma)$ times an integer power of $q^{1/2}$.

Remark 4.20. By comparing the formulae in [11, Proposition 6.6] with (4.12), one sees that our eigenvalues of Frobenius in the case of groups $B_n(q)$ coincide with the eigenvalues of the Frobenius map defined by Lusztig. Similarly, we obtain for $e = 1$ by Lemma 4.13(b) that $\mathrm{Fr}(\mathscr{S}) = 1$ for all symbols $\mathscr{S}$, which tallies with the fact that for $\mathrm{GL}_n(q)$ the eigenvalue of the Frobenius map is 1 for all unipotent characters.

Also, Lusztig's eigenvalue of the Frobenius map is compatible with the usual Harish-Chandra induction: if γ lies in the Harish-Chandra series of the cuspidal character $\boldsymbol{\lambda}$ of a Levi subgroup, we have $\mathrm{Fr}(\gamma) = \mathrm{Fr}(\boldsymbol{\lambda})$. We show that the analogous statement holds for every Φ-Harish-Chandra series.

We first describe the situation for groups of Lie type. If the unipotent character γ lies in the d-Harish-Chandra series of the d-cuspidal character $\boldsymbol{\lambda}$ of a d-split Levi subgroup $\mathbb{L}$ of $\mathbb{G}$, then γ is parametrized by an irreducible character χ of the associated Hecke algebra $\mathscr{H}$ of the relative Weyl group $W_{\mathbb{G}}(\mathbb{L}, \boldsymbol{\lambda})$. If this cyclotomic Hecke algebra is in fact the endomorphism algebra of the associated Deligne–Lusztig induction $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$, as conjectured in [4, 1B], then there is a natural candidate F in $\mathscr{H}$ for the image of the Frobenius map. This is a central element, acting as a scalar on the representation associated to χ, which can be written as a power of q times a root of unity F_χ. Then the eigenvalue F_χ should be linked by the equation $\mathrm{Fr}(\gamma) = F_\chi \cdot \mathrm{Fr}(\boldsymbol{\lambda})$. (In the case $d = 1$ we always have $F_\chi = 1$, and we recover Lusztig's statement.)

One obtains the appropriate candidate for F as follows: let $T_1, \ldots, T_n$ be the standard generators of the Hecke algebra $\mathscr{H}(W_n, \mathbf{u})$, and define U_i via

$$U_1 := T_1, \qquad U_i := T_i U_{i-1} T_i \text{ for } 2 \leqslant i \leqslant n.$$

Then $U := U_1 \ldots U_n$ lies in the centre of $\mathscr{H}$ [2, Theorem 3.20]. By specializing the parameters of $\mathscr{H}$ to the eth roots of unity, U becomes a generator (of order e) of the centre of W_n. The Frobenius map then acts as an appropriate power of U.

The corresponding statement makes sense in the general case for reflection data of type $B_n^{(e)}$. Let Φ be the minimal polynomial over K of a primitive dth root of unity ξ, and define c, d', e', ξ_c as in Theorem 3.14. For $\gamma_{\mathscr{S}} \in \mathrm{Uch}(B_n^{(e)})$, let $\gamma_{\mathscr{S}'}$ be an element of $\mathrm{Uch}_\Phi(\mathbb{L})$, with $\mathscr{S}$ having (d', ξ_c)-core $\mathscr{S}'$ and associated character of the Hecke algebra χ. The parameters of the associated Hecke algebra $\mathscr{H}$ were determined in Theorem 3.14. The image of the Frobenius map in $\mathscr{H}$ should now be the e'th power of the element U described above: $F = U^{e'}$. This is a central element, acting therefore as a scalar on the representation of $\mathscr{H}$ associated to χ. This scalar can always be described as a product of an integer power of q times a root of unity F_χ. (More precisely, F_χ also depends on both e' and the parameters $\mathbf{u}$ of $\mathscr{H}$; these should be clear from the context.)

Simultaneous multiplication of all parameters by a root of unity does not change the structure of $\mathscr{H}$, but rather the root of unity F_χ. This corresponds to simultaneous shifting of the beads on all runners of the abacus diagram of the (d', ξ_c)-quotient $\mathscr{Q}$. In order to obtain a well-defined eigenvalue in every case, we require that the number of beads on all runners of $\mathscr{Q}$ be divisible by c. One can check that this uniquely determines F_χ.

Theorem 4.21. *With the above notation and terms, the eigenvalue of Frobenius of $\gamma_{\mathscr{S}} \in \mathrm{Uch}(B_n^{(e)})$ satisfies the compatibility condition*

$$\mathrm{Fr}(\mathscr{S}) = F_\chi \cdot \mathrm{Fr}(\mathscr{S}'), \quad \textit{where} \quad \chi = \chi(\mathscr{S}, \Phi),$$

for all Φ-Harish-Chandra theories.

Proof. As the first step, we determine the change to the eigenvalue of Frobenius upon removal of a (d', ξ_c)-hook. Let $\mathscr{S}$ be a symbol with $|S_i| = m_i$, with content $\mathrm{ct}(\mathscr{S}) = \sum m_i = em+1 \equiv 1 \bmod e$, and with $\mathrm{def}(\mathscr{S}) = \binom{e}{2}m - \sum i m_i \equiv 0 \bmod e$. Then we have

$$\mathrm{Fr}(\mathscr{S}) = \kappa \prod_{i=0}^{e-1} \zeta^{-(i^2+ei)m_i/2} \qquad \text{where } \kappa = \zeta_{24}^{-2(e^2-1)m}.$$

Removal of a (d', ξ_c)-hook at $\lambda \in S_j$ produces a symbol $\mathscr{T}'$. For this symbol we have $\mathrm{def}(\mathscr{T}') \equiv e' \bmod e$, so to calculate the eigenvalue of Frobenius we must cycle the rows by $-e'$. The resulting symbol will be denoted by $\mathscr{T}$. For this symbol, we have

$$\begin{aligned}\mathrm{Fr}(\mathscr{T}) &= \kappa \prod_{i=0}^{e-1} \zeta^{-(i^2+ei)m_{i+e'}/2} \cdot \zeta^{(j-e')^2/2+e(j-e')/2-j^2/2-ej/2} \\ &= \kappa \prod_{i=0}^{e-1} \zeta^{-(i^2+ei)m_i/2} \cdot \prod_{i=0}^{e-1} \zeta^{ie'm_i} \cdot \prod_{i=0}^{e-1} \zeta^{-(e'^2-ee')m_i/2} \zeta^{(e'^2-ee'-2je')/2} \\ &= \zeta^{-je'} \cdot \mathrm{Fr}(\mathscr{S}).\end{aligned}$$

The (d', ξ_c)-core $\mathscr{S}'$ is obtained from $\mathscr{S}$ by successive removal of (d', ξ_c)-hooks in rows $j_1, \dots, j_w$ of $\mathscr{S}$, so we therefore have by induction

$$\mathrm{Fr}(\mathscr{S}) = \zeta^{e'(\sum j_i - e'\binom{w}{2})} \cdot \mathrm{Fr}(\mathscr{S}'). \tag{4.22}$$

For the proof we must therefore calculate the root of unity F_χ. According to [2, S.29], the central element U acts on the representation of $\mathscr{H}(Z_f \wr S_w, \mathbf{u})$ (with $\mathbf{u} = (u_0; u_1, \dots, u_{f-1})$) parametrized by $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{e-1})$, $\alpha_i \vdash w_i$, $\sum w_i = w$, as the scalar

$$u_0^{\binom{w}{2}} \prod_{i=0}^{f-1} u_i^w u_0^{h(\alpha_i)}.$$

Here, for a partition β, $h(\beta)$ is the sum of the differences $i - j$, over all boxes in column i and row j in the associated Young tableau. The inner product can also be interpreted as follows: for every (d', ξ_c)-hook at position $\lambda \in S_j$, with $\lambda = d'\mu + s$, $j = t - e'\mu$, we obtain an additional factor $u_{es+t} u_0^{\mu+k_{es+t}-r}$, where r denotes the constant number of beads on the runners of the symbol $\mathscr{Q}$ of the (d', ξ_c)-quotient $\mathscr{T}'$. If we substitute in the values for the u_i that were given in Theorem 3.14, then this gives a contribution of a root of unity:

$$\zeta^{es+t+e'k_{es+t}} \xi_c^{-(\mu+k_{es+t}-r)} = \zeta^{t+e'k_{es+t}-e'(\mu+k_{es+t}-r)} = \zeta^j,$$

where here the scaling factor $c \mid r$, hence $e \mid e'r$, was used. Finally, $u_0^{\binom{w}{2}}$ contributes the factor $\zeta^{-e'\binom{w}{2}}$. The statement follows by comparing (4.22) with the e'th power of the root-of-unity factor of U. □

5. Reflection data of type ${}^tI_n^{(e)}$

In this section we deal with the second doubly infinite series of irreducible n-dimensional complex reflection groups that are generated by n reflections. To this end, we define these as appropriate subgroups of $W_n = G(e, 1, n)$. In the investigation of the associated reflection datum, the reflection groups $G(e, p, n)$, $p \mid e$, appear in a natural way as relative Weyl groups, and their Hecke algebras introduced in [1]. Also occurring here are the t-twisted reflection data for each divisor t of e, which represent a direct generalization of the twisted generic groups of type ${}^2D_n$, but also of ${}^2B_2$ and ${}^2G_2$.

5A. **The reflection group $G(e,p,n)$.** Let n and e be fixed, let $W_n = G(e,1,n)$ be as in Section 2 and let $\tilde{W}_n$ be the subgroup of W_n generated by the permutations

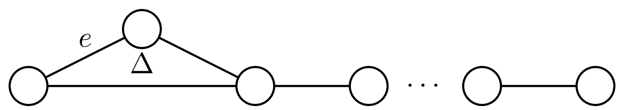
$$\tilde{s}_1 := s_1^{-1} s_2 s_1 = (1, n+2)(n+1, 2n+2)\dots((e-1)n+1, 2)$$

and $s_2, \dots, s_n$, considered as a permutation group on en points. In addition to the relations (2.1), $\tilde{s}_1, s_2, \dots, s_n$ satisfy

$$(\tilde{s}_1 s_2)^e = 1, \qquad \tilde{s}_1 s_3 \tilde{s}_1 = s_3 \tilde{s}_1 s_3, \qquad (\tilde{s}_1 s_2 s_3)^2 = (s_3 \tilde{s}_1 s_2)^2,$$

and this provides a presentation of $\tilde{W}_n$, which is represented by the following diagram (see [5]).

(5.1)



In the faithful n-dimensional irreducible matrix representation of W_n over $K = \mathbb{Q}(\zeta)$, $\zeta = \exp(2\pi \mathrm{i}/e)$, $\tilde{W}_n$ is generated by the permutation matrices of s_i, $2 \leqslant i \leqslant n$, which swap the $(i-1)$th and ith basis vectors, and $\tilde{s}_1$, which acts on the subspace generated by the first two basis vectors as

$$\begin{pmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{pmatrix},$$

and on the remaining basis vectors as the identity. Hence $\tilde{W}_n$ is also an n-dimensional complex reflection group that is generated by n reflections, all of which have order 2. In [16] these reflection groups are denoted by $G(e,e,n)$. Alternatively, $\tilde{W}_n$ can also be defined as the kernel of the linear character $\gamma_1 : W_n \to K$ defined in (2.2). Hence $\tilde{W}_n$ is a normal subgroup of index e in W_n, and has order $e^{n-1} n!$. Clearly for $n = 1$, $\tilde{W}_n$ is the trivial group, while for $e = 1$ it coincides with W_n. We can therefore assume that $n, e \geq 2$ in the following.

To investigate the conjugacy classes of $\tilde{W}_n$ we go back to the description of the classes of W_n in Section 2A. The element

$$\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \zeta^i & & & \end{pmatrix}$$

of W_n lies in the conjugacy class parametrized by $\boldsymbol{\alpha} = (-, \dots, -, (n), -, \dots, -)$, with a single entry $\alpha_i = (n)$. Therefore if the class C of W_n is parametrized by the e-tuple $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{e-1})$, $\alpha_i = (\alpha_{i1}, \dots, \alpha_{im_i})$, $\alpha_i \vdash n_i$ with $\sum n_i = n$, then it lies in $\tilde{W}_n$ if and only if

$$\gamma_1(C) = \prod_{i=0}^{e-1} \zeta^{im_i} = 1,$$

and hence if and only if $\sum_i im_i \equiv 0 \bmod e$. Furthermore, C decomposes in $\tilde{W}_n$ if the image of the centralizer of an element of C under γ_1 is not all of $\langle \zeta \rangle$. In this case, C decomposes into $g(C) := \gcd\{e, i, \alpha_{ij} \mid \alpha_{ij} \neq 0\}$ classes of $\tilde{W}_n$.

In general, set $W_{n,p}$ for $p \mid e$ to be the subgroup of W_n generated by $s_0 := s_1^p, \tilde{s}_1, s_2, \ldots, s_n$. We have $\tilde{W}_n = W_{n,e}$, and the $W_{n,p}$ are the complex reflection groups $G(e, p, n)$ in the notation of [16].

The irreducible characters of $\tilde{W}_n$ can easily be obtained from those of W_n. Let $\chi \in \mathrm{Irr}(W_n)$ be parametrized by $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$. The subgroup $\langle\pi\rangle$ of the symmetric group on $\{0, \ldots, e-1\}$ generated by $\pi = (0, 1, \ldots, e-1)$ acts on the e-tuples $\boldsymbol{\alpha}$ of partitions by cyclic permutation $\pi : \boldsymbol{\alpha} \mapsto (\alpha_1, \ldots, \alpha_{e-1}, \alpha_0)$. According to (2.3), we have $\chi_{\pi(\boldsymbol{\alpha})} \otimes \gamma_1 = \chi_{\boldsymbol{\alpha}}$. Let $s(\boldsymbol{\alpha})$ be the size of the subgroup of $\langle\pi\rangle$ fixing $\boldsymbol{\alpha}$. Since $\tilde{W}_n = \ker \gamma_1$, all χ that are equivalent under the action of $\langle\pi\rangle$ to that indexed by $\boldsymbol{\alpha}$ have the same restriction and decompose into $s(\boldsymbol{\alpha})$ irreducible characters of $\tilde{W}_n$. Hence $\mathrm{Irr}(\tilde{W}_n)$ is parametrized by e-tuples of partitions up to cyclic permutation, with the proviso that every $\boldsymbol{\alpha}$ stands for exactly $s(\boldsymbol{\alpha})$ characters.

In general, since $W_{n,p} = \ker \gamma_1^{e/p}$, an irreducible character $\chi = \chi_{\boldsymbol{\alpha}} \in \mathrm{Irr}(W_n)$ decomposes on restriction to $W_{n,p}$ into $s_p(\boldsymbol{\alpha})$ distinct irreducible constituents, where $s_p(\boldsymbol{\alpha})$ is the order of the subgroup of $\langle\pi^{e/p}\rangle$ that fixes $\boldsymbol{\alpha}$.

5B. **The reflection data ${}^tI_n^{(e)}$.** By the above, $\tilde{W}_n$ is a normal subgroup of index $t := e/p$ in the reflection group $W_{n,p}$, with cyclic quotient generated by s_1^p. Let $I_n^{(e)}(q) = (V, \tilde{W}_n)$, and

$$ {}^tI_n^{(e)}(q) = (V, \tilde{W}_n s_1^p) \tag{5.2} $$

the reflection datum resulting from twisting $I_n^{(e)}$ by s_1^p. This is a generalization of the twisted generic group ${}^2D_n$, which represents the special case $(e, p) = (2, 1)$. The coset $W_n s_1^p$ consists of those elements $w \in W_n$ with $\gamma_1(w) = \zeta^p$. Because $|Z(W_{n,p})| = \gcd(nt, e)$, the normal subgroup $\tilde{W}_n$ of $W_{n,p}$ has a central supplement in the case where $t \mid (e/\gcd(n, e))$. Hence we only have a genuine twisted reflection datum if t does not divide $e/\gcd(n, e)$. In general, for any $1 \leqslant p \leqslant e$, set $t := e/p$ and define the associated twisted reflection datum ${}^tI_n^{(e)}(q)$ via (5.2).

Let Φ be a cyclotomic polynomial over K with a primitive dth root of unity ξ as a zero. We set $c := \gcd(e, d)$, $d' := d/c$ and $\xi_c := \xi^{d'}$. From the description of the coset $W_n s_1^p$ using γ_1 we see that a Sylow Φ-torus $\mathbb{S}$ of $\mathbb{G} = {}^tI_n^{(e)}(q)$ has the same order as that of the reflection datum associated to W_n, except in the case $d \mid en$, $\xi^n \neq \zeta^p$, in which case the exponent is reduced by l. Hence ${}^tI_n^{(e)}$ has order

$$ |{}^tI_n^{(e)}(q)| = (q^e - 1)(q^{2e} - 1) \ldots (q^{(n-1)e} - 1)(q^n - \zeta^p). $$

The Φ-split Levi subgroups have the structure

$$ \mathbb{L} = \mathrm{GL}_{n_1}(\xi_c^{-1} q^{d'}) \times \cdots \times \mathrm{GL}_{n_r}(\xi_c^{-1} q^{d'}) \times {}^sI_m^{(e)}(q), \tag{5.3} $$

where $\sum_{i=1}^r d' n_i = n - m$ and $s = e/p'$ for some $1 \leqslant p' \leqslant e$ determined via γ_1 by the condition

$$ \zeta^{p'} = \xi_c^{-\sum n_i} \zeta^p. $$

The relative Weyl group $W_{\mathbb{G}}(\mathbb{L}) = \mathscr{N}_{\tilde{W}_n}(W_{\mathbb{L}})/W_{\mathbb{L}}$ of such a Levi subgroup is then

$$W_{\mathbb{G}}(\mathbb{L}) = Z_{ed'} \wr S_{k_1} \times \cdots \times Z_{ed'} \wr S_{k_n} \qquad \text{when } m \neq 0, \tag{5.4}$$

where $k_i := k_i(\mathbb{L}) := |\{j \mid n_j = i\}|$, or a subgroup thereof of index e for $m = 0$.

Using (2.8), we can assign an invariant $\delta_\chi(W, q)$ to every irreducible character χ of every finite group W with a representation on a finite-dimensional complex vector space. The polynomial ${}^tR_{\boldsymbol{\alpha}}$ is the analogue of the fake degree defined in (2.9) for ${}^tI_n^{(e)}$, hence ${}^tR_{\boldsymbol{\alpha}}(q) := |{}^tI_n^{(e)}|_q \cdot {}^t\tilde{\delta}_{\boldsymbol{\alpha}}$. With (2.11) we obtain an explicit formula for the ${}^t\tilde{\delta}_{\boldsymbol{\alpha}}$.

Proposition 5.5. *Let $\chi_{\boldsymbol{\alpha}} \in \mathrm{Irr}(\tilde{W}_n)$ be parametrized by $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$, $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im_i}) \vdash n_i$, and let $S_i = (\lambda_{i1}, \ldots, \lambda_{im_i})$ be the associated sequence of β-numbers. Then the fake degree of $\chi_{\boldsymbol{\alpha}}$ for ${}^tI_n^{(e)}$ is*

$${}^t\tilde{\delta}_{\boldsymbol{\alpha}}(q) = \frac{1}{s(\boldsymbol{\alpha})} \sum_{j=0}^{e-1} \zeta^{jp} \prod_{i=0}^{e-1} \frac{\Delta(S_i, q^e)}{\Theta(S_i, q^e)} \frac{\prod_{i=0}^{e-1} \prod_{\lambda \in S_{i+j}} q^{i\lambda}}{q^{\binom{e(m-1)}{2} + \binom{e(m-2)}{2} + \cdots}}. \tag{5.6}$$

We see that $\boldsymbol{\alpha}$ that are equivalent under $\langle \pi \rangle$ actually have the same fake degree ${}^t\tilde{\delta}_{\boldsymbol{\alpha}}(q)$. The order of the zero $\tilde{a}(\chi_{\boldsymbol{\alpha}})$ of ${}^t\tilde{\delta}_{\boldsymbol{\alpha}}$ at $q = 0$ is, by (5.6),

$$\tilde{a}(\chi_{\boldsymbol{\alpha}}) = e \sum_{i=0}^{e-1} \sum_{\substack{\lambda, \lambda' \in S_i \\ \lambda' < \lambda}} \lambda' + \min_j \left\{ \sum_{i=0}^{e-1} i \sum_{\lambda \in S_{i+j}} \lambda \right\} - \sum_{k=1}^{m-1} \binom{ek}{2}, \tag{5.7}$$

where $(S_0, \ldots, S_{e-1})$ denotes a symbol of β-sequences S_i for $\boldsymbol{\alpha}$ with $|S_i| = m$.

5C. **Hecke algebras and generic degrees.** From the structure of the relative Weyl group of the Φ-split Levi subgroups of $\mathbb{G}$ given in (5.4) it is clear that, for the description of the Φ-Harish-Chandra series of $I_n^{(e)}$, we need Hecke algebras $\mathscr{H}(W_{n,p}; \mathbf{u})$ with parameters $\mathbf{u} = (u_0, \ldots, u_{e/p-1})$ for all complex reflection groups $G(e, p, n)$. These were introduced by Ariki in his article [1], generalizing the cases $e = 1, 2$ [8] and $p = 1, 2$ [4]. The definition is based on an appropriate presentation of $W_{n,p}$ on the generators $s_0, \tilde{s}_1, s_2, \ldots, s_n$, in line with Definition 2.18. Here we use an alternative description, from [1, Prop. 1.6], as a subalgebra of $\mathscr{H}(B_n^{(e)})$ with suitably specialized parameters.

Definition 5.8. Define $\mathscr{H}(W_{n,p}; \mathbf{u})$, where $\mathbf{u} = (u_0, \ldots, u_{t-1})$, to be the $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-subalgebra of the cyclotomic Hecke algebra $\mathscr{H}(W_n; \tilde{\mathbf{u}})$ generated by

$$\{T_1^p, T_1^{-1} T_2 T_1, T_2, \ldots, T_n\},$$

with the parameters

$$\tilde{\mathbf{u}} = (u_0, u_0\zeta_p, \ldots, u_0\zeta_p^{p-1}, \ldots, u_{t-1}, \ldots, u_{t-1}\zeta_p^{p-1}), \tag{5.9}$$

where $\zeta_p := \mathrm{e}^{2\pi \mathrm{i}/p}$.

For $p = e$, this algebra $\mathscr{H}(\tilde{W}_n; \mathbf{u})$ agrees with that defined as in Definition 2.18, based on the diagram $I_n^{(e)}$.

We recall (see [4, 1C]) that for a split semisimple algebra $K\mathscr{H}$ over a field K with symmetrizing form $t : K\mathscr{H} \to K$, the relative degrees $\delta_\chi(t)$ of $K\mathscr{H}$ with respect to t are given as the coefficients in the representation

$$t = \sum_{\chi \in \mathrm{Irr}(K\mathscr{H})} \delta_\chi(t)\chi.$$

For the description of the relative degrees of $\mathscr{H}(W_{n,p}; \mathbf{u})$, we use the following simple observation, which was kindly pointed out to me by M. Broué.

Lemma 5.10. *Let $K\mathscr{H}$ and t be as above, $K\mathscr{H}'$ a split semisimple subalgebra of $K\mathscr{H}$, and t' the restriction of t to $K\mathscr{H}'$. Writing*

$$\mathrm{Res}_{K\mathscr{H}'}^{K\mathscr{H}}(\chi) = \sum_{\chi' \in \mathrm{Irr}(K\mathscr{H}')} a_{\chi',\chi}\chi',$$

for $\chi \in \mathrm{Irr}(K\mathscr{H})$, we have

$$\delta_{\chi'}(t') = \sum_{\chi \in \mathrm{Irr}(K\mathscr{H})} a_{\chi',\chi}\delta_\chi(t).$$

In particular, t' is a symmetrizing form for $K\mathscr{H}'$ if and only if all of the $\sum_{\chi \in \mathrm{Irr}(K\mathscr{H})} a_{\chi',\chi}\delta_\chi(t)$ are non-zero.

This allows us to compute the relative degrees of $\mathscr{H}(W_{n,p}; \mathbf{u})$ under the assumptions of Conjecture 2.20. The result can again be understood via symbols; the requisite formulas will be established next.

As stated in Section 5A, the irreducible characters of $W_{n,p}$ are parametrized by e-tuples of partitions $\boldsymbol{\alpha}$ modulo cyclic permutation by π^t, where $t = e/p$. We again normalize the tuples of partitions $\boldsymbol{\alpha}$. By adding zeroes, we can alter $\boldsymbol{\alpha}$ so that α_i has $m+1$ parts for $i \equiv 0 \bmod t$, and m parts for the other i. Let $S_0, \ldots, S_{e-1}$ be the associated sequences of β-numbers, and $\mathscr{S} := (S_0, \ldots, S_{e-1})$ the symbol so formed. Then the class of $\mathscr{S}$ only depends on the partition $\boldsymbol{\alpha}$ modulo cyclic permutation by π^t. In particular, $\mathrm{Irr}(W_{n,p})$ is parametrized by (e-)symbols of rank n, whose ith row has $m+1$ entries for $i \equiv 0 \bmod t$ and m entries otherwise, where each $\mathscr{S}$ stands for $s_p(\mathscr{S}) := s_p(\boldsymbol{\alpha})$ characters.

For a symbol $\mathscr{S}$ for $W_{n,p}$, let $D_{\mathscr{S}}(q; v_0, \ldots, v_{t-1}) := D_{\mathscr{S}}(q; w_0, \ldots, w_{e-1})$ be given by

(5.11)

$$\frac{p}{s_p(\mathscr{S})} \frac{(-1)^{\binom{e}{2}\binom{m}{2}+\binom{p}{2}mt+n(e-1)}(q-1)^n \prod_{i=0}^{e-1}\prod_{j=i}^{e-1}\prod_{\substack{(\lambda,\mu)\in S_i\times S_j \\ \mu<\lambda \text{ if } i=j}} \left(q^\lambda w_i - q^\mu w_j\right) \cdot \prod_{i=0}^{e-1} w_i^n}{q^{\binom{e(m-1)+p}{2}+\binom{e(m-2)+p}{2}+\cdots} \prod_{i,j=0}^{e-1}\prod_{\lambda\in S_i}\prod_{k=1}^{\lambda}(q^k w_i - w_j)\prod_{i=0}^{e-1}\prod_{j=i+1}^{e-1}(w_i - w_j)^{m+\delta(t,i,j)}}$$

where

$$\delta(t,i,j) := \begin{cases} 1 & \text{if } i \equiv j \equiv 0 \bmod t \\ 0 & \text{otherwise,} \end{cases} \qquad w_i := v_i\zeta_p^{\tilde{i}} \quad \text{for } 0 \leqslant i \leqslant e-1, \quad \tilde{i} := \lfloor i/t \rfloor,$$

and where the index of v_i is read modulo t. In the case $p = 1$ we clearly obtain the expression defined in (2.19). In the case $e = 2(= p)$, the above formula becomes just the generic degrees of the groups of Lie type D_n. In general, we have the following.

Theorem 5.12. *Assuming Conjecture 2.20, the relative degree of the character $\chi_{\boldsymbol{\alpha}}$ of the generic Hecke algebra $\mathscr{H}(W_{n,p};\mathbf{u})$ with respect to the restriction of the symmetrizing form t on $\mathscr{H}(W_n;\tilde{\mathbf{u}})$ is given by $D_{\mathscr{S}}(u_0;1,u_1,\ldots,u_{t-1})$, where $\mathscr{S} = (S_0,\ldots,S_{e-1})$ denotes the symbol constructed from $\boldsymbol{\alpha}$ according to the above rule.*

Proof. We saw in Section 5A how the irreducible characters of $W_{n,p}$ can be obtained as constituents of the restriction of irreducible characters of W_n. Based on results of Ariki [1, Theorem 2.6], the irreducible characters of $\mathscr{H}_1 := \mathscr{H}(W_n;\tilde{\mathbf{u}})$ exhibit the same behaviour upon restriction to $\mathscr{H} := \mathscr{H}(W_{n,p};\mathbf{u})$. For $\chi \in \mathrm{Irr}(\mathscr{H}_1)$ let $\chi|_{\mathscr{H}} = \sum_{i=1}^{s(\chi)} \chi^{(i)}$ be the decomposition into irreducibles of the restriction to $\mathscr{H}$. Then we obtain by Lemma 5.10 that the relative degrees of $\mathscr{H}$ are

$$\delta_{\chi^{(i)}}(t) = \frac{p}{s(\chi)}\delta_\chi(t), \qquad 1 \leqslant i \leqslant s(\chi).$$

Next, we have $s(\chi) = s_p(\mathscr{S})$, where χ is parametrized by the tuple $\boldsymbol{\alpha}$ of partitions with associated symbol $\mathscr{S}$. It is easy to see, and left to the reader to verify, that the $D_{\mathscr{S}}(q;v_0,\ldots,v_{t-1})$ in (5.11) is simply that of (2.19) through specializing the parameters as in (5.9). Multiplication by $p/s_p(\mathscr{S})$ and shifting the ith rows of the symbol $\mathscr{S}$, for all $i \equiv 0 \bmod e/p$, yields the result. $\square$

For $p = e$, i.e., the reflection group $\tilde{W}_n$, (5.11) simplifies to only being dependent on the one parameter q, namely

$$D_{\mathscr{S}}(q) = \frac{e}{s(\mathscr{S})} \frac{(-1)^{\binom{e}{2}\binom{m+1}{2}}(q-1)^n \cdot \prod_{i=0}^{e-1}\prod_{j=i}^{e-1} \prod_{\substack{(\lambda,\mu)\in S_i\times S_j \\ \mu<\lambda \text{ if } i=j}} (q^\lambda\zeta^i - q^\mu\zeta^j)}{\tau(e)^{m+1} \cdot q^{\binom{em}{2}+\binom{e(m-1)}{2}+\cdots} \cdot \prod_{i=0}^{e-1}\Theta(S_i,q^e)}, \tag{5.13}$$

assuming the requirement that every row of $\mathscr{S}$ has exactly $m+1$ entries. We change our notation in this case so that m indicates the number of entries in each row of $\mathscr{S}$. Denoting by $a(\mathscr{S})$ the order of the zero at $q = 0$ of $D_{\mathscr{S}}(q)$, and by

$A(\mathscr{S})$ the degree of $|{}^tI_n^{(e)}|_q \cdot D_{\mathscr{S}}(q)$, we have

$$a(\mathscr{S}) = \sum_{\{\lambda,\mu\}} \min(\lambda,\mu) - \sum_{k=1}^{m-1} \binom{ek}{2}$$

$$A(\mathscr{S}) = \sum_{\{\lambda,\mu\}} \max(\lambda,\mu) - \sum_{k=1}^{m-1} \binom{ek}{2} - e \sum_{\lambda\in\mathscr{S}} \binom{\lambda+1}{2} + \frac{n(en-e+2)}{2}, \tag{5.14}$$

where in each case the first sum is taken over all unordered pairs of entries of $\mathscr{S}$. As in the case of W_n, we prove the following lemma, from (5.7) and (5.14).

Lemma 5.15. (a) *We have $\tilde{a}(\chi_{\mathscr{S}}) \geq a(\mathscr{S})$ for all $\chi_{\mathscr{S}} \in \mathrm{Irr}(\tilde{W}_n)$. We have equality if and only if $\mathscr{S}$ is, up to cyclic permutation of the rows, of the form*

$$\mathscr{S} = \begin{pmatrix} \lambda_{01} & \cdots & \lambda_{0m} \\ \vdots & & \vdots \\ \lambda_{e-1,1} & \cdots & \lambda_{e-1,m} \end{pmatrix}$$

with

$$\lambda_{ij} \leqslant \lambda_{kl} \qquad \text{for } j < l, \text{ or } j = l \text{ and } i > k. \tag{5.16}$$

We then call $\mathscr{S}$ a special *symbol.*

(b) *We have $\tilde{A}(\chi_{\mathscr{S}}) \leqslant A(\mathscr{S})$ for all $\chi_{\mathscr{S}} \in \mathrm{Irr}(\tilde{W}_n)$. We have equality exactly when $\tilde{\mathscr{S}}$ is special. Such an $\mathscr{S}$ is called* cospecial.

(c) *For all $\chi_{\mathscr{S}} \in \mathrm{Irr}(\tilde{W}_n)$ we have*

$$\tilde{a}(\chi_{\mathscr{S}}) + \tilde{A}(\chi_{\mathscr{S}}) + \tilde{a}(\chi_{\bar{\mathscr{S}}}) + \tilde{A}(\chi_{\bar{\mathscr{S}}}) = a(\mathscr{S}) + A(\mathscr{S}) + a(\bar{\mathscr{S}}) + A(\bar{\mathscr{S}}).$$

(d) *We have $a(\mathscr{S}) \leqslant N^* = N$ and $A(\mathscr{S}) \leqslant N$. Equality occurs exactly for $\mathscr{S}$ that are equivalent to*

$$\mathscr{S}_{St} := (\,(1,\ldots,n),(0,\ldots,n-1),\ldots,(0,\ldots,n-1)\,).$$

6. Unipotent degrees for ${}^tI_n^{(e)}$

The unipotent degrees for ${}^tI_n^{(e)}$ are parametrized by symbols of content 0, and are given by the same combinatorial formula, as with $B_n^{(e)}$. The degrees satisfy a hook formula and a degree theorem. They are sent by a Fourier transform to the fake degrees of the associated reflection datum. Together with suitable eigenvalues of Frobenius, we obtain a representation of $\mathrm{SL}_2(\mathbb{Z})$. In the untwisted case, the eigenvalue of Frobenius depends only on the equivalence class of the symbol, and thus allows us to assign them to the unipotent degrees. For the case $I_2^{(e)}$ our Fourier transform specializes to the exotic Fourier transform of Lusztig [14].

6A. **Symbols and unipotent degrees for ${}^tI_n^{(e)}$.** The unipotent degrees of the reflection datum ${}^tI_n^{(e)}$ will be parametrized by symbols of content $\mathrm{ct}(\mathscr{S}) \equiv 0 \bmod e$. For such an $\mathscr{S} = (S_0, \ldots, S_{e-1})$, the formula (3.2) for the rank simplifies to

$$\mathrm{rk}(\mathscr{S}) = \sum_{\lambda \in \mathscr{S}} \lambda - e\binom{m}{2} \qquad \text{where} \qquad \mathrm{ct}(\mathscr{S}) = em.$$

We define the *defect* to be

$$\mathrm{def}(\mathscr{S}) := \frac{e-1}{2}\mathrm{ct}(\mathscr{S}) - \sum_{i=0}^{e-1} i|S_i| \mod e. \tag{6.1}$$

Note that because of $\mathrm{ct}(\mathscr{S}) \equiv 0 \bmod e$ this in fact only depends on the equivalence class of $\mathscr{S}$. By removing an (l, ζ^j)-hook, the defect is obviously reduced by j. From the construction of the (l, ζ^j)-quotient we obtain, via the description of the irreducible characters of $W_{n,p}$:

Corollary 6.2. *The equivalence classes of symbols of rank n and content congruent to 0 modulo e with a given (l, ζ^j)-core $\mathscr{S}'$ (and of (l, ζ^j)-weight $w = (n - \mathrm{rk}(\mathscr{S}'))/l$), with each symbol $\mathscr{S}$ occurring $s(\mathscr{S})$ times, are in bijection with the irreducible characters of the reflection group $G(el, s(\mathscr{S}'), w)$.*

Definition 6.3. For an e-symbol of rank n, content $\mathrm{ct}(\mathscr{S}) = em \equiv 0 \bmod e$ and defect $\mathrm{def}(\mathscr{S}) \equiv p \bmod e$, $t := e/p$, we define the degree to be

$$\gamma_{\mathscr{S}}(q) := \frac{e}{s(\mathscr{S})} \frac{(-1)^{\binom{e}{2}\binom{m}{2}} |{}^tI_n^{(e)}|_{q'} \cdot \prod_{i=0}^{e-1}\prod_{j=i}^{e-1} \prod_{\substack{(\lambda,\mu)\in S_i\times S_j \\ \mu<\lambda \text{ if } i=j}} (q^\lambda\zeta^i - q^\mu\zeta^j)}{\tau(e)^m \cdot q^{\binom{e(m-1)}{2}+\binom{e(m-2)}{2}+\cdots} \cdot \prod_{i=0}^{e-1} \Theta(S_i, q^e)}. \tag{6.4}$$

As $\mathscr{S}$ runs through the set of equivalence classes of e-symbols of rank n, content divisible by e and defect $\mathrm{def}(\mathscr{S}) \equiv p \bmod e$, where each symbol $\mathscr{S}$ appears $s(\mathscr{S})$ times, the multiset $\mathrm{Uch}({}^tI_n^{(e)})$ of such $\gamma_{\mathscr{S}}$ is the set of *unipotent degrees* of ${}^tI_n^{(e)}$. (As a consequence, the degree is well defined only up to sign.) The group $W_n/\tilde{W}_n$ acts on $\mathrm{Uch}({}^tI_n^{(e)})$, with the generator s_1 cyclically permuting the $s(\mathscr{S})$ copies of each symbol $\mathscr{S}$.

For symbols $\mathscr{S}$ with e rows of the same length m, $\gamma_{\mathscr{S}}$ is therefore just the (conjectured) relative degree (5.13) for $\mathscr{S}$, multiplied by the Poincaré polynomial of $I_n^{(e)}$. The well definedness up to sign will not cause problems as the theorems are also defined only up to sign. The only exception is Theorem 6.26, in which (well-defined) degrees are attached to each individual symbol.

Remark 6.5. (a) For $e = 2$, $t = 1, 2$, the above definition coincides with the unipotent character degrees for the groups of Lie type D_n and ${}^2D_n$.

(b) One sees that for $n = 2$ the equation (6.4) provides a description based on symbols of the unipotent degrees defined by Lusztig in [13] for the reflection data $I_2(e)$ and ${}^2I_2(e)$. In particular, since $G(e, e, 2) = I_2(e)$, we thus encode the unipotent character degrees of GL_3, Sp_4, G_2, ${}^2B_2$ and ${}^2G_2$ in a single formula.

Example 6.6. (a) For $n = 2$, $e = 3$, $t = 1$, we have that $\tilde{W}_2 = G(3, 3, 2)$. The set $\mathrm{Uch}(I_2^{(3)})$ is given by

$$\begin{pmatrix}2\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \begin{pmatrix}1 & 2\\0 & 1\\0 & 1\end{pmatrix}.$$

One sees that these unipotent degrees are the same as those for the symbols (3), $(1, 3)$ and $(1, 2, 3)$ for $W(A_2) \cong G(3, 3, 2)$.

(b) For $n = 2$, $e = 4$, $t = 1$, we have $\tilde{W}_2 = G(4, 4, 2)$. The set $\mathrm{Uch}(I_2^{(4)})$ is given by

$$\begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\\1\\0\end{pmatrix} (2\times), \quad \begin{pmatrix}0 & 1\\0 & 1\\- & \\- & \end{pmatrix}, \quad \begin{pmatrix}1 & 2\\0 & 1\\0 & 1\\0 & 1\end{pmatrix}.$$

One sees that these give rise to the same unipotent degrees as the symbols

$$\begin{pmatrix}2\\-\end{pmatrix}, \quad \begin{pmatrix}0 & 2\\1 & \end{pmatrix}, \quad \begin{pmatrix}0 & 1\\2 & \end{pmatrix}, \quad \begin{pmatrix}1 & 2\\0 & \end{pmatrix}, \quad \begin{pmatrix}0 & 1 & 2\\ & - & \end{pmatrix}, \quad \begin{pmatrix}0 & 1 & 2\\1 & 2 & \end{pmatrix}.$$

for $W(B_2) \cong G(4, 4, 2)$.

(c) For $n = 2$, $e = 4$, $t = 4$, we have for $\mathrm{Uch}({}^4I_2^{(4)})$ the symbols

$$\begin{pmatrix}0 & 2\\- & \\0 & \\0 & \end{pmatrix}, \quad \begin{pmatrix}0 & 1\\- & \\0 & \\1 & \end{pmatrix}, \quad \begin{pmatrix}0 & 1\\- & \\1 & \\0 & \end{pmatrix}, \quad \begin{pmatrix}0 & 1 & 2\\1 & & \\0 & 1 & \\0 & 1 & \end{pmatrix}.$$

We have $|{}^4I_2^{(4)}| = q^4(q^4 - 1)(q^2 - \zeta_4)$, and Ennola twisting by ζ_8 gives the unipotent characters of ${}^2B_2$. Doing so we transform the two middle principal series degrees into 1-cuspidal degrees.

(d) For $n = 2$, $e = 6$, $t = 1$ we obtain $\tilde{W}_2 = G(6, 6, 2) = W(G_2)$. The set $\mathrm{Uch}(I_2^{(6)})$ is given by

$$\begin{pmatrix}2\\0\\0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\\1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix} (2\times), \quad \begin{pmatrix}1 & 2\\0 & 1\\0 & 1\\0 & 1\\0 & 1\\0 & 1\end{pmatrix}$$

for the principal 1-series, and the 1-cuspidal symbols

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & \\ - & \\ - & \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & \\ 0 & 1 \\ - & \\ 0 & \\ - & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ - & \\ 0 & \\ 0 & \\ - & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & \\ 0 & \\ 0 & 1 \\ - & \\ - & \end{pmatrix}.$$

In the notation of [6, 13.8], these correspond (in order) to the unipotent characters

$$\phi_{1,0},\ \phi_{2,1},\ \phi_{2,2},\ \phi'_{1,3},\ \phi''_{1,3},\ \phi_{1,6},\ G_2[1],\ G_2[-1],\ G_2[\theta],\ G_2[\theta^2].$$

(e) For $n = 2$, $e = 6$, $t = 2$, we have that $\mathrm{Uch}({}^2I_2^{(6)})$ is

$$\begin{pmatrix} 0 & 2 \\ 0 & \\ 0 & \\ - & \\ 0 & \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & \\ 0 & \\ - & \\ 0 & \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & \\ 1 & \\ - & \\ 0 & \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & \\ 0 & \\ - & \\ 1 & \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & \\ 0 & \\ - & \\ 0 & \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & \\ 0 & 1 & \\ 1 & & \\ 0 & 1 & \\ 0 & 1 & \end{pmatrix}$$

for the principal 1-series, and the 1-cuspidal symbols are

$$\begin{pmatrix} 0 & 1 \\ - & \\ 0 & 1 \\ 0 & \\ - & \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & \\ - & \\ 0 & 1 \\ - & \\ 0 & \end{pmatrix}.$$

We have $|{}^2I_2^{(6)}| = q^6(q^6-1)(q^2+1)$, and by Ennola twisting with ζ_4 we obtain the unipotent degrees $\mathrm{Uch}({}^2G_2)$ of ${}^2G_2$.

The unipotent degrees again satisfy the Ennola property and the hook formula:

Corollary 6.7. *For all $\gamma(q) \in \mathrm{Uch}({}^tI_n^{(e)})$ we have $\pm\gamma(\zeta q)$ lies in $\mathrm{Uch}({}^sI_n^{(e)})$, where $s = et/(e-tn)$.*

Proposition 6.8 (Hook formula)**.** *We have*

$$\gamma_{\mathscr{S}}(q) = \frac{e}{s(\mathscr{S})} \frac{b_{\mathscr{S}} q^{a(\mathscr{S})} |{}^tI_n^{(e)}|_{q'}}{\prod_{j,l}(q^l - \zeta^j)},$$

where the product in the denominator runs over all ζ^j-hooks of $\mathscr{S}$ of length l, and $b_{\mathscr{S}}$ is defined exactly as in Proposition 3.12 in terms of $\mathscr{S}$.

Let Φ be a cyclotomic polynomial over K, with the primitive dth root of unity ξ as a zero, $d := \gcd(e,d)$, $d' := d/c$, and $\xi_c := \xi^{d'}$. As in Section 2, we say that a degree $\gamma_{\mathscr{S}} \in \mathrm{Uch}({}^tI_n^{(e)})$ is Φ-*cuspidal* if it is without (d', ξ_c)-hooks. We use the abbreviation $\mathrm{Uch}_\Phi(\mathbb{G})$ for the Φ-cuspidal degrees of $\mathbb{G}$. According to the description of the Φ-split Levi subgroups in (5.3) the Φ-cuspidal pairs of $\mathbb{G}$ have the form $(\mathbb{L}, \boldsymbol{\lambda})$, where

$$\begin{aligned} \mathbb{L} &= \mathrm{GL}_1(\xi_c^{-1}q^{d'}) \times \cdots \times \mathrm{GL}_1(\xi_c^{-1}q^{d'}) \times {}^sI_m^{(e)}(q), \\ \boldsymbol{\lambda} &= 1 \otimes \cdots \otimes 1 \otimes \boldsymbol{\lambda}' \in \mathrm{Uch}_\Phi(\mathbb{L}). \end{aligned} \tag{6.9}$$

Let $\mathscr{S}'$ be the symbol of rank m which parametrizes the ϕ-cuspidal degree $\boldsymbol{\lambda}'$. Then $\mathscr{S}'$ is a (d', ξ_c)-core, and the symbols $\mathscr{S} \in \mathrm{Uch}(\mathbb{G})$ of rank n with (d', ξ_c)-core $\mathscr{S}'$ have (d', ξ_c)-weight $w := (n-m)/d'$. With respect to the action on Uch defined in Definition 6.3, according to (5.4) the associated relative Weyl group has structure

$$W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda}) = G(ed', s(\mathscr{S}'), w).$$

We saw in Corollary 6.2 that the unipotent degrees $\gamma_\mathscr{S} \in \mathrm{Uch}(\mathbb{G})$ with (d', ξ_c)-core $\mathscr{S}'$ are in bijection with the irreducible characters of $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$, via the construction of the (d', ξ_c)-quotient. More precisely, we have as for $B_n^{(e)}$ the following.

Theorem 6.10. *Let $\gamma_\mathscr{S} \in \mathrm{Uch}({}^tI_n^{(e)})$ be parametrized by the symbol $\mathscr{S}$. Let ξ be a primitive dth root of unity with minimal polynomial Φ over K, set $c := \gcd(e, d)$, $d' := d/c$, and define e' via $\xi^{d'} = \zeta^{e'} =: \xi_c$. Let $\mathscr{S}'$ denote the (d', ξ_c)-core, and $\mathscr{Q} = (Q_0, \dots, Q_{ed'-1})$ an ed'-symbol of the (d', ξ_c)-quotient of $\mathscr{S}$. Then we have*

$$\gamma_\mathscr{S}(q) = \pm \frac{|\mathbb{G}(q)|_{q'}}{|\mathbb{L}(q)|_{q'}} D_\mathscr{Q}(\xi_c^{-1}q^{d'}; w_0, \dots, w_{ed'-1}) \gamma_{\mathscr{S}'}(q),$$

where $\mathbb{G} = {}^tI_n^{(e)}$, $\gamma_{\mathscr{S}'} \in \mathrm{Uch}_\Phi(\mathbb{L})$,

$$w_i := q^{\tilde{i} - k_i d'} \zeta^{i + e'k_i} \quad (0 \leqslant i \leqslant ed' - 1) \quad \textit{where } \tilde{i} := \lfloor i/e \rfloor,$$

and $k_i := \max_l\{|Q_l| - \delta(l)\} - |Q_i| + \delta(i)$, with

$$\delta(l) = \begin{cases} 1 & \textit{if } l \equiv 0 \bmod ed'/s(\mathscr{S}'), \\ 0 & \textit{otherwise.} \end{cases}$$

Observe the following: with $s' := e/s(\mathscr{S}')$, the above parameters satisfy $w_{i+js'} = \zeta^{js'} w_i$, $0 \leqslant j \leqslant s(\mathscr{S}') - 1$, and therefore constitute an allowable set of parameters for the degrees $D_\mathscr{Q}$ of the Hecke algebra of the reflection group $G(ed', s(\mathscr{S}'), w)$, according to (5.11).

Proof. We can proceed exactly as in the proof of Theorem 3.14. It suffices to show the analogue of (3.15). The subsequent calculations remain valid until the last chain of equations. There, to the left-hand side a factor of $s(\mathscr{S})/s(\mathscr{R})$, and to the right-hand side a factor of $s_{s'}(\mathscr{T})/s_{s'}(\mathscr{T}')$, must be added, where $s' := s(\mathscr{S}')$ denotes the order of the symmetry group of the core $\mathscr{S}'$, and $\mathscr{T}'$ denotes the ed'-symbol obtained from $\mathscr{T}$ by remval of a 1-hook at position $\nu \in T_j$. The equality of these two factors is clear, however, as the symmetry groups of $\mathscr{S}$ and $\mathscr{R}$ are each subgroups of the symmetry group of $\mathscr{S}'$. $\square$

The above result was again already known in some special cases: for $d = 1$, and either $e = 2$ or $n = 2$, $t \in \{1, e\}$, it is the usual Harish-Chandra theory for the groups of Lie type D_n, ${}^2D_n$, G_2, ${}^2B_2$ and ${}^2G_2$, and for the generic groups of type $I_2(n)$ respectively (see [4, 7.2]). The case $e \in \{1, 2\}$, d arbitrary, $w(\mathscr{S}, d', \xi_c) = 1$, was handled in [4, Bem. 2.14, 2.16, 2.19]. In terms of the conjectures given in [4], we obtain the following.

Corollary 6.11. *Let $\mathbb{G}$ be a generic group of Lie type D_n or ${}^2D_n$. Then, assuming Conjecture 2.20, the Φ_d-blocks of $\mathbb{G}$ satisfy Conjecture (d-HV6) in* [4].

The next statement has an identical proof as for $B_n^{(e)}$.

Corollary 6.12. *The degrees $\gamma_{\mathscr{S}} \in \mathrm{Uch}({}^tI_n^{(e)})$ are polynomials.*

Remark 6.13. Although the definition of unipotent degrees in (3.9) and (6.4) makes sense for arbitrary e-symbols $\mathscr{S}$, one does not obtain in general, that is, for $\mathrm{ct}(\mathscr{S}) \not\equiv 0, 1 \bmod e$, a polynomial using this rule. The smallest example of this is the symbol

$$\mathscr{S} = \begin{pmatrix} 2 \\ - \\ 0 \\ - \end{pmatrix},$$

which using an analogous interpretation, should parametrize a unipotent principal 1-series character for the reflection group $G(4, 2, 2)$ (which is also called $D_2^{(4)}$ in [4]) (see Theorem 5.12). Using the Poincaré polynomial $(q^4 - 1)^2/(q - 1)^2$, one gets

$$\gamma_{\mathscr{S}} = \frac{(q^4 - 1)^2(q^2 + 1)}{(q^4 - 1)(q^8 - 1)} = \frac{q^2 + 1}{q^4 + 1}.$$

Calculations with possible degrees for $G(4, 2, 2)$ make it seem unlikely that there exists for this reflection group a set of polynomials with the properties in Theorem 1.1.

Finally, we obtain (compare with [3] for $e = 2$).

Corollary 6.14. *Under the assumptions of Theorem 6.10, we have*

$$\gamma_{\mathscr{S}}(q)_{q'} \mid \deg(R_{\mathbb{L}}^{\mathbb{G}}(\gamma_{\mathscr{S}}) \text{ in } K[q], \quad \text{where } \deg(R_{\mathbb{L}}^{\mathbb{G}}(\gamma_{\mathscr{S}'})) := \frac{|\mathbb{G}(q)|_{q'}}{|\mathbb{L}(q)|_{q'}} \gamma_{\mathscr{S}'}(q).$$

6B. **An involution on** $\mathrm{Uch}({}^tI_n^{(e)})$**.** In the case of groups of Lie type, Alvis–Curtis duality defines an involution on the set of unipotent characters with good functorial properties. A corresponding involution appears not to exist in the case of the reflection data $B_n^{(e)}$, $e > 2$, because among other reasons the group W_n does not possess a sign character. The group $\tilde{W}_n$ however is generated by involutive reflections and does have a sign character ε. For this an analogue of duality can be defined.

For this, firstly, let $\mathscr{S} = (S_0, \ldots, S_{e-1})$ be an e-symbol. For $k \geq \max\{\lambda \mid \lambda \in \mathscr{S}\}$ we set

$$D(S)_i := \{k - j \mid 0 \leqslant j \leqslant k,\, j \notin S_i\},$$
$$D(\mathscr{S}) := (D(S)_0, \ldots, D(S)_{e-1}).$$

(See [12, 1.4].) Then $D(\mathscr{S})$ is also an e-symbol, of the same rank as $\mathscr{S}$, with content $\mathrm{ct}(D(\mathscr{S})) = e(k+1-m)$ and defect $\mathrm{def}(D(\mathscr{S})) = -\mathrm{def}(\mathscr{S})$. Furthermore, the equivalence class of $D(\mathscr{S})$ does not depends on the choice of k, and by applying the map twice we obtain a symbol from the original class. Thus this mapping induces an involution

$$D : \mathrm{Uch}({}^tI_n^{(e)}) \to \mathrm{Uch}({}^{-t}I_n^{(e)}), \qquad \gamma_{\mathscr{S}} \mapsto D(\gamma_{\mathscr{S}}) := \gamma_{D(\mathscr{S})} \tag{6.15}$$

on the set of unipotent degrees of $\bigcup_t {}^tI_n^{(e)}$.

Lemma 6.16. (a) *For all $\mathscr{S}$ we have $a(D(\mathscr{S})) = N - A(\mathscr{S})$, where $n = \mathrm{rk}(\mathscr{S})$ and $N = e\binom{n}{2}$ is the degree in q of the Poincaré polynomial $P(\tilde{W}_n)$.*

(b) *For all $\gamma \in \mathrm{Uch}({}^tI_n^{(e)})$ we have*

$$D(\gamma)(q) = b(\gamma)q^{e\binom{n}{2}}\gamma(q^{-1}),$$

for some $b(\gamma) \in K$.

Proof. Part (a) follows from (5.14). For the second part we use the hook formula. If $\mathscr{S}'$ arises from $\mathscr{S}$ by the removal of an (l, ζ^j)-hook at $\lambda \in S_i$, then one obtains $D(\mathscr{S}')$ by removing the (l, ζ^{-j})-hook at $k + l - \lambda \in D(S)_{i+j}$. The involution D thus swaps (l, ζ^j)-hooks and (l, ζ^{-j})-hooks, and the hook formula, Proposition 6.8, proves (b) up to a constant $b(\gamma) \in K$. □

In particular, the image of the *trivial degree*

$$\gamma_1(q) = 1, \qquad \text{from the symbol } \mathscr{S}_1 = ((n), (0), \ldots, (0)),$$

is the *Steinberg degree*

$$\gamma_{St}(q) = q^{e\binom{n}{2}}, \quad \text{from} \quad \mathscr{S}_{St} = ((1, \ldots, n), (0, \ldots, n-1), \ldots, (0, \ldots, n-1)),$$

which coincides exactly with the q-part of $|I_n^{(e)}|$. Since (l, ζ^j)-hooks become their complex conjugates via D, we see that D also sends (l, ζ^j)-cores to (l, ζ^{-j})-cores, and so Φ-Harish-Chandra series to $\bar{\Phi}$-Harish-Chandra series, for each cyclotomic polynomial Φ over K. In the case of 1-series, hence the Harish-Chandra series for $\Phi = q - 1$, we can specify the operation more precisely. For $\gamma \in \mathrm{Uch}({}^tI_n^{(e)})$, let $\mathscr{S}'$ be the $(1, \zeta^0)$-core and $\chi_{\boldsymbol{\alpha}}$ the irreducible character of the associated relative Weyl group formed from the $(1, \zeta^0)$-quotient, according to Corollary 6.2.

Remark 6.17. Let $\gamma_{\mathscr{S}} \in \mathrm{Uch}({}^tI_n^{(e)})$ be parametrized by $\chi_{\boldsymbol{\alpha}}$, and lie in the 1-series of the 1-cuspidal degree $\gamma_{\mathscr{S}'}$. Then $D(\gamma_{\mathscr{S}})$ is parametrized by $\varepsilon \otimes \chi_{\boldsymbol{\alpha}}$, and lies in the 1-series of $\gamma_{D(\mathscr{S}')}$, where ε is the sign character defined in Section 2A.

Proof. In the construction of the $(1,\zeta^0)$-quotient, one interprets the rows of $\mathscr{S}$ as sequences of β-numbers: the e-tuple $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{e-1})$ of the associated partitions then parametrizes $\chi_{\boldsymbol{\alpha}}$. Obviously, with this process we obtain the e-tuple $(\alpha_0', \dots, \alpha_{e-1}')$ for $\mathscr{S}'$, where α_i' denotes the conjugate partition of α_i. The assertion follows from the statement $\chi_{\alpha_i'} = \varepsilon \otimes \chi_{\alpha_i}$ for symmetric groups. □

Remark 6.18. The involution D constructed above coincides for $e = 2$ with Alvis–Curtis duality, but can *not* in general be obtained through the usual Alvis–Curtis formula, for if it were, then every 1-cuspidal character would be a fixed point of D. In $I_3^{(3)}$, however, there are the two 1-cuspidal symbols

$$\begin{pmatrix} 0 & 1 & 2 \\ & - & \\ & - & \end{pmatrix}, \quad \begin{pmatrix} & - & \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix},$$

with degrees $-\sqrt{-3}/3q(q-1)^3(q+1)$ and $\sqrt{-3}/3q^4(q-1)^3(q+1)$, which are interchanged by D.

6C. **Fourier transform matrices and eigenvalues of Frobenius.** As in the case of $B_n^{(e)}$, the unipotent degrees fall into families in a natural way, which are related to corresponding fake degrees via a Fourier transform.

Let Y be an em-element, totally ordered set, and $\Psi := \Psi(Y) := \{0, \dots, e-1\}^Y$ the set of functions

$$\psi : Y \to \{0, \dots, e-1\}.$$

Obviously, Ψ is the disjoint union of the sets

$${}^t\Psi := \left\{ \psi \in \Psi \,\middle|\, m\binom{e}{2} - \sum_{y \in Y} \psi(y) \equiv e/t \bmod e \right\},$$

for $t = e/p$, $1 \leqslant p \leqslant e$. Furthermore, let

$$\Psi_0 := \{\psi \in \Psi \mid |\psi^{-1}(i)| = m\},$$

and $^-$ be the involution

$$^- : \Psi \to \Psi, \qquad \psi \mapsto \bar\psi \text{ where } \bar\psi(y) := e - \psi(y) \bmod e.$$

Obviously we have $\overline{{}^t\Psi} = {}^{-t}\Psi$. Denote by $\langle\, , \rangle$ the symmetric pairing on Ψ given by

$$\langle \phi, \psi \rangle := \varepsilon(\phi)\varepsilon(\psi) \prod_{y \in Y} \zeta^{-\phi(y)\psi(y)},$$

where

$$\begin{aligned} &\varepsilon(\psi) := (-1)^{c(\psi)}, \text{ with} \\ &c(\psi) := |\{(y, y') \in Y \times Y \mid y < y',\, \psi(y) < \psi(y')\}|. \end{aligned}$$

As in Section 4A one proves:

Lemma 6.19. *For all $\psi, \phi \in \Psi$ and $1 \leqslant p \leqslant e$, $t := e/p$, we have*

$$\sum_{\nu \in {}^t\Psi} \langle \phi, \nu \rangle \langle \nu, \psi \rangle = \begin{cases} \zeta^{j(p+m\binom{e}{2})} \varepsilon(\phi)\varepsilon(\psi)|{}^t\Psi| & \text{if } \phi(i) \equiv \psi(i) + j \bmod e, \\ 0 & \text{otherwise.} \end{cases}$$

As in Section 4A, we call

$$T : H \to H, \qquad T(f)(\phi) := \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\psi \in \Psi} \langle \phi, \psi \rangle f(\psi)$$

the *associated Fourier transform* on the space of functions $H := R^\psi$ from Ψ to a $\mathbb{C}$-algebra R. By Lemma 6.19, it results that

$$T^2(f)(\psi) = (-1)^{m\binom{e-1}{2}} f(\bar{\psi})$$

for all $\psi \in \Psi$, and T^4 is the identity on H. For a fixed function g from Y to a $\mathbb{C}$-algebra we define

$$\gamma(\psi) := \frac{1}{\tau(e)^m} \prod_{i=0}^{e-1} \prod_{j=i}^{e-1} \prod_{\substack{y,y' \,:\, \psi(y)=i,\, \psi(y')=j \\ y>y' \text{ if } i=j}} \left(g(y)\zeta^{\psi(y)} - g(y')\zeta^{\psi(y')} \right), \tag{6.20}$$

as well as, for $1 \leqslant p \leqslant e$, $t := e/p$,

$${}^t\gamma(\psi) := \begin{cases} \gamma(\psi) & \text{if } \psi \in {}^t\Psi, \\ 0 & \text{otherwise,} \end{cases}$$

and

$${}^t\partial(\psi) := \begin{cases} \dfrac{1}{e} \displaystyle\sum_{j=0}^{e-1} \zeta^{jp} \prod_{i=0}^{e-1} \prod_{y \,:\, \psi(y)=i-j} g(y)^i \prod_{\substack{y' \,:\, \psi(y')=i-j \\ y>y'}} \left(g(y)^e - g(y')^e \right) & \text{if } \psi \in \Psi_0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.21. *The function $\phi \mapsto (-1)^{\binom{e}{2}\binom{m}{2}}\, {}^t\partial(\phi)$ is the Fourier transform of $\phi \mapsto {}^t\gamma(\phi)$.*

Proof. As in the proof of Proposition 4.6, we obtain for $\phi in \Psi$

$$T({}^t\gamma)(\phi) = \frac{(-1)^{m(e-1)}\varepsilon(\phi)}{\tau(e)^{2m}} \sum_{\sigma \in S_{em}} \varepsilon(\sigma) \left(\prod_{y \in Y} g(y)^{\sigma(y)} \right) \sum_{\psi \in {}^t\Psi} \prod_{y \in Y} \zeta^{\psi(y)(\sigma(y)-\phi(y))}.$$

The inner sum does not vanish if and only if $\sigma(y) - \phi(y)$ modulo e is a constant on Y. If, say, $\phi(y) - \sigma(y) \equiv j \bmod e$, then

$$\sum_{\psi \in {}^t\Psi} \prod_{y \in Y} \zeta^{\psi(y)(\sigma(y)-\phi(y))} = \zeta^{j(e/t+\binom{e}{2}m)} |{}^t\Psi| = (-1)^{j(e-1)m} \zeta^{je/t} e^{em-1}.$$

Moreover, ϕ then lies in Ψ_0, and σ induces bijections

$$\sigma_i : \phi^{-1}(i+j) \xrightarrow{\sim} \{i, e+i, \ldots, (m-1)e+i\}, \qquad 0 \leqslant i \leqslant e-1.$$

The corresponding signs satisfy

$$(-1)^{j(e-1)m}\varepsilon(\sigma) = (-1)^{\binom{e}{2}\binom{m}{2}}\varepsilon(\phi)\prod_{i=0}^{e-1}\varepsilon(\sigma_i).$$

The claimed formula follows. $\square$

As before, for $\psi \in \Psi$ we set the *eigenvalue of Frobenius* to be

$$\mathrm{Fr}(\psi) := \zeta_{24}^{-2(e^2-1)m}\prod_{y\in Y}\zeta^{-(\psi(y)^2+e\psi(y))/2}.$$

Lemma 6.22. *For all $\phi, \psi \in \Psi$ and γ as in (6.20) we have:*

1. $\mathrm{Fr}(\psi)^{2e} = 1$*;*
2. $\mathrm{Fr}(\psi) = 1$ *if* $\psi \in \Psi_0$*;*
3. $\mathrm{Fr}(\psi) = \mathrm{Fr}(\bar\psi)$*;*
4. $\sum_{\nu,\mu\in\Psi}\mathrm{Fr}(\psi)\mathrm{Fr}(\nu)\mathrm{Fr}(\mu)\langle\psi,\nu\rangle\langle\nu,\mu\rangle\langle\mu,\phi\rangle = (-1)^{(e-1)m}\tau(e)^{3m}\delta_{\psi,\phi}$*;*
5. $\sum_{\nu\in\Psi}\mathrm{Fr}(\phi)\overline{\langle\phi,\nu\rangle}\mathrm{Fr}(\nu)\gamma(\nu) = (-1)^{(e-1)m}\overline{\tau(e)}^m\gamma(\phi)$.

Proof. This follows as for Lemma 4.13. For part (d), one also uses the equality

$$\begin{aligned}\left(\sum_{i=0}^{e-1}\zeta^{-(i^2+ei)/2}\right)^{em} &= \sum_{\psi\in\Psi}\prod_{y\in Y}\zeta^{-(\psi(y)^2+e\psi(y))/2}\\ &= \sum_{p=0}^{e-1}\sum_{\substack{\psi\in{}^t\Psi\\ t=e/p}}\prod_{y\in Y}\zeta^{-(\psi(y)^2+e\psi(y))/2},\end{aligned}$$

and the observation

$$\sum_{\psi\in{}^t\Psi}\mathrm{Fr}(\psi) = 0 \qquad \text{if} \qquad t \neq 1.$$ $\square$

As in Section 3A, a function $\pi : Y \to \mathbb{N}$ induces an equivalence relation $\sim_\pi$ on Ψ via

$$\phi \sim_\pi \psi \text{ if } \pi\circ\phi^{-1}(i) = \pi\circ\psi^{-1}(i) \text{ for } 0 \leqslant i \leqslant e-1.$$

The class of ψ in Ψ with respect to $\sim_\pi$ will be denoted by $[\psi]$. A $\psi \in \Psi$ is called π-*admissible* if there is no $y \neq y' \in Y$ with $\pi(y) = \pi(y')$ and $\psi(y) = \psi(y')$. Denote by H_π the subspace of H generated by those $f \in H$ such that

$$f(\phi) = f(\psi) \text{ if } [\phi] = [\psi], \qquad f(\psi) = 0 \text{ if } \psi \text{ is not } \pi\text{-admissible}, \tag{6.23}$$

with basis

$$\left\{f_{[\phi]} \mid \phi\ \pi\text{-admissible},\ f_{[\phi]}(\psi) := \delta_{[\phi],[\psi]}\right\}. \tag{6.24}$$

This is again invariant under Fourier transform, whence T induces a Fourier transform on H_π. Let, as above, g be a function on Y that factors over $\pi(Y)$ via π. In this case, the functions ${}^t\gamma$ and ${}^t\partial$ introduced in Section 6C lie in H_π, and are exchanged via Fourier transform by Proposition 6.21.

Call the *Fourier transform matrix of* (Y, π), the representing matrix $T = T(Y, \pi)$ of T with respect to the basis (6.24).

The eigenvalue of Frobenius Fr is constant on the equivalence classes $[\psi]$. The associated representing matrix will be denoted by $F(Y,\pi)$. As before, let

$$\mathrm{Sh} := F \circ T^3 \circ F = F \circ \bar{T} \circ F$$

be the *Shintani operator* on H and on H_π, and denote by $\mathrm{Sh}(Y,\pi)$ the associated matrix representation with respect to the basis (6.24). One obtains by Lemmas 6.19 and 6.22(c)–(e) the following (compare with [7, VII.3] for $e=2$, $t=1$).

Corollary 6.25. *The function $\gamma \in H$ is a fixed point of* Sh. *The matrices $T := T(Y,\pi)$ and $U := T \cdot F(Y,\pi)$ satisfy*

$$T^4 = 1, \qquad U^3 = 1, \qquad [T^2, U] = 1.$$

In particular, T and U yield a representation of $\mathrm{SL}_2(\mathbb{Z})$ via

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto T, \qquad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mapsto U.$$

The preimages of $F(Y,\pi)$ and $\mathrm{Sh}(Y,\pi)$ under this representation are

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto F(Y,\pi), \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \mathrm{Sh}(Y,\pi).$$

6D. **Families.** As discussed in Section 3C, we can divide the symbols of of $\bigcup_t \mathrm{Uch}({}^tI_n^{(e)})$ into families, so that two symbols $\mathscr{S}$ and $\mathscr{S}'$ lie in the same family if the multisets of entries of $\mathscr{S}$ and $\mathscr{S}'$ coincide. This in turn induces a partition of the unipotent degrees into families. The unipotent degrees in a family $\mathscr{F}$ are closely related to a suitable set $\Psi(Y,\pi)$. The symbols of $\mathscr{F}$ each have em entries, so let Y be an em-element totally ordered set and $\pi : Y \to \mathbb{N}$ a function compatible with the order structure, with $|\pi^{-1}(n)| = |\{i \mid n \in S_i\}|$. We associate to a class of $\psi \in \Psi$ the symbol $\mathscr{S}$ with $S_i = \pi(\psi^{-1}(i))$, which does not depend on the chosen representative $\psi \in [\psi]$. The image of $\psi \in {}^t\Psi$ then lies in $\mathrm{Uch}({}^tI_n^{(e)})$. However, in this way we obtain each class of symbols exactly $e/s(\mathscr{S})^2$ times. By $\mathscr{F}_0$ we denote the image in $\mathscr{F}$ of Ψ_0. For $t := e/p$, $1 \leqslant p \leqslant e$, and a symbol $\mathscr{S}$ of content $\mathrm{ct}(\mathscr{S}) \equiv 0 \bmod e$, we set

$${}^t\gamma_{\mathscr{S}} := \begin{cases} \gamma_{\mathscr{S}} & \text{if } \mathrm{def}(\mathscr{S}) = p \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.26. *For every $t := e/p$, $1 \leqslant p \leqslant e$, the function*

$$\mathscr{S} \mapsto {}^tR_{\mathscr{S}}(q)$$

is the Fourier transform of

$$\mathscr{S} \mapsto {}^t\gamma_{\mathscr{S}}(q).$$

In particular, we have for every family $\mathscr{F}$ of symbols of defect $p = e/t$,

$$\sum_{\mathscr{S} \in \mathscr{F}} {}^t\gamma_{\mathscr{S}}(q)\overline{{}^t\gamma_{\mathscr{S}}(q)} = \sum_{\mathscr{S} \in \mathscr{F}_0} {}^tR_{\mathscr{S}}(q)\overline{{}^tR_{\mathscr{S}}(q)}.$$

Proof. This is obtained by comparing the definitions in (5.6) and (6.4) with (6.20), by using Proposition 6.21 as in Theorem 4.17. It only remains to verify that the prefactors occurring through the non-trivial symmetry groups cancel each other out exactly. □

Remark 6.27. Let Y be as above, with em odd. Let $\psi, \psi' \in \Psi(Y)$ with

$$\psi'(y) = \psi(y) + 1 \qquad \text{for all } y \in Y. \tag{6.28}$$

One convinces oneself that then $\varepsilon(\psi) = \varepsilon(\psi')$, so that $\langle \phi, \psi \rangle = \langle \phi, \psi' \rangle$ for all $\phi \in {}^1\Psi$. Let $\tilde{\Psi}$ be the set of all $\psi \in {}^1\Psi$ modulo the equivalence relation (6.28). Then T induces a Fourier transform

$$T(f)(\tilde{\phi}) := \sum_{\tilde{\psi} \in \tilde{\Psi}} \langle \tilde{\phi}, \tilde{\psi} \rangle f(\tilde{\psi})$$

on the space $\tilde{H} := R^{\tilde{\Psi}}$. We see also that the eigenvalue of Frobenius Fr is therefore constant on the class $\tilde{\psi}$. Clearly the $\tilde{\psi}$-classes are compatible with the $[\tilde{\psi}]_\pi$-classes, so we obtain in this case an alternative description of the Fourier transform and the representation of $\mathrm{SL}_2(\mathbb{Z})$ on the space $\tilde{\Psi}$, hence for the untwisted reflection datum $I_n^{(e)}$.

Example 6.29. If $m = 1$ and $\pi : Y \to \mathbb{N}$ has $|\pi^{-1}(0)| = e - 2$, $|\pi^{-1}(1)| = 2$, then we get that the $[\tilde{\psi}]_\pi$-classes of symbols $\mathscr{S} = (S_0, \ldots, S_{e-1})$ are

$$S_i = \{0\} \text{ for } i \neq 0, k, l, k+l, \qquad S_0 = S_{k+l} = \{0, 1\}, \qquad S_k = S_l = \emptyset,$$

or

$$S_i = \{0\} \text{ for } i \neq 0, l \qquad S_0 = S_l = \{1\}.$$

The classes can thus also be parametrized by pairs (k, l) of non-negative integers with $0 < k < l < e$, $k + l < e$, or $0 = k < l < e/2$. We see that

$$\frac{\langle (k_1, l_1), (k_2, l_2) \rangle}{\tau(e)} = v(k_1, k_2) \frac{\zeta^{k_1 l_2 + k_2 l_1} + \zeta^{-k_1 l_2 - k_2 l_1} - \zeta^{k_1 k_2 + l_1 l_2} - \zeta^{-k_1 k_2 - l_1 l_2}}{e},$$

where the sign $v(k_1, k_2)$ is -1 for $k_1 = 0$, $k_2 \neq 0$, or $k_1 \neq 0$, $k_2 = 0$, and equal to 1 otherwise. Furthermore, one immediately sees

$$\mathrm{Fr}((k, l)) = \zeta^{-kl}.$$

Since the symbols $\mathscr{S}$ have rank 2, they parametrize the unipotent degrees in a family of $I_2^{(e)}$. Comparing shows that the above matrices of the Fourier transform (up to the sign $v(k_1, k_2)$) and the eigenvalue of Frobenius coincide with the corresponding exotic Fourier transform introduced in [14, 3.1]. The result of Proposition 3.2 in [14] is therefore proved by the above, up to the validity of the property 1.1(f). Since our unipotent degrees for $I_2^{(e)}$ differ from those of Lusztig in [13, 4.1] just by the sign $v(k) = -1$ if $k \neq 0$, we can conclude from Theorem 6.26, that the exotic Fourier transform in [14] sends the unipotent degrees of $I_2(e)$ to the fake degrees.

The case of even e is somewhat more complicated and left to the reader.

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