BLOCKS WITH ABELIAN DEFECT GROUPS OF FINITE REDUCTIVE GROUPS

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Summary

0. Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group over the algebraic closure $\bar{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$ (where q is a power of a prime p), equipped with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ that gives $\mathbf{G}$ an $\mathbb{F}_q$-rational structure, and let $\mathbf{G}^F$ be the group of rational points of $\mathbf{G}$ ($\mathbf{G}^F$ is a *finite group of Lie type*). Let ℓ be a prime different from p, and good for $\mathbf{G}$. We also suppose that ℓ satisfies certain technical conditions: ℓ does not divide $|(Z(\mathbf{G})/Z^\circ(\mathbf{G}))^F|$ or $|(Z(\mathbf{G}^*)/Z^\circ(\mathbf{G}^*))^{F^*}|$ (where $\mathbf{G}^*$ is the *Langlands dual* of $\mathbf{G}$ taken over the same field $\bar{\mathbb{F}}_q$), and if $\mathbf{G}^F$ has a component of type ${}^3D_4$, then $\ell \neq 3$.

The aim of this article is to classify the unipotent ℓ-blocks of $\mathbf{G}^F$ with abelian defect groups, and to demonstrate in each case that a general conjecture made in [4] holds: if e is an ℓ-block of $\mathbf{G}^F$ with abelian defect group D, and (D, f) is a maximal e-Brauer pair of $\mathbf{G}^F$, then the blocks e of $\mathbf{G}^F$ and f of $N_{\mathbf{G}^F}(D, f)$ are isotypical. Along the way, we demonstrate a general result concerning the π-blocks of $\mathbf{G}^F$ (where π is a set of primes ℓ possessing in particular the above properties). The part concerning the classification of blocks is also obtained in [8] by other methods.

This paper originally appeared as *Blocs à groupes de défaut abéliens des groupes réductifs finis*, Astérisque **212** (1993) 93–117.

In [6], the analogous results are demonstrated under the stronger hypothesis that ℓ is 'large' (i.e., essentially that ℓ doesn't divide the order of the Weyl group of $\mathbf{G}$). In this case, these results are obtained by an application of the *generic methods* of [6]. To extend to the case of small primes (the non-generic case) we need to develop techniques combining methods from [6] and [7].

If $\ell \neq p$, the conditions on ℓ given above automatically hold when the Sylow ℓ-subgroups of $\mathbf{G}^F$ are abelian [11], so we have in particular:

Theorem. *If a Sylow ℓ-subgroup D of $\mathbf{G}^F$ is abelian, the principal ℓ-blocks of $\mathbf{G}^F$ and $N_{\mathbf{G}^F}(D)$ are isotypical.*

1. Sets of Prime Numbers, π-Series

In the first section we introduce the main notation and recall, with some extensions, the main results from [7]. We determine the main properties of the primes considered in the sequel.

From now on, we use the following notation: $\mathbf{G}$ is a connected reductive algebraic group over the algebraic closure $\bar{\mathbb{F}}_q$ of the field $\mathbb{F}_q$ of characteristic p, equipped with an $\mathbb{F}_q$-rational structure. Write $F : \mathbf{G} \to \mathbf{G}$ for the corresponding Frobenius endomorphism, and $\mathbf{G}^F$ for the group of rational points of $\mathbf{G}$. By a *Levi subgroup of* $\mathbf{G}$ we mean a Levi subgroup $\mathbf{L}$ of a parabolic subgroup of $\mathbf{G}$, and if $\mathbf{L}$ is rational, we write $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ for the associated Deligne–Lusztig induction and restriction respectively (see, for example, [10]).

1.A. **Background.** Let π be a set of primes not containing p, and let π' be all primes not in π. If n is a natural number, write n_π for the largest divisor of n that is a product of elements of π.

Definition 1.1. Let $\sigma_{\pi'}^{\mathbf{G}^F}$ be the class function on $\mathbf{G}^F$ given by

$$\sigma_{\pi'}^{\mathbf{G}^F}(g) = \begin{cases} |\mathbf{G}^F|_\pi & g \text{ is a } \pi'\text{-element} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\sigma_{\pi'}^{\mathbf{G}^F}(g)$ only depends on the semisimple part of g.

If γ is a unipotent character of $\mathbf{G}^F$, we write $\deg(\gamma)$ for its degree, and $\operatorname{Deg}(\gamma)$ for its generic degree. Thus $\operatorname{Deg}(\gamma) \in \mathbb{Q}[x]$ and $\deg(\gamma) = \gamma(1) = \operatorname{Deg}(\gamma)(q)$.

The next facts follow from [7, 2.5].

(1.2)

(1) *For every F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and every class function ψ on $\mathbf{L}^F$, we have*

$${}^*R_{\mathbf{L}}^{\mathbf{G}}(\sigma_{\pi'}^{\mathbf{G}^F}\psi) = \deg(R_{\mathbf{L}}^{\mathbf{G}}(1))_\pi\, \sigma_{\pi'}^{\mathbf{L}^F}\, {}^*R_{\mathbf{L}}^{\mathbf{G}}(\psi)$$

and

$$\sigma_{\pi'}^{\mathbf{G}^F} R_{\mathbf{L}}^{\mathbf{G}}(\psi) = \deg(R_{\mathbf{L}}^{\mathbf{G}}(1))_\pi\, R_{\mathbf{L}}^{\mathbf{G}}(\sigma_{\pi'}^{\mathbf{L}^F}\psi)$$

(2) $\sigma_{\pi'}^{\mathbf{G}^F}$ *is uniform, and*

$$\sigma_{\pi'}^{\mathbf{G}^F} = \sum_{[\mathbf{T}]_{\mathbf{G}^F}} \frac{\deg(R_{\mathbf{T}}^{\mathbf{G}}(1))_\pi}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(\sigma_{\pi'}^{\mathbf{T}^F}),$$

where the sum is taken over a set of representations of the $\mathbf{G}^F$*-conjugacy classes of maximal* F*-stable tori of* $\mathbf{G}$*, and where* $W_{\mathbf{G}^F}(\mathbf{T}) = N_{\mathbf{G}^F}(\mathbf{T})/\mathbf{T}$.

Remark. By convention, the π-part of a number is always positive. In particular,

$$\deg(R_{\mathbf{L}}^{\mathbf{G}}(1))_\pi \deg(R_{\mathbf{L}}^{\mathbf{G}}(1))_{\pi'} = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}} \deg(R_{\mathbf{L}}^{\mathbf{G}}(1)),$$

where $\varepsilon_{\mathbf{G}} = (-1)^i$ and i is the semisimple $\mathbb{F}_q$-rank of $\mathbf{G}$.

We have $\sigma_{\pi'}^{\mathbf{T}^F} = \sum \theta$, where θ ranges over the set of characters of $\mathbf{T}^F$ whose order is a π-number. If $\mathrm{u}\sigma_{\pi'}^{\mathbf{G}^F}$ denotes the projection of $\sigma_{\pi'}^{\mathbf{G}^F}$ onto the subspace of class functions generated by the unipotent characters, we deduce that

$$\mathrm{u}\sigma_{\pi'}^{\mathbf{G}^F} = \sum_{[\mathbf{T}]_{\mathbf{G}^F}} \frac{\deg(R_{\mathbf{T}}^{\mathbf{G}}(1))_\pi}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(1). \tag{1.3}$$

As is standard, we write $\mathcal{E}(\mathbf{G}^F, (s))$ for the *Lusztig series* associated to the conjugacy class of the semisimple element $s \in \mathbf{G}^{*F^*}$ and set $\mathcal{E}_\pi(\mathbf{G}^F, 1) := \bigcup_{s\in(\mathbf{G}^{*F^*})_\pi} \mathcal{E}(\mathbf{G}^F, (s))$ (see [7]). We write $\mathrm{pr}_\pi^{\mathbf{G}^F}$ for the projection from the space of class functions on $\mathbf{G}^F$ to the subspace generated by the elements of $\mathcal{E}_\pi(\mathbf{G}^F, 1)$, and we set

$$\mathrm{Reg}_\pi^{\mathbf{G}^F} := \mathrm{pr}_\pi^{\mathbf{G}^F} \mathrm{Reg}^{\mathbf{G}^F} = \sum_{\chi\in\mathcal{E}_\pi(\mathbf{G}^F,1)} \chi(1)\chi.$$

Finally, we write $\mathrm{UReg}_\pi^{\mathbf{G}^F}$ for the projection of $\mathrm{Reg}_\pi^{\mathbf{G}^F}$ onto the subspace generated by the unipotent characters.

The second assertion in the next proposition has already been proved in [7].

Proposition 1.4.

(1) $\mathrm{Reg}_\pi^{\mathbf{G}^F} = \sum_{[\mathbf{T}]_{\mathbf{G}^F}} \frac{\deg(R_{\mathbf{T}}^{\mathbf{G}}(1))_\pi}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Reg}_\pi^{\mathbf{T}^F})$,

(2) $\mathrm{Reg}_\pi^{\mathbf{G}^F} = \sigma_{\pi'}^{\mathbf{G}^F} D(\mathrm{u}\sigma_{\pi\cup\{p\}}^{\mathbf{G}^F})$, *where D denotes Alvis–Curtis duality.*

Proof. We obtain (1) by applying $\mathrm{pr}_\pi^{\mathbf{G}^F}$ to the formula

$$\mathrm{Reg}^{\mathbf{G}^F} = \sum_{[\mathbf{T}]_{\mathbf{G}^F}} \frac{\deg R_{\mathbf{T}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Reg}^{\mathbf{T}^F}),$$

since
$$\mathrm{pr}_\pi^{\mathbf{G}^F}(R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Reg}^{\mathbf{T}^F})) = R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{pr}_\pi^{\mathbf{T}^F}(\mathrm{Reg}^{\mathbf{T}^F})) = R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Reg}_\pi^{\mathbf{T}^F}).$$
We now prove (2). By (1.3), we have that

$$\mathrm{u}\sigma_{\pi\cup\{p\}}^{\mathbf{G}^F} = \sum_{[\mathbf{T}]_{\mathbf{G}^F}} \frac{\deg(R_{\mathbf{T}}^{\mathbf{G}}(1))_{\pi'}}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(1). \tag{1.5}$$

Multiplying both sides of (1.3) by $\sigma_{\pi'}^{\mathbf{G}^F}$ and using (1.2(1)), together with the equality $D(R_{\mathbf{T}}^{\mathbf{G}}(1)) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(1)$, we obtain the result. □

As in [7], we see from Proposition 1.4(2) that $\mathrm{Reg}_\pi^{\mathbf{G}^F}$ is the regular character associated to a central π-idempotent of $\bar{\mathbb{Q}}\mathbf{G}^F$, which we denote by $e_\pi^{\mathbf{G}^F}$.

1.B. **The case where π consists of good primes.** Suppose now that π only consists of primes that are good for $\mathbf{G}$. Hence, for every abelian π-subgroup S of $\mathbf{G}^F$, the group $C_{\mathbf{G}}^\circ(S)$ is an F-stable Levi subgroup of $\mathbf{G}$ (see for example [13, 2.1]).

Definition 1.6. A Levi subgroup of $\mathbf{G}$ is *π-split* if it is a connected centralizer of an abelian π-subgroup of $\mathbf{G}^F$.

A group $\mathbf{M}$ is a π-split Levi subgroup of $\mathbf{G}$ if and only if there exists an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that $\mathbf{M} = C_{\mathbf{G}}^\circ(\mathbf{T}_\pi^F)$, where we write $\mathbf{T}_\pi^F$ for a Sylow π-subgroup of $\mathbf{T}^F$.

Definition 1.7.

(1) An F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ is *π-anisotropic* if the Sylow π-subgroup $\mathbf{T}_\pi^F$ of $\mathbf{T}^F$ is contained in $Z^\circ(\mathbf{G})^F$. We write $\mathcal{T}_\pi(\mathbf{G})$ for the set of π-anisotropic maximal tori of $\mathbf{G}$.

(2) For every uniform class function ψ on $\mathbf{G}^F$, the *π-cuspidal projection* of ψ, denoted by $c_\pi(\psi)$, is defined by
$$c_\pi(\psi) := \sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{G})]_{\mathbf{G}^F}} \frac{1}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}({}^*R_{\mathbf{T}}^{\mathbf{G}}(\psi)).$$

In what follows, the symbols $=_{\mathbf{G}^F}$, $\leqslant_{\mathbf{G}^F}$, $\subseteq_{\mathbf{G}^F}$ and $\preccurlyeq_{\mathbf{G}^F}$, represent the symbols $=$, $\leqslant$, $\subseteq$ and $\preccurlyeq$ modulo $\mathbf{G}^F$-conjugation.

The following analogue of [6, Lemma 2.14] will be useful.

Lemma 1.8. *Let ϕ be a function on the set of all F-stable maximal tori of $\mathbf{G}$, invariant under $\mathbf{G}^F$-conjugation. Let $\mathbf{M}$ be a π-split Levi subgroup of $\mathbf{G}$. We have*
$$\sum_{[\mathbf{T}:\mathbf{T}_\pi^F =_{\mathbf{G}^F} Z^\circ(\mathbf{M})_\pi^F]_{\mathbf{G}^F}} \phi(\mathbf{T}) = \frac{1}{|W_{\mathbf{G}^F}(\mathbf{M})|} \sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{M})]_{\mathbf{M}^F}} \frac{\phi(\mathbf{T})}{|W_{\mathbf{M}^F}(\mathbf{T})|}.$$

Proof. It suffices to follow the proof of [6, Lemma 2.14], replacing d by π. □

As in [6, §2.C], we write $\mathrm{AbIrr}(\mathbf{G}^F)$ for the character group of the abelian group $\mathbf{G}^F/[\mathbf{G},\mathbf{G}]^F$ (seen as characters of $\mathbf{G}^F$), and we write $\mathrm{Ab}_\pi\mathrm{Irr}(\mathbf{G}^F)$ for the subgroup of elements of $\mathrm{AbIrr}(\mathbf{G}^F)$ whose order is a π-number. We set

$$\mathrm{Ab}_\pi\mathrm{Reg}(\mathbf{G}^F) := \sum_{\theta\in\mathrm{Ab}_\pi\mathrm{Irr}(\mathbf{G}^F)} \theta.$$

Proposition 1.9.

(1) $$\mathrm{Reg}_\pi^{\mathbf{G}^F} = \sum_{[\mathbf{M}\ \pi\text{-split}]_{\mathbf{G}^F}} \frac{\deg R_{\mathbf{M}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M})|} R_{\mathbf{M}}^{\mathbf{G}}(\mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{M}^F} c_\pi(\mathrm{UReg}^{\mathbf{M}^F})).$$

(2) $$\mathrm{UReg}^{\mathbf{G}^F} = \sum_{[\mathbf{M}\ \pi\text{-split}]_{\mathbf{G}^F}} \frac{\deg R_{\mathbf{M}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M})|} R_{\mathbf{M}}^{\mathbf{G}}(c_\pi(\mathrm{UReg}^{\mathbf{M}^F})).$$

Proof. The proof is similar to that of [6, 2.16,2.33]. In the formula for $\mathrm{Reg}_\pi^{\mathbf{G}^F}$ in Proposition 1.4(1), grouping the tori $\mathbf{T}$ according to whether the subgroups $\mathbf{T}_\pi^F$ are $\mathbf{G}^F$-conjugate, and applying Lemma 1.8 to each grouping, we obtain

$$\begin{aligned}
\mathrm{Reg}_\pi^{\mathbf{G}^F} &= \sum_{[\mathbf{M}]_{\mathbf{G}^F}} \frac{1}{|W_{\mathbf{G}^F}(\mathbf{M})|} \sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{M})]_{\mathbf{M}^F}} \frac{\deg R_{\mathbf{T}}^{\mathbf{G}}(1)}{|W_{\mathbf{M}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Reg}_\pi^{\mathbf{T}^F})\\
&= \sum_{[\mathbf{M}]_{\mathbf{G}^F}} \frac{\deg R_{\mathbf{M}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M})|} R_{\mathbf{M}}^{\mathbf{G}} \left(\sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{M})]_{\mathbf{M}^F}} \frac{\deg R_{\mathbf{T}}^{\mathbf{M}}(1)}{|W_{\mathbf{M}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{M}}(\mathrm{Reg}_\pi^{\mathbf{T}^F}) \right)\\
&= \sum_{[\mathbf{M}]_{\mathbf{G}^F}} \frac{\deg R_{\mathbf{M}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M})|} R_{\mathbf{M}}^{\mathbf{G}} \left(c_\pi(\mathrm{Reg}_\pi^{\mathbf{M}^F}) \right)
\end{aligned}$$

where each sum is over a set of representatives $\mathbf{M}$ of $\mathbf{G}^F$-classes of π-split Levi subgroups. (1) is therefore a consequence of the following analogue of [6, 2.34].

Lemma 1.10.

$$c_\pi(\mathrm{Reg}_\pi^{\mathbf{G}^F}) = \mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{G}^F} c_\pi(\mathrm{UReg}^{\mathbf{G}^F}).$$

Proof of 1.10. According to Proposition 1.4(1) and Definition 1.7(2), we see that

$$c_\pi(\mathrm{Reg}_\pi^{\mathbf{G}^F}) = \sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{G})]_{\mathbf{G}^F}} \frac{\deg R_{\mathbf{T}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Reg}_\pi^{\mathbf{T}^F})$$

$$c_\pi(\mathrm{UReg}_\pi^{\mathbf{G}^F}) = \sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{G})]_{\mathbf{G}^F}} \frac{\deg R_{\mathbf{T}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{T})|} R_{\mathbf{T}}^{\mathbf{G}}(1^{\mathbf{T}^F})$$

It therefore suffices to demonstrate that, for all $\mathbf{T} \in \mathcal{T}_\pi(\mathbf{G})$, we have $R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Reg}_\pi^{\mathbf{T}^F}) = \mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{G}^F} R_{\mathbf{T}}^{\mathbf{G}}(1^{\mathbf{T}^F})$. Since the function $\mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{G}^F}$ is 'p-constant' (see [7, 2.5]) we have that $\mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{G}^F} R_{\mathbf{T}}^{\mathbf{G}}(1^{\mathbf{T}^F}) = R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Res}_{\mathbf{T}^F}^{\mathbf{G}^F}(\mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{G}^F})$.

Therefore it suffices to verify that

$$(*) \qquad \mathrm{Res}_{\mathbf{T}^F}^{\mathbf{G}^F}(\mathrm{Ab}_\pi \mathrm{Reg}^{\mathbf{G}^F}) = \mathrm{Reg}_\pi^{\mathbf{T}^F}.$$

The function $\mathrm{Res}_{\mathbf{T}^F}^{\mathbf{G}^F}(\mathrm{Ab}_\pi \mathrm{Reg}^{\mathbf{G}^F})$ is a function on $\mathbf{T}^F$ that takes the value $|\mathbf{G}^F/[\mathbf{G},\mathbf{G}]^F|_\pi$ on the elements $t \in \mathbf{T}^F$ whose image modulo $[\mathbf{G},\mathbf{G}]^F$ is a π-regular element, and value 0 on the other elements of $\mathbf{T}^F$.

The group $\mathbf{G}^F/[\mathbf{G},\mathbf{G}]^F$ is a quotient of $\mathbf{T}^F$ (see for example [6, §2]) and $|\mathbf{G}^F/[\mathbf{G},\mathbf{G}]^F| = |Z^\circ(\mathbf{G})^F|$. By the hypothesis on $\mathbf{T}^F$ we have $|\mathbf{T}^F|_\pi = |Z^\circ(\mathbf{G})^F|_\pi$, therefore $|\mathbf{T}^F|_\pi = |\mathbf{G}^F/[\mathbf{G},\mathbf{G}]^F|_\pi$; we see that $\mathbf{T}^F \cap [\mathbf{G},\mathbf{G}]^F$ is a π'-group. As a result, the function $\mathrm{Res}_{\mathbf{T}^F}^{\mathbf{G}^F}(\mathrm{Ab}_\pi \mathrm{Reg}^{\mathbf{G}^F})$ takes the value 0 on the π-singular elements of $\mathbf{T}^F$. This completes the proof of equality (*). □

The proof of (2) is identical, starting instead with (1.3). □

Remark. The proof of Lemma 1.10 above shows that there exist π-anisotropic tori, so $Z([\mathbf{G},\mathbf{G}])^F$ is a π'-group. This hypothesis is precisely that of Proposition 1.19(1) below.

1.C. **F-excellent primes.** For reasons that will become clear later (see for example Propositions 1.15 and 1.16), we must impose some restrictions on the set of primes that we use (see [13, 2.3] and also [8]).

Definition 1.11. A prime ℓ is $(\mathbf{G}, F)$-*excellent* if and only if

(1) ℓ is good for $\mathbf{G}$,
(2) $\ell \neq p$,
(3) ℓ does not divide either $|(Z(\mathbf{G})/Z^\circ(\mathbf{G}))^F|$ or $|(Z(\mathbf{G}^*)/Z^\circ(\mathbf{G}^*))^{F^*}|$,
(4) ℓ is not equal to 3 if $\mathbf{G}^F$ possesses a component of type ${}^3D_4$.

The 'large' primes ('generic') of [6, §5] satisfy these conditions.

Proposition 1.12. *Suppose that* $\mathbf{G}$ *is simple. If* $\ell \neq p$ *and* $\ell \nmid |W|$ *then* ℓ *is excellent for* $(\mathbf{G}, F)$.

Proof. Since the bad primes divide $|W|$, and since 3 divides the order of the Weyl group of type D_4, we just have to prove condition (3).

Following the notation of [6], we write R for the root system of $\mathbf{G}$, a subset of the cocharacter group X of $\mathbf{T}$, and $R^\vee$ for the dual system, a subset of the character group Y of $\mathbf{T}$. Additionally, we write $Q(R)$ and $Q(R^\vee)$ for the $\mathbb{Z}$-submodule of X and Y respectively, generated by R and $R^\vee$ respectively, and let $P(R)$ be the dual of $Q(R^\vee)$ in $\mathbb{Q} \otimes X$.

The character group of $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ is isomorphic to a p'-torsion subgroup of $X/Q(R)$. Since the torsion subgroup of $X/Q(R)$, namely the group $Q(R)^{\perp\perp}/Q(R)$, is a subgroup of $P(R)/Q(R)$, it suffices to verify that $|P(R)/Q(R)|$ and $|P(R^\vee)/Q(R^\vee)|$ (the latter must be considered to obtain the result for $\mathbf{G}^*$) divides $|W|$. But, by definition $|P(R)/Q(R)|$ is the connection index f_R of R, and $f_R = f_{R^\vee}$ divides $|W|$ by [2, Proposition VI.2.7]. □

The next property of excellence is fundamental (it has already been observed in [8, 1.2]).

Proposition 1.13. *Let* $\mathbf{L}$ *be an* F*-stable Levi subgroup of* $\mathbf{G}$*. If* ℓ *is excellent for* $(\mathbf{G}, F)$*, it is also excellent for* $(\mathbf{L}, F)$*.*

Proof. As before, only condition (3) needs verification. By [13, 2.4] the group $H^1(F, Z(\mathbf{L})/Z^\circ(\mathbf{L}))$ is isomorphic to a direct factor of $H^1(F, Z(\mathbf{G})/Z^\circ(\mathbf{G}))$. This gives the result because for any finite group A with an automorphism F we have the short exact sequence

$$1 \longrightarrow A^F \xrightarrow{F-1} A \longrightarrow H^1(F, A) \longrightarrow 1 ,$$

which shows that $|H^1(F, Z(\mathbf{G})/Z^\circ(\mathbf{G}))| = |(Z(\mathbf{G})/Z^\circ(\mathbf{G}))^F|$. The same applies to $\mathbf{L}$. □

We need some concepts defined in [5] and [6]. Let $\mathbf{T}$ be an $\bar{\mathbb{F}}_q$-torus, defined over $\mathbb{F}_q$. The associated Frobenius endomorphism induces on $Y(\mathbf{T})$ an operator of the form $q\phi$, where ϕ is an endomorphism of finite order. We call the data $\mathbb{T} = (X(\mathbf{T}), Y(\mathbf{T}), \phi)$ a *generic torus*. We write $|\mathbb{T}|$ for the characteristic polynomial of ϕ; the notation is 'justified' by the fact that $|\mathbf{T}^F| = |\mathbb{T}|(q)$. Given an integer d, $\mathbf{T}$ possesses a unique F-stable subtorus $\mathbf{T}_d$ such that $|\mathbb{T}_d|$ is a power of Φ_d (the dth cyclotomic polynomial). Such a subtorus is called a Φ_d*-group*, and $\mathbf{T}_d$ is called the *Sylow* Φ_d*-subgroup* of $\mathbf{T}$. The Levi subgroups of $\mathbf{G}$ that are the centralizers of Φ_d-subgroups are called *d-split*.

Definition 1.14. Let $\mathbf{S}$ be an F-stable torus, and let $d \in \mathbb{N}$. A prime ℓ is $(\mathbf{S}, F, d)$*-adapted* if $|\mathbf{S}^F|_\ell = |\mathbf{S}_d^F|_\ell$ (i.e., the Sylow ℓ-subgroup of $\mathbf{S}^F$ is contained in the Sylow Φ_d-subgroup of $\mathbf{S}^F$). A set π of primes is $(\mathbf{S}, F, d)$*-adapted* if all of the elements in it are.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$. We write $\mathbf{G}(\mathbf{T}, \theta)$ (see [6, 2.C]) for the group generated by $\mathbf{T}$, together with the radical subgroups $\mathbf{U}_\alpha$ (relative to $\mathbf{T}$) with $\theta(\alpha^\vee) = 1$.

Proposition 1.15. *Let* $\mathbf{L}$ *be an* F*-stable Levi subgroup of* $\mathbf{G}$*. Let* π *be a set of primes that are excellent for* $(\mathbf{G}, F)$*, and* $(Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G}), F, d)$*-adapted.*

1. *If* S *is a* π*-subgroup of* $(Z^\circ(\mathbf{L}))^F$ *then* $C^\circ_{\mathbf{G}}(S)$ *is a d-split Levi subgroup of* $\mathbf{G}$*.*
2. *Let* θ *be a* π*-character of* $(\mathbf{L}/[\mathbf{L}, \mathbf{L}])^F$*, and let* $\mathbf{T}$ *be an* F*-stable maximal torus of* $\mathbf{L}$*. Then* $\mathbf{G}(\mathbf{T}, \theta)$ *is a d-split Levi subgroup of* $\mathbf{G}$*.*

Proof. For every element of π that is good for $\mathbf{G}$, the group $C^\circ_{\mathbf{G}}(S)$ is a Levi subgroup of $\mathbf{G}$ (see for example [13, 2.1]); we have $S \subseteq C^\circ_{\mathbf{G}}(S)$ as $S \subseteq \mathbf{L} \subseteq C^\circ_{\mathbf{G}}(S)$ and, as every element of π is still excellent for $(C^\circ_{\mathbf{G}}(S), F)$ by Proposition 1.13, we have $S \subseteq Z^\circ(C^\circ_{\mathbf{G}}(S))$. As $Z^\circ(C^\circ_{\mathbf{G}}(S)) \subseteq Z^\circ(\mathbf{L})$ (because $C^\circ_{\mathbf{G}}(S)$ contains $\mathbf{L}$), π is still $(Z^\circ(C^\circ_{\mathbf{G}}(S))/Z^\circ(\mathbf{G}), F, d)$-adapted. As a result, the quotient $(Z^\circ(C^\circ_{\mathbf{G}}(S)))^F/(\mathbf{S}^F \cdot Z^\circ(\mathbf{G}))^F)$ is a π'-group, if $\mathbf{S}$ is the Sylow Φ_d-subgroup of $Z^\circ(C^\circ_{\mathbf{G}}(S))$; therefore S is contained in $\mathbf{S}\cdot Z^\circ(\mathbf{G})$. Hence we have $C^\circ_{\mathbf{G}}(S)=C^\circ_{\mathbf{G}}(\mathbf{S})$, demonstrating that $C^\circ_{\mathbf{G}}(S)$ is a d-split Levi subgroup of $\mathbf{G}$.

The second assertion follows from the first, applied to the group $\mathbf{G}^*$, which is possible because the primes that are excellent for $(\mathbf{G},F)$ are also excellent for $(\mathbf{G}^*,F^*)$, and $Z^\circ(\mathbf{G})$ and $Z^\circ(\mathbf{G}^*)$ have the same generic order, as do $Z^\circ(\mathbf{L})$ and $Z^\circ(\mathbf{L}^*)$. □

The following technical result will be used in Section 2 for the description of ℓ-blocks of abelian defect group of $\mathbf{G}^F$.

Proposition 1.16. *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$, and let ℓ be a prime that is good for $\mathbf{G}$, and not dividing $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ for some unipotent representation λ of $\mathbf{L}^F$. If $\ell=3$, suppose in addition that $\mathbf{G}^F$ does not have a component of type ${}^3D_4$. There exists d such that ℓ is $(Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G}),F,d)$-adapted.*

Proof. To prove the proposition, we may suppose that $\mathbf{G}$ is adjoint. Indeed, if $\mathbf{L}'$ is the image of $\mathbf{L}$ in $\mathbf{G}_{\mathrm{ad}}$, then $Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G})$ and $Z^\circ(\mathbf{L}')$ have the same generic order. The character λ, being unipotent, factors through a unipotent character λ' of $\mathbf{G}_{\mathrm{ad}}^F$ and $W_{\mathbf{G}_{\mathrm{ad}}^F}(\mathbf{L}',\lambda')=W_{\mathbf{G}^F}(\mathbf{L},\lambda)$.

We therefore suppose that $\mathbf{G}$ is adjoint. Since an adjoint group is a product of the restriction of scalars of simple groups, $\mathbf{G}$ is a product of groups of the form $(\mathbf{G}_i^{(a)},F_i^{(a)})$ (the restriction of scalars of $\mathbb{F}_{q^a}$ to $\mathbb{F}_q$ of $(\mathbf{G}_i,F_i)$), and $(\mathbf{L},\lambda)$ is a product of $(\mathbf{L}_i^{(a)},F_i^{(a)})$. As $(\mathbf{L}_i^{(a)})^F=\mathbf{L}_i^{F^a}$ and

$$W_{\mathbf{G}_i^{(a)F^{(a)}}}(\mathbf{L}_i^{(a)},\lambda_i^{(a)})=W_{\mathbf{G}_i^{F^a}}(\mathbf{L}_i,\lambda_i),$$

we can even replace $(\mathbf{G}_i^{(a)},F_i^{(a)},\mathbf{L}_i^{(a)},\lambda_i^{(a)})$ with $(\mathbf{G}_i,F_i^a,\mathbf{L}_i,\lambda_i)$, and suppose that $\mathbf{G}$ is a product of simple groups, each F-stable.

We now use the following lemma, which has appeared in [12, 3.1].

Lemma 1.17. *If $\mathbb{T}=(X,Y,\phi)$ is a generic torus, and $(\mathbf{T},F)$ is the algebraic torus corresponding to a choice of q, then a prime ℓ that does not divide the order of ϕ is $(\mathbf{T},F)$-adapted.*

Proof of 1.17. Let $\prod_{i=1}^r \Phi_{d_i}^{n_i}$ be the polynomial order of $\mathbb{T}$. As it is the characteristic polynomial of ϕ on Y, the order of ϕ is the lowest common multiple of the orders (as roots of unity) of the zeros of Φ_d, which is the lowest common multiple of the d_i. Moreover, if ℓ is not $(\mathbf{T},F)$-adapted, it divides two of the cyclotomic factors, which we write $\Phi_d(q)$ and $\Phi_{d'}(q)$. It therefore divides (see, for example [5, Appendix 2, (3)]) the lowest common multiple of d and d', so therefore divides the order of ϕ, and we have the lemma. □

The following paragraph freely uses concepts and notation of [6]. Suppose that $(\mathbf{G},\mathbf{T}_0,F)$ is a triple associated to the generic group $\mathbb{G}=((X,R,Y,R^\vee),W\phi)$, where $\mathbf{T}_0$ is chosen to be quasi-split, i.e., such that there exists an F-stable

Borel subgroup $\mathbf{B}$ containing $\mathbf{T}_0$. Suppose also that $\mathbf{L}$ is associated to $\mathbb{L} = ((X, R', Y, R'^\vee), W_\mathbb{L} w\phi)$. We can identify $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ with $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$ where $\boldsymbol{\lambda} \in \mathrm{Uch}(\mathbb{G})$ is such that $\rho_{\boldsymbol{\lambda}}^{\mathbf{L}^F} = \lambda$ (see [6, 1.26]). Let R^+ be the positive roots of the root system corresponding to the choice of $\mathbf{B}$. We therefore have a decomposition as a semidirect product $N_{W_\mathbb{G}}(\mathbb{L}, \boldsymbol{\lambda}) = W_\mathbb{L} \rtimes W'$ where $W' \cong W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$ consists of those $v \in N_{W_\mathbb{G}}(\mathbb{L}, \boldsymbol{\lambda})$ such that $v(R'^+) = R'^+$. We may assume that, in the class $W_\mathbb{L} w\phi$ that defines $\mathbb{L}$, w was chosen with $w\phi(R'^+) = R'^+$. Let δ be the order of ϕ: we have $(w\phi)^\delta \in N_{W_\mathbb{G}}(\mathbb{L}, \boldsymbol{\lambda})$ and, since we again have $(w\phi)^\delta(R'^+) = R'^+$, we similarly obtain $(w\phi)^\delta \in W'$. Therefore, by the hypothesis of the proposition, ℓ does not divide the order of $(w\phi)^\delta$.

If ℓ does not divide δ, then ℓ doesn't divide the order of $w\phi$, and by Lemma 1.17 we see that there is d such that ℓ is $(\mathbf{T}, F, d)$-adapted, where $\mathbf{T}$ is a maximal tori of $\mathbf{L}$ corresponding to a generic torus $(X, Y, w\phi)$. This implies in particular that ℓ is $(Z^\circ(\mathbf{L}), F, d)$-adapted, yielding the proposition in this case.

Now suppose that ℓ divides δ. As we have chosen $\mathbf{T}_0$ to be quasi-split, ϕ is a diagram automorphism, of order 2 or 3 on each non-split component $\mathbb{G}_i$ of $\mathbb{G}$ because they are simple and F-stable. Therefore $\ell = 2$ or $\ell = 3$. We even have, since we assumed that ℓ is good for $\mathbb{G}$, and excluded components of type ${}^3D_4$ for $\ell = 3$, that $\delta = \ell = 2$ and that every non-split component of $\mathbf{G}$ is of type A_n. Let $\mathbf{G}_i$ be such a component; then $\mathbb{G}_i^-$ is split and, as $W_{\mathbb{G}_i^-}(\mathbb{L}_i^-, \sigma^{\mathbb{L}_i}(\boldsymbol{\lambda}_i)) = W_{\mathbb{G}_i}(\mathbb{L}_i, \boldsymbol{\lambda}_i)$ (see [6, 3.3] for the definition of $\sigma^{\mathbb{L}_i}$) we know, using the result already proved for $\delta = 1$ for $\mathbb{G}_i^-$, that 2 divides at most one of the cyclotomic factors of $|\mathrm{Rad}(\mathbb{L}_i^-)|(q) = \pm|\mathrm{Rad}(\mathbb{L}_i)|(-q)$. This gives the result since $\Phi_d(q) \equiv \Phi_d(-q) \pmod 2$. □

Remark. Let $\mathbf{G}$ be a group of type ${}^3D_4$, and set $(\mathbf{L}, \lambda) = (\mathbf{T}, 1)$ where,

- if $q \equiv 1 \pmod 3$, $\mathbf{T}$ is a maximal torus such that

$$|\mathbb{T}| = (x^2 + x + 1)(x - 1)(x + 1)$$

- and if $q \equiv -1 \pmod 3$, $\mathbf{T}$ is a maximal torus such that

$$|\mathbb{T}| = (x^2 - x + 1)(x + 1)(x - 1).$$

Hence 3 is not $(\mathbf{T}, F, d)$-adapted for these tori, but, the two tori have a Weyl group $W_{\mathbf{G}^F}(\mathbf{T}, 1)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (see [9]): this shows that excluding the type ${}^3D_4$ in the statement of the proposition is necessary.

Proposition 1.18.

(1) *Let π be a set of primes different from p, and not dividing the order of $(Z(\mathbf{G}^*)/Z^\circ(\mathbf{G}^*))^F$, and let s be a π-element of $\mathbf{G}^F$. Then $C_\mathbf{G}(s)^F = C^\circ_\mathbf{G}(s)^F$.*

(2) *Let π be a set of primes that are good for $\mathbf{G}$, and not dividing the order of $(Z(\mathbf{G})/Z^\circ(\mathbf{G}))^F$. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ and let $\theta \in \mathrm{Irr}(\mathbf{T}^F)_\pi$. Then $N_{\mathbf{G}^F}(\mathbf{T}, \theta) \subseteq \mathbf{G}(\mathbf{T}, \theta)$.*

Proof. (1) is in [13, 2.5] (see also [8, 4.4(i)]). (2) can be deduced by applying (1) to $\mathbf{G}^*$ and using the fact that, if $(\mathbf{T}^*, s)$ is dual to $(\mathbf{T}, \theta)$, then, as π only contains primes that are good for $\mathbf{G}$, we have that the group $\mathbf{G}(\mathbf{T}, \theta)$ is dual to $C^\circ_{\mathbf{G}^*}(s)$. □

The following result will be used in Section 3 for a description of isotypies.

Proposition 1.19.

(1) *Let π be a set of primes, none of which divides the order of $Z([\mathbf{G}, \mathbf{G}])^F$. The restriction of $\mathbf{G}^F$ to $Z^\circ(\mathbf{G})^F_\pi$ induces an isomorphism*

$$\mathrm{Ab}_\pi\mathrm{Irr}(\mathbf{G}^F) \xrightarrow{\sim} \mathrm{Irr}(Z^\circ(\mathbf{G})^F_\pi).$$

(2) *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$, let $\mathbf{T}$ be a maximal torus of $\mathbf{L}$, and let π be a set of primes that are excellent for $(\mathbf{G}, F)$, and that do not divide $Z([\mathbf{L}, \mathbf{L}])^F$. Let $\theta \in \mathrm{Ab}_\pi\mathrm{Irr}(\mathbf{L}^F)$, and let $\theta_\mathbf{T}$ be the restriction of θ to $\mathbf{T}^F$, and θ_Z the restriction to $Z^\circ(\mathbf{L})^F_\pi$. We have*

$$N_{\mathbf{G}^F}(\mathbf{L}, \theta_Z) = N_{\mathbf{G}(\mathbf{T},\theta_\mathbf{T})^F}(\mathbf{L}).$$

Proof. The exact sequence

$$0 \to Z([\mathbf{G}, \mathbf{G}]) \cap Z^\circ(\mathbf{G}) \to Z^\circ(\mathbf{G}) \to \mathbf{G}/[\mathbf{G}, \mathbf{G}] \to 0$$

gives the exact sequence of Galois cohomology

$$0 \to (Z([\mathbf{G}, \mathbf{G}]) \cap Z^\circ(\mathbf{G}))^F \to (Z^\circ(\mathbf{G}))^F \\ \to (\mathbf{G}/[\mathbf{G}, \mathbf{G}])^F \to H^1(F, Z([\mathbf{G}, \mathbf{G}]) \cap Z^\circ(\mathbf{G})) \to 0$$

and, by assumption, the terms at either end are π'-groups, whence we get (1).

Under the assumptions of the statement, $\mathbf{G}(\mathbf{T}, \theta_\mathbf{T})$ is the largest F-stable Levi subgroup $\mathbf{M}$ containing $\mathbf{L}$ such that $\theta_\mathbf{T}$ is the restriction of a character of $\mathrm{AbIrr}(\mathbf{M}^F)$. Moreover, if $\mathbf{T}'$ is another rational maximal torus of $\mathbf{L}$, we have $\mathbf{G}(\mathbf{T}, \theta_\mathbf{T}) = \mathbf{G}(\mathbf{T}', \theta_{\mathbf{T}'})$. Therefore we may suppose that we have chosen $\mathbf{T}$ to be a quasi-split torus of $\mathbf{L}$. Two quasi-split tori are conjugate in $\mathbf{L}^F$, and we can therefore find representatives of $N_{\mathbf{G}^F}(\mathbf{L}, \theta_Z)/\mathbf{L}^F$ in $N_{\mathbf{G}^F}(\mathbf{L}, \mathbf{T})$ (and by (1) these representatives are in $N_{\mathbf{G}^F}(\mathbf{L}, \mathbf{T}, \theta)$, hence in $N_{\mathbf{G}^F}(\mathbf{L}, \mathbf{T}, \theta_\mathbf{T})$); hence $N_{\mathbf{G}^F}(\mathbf{L}, \theta_Z) = N_{\mathbf{G}^F}(\mathbf{T}, \theta_\mathbf{T}, \mathbf{L}) \cdot \mathbf{L}^F$, which is equal to $N_{\mathbf{G}(\mathbf{T},\theta_\mathbf{T})^F}(\mathbf{L})$ by (1) of the proposition. □

2. Unipotent blocks with abelian defect groups

2.A. **Necessary conditions.** *We analyze the structure of maximal pairs of unipotent ℓ-blocks of $\mathbf{G}^F$ with abelian defect group, when ℓ is excellent for $(\mathbf{G}, F)$.*

Let ℓ be a prime that is excellent for $(\mathbf{G}, F)$. Let K be a finite extension of the field of ℓ-adic rationals $\mathbb{Q}_\ell$ that is 'large enough' for $\mathbf{G}^F$, and let $\mathcal{O}$ be its ring of integers, a finite extension of the ring of ℓ-adic integers $\mathbb{Z}_\ell$.

Let e be a unipotent block of $\mathcal{O}\mathbf{G}^F$, i.e., a primitive central idempotent of $\mathcal{O}\mathbf{G}^F$ such that $ee_\ell^{\mathbf{G}^F} = e$, where $e_\ell^{\mathbf{G}^F}$ is the central idempotent corresponding to $\mathcal{E}_\ell(\mathbf{G}^F, 1)$ (see Section 1). Let D be a defect group of e, which we suppose is abelian. Set $\mathbf{L} := C^\circ_{\mathbf{G}}(D)$. Since ℓ is good, there is an F-stable Levi subgroup of $\mathbf{G}$, and we have (see Lemma 1.17(2) above) $\mathbf{L}^F = C_{\mathbf{G}^F}(D)$. Let f be an ℓ-block of $\mathbf{L}^F$ such that $\mathrm{Br}_D(e)\cdot f = f$, where Br_D denotes the Brauer morphism (in other words, f is a root of e, and (D, f) is a maximal e-Brauer pair). Let λ be the canonical character of $\mathbf{L}^F$.

Theorem 2.1.

(1) *D is a Sylow ℓ-subgroup of $Z(\mathbf{L}^F)$, and $D \subseteq Z^\circ(\mathbf{L})^F$.*
(2) *λ is a character of ℓ-defect zero of $\mathbf{L}^F/D$, i.e., $\deg(\lambda)_\ell = |\mathbf{L}^F_{ss}|_\ell$.*
(3) *$N_{\mathbf{G}^F}(D,\lambda) = N_{\mathbf{G}^F}(\mathbf{L},\lambda)$, and $\ell \nmid |W_{\mathbf{G}^F}(\mathbf{L},\lambda)|$.*
(4) *λ is a unipotent character of $\mathbf{L}^F$ and it is the unique unipotent character of $\mathbf{L}^F$ in the block f.*
(5) *$\ell \nmid |Z([\mathbf{L},\mathbf{L}]^F)|$.*
(6) *There exists an integer d such that $\ell \mid \Phi_d(q)$ and $(\mathbf{L},\lambda)$ is a d-cuspidal pair for $\mathbf{G}$. The integer d is uniquely determined if $\mathbf{L}$ is a proper subgroup of $\mathbf{G}$, and hence ℓ is $(Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G}), F, d)$-adapted.*

Remark. The group $\mathrm{GL}_{15}(13)$ has a unipotent character of 7-defect 0, which is both 2-cuspidal and 14-cuspidal. We therefore see that the uniqueness of d in assertion (6) requires that $\mathbf{L}$ is proper.

Proof. The first three assertions are classical results of block theory (the inclusion $D \subseteq Z^\circ(\mathbf{L})^F$ follows from the fact that ℓ is excellent).

To demonstrate (4), we first check that the block f is unipotent, i.e., that $fe_\ell^{\mathbf{L}^F} = f$. By iterating the formula $\mathrm{Br}_{\langle x\rangle}(e_\ell^{\mathbf{G}^F}) = e_\ell^{C^\circ_{\mathbf{G}}(x)^F}$ (see [7, 3.2]), where x denotes an ℓ-element of $\mathbf{G}^F$, we see that $\mathrm{Br}_D(e_\ell^{\mathbf{G}^F}) = e_\ell^{\mathbf{L}^F}$. It therefore suffices to apply the Brauer morphism Br_D to the equality $ee_\ell^{\mathbf{G}^F} = e$ to get that all constituents of $\mathrm{Br}_D(e)$ (and in particular f) are unipotent.

The following lemma follows immediately from [14, 3.1].

Lemma 2.2 (G. Hiss)**.** *Every unipotent block contains a unipotent character.*

As λ is the unique character of f that is trivial on D, and as all unipotent characters of $\mathbf{L}^F$ are trivial on $Z(\mathbf{L}^F)$, we deduce assertion (4).

To demonstrate (5) note that, since λ is unipotent, its restriction to $[\mathbf{L},\mathbf{L}]^F$ is again an irreducible character. However, $|[\mathbf{L},\mathbf{L}]^F| = |(\mathbf{L}/Z(\mathbf{L}))^F| = |\mathbf{L}^F/Z(\mathbf{L})^F|\cdot|H^1(F, Z(\mathbf{L}))|$, therefore $(\deg\lambda)_\ell = |[\mathbf{L},\mathbf{L}]^F|_\ell$. Hence λ is a character of ℓ-defect 0 of $[\mathbf{L},\mathbf{L}]^F$, and this implies that this group does not have a non-trivial central ℓ-subgroup, demonstrating (5).

We prove (6). Suppose first that $\mathbf{L} \neq \mathbf{G}$. There therefore exists $d \in \mathbb{N}$ such that ϕ_d divides the generic order of $Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G})$ and $\ell \mid \Phi_d(q)$. As ℓ does not divide $W_{\mathbf{G}^F}(\mathbf{L},\lambda)|$, it results from Proposition 1.16 that d is uniquely determined by these conditions. In other words, ℓ is $(Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G}), F, d)$-adapted. We obtain therefore by Proposition 1.15(i) that $\mathbf{L}$ is d-split.

Moreover, there exists a product c of bad primes for $\mathbf{L}$ such that $c \cdot \operatorname{Deg}(\lambda) \in \mathbb{Z}[x]$ is a monic polynomial (see for example, [6, 1.32]). Therefore $|\mathbb{L}_{ss}|/c\cdot\operatorname{Deg}(\lambda)$ is a product of cyclotomic polynomials multiplied by a power of x, whose value at q is not divisible by ℓ. As ℓ divides Φ_d, we have that the contribution of the cyclotomic polynomial Φ_d to $\operatorname{Deg}(\lambda)$ and $|\mathbb{L}_{ss}|$ are equal; this proves that λ is d-cuspidal by [6, 2.9], and completes the proof of Theorem 2.1. □

Remark. Note that, if $\mathbf{L} = \mathbf{G}$, the proof above shows that λ is a d-cuspidal character of $\mathbf{G}^F$ for all d such that $\ell \mid \Phi_d(q)$ and $\Phi_d \mid |\mathbb{G}_{ss}|$.

2.B. **The unipotent π-blocks with abelian defect groups.** *In this section, for some sets π of primes, we define subsets of the π-series $\mathcal{E}_\pi(\mathbf{G}^F, 1)$, which we show correspond to π-idempotents of $Z\bar{\mathbb{Q}}\mathbf{G}^F$, and for $\pi = \{\ell\}$ they correspond to the ℓ-blocks of abelian defect group in $\mathbf{G}^F$.*

The following notation and hypotheses will be in effect for the rest of this section:

(Hd) $(\mathbf{L}, \lambda)$ is a d-cuspidal pair for $\mathbf{G}$.
(Hπ1) π is a set of primes excellent for $(\mathbf{G}, F)$.
(Hπ2) For all $\ell \in \pi$, $\mathbf{L} = C_{\mathbf{G}}(Z^\circ(\mathbf{L})^F_\ell)$.
(Hπ3) $\deg(\lambda)_\pi = |\mathbf{L}^F_{ss}|_\pi$.
(Hπ4) $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ is a π'-group.
(H$d\pi$) π is $(Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G}), F, d)$-adapted.

Remark. If π consists only of primes ℓ that are excellent for $(\mathbf{G}, F)$, we have (see Theorem 2.1) that the above hypotheses are satisfied for a suitable choice of $(\mathbf{L}, \lambda)$ if $\mathbf{G}^F$ has an ℓ-block with abelian defect group.

Notation.

- From now on, we assume that we have chosen an F-stable maximal torus $\mathbf{T}_0$ of $\mathbf{L}$.
- Let $\ell \in \pi$. We write $e^{\mathbf{L}^F}_{\ell,(\mathbf{L},\lambda)}$ for the primitive ℓ-idempotent of $Z\bar{\mathbb{Q}}\mathbf{L}^F$ defined by λ. Hence $e^{\mathbf{L}^F}_{\ell,(\mathbf{L},\lambda)}$ is an ℓ-block of $\mathbf{L}^F$ with defect group $Z^\circ(\mathbf{L})^F_\ell$.
- We write $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$ for the unique block of $\mathbf{G}^F$ corresponding to $e^{\mathbf{L}^F}_{\ell,(\mathbf{L},\lambda)}$ under the Brauer correspondence – in other words, using the notation of [1], we have
$$(\{1\}, e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}) \subseteq (Z^\circ(\mathbf{L})^F_\ell, e^{\mathbf{L}^F}_{\ell,(\mathbf{L},\lambda)}).$$

As $N_{\mathbf{G}^F}(Z^\circ(\mathbf{L})^F_\ell, e^{\mathbf{L}^F}_{\ell,(\mathbf{L},\lambda)}) = N_{\mathbf{G}^F}(\mathbf{L},\lambda)$ and ℓ doesn't divide $|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|$, it follows from Brauer's first main theorem (see for example [1]) that $(Z^\circ(\mathbf{L})^F_\ell, e^{\mathbf{L}^F}_{\ell,(\mathbf{L},\lambda)})$ is a maximal $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$-Brauer pair. In particular,

(2.3) *$Z^\circ(\mathbf{L})^F_\ell$ is a defect group of $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$.*

We will provide, among other things, (see Theorem 2.8 below) a complete description of the characters of $\mathbf{G}^F$ in $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$, demonstrating that they are exactly the characters of $\mathcal{E}_\ell(\mathbf{G}^F,1)$ that lie above $(\mathbf{L},\lambda)$. We proceed in several stages.

Studying $\mathcal{E}(\mathbf{G}^F,1,(\mathbf{L},\lambda))$.

Let $\mathcal{E}(\mathbf{G}^F,1,(\mathbf{L},\lambda))$ the set of unipotent characters γ of $\mathbf{G}^F$ such that $(\gamma, R^{\mathbf{G}}_{\mathbf{L}}(\lambda))_{\mathbf{G}^F} \neq 0$. We write $\mathrm{Irr}(\mathbf{G}^F, e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)})$ for the set of irreducible characters of $\mathbf{G}^F$ in $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$.

Lemma 2.4. $\mathcal{E}(\mathbf{G}^F,1,(\mathbf{L},\lambda)) \subseteq \mathrm{Irr}(\mathbf{G}^F, e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)})$.

Proof. We go by induction on $|\mathbf{G}^F : \mathbf{L}^F|$. If $\mathbf{G}^F = \mathbf{L}^F$, then the lemma is trivial. Otherwise, there exists $x \in Z^\circ(\mathbf{L})^F_\ell$, $x \notin Z(\mathbf{G}^F)$. Let $\mathbf{G}(x) := C^\circ_{\mathbf{G}}(x)$ and $\mathbf{G}^F(x) := \mathbf{G}(x)^F$. By induction, the lemma is true if we replace $\mathbf{G}$ by $\mathbf{G}(x)$.

Let $\gamma \in \mathcal{E}(\mathbf{G}^F,1,(\mathbf{L},\lambda))$. By the Curtis type formula described as in [3, 4.3], we have

$$\mathrm{dec}^{x,\mathbf{G}^F}_\ell(\gamma) = \mathrm{dec}^{x,\mathbf{G}^F(x)}_\ell({}^*R^{\mathbf{G}}_{\mathbf{G}(x)}(\gamma)).$$

Since, by [6, 4.5], the character ${}^*R^{\mathbf{G}}_{\mathbf{G}(x)}(\gamma)$ is afforded by the free $\mathbb{Z}$-module generated by

$$\bigcup_{(\mathbf{L}',\lambda')\sim_{\mathbf{G}^F}(\mathbf{L},\lambda)} \mathcal{E}(\mathbf{G}^F(x),1,(\mathbf{L}',\lambda'))$$

(where $(\mathbf{L}',\lambda')$ runs through the set of pairs of $\mathbf{G}(x)$ that are $\mathbf{G}^F$-conjugate to $(\mathbf{L},\lambda)$), we see that

$$\mathrm{dec}^{x,\mathbf{G}}_\ell(\gamma) \in \bar{\mathbb{Z}} \bigcup_{(\mathbf{L}',\lambda')\sim_{\mathbf{G}^F}(\mathbf{L},\lambda)} \mathcal{E}(\mathbf{G}^F(x),1,(\mathbf{L}',\lambda')),$$

and by induction hypothesis

$$\mathrm{dec}^{x,\mathbf{G}}_\ell(\gamma) \in \bar{\mathbb{Z}} \bigcup_{(\mathbf{L}',\lambda')\sim_{\mathbf{G}^F}(\mathbf{L},\lambda)} \mathrm{Irr}(\mathbf{G}^F(x), e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}).$$

As $(\langle x\rangle, e^{\mathbf{G}^F(x)}_{\ell,(\mathbf{L}',\lambda')}) \supseteq (\{1\}, e^{\mathbf{G}^F}_{\ell,(\mathbf{L}',\lambda')})$ for all $(\mathbf{L}',\lambda') \sim_{\mathbf{G}^F} (\mathbf{L},\lambda)$, to prove the lemma it now suffices (by Brauer's second main theorem) to check that $\mathrm{dec}^{x,\mathbf{G}^F}_\ell(\gamma) \neq 0$.

Applying again the Curtis type formula ([3, 4.3]) we obtain

$${}^*R^{\mathbf{G}(x)}_{\mathbf{L}}(\mathrm{dec}^{x,\mathbf{G}^F}_\ell(\gamma)) = \mathrm{dec}^{x,\mathbf{L}^F}_\ell({}^*R^{\mathbf{G}}_{\mathbf{L}}(\gamma)),$$

and it remains to check that $\mathrm{dec}^{x,\mathbf{L}^F}_\ell({}^*R^{\mathbf{G}}_{\mathbf{L}}(\gamma)) \neq 0$. Since, by [6, 3.15],

$${}^*R^{\mathbf{G}}_{\mathbf{L}}(\gamma) = (\gamma, R^{\mathbf{G}}_{\mathbf{L}}(\lambda)) \sum_{w \in W_{\mathbf{G}^F}(\mathbf{L})/W_{\mathbf{G}^F}(\mathbf{L},\lambda)} \lambda^w,$$

it remains to check that $\mathrm{dec}_\ell^{x,\mathbf{L}^F}(\lambda) \neq 0$. But, as λ is unipotent (therefore trivial on $Z^\circ(\mathbf{L})^F$) and is therefore of defect zero, we have $\mathrm{dec}_\ell^{x,\mathbf{L}^F}(\lambda) = \lambda$. □

The following characterization of abelian defect groups is analogous to that in [6, 4.8]. Note that, in the case where $\pi = \{\ell\}$, it provides a characterization of defect groups in terms of the application of the Deligne–Lusztig functor $R_{\mathbf{T}}^{\mathbf{G}}$ (such a characterization has been suggested by G. Hiss).

Proposition 2.5. *Let $\gamma \in \mathcal{E}(\mathbf{G}^F, 1, (\mathbf{L}, \lambda))$. If $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, and if $(R_{\mathbf{T}}^{\mathbf{G}}(1), \gamma)_{\mathbf{G}^F} \neq 0$, then the Hall π-subgroup $\mathbf{T}_\pi^F$ of $\mathbf{T}^F$ is $\mathbf{G}^F$-conjugate to a subgroup of $Z^\circ(\mathbf{L})^F$. Moreover, there exists an F-stable maximal torus $\mathbf{T}$ such that $\mathbf{T}_\pi^F = Z^\circ(\mathbf{L})_\pi^F$.*

Proof. Suppose that $(\gamma, R_{\mathbf{T}}^{\mathbf{G}}(1))_{\mathbf{G}^F} \neq 0$, and let $x \in \mathbf{T}_\ell^F$ for some $\ell \in \pi$. By the Curtis type formula, we have

$$ {}^*R_{\mathbf{T}}^{C^\circ_{\mathbf{G}}(x)} \mathrm{dec}_\ell^{x,\mathbf{G}^F}(\gamma) = \mathrm{dec}_\ell^{x,\mathbf{T}^F}({}^*R_{\mathbf{T}}^{\mathbf{G}}(\gamma)), $$

hence in particular $\mathrm{dec}_\ell^{x,\mathbf{G}^F}(\gamma) \neq 0$ since ${}^*R_{\mathbf{T}}^{\mathbf{G}}(\gamma) = (R_{\mathbf{T}}^{\mathbf{G}}(1), \gamma)_{\mathbf{G}^F} \cdot 1_{\mathbf{T}^F}$ and $\mathrm{dec}_\ell^{x,\mathbf{T}^F}(1_{\mathbf{T}^F}) \neq 0$. This implies that x is $\mathbf{G}^F$-conjugate to an element of the defect group of the ℓ-block of γ, which by Lemma 2.4 is $Z^\circ(\mathbf{L})^F$.

We now show the first part of the proposition by induction on $\dim \mathbf{G}$. If $\mathbf{T}_\pi^F \subseteq Z^\circ(\mathbf{G})^F$, nothing needs to be proved. Otherwise, there exists $\ell \in \pi$ and $x \in \mathbf{T}_\ell^F \setminus Z^\circ(\mathbf{G})^F$. By the reasoning above, x is $\mathbf{G}^F$-conjugate to some $x' \in Z^\circ(\mathbf{L})^F$; this conjugation sends $\mathbf{T}$ to some maximal torus $\mathbf{T}'$ of $\mathbf{M} := C^\circ_{\mathbf{G}}(x')$. As π is $(Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G}), F, d)$-adapted, it follows from Proposition 1.15(1) that $\mathbf{M}$ is a d-split Levi subgroup of $\mathbf{G}$. As $(R_{\mathbf{T}'}^{\mathbf{G}}(1), \gamma)_{\mathbf{G}^F} = (R_{\mathbf{T}}^{\mathbf{G}}(1), \gamma)_{\mathbf{G}^F} \neq 0$, it follows that there exists $\mu \in \mathcal{E}(\mathbf{M}^F, 1)$ with ${}^*R_{\mathbf{T}'}^{\mathbf{M}}(\mu) \neq 0$ and $(\mathbf{M}, \mu) \preccurlyeq (\mathbf{G}, \gamma)$. Let $(\mathbf{L}', \lambda')$ be a d-cuspidal pair such that $(\mathbf{L}', \lambda') \preccurlyeq (\mathbf{M}, \mu)$. Then $(\mathbf{L}', \lambda')$ is $\mathbf{G}^F$-conjugate to $(\mathbf{L}, \lambda)$, and by induction hypothesis we have

$$ \mathbf{T}'_\pi \leqslant_{\mathbf{M}^F} Z^\circ(\mathbf{L}') \qquad \text{and} \qquad Z^\circ(\mathbf{L}') =_{\mathbf{G}^F} Z^\circ(\mathbf{L}), $$

hence the first part of the proposition holds.

For every F-stable maximal tori $\mathbf{T}$ of $\mathbf{L}$ such that $(\lambda, R_{\mathbf{T}}^{\mathbf{L}}(1)) \neq 0$, by [6, 1.38] we have that $(\deg \lambda)_\pi$ divides $|\mathbf{L}^F : \mathbf{T}^F|_\pi$. As $(\deg \lambda)_\pi = |\mathbf{L}_{ss}^F|_\pi$, we have that $\mathbf{T}_\pi^F \leqslant Z^\circ(\mathbf{L})$, whence $\mathbf{T}_\pi^F = Z^\circ(\mathbf{L})_\pi^F$. □

The next result shows that the π-local subgroups used in our context are, as in [6], d-split groups.

Lemma 2.6. *A Levi subgroup of $\mathbf{G}$ containing $\mathbf{L}$ is π-split if and only if it is d-split.*

Proof. A π-split Levi subgroup $\mathbf{M}$ containing $\mathbf{L}$ is the connected centralizer of a π-subgroup of $Z(\mathbf{L})^F$. This π-subgroup is necessarily contained in

$Z^\circ(\mathbf{L})^F$ since π is excellent for $(\mathbf{G}, F)$ (hence excellent for $(\mathbf{L}, F)$). By Proposition 1.15 and by hypothesis (H$d\pi$), $\mathbf{M}$ is d-split.

Conversely, if $\mathbf{M} \supseteq \mathbf{L}$ is d-split, then $\mathbf{M}$ is the centralizer of the π-part of its centre. Indeed, let $\mathbf{M}'$ be the centralizer of the π-part of the centre of $\mathbf{M}'$. By the above, $\mathbf{M}'$ is d-split. If $\mathbf{M}'$ strictly contains $\mathbf{M}$, the d-part of the centre of $\mathbf{M}'$ is strictly smaller than the d-part of the centre of $\mathbf{M}$, and by (H$d\pi$) it follows that it must be true of the π-part, a contradiction. □

As in [6], we set

$$\mathrm{UReg}^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} := \sum_{\gamma \in \mathcal{E}(\mathbf{G}^F, 1, (\mathbf{L},\lambda))} \deg \gamma \cdot \gamma.$$

Lemma 2.7.

(1) *Let $\gamma \in \mathcal{E}(\mathbf{G}^F, 1, (\mathbf{L}, \lambda))$, and let $\boldsymbol{\psi}$ be a uniform unipotent function on $\mathbf{G}^F$. Hence*

$$(\gamma, \boldsymbol{\psi})_{\mathbf{G}^F} = \sum_{[\mathbf{M}]_{\mathbf{G}^F}} \frac{1}{|W_{\mathbf{G}^F}(\mathbf{M})|} (\gamma, R^{\mathbf{G}}_{\mathbf{M}}(c_\pi {}^*R^{\mathbf{G}}_{\mathbf{M}}(\boldsymbol{\psi})))_{\mathbf{G}^F},$$

where $\mathbf{M}$ runs through the set of all d-split Levi subgroups of $\mathbf{G}$ containing $\mathbf{L}$.

(2) *We have*

$$\mathrm{UReg}^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} = \sum_{[(\mathbf{M},\mu)]_{\mathbf{G}^F}} \frac{\deg(R^{\mathbf{G}}_{\mathbf{M}}(1))}{|W_{\mathbf{G}^F}(\mathbf{M}, \mu)|} R^{\mathbf{G}}_{\mathbf{M}}(\deg c_\pi(\boldsymbol{\mu}) \cdot \boldsymbol{\mu}),$$

where $(\mathbf{M}, \mu)$ runs through the set of all d-split pairs such that $(\mathbf{L}, \lambda) \preccurlyeq (\mathbf{M}, \mu)$.

Proof. We start with (1). As $\boldsymbol{\psi}$ is uniform, we have

$$(\gamma, \boldsymbol{\psi})_{\mathbf{G}^F} = \left(\gamma, \sum_{[\mathbf{T}]_{\mathbf{G}^F}} \frac{1}{|W_{\mathbf{G}^F}(\mathbf{T})|} R^{\mathbf{G}}_{\mathbf{T}} {}^*R^{\mathbf{G}}_{\mathbf{T}} \boldsymbol{\psi} \right)_{\mathbf{G}^F}.$$

We continue as in the proof of Proposition 1.9(1). Here we know that the terms where $\mathbf{M}$ is not π-split and containing $\mathbf{L}$ (which is equivalent to being d-split and containing $\mathbf{L}$ by Lemma 2.6) are zero by Proposition 2.5.

For (2), we have

$$\mathrm{UReg}^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} = \sum_{\gamma \in \mathcal{E}(\mathbf{G}^F, 1, (\mathbf{L},\lambda))} (\gamma, \mathrm{UReg}^{\mathbf{G}^F})_{\mathbf{G}^F} \gamma.$$

Applying (1) with $\boldsymbol{\psi} = \mathrm{UReg}^{\mathbf{G}^F}$, exchanging the summations, and using the equality

$${}^*R^{\mathbf{G}}_{\mathbf{M}}(\mathrm{UReg}^{\mathbf{G}^F}) = \deg(R^{\mathbf{G}}_{\mathbf{M}}(1)) \mathrm{UReg}^{\mathbf{M}^F},$$

we obtain

$$\mathrm{UReg}^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} = \sum_{[\mathbf{M}]_{\mathbf{G}^F}} \frac{\deg(R^{\mathbf{G}}_{\mathbf{M}}(1))}{|W_{\mathbf{G}^F}(\mathbf{M})|} \sum_{\gamma\in\mathcal{E}(\mathbf{G}^F,1,(\mathbf{L},\lambda))} \Big(\gamma, R^{\mathbf{G}}_{\mathbf{M}}(c_\pi \mathrm{UReg}^{\mathbf{M}^F})\Big)_{\mathbf{G}^F}\gamma,$$

where $\mathbf{M}$ runs though the π-split Levi subgroups of $\mathbf{G}^F$. Expanding $c_\pi\mathrm{UReg}^{\mathbf{M}^F} = \sum_{\boldsymbol{\mu}\in\mathcal{E}(\mathbf{M}^F,1)} \deg c_\pi(\boldsymbol{\mu})\cdot\boldsymbol{\mu}$, there are only representatives of $\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{M},\mu)$ (which are necessarily up to $\mathbf{G}^F$-conjugation above $(\mathbf{L},\lambda)$), so we obtain the lemma. □

Descriptions of the blocks

Recall the notation used in [6] for Lusztig's indexing of the irreducible characters of $\mathbf{G}^F$. Let $\mathbf{T}$ be a rational maximal torus of $\mathbf{G}^F$. For θ a character of $\mathbf{T}^F$ and ν a unipotent character of $\mathbf{G}^F(\mathbf{T},\theta)$, we write $\chi^{\mathbf{G}^F}_{(\mathbf{G}(\mathbf{T},\theta),\theta,\nu)}$ for the irreducible character of $\mathbf{G}^F$ that corresponds to the pair (θ,ν) via the Jordan decomposition of characters.

Theorem 2.8. *We set*

$$\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\lambda)} = \sum_{[(\mathbf{N},\theta,\boldsymbol{\nu})]_{\mathbf{G}^F}} \deg(\chi^{\mathbf{G}^F}_{(\mathbf{N},\theta,\boldsymbol{\nu})})\chi^{\mathbf{G}^F}_{(\mathbf{N},\theta,\boldsymbol{\nu})}$$

where θ ranges over $\mathrm{Ab}_\pi\mathrm{Irr}(\mathbf{L}^F)$, $\mathbf{N} := \mathbf{G}(\mathbf{T},\theta)$, and $\nu\in\mathcal{E}(\mathbf{N}^F,1,(\mathbf{L},\lambda))$.

(1) *We have*

$$\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\lambda)} = \sum_{[(\mathbf{M},\mu)]_{\mathbf{G}^F}} \frac{\deg R^{\mathbf{G}}_{\mathbf{M}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M},\mu)|} R^{\mathbf{G}}_{\mathbf{M}}(\mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{M}^F}\cdot \deg c_\pi(\boldsymbol{\mu})\boldsymbol{\mu})$$

where $(\mathbf{M},\mu)$ runs through the set of d-split pairs of $\mathbf{G}^F$ such that $(\mathbf{L},\lambda)\preccurlyeq_{\mathbf{G}^F}(\mathbf{M},\mu)$.

(2) *The character $\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\lambda)}$ is the regular character associated to a central π-idempotent of $\bar{\mathbb{Q}}\mathbf{G}^F$. In particular, its values are divisible by $|\mathbf{G}^F|_\pi$.*

Remark. For $\pi=\{\ell\}$, the idempotent corresponding to $\mathrm{Reg}^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$ is the one we have already designated by $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$. In the general case, we denote by $e^{\mathbf{G}^F}_{\pi,(\mathbf{L},\lambda)}$ the idempotent corresponding to $\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\lambda)}$.

Proof. We set $\Theta^{\mathbf{N}^F}_{\mathbf{G},\pi} := \sum\theta$, where the sum runs over all elements θ of $\mathrm{Ab}_\pi\mathrm{Irr}(\mathbf{L}^F)$ such that $\mathbf{G}(\mathbf{T},\theta)=\mathbf{N}$. Then

$$\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\lambda)} = \sum_{[(\mathbf{N},\nu)]_{\mathbf{G}^F}} \frac{\deg R^{\mathbf{G}}_{\mathbf{N}}(1)}{|W_{\mathbf{G}^F}(\mathbf{N},\nu)|} R^{\mathbf{G}}_{\mathbf{N}}(\Theta^{\mathbf{N}^F}_{\mathbf{G},\pi}\cdot\deg(\boldsymbol{\nu})\boldsymbol{\nu})$$

where $(\mathbf{N},\nu)$ ranges over the set of d-split pairs such that $(\mathbf{L},\lambda)\preccurlyeq_{\mathbf{G}^F}(\mathbf{N},\nu)$.

We will use the next technical lemma twice.

Lemma 2.9. *Let ϕ be a $\mathbf{G}^F$-invariant function on the set of d-split pairs $(\mathbf{N},\nu)$ such that $(\mathbf{L},\lambda) \leqslant_{\mathbf{G}^F} (\mathbf{N},\nu)$. Then*

$$\sum_{[(\mathbf{N},\nu):(\mathbf{L},\lambda)\leqslant_{\mathbf{G}^F}(\mathbf{N},\nu)]_{\mathbf{G}^F}} \frac{\phi(\mathbf{N},\nu)}{|W_{\mathbf{G}^F}(\mathbf{N},\nu)|} = \sum_{\{(\mathbf{N},\nu):(\mathbf{L},\lambda)\leqslant_{\mathbf{G}^F}(\mathbf{N},\nu)\}} \frac{|W_{\mathbf{N}^F}(\mathbf{L},\lambda)|}{|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|}\phi(\mathbf{N},\nu).$$

Proof of 2.9. Note that the first sum covers the $\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{N},\nu)$ while the second sum covers all pairs $(\mathbf{N},\nu)$. The lemma easily follows from [6, 3.14]. □

By Lemma 2.9, we see that

$$\begin{aligned}\mathrm{Reg}^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} &= \sum_{\{(\mathbf{N},\nu):(\mathbf{L},\lambda)\leqslant_{\mathbf{G}^F}(\mathbf{N},\nu)\}} \frac{|W_{\mathbf{N}^F}(\mathbf{L},\lambda)|}{|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|} R^{\mathbf{G}}_{\mathbf{N}}(\Theta^{\mathbf{N}^F}_{\mathbf{G},\pi}\cdot \deg(\boldsymbol{\mu})\boldsymbol{\mu}) \\ &= \sum_{\{\mathbf{N}:\mathbf{L}\leqslant\mathbf{N}\}} \frac{|W_{\mathbf{N}^F}(\mathbf{L},\lambda)|}{|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|} R^{\mathbf{G}}_{\mathbf{N}}(\Theta^{\mathbf{N}^F}_{\mathbf{G},\pi}\cdot \mathrm{UReg}^{\mathbf{N}^F}_{(\mathbf{L},\lambda)})\end{aligned}$$

By Lemma 2.9, and by Lemma 2.7(2), we have that

$$\mathrm{UReg}^{\mathbf{N}^F}_{(\mathbf{L},\lambda)} = \sum_{\{\mathbf{M};\mathbf{L}\leqslant\mathbf{M}\leqslant\mathbf{N}\}} \frac{|W_{\mathbf{M}^F}(\mathbf{L},\lambda)|}{|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|} R^{\mathbf{G}}_{\mathbf{N}}(\sum_{\boldsymbol{\mu}} \deg c_\pi(\boldsymbol{\mu})\boldsymbol{\mu})$$

where $(\mathbf{M},\mu)$ is d-split and such that $(\mathbf{L},\lambda) \preccurlyeq (\mathbf{M},\mu)$. By virtue of the formula

$$\sum_{\mathbf{N}\geq\mathbf{M}} \Theta^{\mathbf{N}^F}_{\mathbf{M},\pi}\Big|_{\mathbf{M}^F} = \mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{M}^F},$$

we obtain

$$\mathrm{Reg}^{\mathbf{G}^F}_{(\mathbf{L},\lambda)} = \sum_{\{(\mathbf{M},\mu);(\mathbf{L},\lambda)\leqslant_{\mathbf{G}^F}(\mathbf{M},\mu)\}} \frac{|W_{\mathbf{M}^F}(\mathbf{L},\lambda)|}{|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|} R^{\mathbf{G}}_{\mathbf{M}}(\mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{M}^F}\cdot \deg c_\pi(\boldsymbol{\mu})\boldsymbol{\mu})$$

where $(\mathbf{M},\mu)$ runs through the set of d-split pairs. The formula we need results by another application of Lemma 2.9. This proves (1).

We move to (2). It is necessary and sufficient to prove that $|\mathbf{G}^F|_\pi$ divides all of the values of $\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\lambda)}$. By (1), we see that it suffices to prove that for all d-split pairs $(\mathbf{M},\mu)$ such that $(\mathbf{L},\lambda) \preccurlyeq (\mathbf{M},\mu)$,

$$|\mathbf{G}^F|_\pi \text{ divides } \frac{\deg R^{\mathbf{G}}_{\mathbf{M}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M},\mu)|} R^{\mathbf{G}}_{\mathbf{M}}(\mathrm{Ab}_\pi\mathrm{Reg}^{\mathbf{M}^F}\cdot \deg c_\pi(\boldsymbol{\mu})\boldsymbol{\mu}).$$

As $W_{\mathbf{G}^F}(\mathbf{M},\mu)$ is a quotient of a subgroup of $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ (see [6, 3.14]), we see that $W_{\mathbf{G}^F}(\mathbf{M},\mu)$ is a π'-group, and so it suffices to verify that $|\mathbf{M}^F_{ss}|_\pi$ divides $\deg c_\pi(\boldsymbol{\mu})$. This is a consequence of the following proposition.

Proposition 2.10. *For $\gamma \in \mathcal{E}(\mathbf{G}^F, 1, (\mathbf{L},\lambda))$, $|\mathbf{G}^F_{ss}|_\pi$ divides $\deg c_\pi(\gamma)$.*

Proof of 2.10. We first prove the following technical lemma, a corollary of Lemma 1.8.

Lemma 2.11. *Let $\gamma \in \mathcal{E}(\mathbf{G}^F, 1)$. Then*

$$(\gamma, D(\mathbf{u}\sigma^{\mathbf{G}^F}_{\pi\cup\{p\}}))_{\mathbf{G}^F} = \sum_{[(\mathbf{M},\mu)]_{\mathbf{G}^F}} \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}}|\mathbf{G}^F : \mathbf{M}^F|_{\pi'}}{|W_{\mathbf{G}^F}(\mathbf{M}, \boldsymbol{\mu})|}(R^{\mathbf{G}}_{\mathbf{M}}(\boldsymbol{\mu}), \gamma)_{\mathbf{G}^F} \frac{\mathrm{Deg}(c_\pi(\boldsymbol{\mu}))}{|\mathbf{M}^F_{ss}|_\pi}$$

where the sum is over representatives of $\mathbf{G}^F$-conjugacy classes of pairs consisting of a π-split Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ and where $\boldsymbol{\mu} \in \mathcal{E}(\mathbf{M}^F, 1)$.

Proof of 2.11. By (1.5) and Lemma 1.8 we deduce that

$$D(\mathbf{u}\sigma^{\mathbf{G}^F}_{\pi\cup\{p\}}) = \sum_{[\mathbf{M}\ \pi\text{-split}]_{\mathbf{G}^F}} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}} \frac{(\mathrm{Deg} R^{\mathbf{G}}_{\mathbf{M}}(1))_{\pi'}}{|W_{\mathbf{G}^F}(\mathbf{M})|} R^{\mathbf{G}}_{\mathbf{M}}(c_\pi(\mathbf{u}\sigma^{\mathbf{M}^F}_{\pi\cup\{p\}})).$$

Therefore

$$(\gamma, D(\mathbf{u}\sigma^{\mathbf{G}^F}_{\pi\cup\{p\}}))_{\mathbf{G}^F} = \sum_{[\mathbf{M}\ \pi\text{-split}]_{\mathbf{G}^F}} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}} \frac{(\mathrm{Deg} R^{\mathbf{G}}_{\mathbf{M}}(1))_{\pi'}}{|W_{\mathbf{G}^F}(\mathbf{M})|} (c_\pi(\mathbf{u}\sigma^{\mathbf{M}^F}_{\pi\cup\{p\}}), {}^*R^{\mathbf{G}}_{\mathbf{M}}(\gamma))_{\mathbf{M}^F}.$$

On the other hand, again using (1.5), we also have:

$$\begin{aligned}(\gamma, D(c_\pi(\mathbf{u}\sigma^{\mathbf{G}^F}_{\pi\cup\{p\}})))_{\mathbf{G}^F} &= \sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{G})]_{\mathbf{G}^F}} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} \frac{|\mathbf{G}^F : \mathbf{T}^F|_{\pi'\setminus\{p\}}}{|W_{\mathbf{G}^F}(\mathbf{T})|}(\gamma, R^{\mathbf{G}}_{\mathbf{T}}(1))_{\mathbf{G}^F} \\ &= \sum_{[\mathbf{T}\in\mathcal{T}_\pi(\mathbf{G})]_{\mathbf{G}^F}} \frac{\mathrm{Deg} R^{\mathbf{G}}_{\mathbf{T}}(1)}{|W_{\mathbf{G}^F}(\mathbf{T})|} \frac{(\gamma, R^{\mathbf{G}}_{\mathbf{T}}(1))_{\mathbf{G}^F}}{|\mathbf{G}^F_{ss}|_\pi} \\ &= \frac{\mathrm{Deg} c_\pi(\gamma)}{|\mathbf{G}^F_{ss}|_\pi}.\end{aligned}$$

Applying this last formula for $\mathbf{M}$, we obtain

$$(\gamma, D(\mathbf{u}\sigma^{\mathbf{G}^F}_{\pi\cup\{p\}}))_{\mathbf{G}^F} = \sum_{[\mathbf{M}\ \pi\text{-split}]_{\mathbf{G}^F}} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}} \frac{|\mathbf{G}^F : \mathbf{M}^F|_{\pi'}}{|W_{\mathbf{G}^F}(\mathbf{M})|} \frac{\mathrm{Deg}(c_\pi({}^*R^{\mathbf{G}}_{\mathbf{M}}(\gamma)))}{|\mathbf{M}^F_{ss}|_\pi}$$

and the lemma follows easily. □

This lemma allows us to prove the proposition, by induction on $|\mathbf{G}^F : L^F|$. Indeed, the left-hand side of Lemma 2.11 is an integer, since $\sigma^{\mathbf{G}^F}_{\pi\cup\{p\}}$, and hence the dual of the unipotent projection $D(\mathbf{u}\sigma^{\mathbf{G}^F}_{\pi\cup\{p\}})$, is a virtual character. If $\mathbf{L} = \mathbf{G}$, the sum of the right-hand side of Lemma 2.11 only contains terms equal to $\deg c_\pi(\gamma)/|\mathbf{G}^F_{ss}|_\pi$, whence the result holds in this case. In the general case, the right-hand side includes only one term where $\mathbf{M} = \mathbf{G}$, equal to $\deg c_\pi(\gamma)/|\mathbf{G}^F_{ss}|_\pi$ and, since by the induction hypothesis $\deg(c_\pi(\mu))/|\mathbf{M}^F_{ss}|_\pi$ is an integer, all of the other terms are π-numbers. Indeed, by [6, 3.3(2)(a)], $(R^{\mathbf{G}}_{\mathbf{M}}(\mu), \gamma)_{\mathbf{G}^F}$ divides $|W_{\mathbf{G}^F}(\mathbf{M}, \mu)|$ and the latter number is prime to π by (Hπ4). We deduce that the term for $\mathbf{M} = \mathbf{G}$ is also a π-number, yielding the proposition. □

□

3. The Isotypies

Let ℓ be a prime that is excellent for $(\mathbf{G}, F)$. Let $e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F}$ be an ℓ-block of $\mathbf{G}^F$ with abelian defect group D (see Theorem 2.1 for the properties and notation used here), where $\mathbf{L} = C^\circ_{\mathbf{G}}(D)$, and where λ is the canonical character of $C_{\mathbf{G}^F}(D)$ in the block corresponding to $e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F}$ under the Brauer correspondence. Recall the following properties (see Theorems 2.1 and 2.8; see also [8, 4.2]).

- $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$, and $\mathbf{L}^F = C_{\mathbf{G}^F}(D)$. Moreover, there exists d such that $\mathbf{L}$ is d-split and $(\mathbf{L}, \lambda)$ is a d-cuspidal pair, and ℓ is $Z^\circ(\mathbf{L})/Z^\circ(\mathbf{G})$-adapted.
- $\mathrm{Irr}(e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F}, \mathbf{G}^F)$ is the set of constituents of the characters $R_{\mathbf{L}}^{\mathbf{G}}(\theta\lambda)$, where $\theta \in \mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$.

We fix once and for all an F-stable maximal torus $\mathbf{T}$ of $\mathbf{L}$. If θ is a linear character of $\mathbf{L}^F$, we set (see Section 1 above)

$$\mathbf{G}(\theta) := \mathbf{G}(\mathbf{T}, \theta|_{\mathbf{T}^F}) \qquad \text{and} \qquad \mathbf{G}^F(\theta) := \mathbf{G}(\theta)^F.$$

By Proposition 1.18(2), we know that the set of irreducible characters of the group $Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ is therefore

$$\mathrm{Irr}(Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \lambda) = \{\mathrm{Ind}_{Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F(\theta)}(\mathbf{L},\lambda)}^{Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\lambda)}(\theta \cdot \tau)\},$$

where θ runs through a system of representatives of the orbits of $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ on $\mathrm{Irr}(Z^\circ(\mathbf{L})_\ell^F)$, and where θ is identified with its extension to $\mathbf{T}^F$.

Theorem.

(1) *For every subgroup S of $Z^\circ(\mathbf{L})_\ell^F$, the group $\mathbf{M} = C^\circ_{\mathbf{G}}(S)$ is a d-split Levi subgroup of $\mathbf{G}$. We have $\mathbf{M}^F = C_{\mathbf{G}^F}(S)$, and the Brauer correspondent $\mathrm{Br}_S(e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F})$ of $e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F}$ is given by the formula*

$$\mathrm{Br}_S(e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F}) = \sum_{(\mathbf{L}',\lambda')} e_{\ell,(\mathbf{L}',\lambda')}^{\mathbf{M}^F}$$

where $(\mathbf{L}', \lambda')$ ranges over a set of representatives of the $\mathbf{M}^F$-conjugacy classes of d-cuspidal pairs of $\mathbf{M}^F$ that are $\mathbf{G}^F$-conjugate to $(\mathbf{L}, \lambda)$.

(2) *Applying*

$$\mathbf{I}_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F} : \mathbb{Z}\mathrm{Irr}\left(Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \lambda)\right) \to \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F, e_{\ell,(\mathbf{L}',\lambda')}^{\mathbf{G}^F})$$

given by

$$\mathrm{Ind}_{Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F(\theta)}(\mathbf{L},\lambda)}^{Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\lambda)}(\theta \cdot \tau) \mapsto R_{\mathbf{G}(\theta)}^{\mathbf{G}}(\theta \cdot I_{(\mathbf{L},\lambda)}^{\mathbf{G}(\theta)}(\tau))$$

yields an isotypy between $(Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \lambda), 1)$ and $(\mathbf{G}^F, e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F})$.

Proof. Assertion (1) is proved in [6, 5.8] under more restrictive hypotheses on ℓ. These hypotheses ensure that $\mathbf{M}^F = C_{\mathbf{G}^F}(S)$, and that $e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F}$ is an ℓ-idempotent. However, in our case the first property is assured by Proposition 1.18(1), and the second is assured by the properties mentioned before the statement of the theorem.

Similarly, for the proof of assertion (2), we can follow some of the details of the proof in [6], whose steps we outline.

Recall that, for an ℓ-element x of $\mathbf{G}^F$, we set $\mathbf{G}(x) := C^\circ_{\mathbf{G}}(x)$ and $\mathbf{G}^F(x) := \mathbf{G}(x)^F$.

We first show that applying $\mathbf{I}_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F}$ commutes with applying the decomposition:

(a) $$\operatorname{dec}_{\ell,(\mathbf{L},\lambda)}^{x,\mathbf{G}^F} \cdot \mathbf{I}_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F} = \mathbf{I}_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F(x)} \cdot \operatorname{dec}_\ell^{x,Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\lambda)},$$

where

- For a class function ψ on $\mathbf{G}^F$ the class function $\operatorname{dec}_{\ell,(\mathbf{L},\lambda)}^{x,\mathbf{G}^F}(\psi)$ on $\mathbf{G}^F(x)$ is the function that vanishes outside the ℓ-regular elements and takes the value $\psi(xx' e_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F(x)})$ on $x' \in \mathbf{G}^F(x)_{\ell'}$.
- Similarly, for ϕ a class function on $Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\lambda)$, we write $\operatorname{dec}_\ell^{x,Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\lambda)}(\phi)$ for the class function on $Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\lambda)$ that vanishes outside the set of ℓ-regular elements and takes the value $\phi(xx')$ on $x' \in Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\lambda)_{\ell'}$.

To prove (a), using the Curtis type formula and the fundamental Theorem 3.2 of [6], we reduce to the case where x is central in $\mathbf{G}$, then, using the character formula for Deligne–Lusztig induction, to the case where $x = 1$ (see [6, 5.17] and after). We are therefore reduced to proving

(e) $$\operatorname{dec}_{\ell,(\mathbf{L},\lambda)}^{1,\mathbf{G}^F} \cdot \mathbf{I}_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F} = \mathbf{I}_{\ell,(\mathbf{L},\lambda)}^{\mathbf{G}^F} \cdot \operatorname{dec}_\ell^{1,Z^\circ(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\lambda)},$$

We proceed therefore by induction on $\dim(\mathbf{G}) - \dim(\mathbf{L})$. If $\mathbf{L} = \mathbf{G}$, then we use the following lemma (here we differ from [6], in replacing the property 'd-cuspidal' by 'of central defect').

Lemma 3.1.

(1) *Let λ be a character of central ℓ-defect in $\mathbf{L}^F$ and let $\theta \in \mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$. We have*

$$\operatorname{dec}_{\ell,(\mathbf{L},\lambda)}^{1,\mathbf{L}^F}(\theta\cdot\lambda) = \frac{1}{|Z^\circ(\mathbf{L})_\ell^F|}\mathrm{Ab}_\ell\mathrm{Reg}^{\mathbf{L}^F}\cdot\lambda = \frac{1}{|Z^\circ(\mathbf{L})_\ell^F|}\sum_{\eta\in\mathrm{Irr}(Z^\circ(\mathbf{L})_\ell^F)}\eta\lambda.$$

(2) *For $\theta \in \mathrm{Irr}(Z^\circ(\mathbf{L})_\ell^F)$, we have*

$$\operatorname{dec}_\ell^{1,Z^\circ(\mathbf{L})_\ell^F(\theta)} = \frac{1}{|Z^\circ(\mathbf{L})_\ell^F|}\mathrm{Reg}^{Z^\circ(\mathbf{L})_\ell^F} = \frac{1}{|Z^\circ(\mathbf{L})_\ell^F|}\sum_{\eta\in\mathrm{Irr}(Z^\circ(\mathbf{L})_\ell^F)}\eta.$$

Contrary to the situation in [6], the two assertions are evident (the first reflects the fact that a character of central ℓ-defect vanishes on the elements whose ℓ-part is central).

The induction hypothesis then initially gives us property (a) for all elements $x \in Z^\circ(\mathbf{L})^F_\ell$ that are not central in $\mathbf{G}$. Using the fact that any class function in the block b is a sum of its decomposition for the various $x \in Z^\circ(\mathbf{L})^F_\ell$, we deduce (a) for the remaining elements (see [6, 5.20]).

To demonstrate assertion (2) of the theorem, it remains to show that $\mathbf{I}^{\mathbf{G}^F}_{\ell,(\mathbf{L},\lambda)}$ realizes a perfect isometry. We proceed again by induction on $\dim(\mathbf{G})-\dim(\mathbf{L})$. We do not reproduce the details of the proof of [6], but the reader can easily verify that the only argument used there that needs to be changed is [6, Lemma 5.21], which must be replaced by the following lemma.

Lemma 3.2. *Let γ be a character of central ℓ-defect of $\mathbf{G}^F$. For all $g \in \mathbf{G}^F$, $|C_{\mathbf{G}^F}(g) : Z^\circ(\mathbf{G})^F_\ell|$ divides $\gamma(g)$.*

Proof of 3.2. If we replace $Z^\circ(\mathbf{G})^F$ in the statement by $Z(\mathbf{G})^F = Z(\mathbf{G}^F)$ (this is allowed since it was assumed ℓ is prime to the order of $Z(\mathbf{G})^F/Z^\circ(\mathbf{G})^F$), this is a well-known property of characters of central defect. □

□

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