ON DEVANEY'S DEFINITION OF CHAOS AND DENSE PERIODIC POINTS

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Abstract. We look again at density of periodic points and Devaney Chaos. We prove that if f is Devaney Chaotic on a compact metric space with no isolated points, then the set of points with prime period at least n is dense for each n. Conversely we show that if f is a continuous function from a closed interval to itself, then there is a decomposition of the interval into closed subintervals on which either f or f^2 are Devaney Chaotic.

There are a number of different definitions of what it means for a function, f, from a compact metric space, X, to itself, to be *chaotic* [1]. One of the most frequently discussed is Devaney's [2], which isolates three essential characteristics of a chaotic function. A function is said to be Devaney Chaotic provided:

- it is *(topologically) transitive*, that is for any two open non-empty sets U and V, there is some $n \in \mathbb{N}$ such that $f^n(U) \cap V$ is non-empty;
- has a *dense set of periodic points*, i.e. that every open set contains a periodic point;
- and is *sensitive to initial conditions* (or has sensitive dependence on initial conditions), so that there is a sensitivity constant $\delta > 0$ such that for every $x \in X$ and every $\epsilon > 0$, there is some point y and some $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ but $d(f^n(x), f^(y)) \geq \delta$.

Banks *et al* [3], prove that the sensitivity condition is unnecessary; if $f : X \to X$ is a transitive continuous map on a compact metric space X that has a dense set of periodic points, then f has sensitive dependence on initial conditions. Assif and Gadbois [4] show that, for general X, neither transitivity nor dense periodicity are implied by the other two condiions. However, Vellekoop and Berglund [5] prove that, for a continuous map of a compact interval, transitivity implies a dense set of periodic points, so that transitive maps of compact intervals are Devaney Chaotic. Silverman [6] proves equivalent results, at least in the case of compact metric spaces without isolated points where transitivity is equivalent to the existence of a dense orbit.

As simple examples, we note that a rational rotation of the circle has a dense set of a dense set of periodic points but is not transitive and an irrational rotation has no periodic points but is transitive. Neither of these functions are sensitive to initial conditions. It turns out, however, that a notion somewhat stronger than dense periodic points does imply Devaney Chaos in some situations. Given $f : X \to X$, for each $n \in \mathbb{N}$ we define

$$P_n(f) = \{x : x \text{ is a periodic point of } f \text{ with prime period } m \geq n\}.$$

We write P_n when f is clear from the context.

Theorem 1. *Let X be a compact metric space without isolated points. If $f : X \to X$ is a continuous, Devaney Chaotic function, then for all $n \in \mathbb{N}$,*

$$P_n = \{x : x \text{ is periodic with prime period } m \geq n\}$$

is dense in X.

Proof. Suppose that, for some n, P_n is not dense. Let U be the (non-empty) complement of the closure of P_n. Since X has no isolated points, U is infinite (in fact it has the cardinality of the continuum). Since the periodic points are dense, every non-empty open subset of U must contain periodic points, but, by the definition of P_n, they all have periods strictly less than n. By the result of Silverman mentioned above, there is a point z with a dense, hence infinite, orbit. It follows that there is some fragment of the orbit of z of length n starting with $z_0 = f^k(z)$ in U and ending with $f^{k+n-1}(z)$ (it need not stay in U). Let ϵ be the minimum of the distances $d(z_i, z_j)$, $0 \leq i \neq j < n$. Then the open balls $B_i = B_{\epsilon/3}(z_i)$, $i < n$, are pairwise disjoint. Let $V_{n-1} = B_{n-1}$, and given V_{n-j}, define $V_{n-j-1} = B_{n-j-1} \cap f^{-1}(V_{n-j})$, for each $j < n$. Then each V_i is open, contains z_i and $f(V_i)$ is a subset of V_{i+1}, which is a subset of B_{i+1}. But then $V_1 \cap U$ is non-empty (it contains z_0), so there is some periodic point $y \in V_1 \cap U$. Since $y \in U$, y has period no more than $n - 1$, but $y \in V_1$, so $f^i(y) \in V_{1+i}$, for each $i \leq n - 1$, so that period of y is at least n, which is a contradiction. □

The following example shows that strengthening the property of having a dense set of periodic points to P_n being dense for all $n \in \mathbb{N}$ still does not imply transitivity or sensitivity in general.

Example 2. Let $D = \{re^{i\theta} \in \mathbb{C} : r \in [0, 1], \theta \in [0, 2\pi)\}$ be the closed unit disk in the complex plane. Define $f : D \to D$ by $f(re^{i\theta}) = re^{i(\theta+2r\pi)}$. Then f is a homeomorphism of D and the restriction f_r of f to $C_r = \{z : |z| = r\}$ is a rotation of order r. If r is irrational, then each point of C_r has infinite orbit and f_r is transitive. If $r = p/q$ is rational, with p and q in lowest terms, then every point in C_r has period q. Since the set of rationals with denominator at least n is dense in $[0, 1]$, $P_n(f)$ is dense for all n. Since $d(f^n(0), f^n(z)) = |z|$ for any z and any n, f is not sensitive to initial conditions (points cannot move away from 0). It is also clear that f is not transitive; for example, for no n does $f^n(B_{1/8}(1/4)$ meet $B_{1/8}(3/4)$. If g is the restriction of f to $\{z : 1/2 \leq |z| \leq 1\}$, then $P_n(g)$ is dense for all n and g is sensitive to initial conditions but not transitive.

It turns out, however, that if f is a map of the interval and P_3 is dense, then the interval can be written as a union of countably many closed nondegenerate subintervals, which intersect only at their end points, on each of which either f or f^2 is Devaney Chaotic.

Barge and Martin [7] prove related results describing the structure of a continuous function, f, of a compact interval, I, for which there us a dense set of periodic points. The prove that for such an f there is a (possibly finite or empty) collection of closed subintervals $\{J_1, J_2, \cdots\}$ meeting only at their end points such that: for each i, $f^2(J_i) = J_i$; for each i, there is $x \in J_i$ such that $\{f^{4n}(x) : n \geq 0\}$ is dense in J_i; for each $x \in I - \bigcup_i J_i$, $f^2(x) = x$. Their results, however, make use of the structure of the inverse limit space; here our proofs are entirely elementary.

Our proof is based on the following simple observation, which we extend in Lemmas 3, 4, and 5. Suppose that $f : [0, 1] \to [0, 1]$ has a dense set of periodic points. Let $0 < a < b < 1$ and let $I = [0, a]$, $J = [a, b]$ and $K = [b, 1]$ and suppose that the interval J is invariant, i.e. $f(J) = J$, see Figure 1. The graph of f must be contained in the boxes $(I \times I) \cup (J \times J) \cup (K \times K)$ or in the boxes $(I \times K) \cup (J \times J) \cup (K \times I)$. For example, if there is point of the graph in (the interior of) $I \times J$, then some open subinterval I' of I is mapped into J, which is invariant, so points of I' map into J and stay there, i.e. I' cannot contain any periodic points.

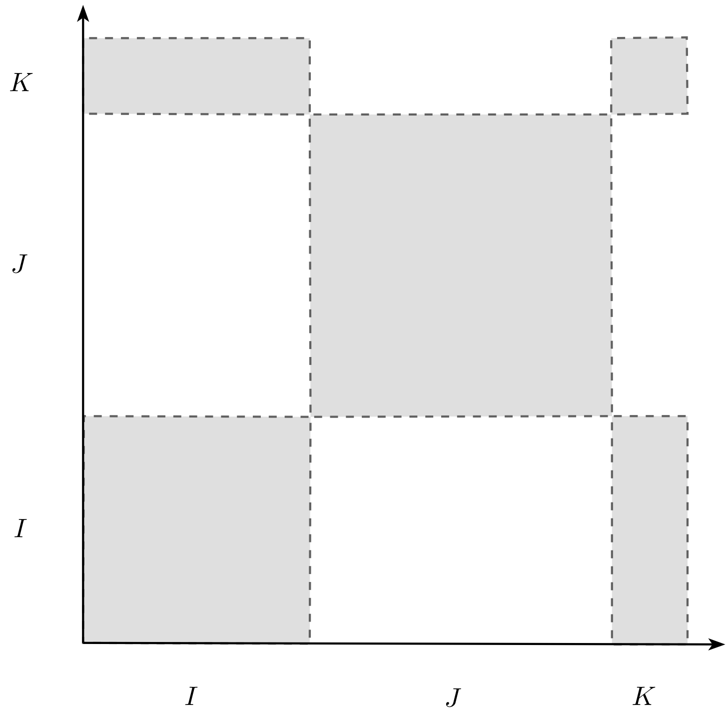

FIGURE 1. The basic idea

Recall that any union of intervals can be written as a countable collection of disjoint intervals (by joining together intervals that overlap). Note that if the periodic points are dense, the continuous image of any interval is a non-degenerate interval, i.e. is not a single point. In general, any function on compact metric space with a dense set of periodic points is a surjection as the image $f(X)$ is compact, hence closed and contains the dense set of periodic points).

Lemma 3. *Let $f : [0, 1] \to [0, 1]$ be a continuous interval map with a dense set of periodic points. Suppose that $0 \le a < b \le c < d \le 1$ and that f permutes the intervals $[a, b]$ and $[c, d]$. Then $f(b) = c$, $f(c) = b$ and f fixes $[b, c]$.*

Moreover, $0 < a$ if and only if $d < 1$, in which case $f(a) = d$, $f(d) = a$ and f permutes the intervals $[0, a]$ and $[d, 1]$.

Proof. Since $[a, b]$ and $[c, d]$ are permuted by f, if x is a point that is mapped from the complement of $[a, b] \cup [c, d]$ into $[a, b] \cup [c, d]$, then x cannot be periodic. So, if any point of (b, c) were mapped below b or above c, then some subinterval of $[b, c]$ would be mapped into $[a, b] \cup [c, d]$. Since the set of periodic points is dense, this is impossible and f must map $[b, c]$ to itself. (If $b = c$, then $f(b) = b$, so $[b, c]$ is still fixed.) Since $f(b)$ must lie both in the image of $[b, c]$ and in the image $[c, d] = f([a, b])$, so $f(b) = c$. Similarly $f(c) = b$.

If $0 < a$, then (arguing as above) $f(a)$ is either c or d, but if $f(a) = c$, then some subinterval J of $[0, a]$ maps to $[b, c]$, which maps to itself, or $[a, b]$, which permutes with $[c, d]$. This would imply that J has no periodic points. Hence $f(a) = d$. Since $[0, a]$ contains periodic points and f maps $[a, d]$ to itself, we see that $d < 1$ and f permutes $[0, a]$ and $[d, 1]$ with $f(d) = a$. □

Our main theorem partitions the interval into subintervals that are invariant under f or f^2 via the following two lemmas.

Lemma 4. *Let $f : [0, 1] \to [0, 1]$ be a continuous function with a dense set of periodic points. If f has a proper invariant subinterval, $[a, b]$, then either:*

(1) *$0 = a < b < 1$, $f(b) = b$ and $[b, 1]$ is also invariant; or*
(2) *$0 < a < b = 1$, $f(a) = a$ and $[0, a]$ is also invariant; or*
(3) *$0 < a < b < 1$, $f(a) = a$, $f(b) = b$ and the intervals $[0, a]$ and $[b, 1]$ are also invariant under f; or*
(4) *$0 < a < b < 1$, $f(a) = b$, $f(b) = a$ and f permutes the intervals $[0, a]$ and $[b, 1]$, which are also invariant under f^2.*

Proof. Suppose that $[a, b]$ is a proper invariant subinterval. One of $0 = a < b < 1$, $0 < a < b = 1$ or $0 < a < b < 1$ must hold. The arguments for cases (1) and (2) are is similar to that for (3), so we suppose that $0 < a < b < 1$. If $a < f(a) < b$, then by continuity, there is some interval $(a - \epsilon, a)$ whose image is a subset of (a, b). But since f fixes $[a, b]$, no point of $(a - \epsilon, a)$ can be periodic, which is a contradiction. Hence $f(a)$ is either a or b. Similarly $f(b)$ is either a or b.

Suppose that $f(a) = a$. Then f must fix $[0, a]$, otherwise some subinterval of $[0, a]$ must map into $[a, b]$, contradicting the fact that $[0, a]$ has a dense set of periodic points. But now if $f(b) = a$, then some subinterval of $[b, 1]$ maps either in to $[0, a]$ or into $[a, b]$, neither of which are possible, so $f(b) = b$ and f fixes $[b, 1]$. This is case (3). On the other hand, case (4) follows if $f(a) = b$, since then f must map $[0, a]$ to the interval $[b, 1]$ (else some subinterval of $[0, a]$ maps into $[a, b]$). But as $[0, a]$ contains periodic points, f must map $[b, 1]$ back to $[0, a]$, and we conclude as well that $f(b) = a$. □

Lemma 5. *Let $f : [0, 1] \to [0, 1]$ be a continuous function with a dense set of periodic points. If f has no proper invariant subinterval, then either:*

(1) *f is transitive; or*
(2) *f permutes two subintervals $[0, a]$ and $[a, 1]$ for some $0 < a < 1$.*

Proof. Suppose now that f has no proper invariant subinterval. Certainly it has no interval of fixed points. Suppose that f is not transitive. Then there are two non-empty open subintervals U and V of $[0, 1]$ such that $U' = \bigcup_{0 \leq i} f^i(U)$ and V are disjoint. U' can be written a union of at most countably disjoint, non-degenerate intervals. If one of these intervals is, say, $J = \bigcup_{i\in N} f^i(U)$, for some subset N of $\mathbb{N}$, then its image under f, $f(J) = \bigcup_{i\in N} f^{i+1}(U)$, is a subset of another of these intervals. But since J must contain a periodic point of period k say, $f^k(J)$ must be a subset of J. Moreover, $f^k(J)$ must be dense in J, since otherwise $J \setminus f^k(J)$ contains some open subinterval that contains no periodic points. We may, therefore, express the closure of U' as a finite union of closed, non-degenerate, intervals $I_i = [r_i, s_i]$, with $r_i < s_i \leq r_{i+1}$, $1 \leq i \leq n$, that are permuted by f. Since U is not equal to I and f has no proper invariant subintervals, $n > 1$.

Consider the least k such that $f(I_k) = I_j$ and $j < k$. Clearly $k > 1$ and $f(I_{k-1}) = I_m$ for some $m > k-1$. We consider three cases: (a) $j = k-1$ and $k = m$; (b) $j \leq k-1$ and $m > k$; (c) $j < k$ and $m \geq k$. We claim that the second two cases are impossible. The arguments are similar in both cases so we consider case (b). In this case, we have that $f(s_{k-1}) \in [r_m, s_m] = I_m$, so that $f(s_{k-1}) \geq s_k$ and, similarly, $f(r_k) \leq s_{k-1}$. This implies that the image of the interval $L = [s_{k-1}, r_k]$ under f contains the interval I_k. But no point of I_k is mapped into L, so some subinterval of L does not contain any periodic points, which is a contradiction. So the only possibility is case (a). Since f permutes the intervals I_i, $i \leq n$, (a) implies that $n = 2$. By Lemma 3, we have that f permutes $[0, r_1]$ and $[s_2, 1]$, but then f permutes $[0, s_1]$ and $[r_2, 1]$ and fixes $[s_1, r_2]$, which implies that $s_1 = r_2$ as f has no proper invariant subintervals. $\square$

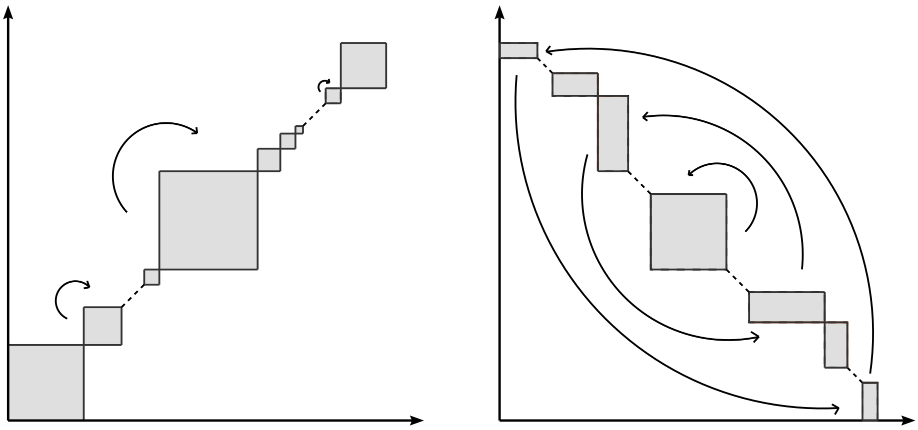

FIGURE 2. Schematic of Theorem 6: the graph of f is contained inside boxes; Case (1) on the left, f is Devaney chaotic on each subinterval; Case (2) on the right, the intervals are permuted by f about the central invariant interval A, f^2 is Devaney Chaotic on each interval.

Our main theorem says that if $f : [0,1] \to [0,1]$ is a continuous function and that P_n is dense for all n, then there is a collection collection of closed intervals whose union is dense in $[0,1]$ on which either f or f^2 is Devaney Chaotic. More precisely we prove the following:

Theorem 6. *Suppose that $f : [0,1] \to [0,1]$ is a continuous function and that $P_3 = \{x : x$ is periodic with prime period $m \geq 3\}$ is dense. One of the two following conditions must hold.*

(1) *There is a finite or countably infinite collection, $\mathcal{I}$, of closed non-degenerate subintervals of $[0,1]$ such that:*
   (a) *the union of all intervals in $\mathcal{I}$, $\bigcup_{I\in\mathcal{I}} I$, is dense in $[0,1]$;*
   (b) *the intervals in $\mathcal{I}$ meet only at their end points;*
   (c) *each interval in $\mathcal{I}$ is invariant under f and its end points are fixed, except possibly if the end point is 0 or 1;*
   (d) *f is Devaney Chaotic on each interval in $\mathcal{I}$.*

(2) *There is a finite or countably infinite collection, $\mathcal{I}$, of closed non-degenerate subintervals of $[0,1]$, and a 'central' interval A, which which might be degenerate, such that:*
   (a) *f^2 and $\mathcal{J}$ satisfy conditions (1)(a) to (1)(d) above, where $\mathcal{J}$ is $\mathcal{I}$, if A is degenerate, or $\mathcal{I} \cup \{A\}$, if A is non-degenerate;*
   (b) *f fixes A (whether it is degenerate or not) and acts as an order-reversing permutation of the intervals in $\mathcal{I}$, transposing them in pairs about A.*

Proof. We first observe the following. If $x \notin P_n(f^k)$, then $f^{km}(x) = x$ for some $m < n$, so that $x \notin P_{nk}(f)$. Hence $P_{nk}(f)$ is a subset of $P_n(f^k)$. It follows in particular that if $P_n(f)$ is dense for all n, then $P_n(f^k)$ is dense for all n. $P_3(f)$ is the union of $P_4(f)$, which is a subset of $P_2(f^2)$, and the set of points of period 3 under f. But if x has period 3 under f, then it has period 3 under f^2. It follows that $P_3(f)$ is a subset of $P_2(f^2)$, so that if $P_3(f)$ is dense, then so is $P_2(f^2)$. and both f and f^2 have a dense set of periodic points that are not fixed.

Suppose that f has a proper invariant subinterval $[a,b]$. By Lemma 4, either: (1) both $[0,b]$ and $[b,1]$ are invariant; or (2) $[0,a]$ and $[a,1]$ are invariant; or (3) $[0,a]$, $[a,b]$ and $[b,1]$ are invariant; or (4) $[a,b]$ is invariant $[a,b]$ and f permutes $[0,a]$ and $[b,1]$. In case (4), note that $[0,a]$, $[a,b]$ and $[b,1]$ are invariant under f^2, so that case (4) reduces to case (3) for f^2 and we need only consider the first three cases.

Note that in each of cases (1), (2) and (3), the end points of each interval are fixed (except possibly in the case it is 0 or 1), so that by Lemma 3 f does not permute any subintervals of $[0,1]$ in pairs. We may therefore consider the restriction of f to which ever of the intervals $[0,a]$, $[a,b]$ and $[b,1]$ are non-degenerate. If every invariant subinterval of a given interval had a proper invariant subinterval, then the end points (except possibly 0 and 1) of these intervals would be fixed points and form a dense subset, contradicting the fact that non-fixed periodic points are dense. It follows that there is a collection, $\mathcal{I}$, of closed non-degenerate minimal proper invariant subsets whose union is dense in $[0,1]$. Since f does not permute any intervals in pairs, Lemma 5 implies that f is transitive and hence Devaney Chaotic on each $I \in \mathcal{I}$, by the result of Vellekoop and Berglund mentioned above. We note that case (4) might give rise to a central invariant interval with no proper invariant subinterval or a central fixed point.

Suppose now that f has no proper invariant subinterval. By Lemma 5, either f is transitive and we are done setting $F = \{0,1\}$ and $\mathcal{I} = \{[0,1]\}$ or f permutes the intervals $[0,a]$ and $[a,1]$, for some $a \in (0,1)$, in which case $[0,a]$ and $[a,1]$ are invariant under f^2 and we may argue as above. In this case the central invariant interval is the degenerate interval $[a,a]=\{a\}$. $\square$

An example of a function on $[0,1]$ with P_3 dense and 0 not a fixed point is the tent map core

$$f(x)=\begin{cases} \frac{4}{3}x+\frac{2}{3}, & \text{if } x\in\left[0,\frac{1}{4}\right],\\ -\frac{4}{3}x+\frac{4}{3}, & \text{if } x\in\left(\frac{1}{4},1\right]. \end{cases}$$

An example of a function $[0,1]$ with P_3 dense and an invariant central interval that has no proper invariant subinterval is

$$g(x)=\begin{cases} -3x+1, & \text{if } x\in\left[0,\frac{1}{3}\right],\\ 3x-1, & \text{if } x\in\left[\frac{1}{3},\frac{2}{3}\right],\\ -3x+3, & \text{if } x\in\left[\frac{2}{3},1\right]. \end{cases}$$

To see that these functions do indeed have P_3 dense, note that the modulus of the gradient is strictly greater than 1 in each case, so that f and g are transitive on $[0,1]$, hence Devaney Chaotic, so that Theorem 1 applies.

The interiors of the intervals in $\mathcal{I}$ form a pairwise disjoint collection of open intervals whose union is dense. The complement, F, of their union is a closed nowhere dense subset of $[0,1]$. F is therefore finite, countable or cardinality of the continuum. In this last case, F must contain a copy of the Cantor set. It is clear that any closed nowhere dense subset F of $[0,1]$ that contains 0 and 1 can be realized as such a set by defining the function h as follows. For each $x\in F$, $h(x)=x$. If (a,b) is a maximal open subinterval (a,b) of $[0,1]-F$, define

$$h(x)=\begin{cases} 3x-2a, & \text{if } x\in\left(a,\frac{2}{3}a+\frac{1}{3}b\right],\\ -3x+2(a+b), & \text{if } x\in\left(\frac{2}{3}a+\frac{1}{3}b,\frac{1}{3}a+\frac{2}{3}b\right],\\ 3x-2b, & \text{if } x\in\left(\frac{1}{3}a+\frac{2}{3}b,b\right), \end{cases}$$

so that h fixes a and b and is transitive on $[a,b]$ which is invariant.

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