"The Lindelöf Property"

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A topological space is said to be **Lindelöf**, or have the Lindelöf property, if every *open cover* of X has a countable *subcover*. The Lindelöf property was introduced by Alexandroff and Urysohn in 1929, the term 'Lindelöf' referring back to Lindelöf's result that any family of open subsets of Euclidean space has a countable sub-family with the same union. Clearly, a space is *compact* if and only if it is both Lindelöf and *countably compact*, though weaker properties, for example *pseudocompactness*, imply compactness in the presence of the Lindelöf property. The real line is a Lindelöf space that is not compact and the space of all countable ordinals ω_1 with the order topology is a countably compact space that is not Lindelöf. It should be noted that some authors require the *Hausdorff* or *regular* (which we take to include T_1) separation axioms as part of the definition of many open covering properties (c.f. [E]). For any unreferenced results in this article we refer the reader to [E].

There are a number of equivalent formulations of the Lindelöf property: (a) the space X is Lindelöf; (b) X is $[\omega_1, \infty]$-*compact* (see the article by Vaughan in this volume); (c) every open cover has a countable *refinement*; (d) every family of closed subspaces with the **countable intersection property**[1] has non-empty intersection; (e) (for regular spaces) every open cover of X has a countable subcover $\mathcal{V}$ such that $\{\overline{V} : V \in \mathcal{V}\}$ covers X (where $\overline{A}$ denotes the closure of A in X). In the class of *locally compact* spaces, a space is Lindelöf if and only if it is **σ-compact** (i.e., is a countable union of compact spaces) if and only if it can be written as an increasing union of countably many open sets each of which has compact closure.

It is an important result that regular Lindelöf spaces are *paracompact*, from which it follows that they are (*collectionwise*) *normal*. Conversely, every paracompact space with a *dense* Lindelöf subspace is Lindelöf (in particular, every separable paracompact space is Lindelöf) and every locally compact, paracompact space is a disjoint sum of *clopen* Lindelöf subspaces. A related result is that any *locally finite* family of subsets of a Lindelöf space is countable.

Closed subspaces and countable unions of Lindelöf spaces are Lindelöf. Continuous images of Lindelöf spaces are Lindelöf and inverse images of Lindelöf spaces under *perfect mappings*, or even *closed mappings* with Lindelöf fibres, are again Lindelöf. In general, the Lindelöf property is badly behaved on taking either *(Tychonoff) products* or *inverse limits*.

The *Tychonoff product* of two Lindelöf spaces need not be Lindelöf or even normal, although any product of a Lindelöf space and a compact space is Lindelöf and countable products of Lindelöf *scattered* spaces are Lindelöf [HvM, Chapter 18, Theorem 9.33]. It is also true that both the class of *Čech complete* Lindelöf and Lindelöf Σ-spaces are closed under countable products. The **Sorgenfrey line**, which one obtains from the real line by declaring every interval of the form $(a, b]$ to be open, is a simple example of a Lindelöf space with non-normal square. Even more pathological examples are possible: Michael constructs a Lindelöf space, similar to the Michael line, which has non-normal product with a subset of the real line and, assuming the Continuum Hypothesis,

[1] A family of sets has the *countable intersection property* if every countable sub-family has non-empty intersection.

constructs a Lindelöf space whose product with the irrationals is non-normal. Details and further results may be found in [KV, Chapter 18] and Section 9 of [HvM, Chapter 18]. A space is said to be *realcompact* if it is homeomorphic to a closed subspace of the Tychonoff product $\mathbb{R}^\kappa$ for some κ. Every regular Lindelöf space is realcompact and, whilst the *inverse limit* of a sequence of Lindelöf spaces need not be normal, both inverse limits and arbitrary products of realcompact spaces are realcompact. Hence arbitrary products and inverse limits of regular Lindelöf spaces are realcompact. In fact a space is realcompact if and only if it is the inverse limit of a family of regular Lindelöf spaces.

Second countable spaces (i.e., spaces with a countable base to the topology) are both Lindelöf and *separable*. The Sorgenfrey line is an example of a separable, Lindelöf space that is not second countable. On the other hand, if X is *metrizable* (or even *pseudometrizable*), then X is second countable if and only if it is separable if and only if it has the *countable chain condition* if and only if it is Lindelöf. By *Urysohn's Metrization Theorem*, a space is second countable and regular if and only if it is a Lindelöf metrizable space if and only if it can be embedded as subspace of the *Hilbert cube*.

A space X is said to be **hereditarily Lindelöf** if every subspace of X is Lindelöf. Since any space can be embedded as a dense subspace of a (not necessarily Hausdorff) compact space, not every Lindelöf space is hereditarily Lindelöf. However, a space is hereditarily Lindelöf if and only if every open subspace is Lindelöf if and only if every uncountable subspace Y of X contains a point y whose every neighbourhood contains uncountably many points of Y. A regular Lindelöf space is hereditarily Lindelöf if and only of it is *perfect* and hereditarily Lindelöf spaces have the *countable chain condition* but need not be separable.

In fact, for regular spaces there is a complex and subtle relationship between the hereditary Lindelöf property and **hereditary separability**[2] (both of which follow from second countability). An hereditarily Lindelöf regular space that is not (hereditarily) separable is called an L-space; an hereditarily separable regular space that is not (hereditarily) Lindelöf is called an S-space. The existence of S- and L-spaces is, to a certain extent, dual and depends strongly on the model of set theory. For example, the existence of a *Souslin line* implies the existence of both S- and L-spaces, $\mathsf{MA} + \neg\mathsf{CH}$ is consistent with the existence of S- and L-spaces but implies that neither compact S- nor compact L-spaces exist. However, the duality is not total: Todorčević [11] has shown that it is consistent with MA that there are no S-spaces but that there exists an L-space, i.e., that every regular hereditarily separable space is hereditarily Lindeöf but that there is a non-separable, hereditarily Lindelöf regular space. It is currently an open question whether it is consistent that there are no L-spaces. For further details about S and L see Roitman's article [KV, Chapter 7], or indeed [11]. It is fair to say that the S/L pathology, along with Souslin's Hypothesis and the Normal Moore Space Conjecture, has been one of the key motivating questions of set-theoretic topology and it crops up frequently in relation to other problems in general topology, such as: the metrizability of *perfectly normal manifolds* [10]; Ostaszewski's construction of a countably compact, perfectly normal non-compact space [9]; and the existence of a counter-example to Katětov's problem 'if X is compact and X^2 is hereditarily normal, is X metrizable?' [5].

[2]A space is hereditarily separable if each of its subspaces is separable.

The **Lindelöf degree** or **number**, $L(X)$, of a space X is the smallest infinite cardinal κ for which every open cover has a subcover of cardinality at most κ. The **hereditary Lindelöf degree**, $hL(X)$, of X is the supremum of the cardinals $L(Y)$ ranging over subspaces Y of X. The Lindelöf degree of a space is one of a number of cardinal invariants or *cardinal functions* one might assign to a space. Cardinal functions are discussed in the article by Tamano in this volume, however, one result due to Arkhangel′skiĭ [1] is worth particular mention here. The **character** $\chi(x, X)$ of a point x in the space X is smallest cardinality of a *local base* at x and the **character** $\chi(X)$ of the space X is the supremum $\sup\{\chi(x, X) : x \in X\}$. A space with countable character is said to be **first-countable**. Arkhangel′skiĭ's result says that the cardinality of a Hausdorff space X is at most $2^{L(X)\cdot\chi(X)}$. In the countable case this theorem tells us that the cardinality of a first-countable, Lindelöf Hausdorff space is at most the continuum, $2^{\aleph_0}$, and that, in particular, the cardinality of a first-countable, compact Hausdorff space is at most the continuum.[3] This impressive result solved a problem posed thirty years earlier by Alexandroff and Urysohn (whether a first-countable compact space could have cardinality greater than that of the continuum), but was, moreover, a model for many other results in the field. The theorem does not remain true if we weaken first-countability, since it is consistent that the cardinality of a regular, (*zero-dimensional* even) Lindelöf Hausdorff space with countable *pseudo-character* can be greater than that of the continuum [12], and Lindelöf spaces can have arbitrary cardinality. However, de Groot has shown that the cardinality of a Hausdorff space X is at most $2^{hL(X)}$ [KV, Chapter 1, Cor. 4.10]. For a much more modern proof of Arkhangel′skiĭ's theorem than the ones given in [1] or [KV, Chapter 1], we refer the reader to Theorem 4.1.8 of the article by Watson in [HvM].

A space is compact if and only if every infinite subset has a *complete accumulation point* if and only if every increasing open cover has a finite subcover and a space is countably compact if and only if every countably infinite subset has a complete accumulation point. However, the requirement that every uncountable subset has a complete accumulation point is implied by, but does not characterize the Lindelöf property. Spaces satisfying this property are called **linearly Lindelöf**, since they turn out to be precisely those spaces in which every open cover that is linearly ordered by inclusion has a countable subcover. Surprisingly little is known about such spaces. There are (somewhat complex) examples of regular linearly Lindelöf, non-Lindelöf spaces in ZFC, but there is, at present, no known example of a normal linearly Lindelöf, non-Lindelöf spaces under any set theory. Such a space would be highly pathological: the problem intrinsically involves singular cardinals and any example is a **Dowker space**, that is, a normal space which has non-normal product with the closed unit interval $[0, 1]$. Nevertheless one can prove some interesting results about linearly Lindelöf spaces, for example every first-countable, linearly Lindelöf Tychonoff space has cardinality at most that of the continuum, generalizing the theorem of Arhangel'ski's result mentioned above. For more on linearly Lindelöf spaces see the paper by Arkhangel′skiĭ and Buzyakova [2].

One important sub-class of Lindelöf spaces, the **Lindelöf** Σ spaces, deserves mention. The notion of a Σ-space was introduced by Nagami [8], primarily

[3] In fact it turns out that first-countable, compact Hausdorff spaces are either countable of have cardinality exactly $2^{\aleph_0}$.

to provide a class of space in which covering properties behave well on taking products. It turns out that there are a number of characterizations of Lindelöf Σ-spaces, two of which we mention here. A Tychonoff space is Lindelöf Σ if it is the continuous image of the pre-image of a separable metric space under a perfect map. An equivalent (categorical) definition is that the class of Lindelöf Σ-spaces is the smallest class containing all compact spaces and all separable metrizable spaces that is closed under countable products, closed subspaces and continuous images. So, as mentioned above, countable products of Lindelöf Σ-spaces are Lindelöf Σ. Every σ-compact space, and hence every locally compact Lindelöf space, is a Lindelöf Σ-space. Lindelöf Σ-spaces play an important rôle in the study of function spaces (with the topology of pointwise convergence). For details, see the article on *C_p-theory* by Arkhangel′skiĭ [HvM, Chapter 1].

There are several strengthenings and weakenings of the Lindelöf property in the literature for example: *almost Lindelöf, n-starLindelöf, totally Lindelöf, strongly Lindelöf, Hurewicz, subbase Lindelöf.* We mention one in passing. A space is **weakly Lindelöf** if any open cover has a countable subfamily $\mathcal{V}$ such that $\bigcup\{V : V \in \mathcal{V}\}$ is *dense* in X. Weakly Lindelöf spaces are of some interest in Banach space theory [HvM, Chapter 16] and, assuming CH, the weakly Lindelöf subspaces of $\beta\mathbb{N}$ are precisely those which are *C^*-embedded* into $\beta\mathbb{N}$ (1.5.3 of [KV, Chapter 11]). Covering properties such as *para-* or *metaLindelöf* belong more properly to a discussion of generalizations of paracompactness.

Finally, we list a number of interesting results concerning the Axiom of Choice and the Lindelöf property. The Countable Axiom of Choice is strictly stronger than either of the statements 'Lindelöf metric spaces are second countable' or 'Lindelöf metric spaces are separable' [7]. In Zermelo-Fraenkel set theory (without choice) the following conditions are equivalent: (a) $\mathbb{N}$ is Lindelöf; (b) $\mathbb{R}$ is Lindelöf; (c) every second countable space is Lindelöf; (d) $\mathbb{R}$ is hereditarily separable; (e) $f : \mathbb{R} \to \mathbb{R}$ is continuous at x iff it is sequentially continuous at x; and (f) the axiom of countable choice holds for subsets of $\mathbb{R}$ [6]. There are models of ZF in which every Lindelöf T_1-space is compact [3] and models in which the space ω_1 is Lindelöf but not countably compact [4].

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