TILINGS IN VERTEX ORDERED GRAPHS

JÓZSEF BALOGH, LINA LI, AND ANDREW TREGLOWN

Abstract. Over recent years there has been much interest in both Turán and Ramsey properties of *vertex ordered graphs*. In this paper we initiate the study of embedding spanning structures into vertex ordered graphs. In particular, we introduce a general framework for approaching the problem of determining the minimum degree threshold for forcing a *perfect H-tiling* in an ordered graph. In the (unordered) graph setting, this problem was resolved by Kühn and Osthus [The minimum degree threshold for perfect graph packings, Combinatorica, 2009]. We use our general framework to resolve the perfect H-tiling problem for all ordered graphs H of interval chromatic number 2. Already in this restricted setting the class of extremal examples is richer than in the unordered graph problem.

1. Introduction

Over recent years there has been interest in extending classical graph theory results to the setting of *vertex ordered graphs*. A *(vertex) ordered graph* or *labelled graph H* on h vertices is a graph whose vertices have been labelled with $[h] := \{1, \dots, h\}$. An ordered graph G with vertex set $[n]$ *contains* an ordered graph H on $[h]$ if (i) there is a mapping $\phi : [h] \to [n]$ such that $\phi(i) < \phi(j)$ for all $1 \le i < j \le h$ and (ii) $\phi(i)\phi(j)$ is an edge in G whenever ij is an edge in H.

A foundation stone in extremal graph theory is Turán's theorem which determines the number of edges in the densest K_r-free graph on n vertices. Furthermore, for every graph H, the Erdős–Stone–Simonovits theorem [5, 6] determines, up to a quadratic error term, the number of edges in the densest H-free n-vertex graph. It is natural to seek Turán-type results in the setting of ordered graphs. Indeed, this question was first raised in a paper of Füredi and Hajnal [7], and there are now many results in the area; see the paper of Tardos [24] for a survey of such results (and the related problem of Turán-type results for *edge* ordered graphs). In particular, Pach and Tardos [21] proved an analogue of the Erdős–Stone–Simonovits theorem in the setting of ordered graphs. In their result they show that the so-called *interval chromatic number* governs the threshold (for graphs H of interval chromatic number at least 3), rather than the chromatic number (as is the case in the unordered setting). There are several Turán-type results for ordered graphs of interval chromatic number 2; see e.g. [9, 10, 16, 20, 21, 24], as well as Turán-type results for ordered hypergraphs (see [8]).

There have also been a number of recent results concerning Ramsey theory for ordered graphs, for example see the work of Balko, Cibulka, Král and Kynčl [2] and of Conlon, Fox, Lee and Sudakov [4].

In this paper we initiate the study of embedding *spanning* structures in ordered graphs. In particular, we study the minimum degree required to ensure an ordered graph has a *perfect H-tiling*. In both the ordered and unordered settings, an *H-tiling* in a graph G is a collection of vertex-disjoint copies of H contained in G. An H-tiling is *perfect* if it covers all the vertices of G. Perfect H-tilings are also often referred to as *H-factors*, *perfect H-packings* or *perfect H-matchings*. H-tilings can be viewed as generalisations of both the notion of a matching (which corresponds to the case when H is a single edge) and the Turán problem (i.e. a copy of H in G is simply an H-tiling of size one).

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A central result in the area is the Hajnal–Szemerédi theorem [11] from 1970, which characterises the minimum degree that ensures a graph contains a perfect K_r-tiling.

Theorem 1.1 (Hajnal and Szemerédi [11])**.** *Every graph G whose order n is divisible by r and whose minimum degree satisfies $\delta(G) \geq (1-1/r)n$ contains a perfect K_r-tiling. Moreover, there are n-vertex graphs G with $\delta(G) = (1-1/r)n - 1$ that do not contain a perfect K_r-tiling.*

There has also been significant interest in the minimum degree threshold that ensures a perfect H-tiling for an arbitrary graph H. After earlier work on this topic (see e.g. [1, 14]), Kühn and Osthus [17, 18] determined, up to an additive constant, the minimum degree that forces a perfect H-tiling for *any* fixed graph H. In particular, they showed that, depending on H, the minimum degree threshold is governed by either the chromatic number $\chi(H)$ of H or the so-called *critical chromatic number of H*.

Definition 1.2 (Critical chromatic number)**.** The *critical chromatic number* $\chi_{cr}(F)$ of an unordered graph F is defined as

$$\chi_{cr}(F) := (\chi(F)-1)\frac{|F|}{|F|-\sigma(F)},$$

where $\sigma(F)$ denotes the size of the smallest possible color class in any $\chi(F)$-coloring of F.

Theorem 1.3 (Kühn and Osthus [18])**.** *Let $\delta(H,n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G) \geq k$ contains a perfect H-tiling. For every unordered graph H,*

$$\delta(H,n) = \left(1 - \frac{1}{\chi^*(H)}\right)n + O(1),$$

where $\chi^*(H) := \chi_{cr}(H)$ if hcf$(H) = 1$[1] *and $\chi^*(H) := \chi(H)$ otherwise.*

The dichotomy in Theorem 1.3 arises as there are two types of extremal construction for this problem: so-called *space* and *divisibility barriers*.

In this paper we show that the corresponding problem for ordered graphs has a rich behaviour. Indeed, our main result resolves the problem for all ordered graphs H of interval chromatic number 2. Even in this restricted case the nature of the minimum degree threshold is diverse, with a range of extremal examples coming into play, including a construction which is neither a divisibility nor space barrier. Whilst we do not resolve the problem for all ordered graphs H, in Section 4 we introduce a framework that can be used to attack the problem in general.

1.1. Our results.

Definition 1.4 (Interval chromatic number)**.** The *interval chromatic number* $\chi_<(H)$ of an ordered graph H is the minimum number of intervals the vertex set $[h]$ of H can be partitioned into, so that no two vertices belonging to the same interval are adjacent in H.

Let $\alpha_0^+(H) := 0$. For every $1 \leq i \leq \chi_<(H)$, we let

(1) $\alpha_i^+(H) :=$ the largest $k \in \mathbb{N}$ such that $[\alpha_{i-1}^+(H)+1,\ k]$ is an independent set in H.

By the definition of interval chromatic number, we always have $\alpha^+_{\chi_<(H)}(H) = h$ and therefore $\bigcup_{i=1}^{\chi_<(H)}[\alpha_{i-1}^+(H)+1,\ \alpha_i^+(H)]$ is a natural partition of $[h]$ into intervals, each spanning an independent set. We also define such parameters in the reverse order. Let $\alpha_0^-(H) := h+1$. For every $1 \leq i \leq \chi_<(H)$, we let

$$\alpha_i^-(H) := \text{ the smallest } k \in \mathbb{N} \text{ such that } [k,\ \alpha_{i-1}^-(H)-1] \text{ is an independent set in } H.$$

[1]See the original paper [18] for the definition of hcf(H).

Similarly, we have $\alpha^-_{\chi_<(H)}(H) = 1$ and therefore $\bigcup_{i=1}^{\chi_<(H)}[\alpha^-_i(H),\ \alpha^-_{i-1}(H) - 1]$ is a natural partition of $[h]$. We then define

$$\alpha^*(H) := \min_{1 \le \ell < \chi_<(H)} \left\{ \frac{\alpha^+_\ell}{\ell \cdot h},\ \frac{h - \alpha^-_\ell + 1}{\ell \cdot h} \right\}. \tag{2}$$

When the underlying graph is clear, we simply write α^+_i, α^-_i and α^*.

Denote by $\delta_<(H, n)$ the smallest integer k such that every ordered graph G whose order n is divisible by $|H|$ and with $\delta(G) \ge k$ contains a perfect H-tiling. The following proposition shows that for any ordered graph H, the parameter $\alpha^*(H)$ provides a lower bound for $\delta_<(H, n)$.

Proposition 1.5. *Let H be an ordered graph on h vertices. For every $n \in \mathbb{N}$ with $h|n$, there is an n-vertex ordered graph G with $\delta(G) \ge \lfloor (1 - \alpha^*(H))n \rfloor - 1$ that does not contain a perfect H-tiling.*

The main goal of this paper is to determine the asymptotics of $\delta_<(H, n)$ for graphs H with interval chromatic number 2. It turns out the value of $\delta_<(H, n)$ in this case depends on structural properties of H encapsulated by the following three definitions.

Definition 1.6 (Property A)**.** An ordered graph H on h vertices is said to have *Property A* if H has no edges in the intervals $[1, \lfloor h/2 \rfloor + 1]$ and $[\lceil h/2 \rceil, h]$.

Note that an ordered graph H has Property A if and only if $\alpha^*(H) > 1/2$.

Definition 1.7 (Property B)**.** An ordered graph H on h vertices is said to have *Property B* if for all partitions of $[h]$ into two non-empty intervals $[1, i]$ and $[i + 1, h]$, there is an edge between these two intervals.

For an ordered graph H on h vertices, let $s(H)$ be the smallest vertex in H that is adjacent to h, and $l(H)$ be the largest vertex in H that is adjacent to 1. If 1 and h have no neighbors in H, then we set $s(H) := 0$ and $l(H) := h + 1$.

Definition 1.8 (Property C)**.** For an ordered graph H on h vertices, the vertex h is said to have *Property C* if $s(H) \ge 1$, and there exists an edge in the interval $[s(H), h - 1]$. Similarly, the vertex 1 is said to have *Property C* if $l(H) \le h$ and there exists an edge in the interval $[2, l(H)]$.

Our main theorem shows that for any ordered graph H with interval chromatic number 2, either its interval chromatic number $\chi_<(H)$ or the new graph parameter $\alpha^*(H)$ governs the minimum degree threshold that forces the existence of a perfect H-tiling in ordered graphs of large minimum degree.

Theorem 1.9. *Let H be an ordered graph on h vertices with $\chi_<(H) = 2$.*

(i) *Suppose that H does not have Property A. Then*

$$\delta_<(H, n) = (1 - \alpha^*(H) + o(1))n.$$

(ii) *Suppose that H has both Property A and Property B. Then*

$$\delta_<(H, n) = (1/2 + o(1))n.$$

(iii) *Suppose that H has Property A but not Property B, and one of the vertices $1, h$ has Property C. Then*

$$\delta_<(H, n) = (1/2 + o(1))n.$$

(iv) *Suppose that H has Property A but not Property B, and neither of the vertices $1, h$ has Property C. Then*

$$\delta_<(H, n) = (1 - \alpha^*(H) + o(1))n.$$

In all cases of Theorem 1.9, except Case (iv), the minimum degree threshold is at least $(1/2 + o(1))n$. Furthermore, although the degree threshold for graphs in Cases (ii) and (iii) are same, they have different types of extremal examples at work. Shortly in Section 2, we will show that there are three types of extremal examples, that is, space barriers for Cases (i) and (iv), divisibility barriers for Case (ii) and local barriers for Case (iii), while recall for unordered graphs there are only space and divisibility barriers (see [18]).

The proof of Theorem 1.9 applies Szemerédi's regularity lemma [23]. As we will discuss in Section 3, a key property of this tool – which is regularly used to help embed (spanning) subgraphs in the unordered setting – breaks down for ordered graphs. We introduce a new approach to overcome this, which we believe will be useful for other embedding problems in the ordered setting. We also provide the first application of the *absorbing method* in the ordered setting; again, the order means one requires more care than in the unordered setting.

1.2. **Notation.** Given integers $n \geq m \geq 1$, let $[m, n] := \{m, \dots, n\}$ and $[n] := \{1, \dots, n\}$. For two subsets X, Y of $[n]$, we write $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$. When X consists of a single element x, we simply write $x < Y$.

A vertex is *isolated* if it has no neighbors. An *empty graph* on n vertices consists of n isolated vertices with no edges. The empty graph on 0 vertices is called the *null graph*. For an ordered graph G and a linearly ordered set $A \subseteq V(G)$, the induced subgraph $G[A]$ is the subgraph of G whose vertex set is A and whose edge set consists of all of the edges with both endpoints in A. For two disjoint subsets $A, B \subseteq V(G)$, the induced bipartite subgraph $G[A, B]$ is the subgraph of G whose vertex set is $A \cup B$ and whose edge set consists of all of the edges with one endpoint in A and the other endpoint in B. For convenience, we also write $G[A, A] := G[A]$.

Given an (ordered) graph G, a vertex $x \in V(G)$ and a set $X \subseteq V(G)$, we define $d_G(x, X)$ to be the number of neighbors that x has in X.

The *join* of two ordered graphs G_1 and G_2, denoted by $G_1 + G_2$, is an ordered graph obtained from G_1 and G_2 by joining each vertex of G_1 to all vertices of G_2. Given an unordered graph G and a positive integer t, let $G(t)$ be the graph obtained from G by replacing every vertex $x \in V(G)$ by a set V_x of t independent vertices, and joining $u \in V_x$ to $v \in V_y$ precisely when xy is an edge in G. That is we replace the edges of G by copies of $K_{t,t}$. We will refer to $G(t)$ as a *blown-up* copy of G.

Throughout the paper, we omit all floor and ceiling signs whenever these are not crucial. The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < a \ll b \ll c \leq 1$, then there are non-decreasing functions $f : (0, 1] \to (0, 1]$ and $g : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ with $b \leq f(c)$ and $a \leq g(b)$. Note that $a \ll b$ implies that we may assume in the proof that e.g. $a < b$ or $a < b^2$.

1.3. **Organisation of paper.** In the next section we describe the extremal examples which show that our main result is best possible. In Section 3 we give a high-level overview of our approach to the regularity method in the ordered setting. In Section 4, we introduce a general framework for attacking ordered tiling problems, and show how to use it to prove Theorem 1.9. In Section 5, we formally state Szemerédi's regularity lemma and introduce related tools. We then prove Theorem 4.3 in Section 6, and prove Theorems 4.1 and 4.6 in Section 7. We close the paper with some concluding remarks in Section 8.

2. Extremal examples

2.1. **Space barriers.** We begin this section with a proof of Proposition 1.5 which provides a general lower bound on $\delta_<(H, n)$ for *all* ordered graphs H.

Proof of Proposition 1.5. By the definition and the symmetry of α^*, it is sufficient to prove that for every $1 \le \ell < \chi_<(H)$, there is an n-vertex graph G with $\delta(G) \ge \left\lfloor \left(1 - \frac{\alpha_\ell^+}{\ell \cdot h}\right) n \right\rfloor - 1$ that does not contain a perfect H-tiling.

For simplicity, we set $s := (\alpha_\ell^+ \cdot n)/h$. Let $A_1 \cup A_2 \cup \ldots \cup A_\ell$ be a partition of the interval $[s+1]$ such that $A_1 < A_2 < \ldots < A_\ell$ and $||A_i| - |A_j|| \le 1$ for every $1 \le i, j \le \ell$. Define

$$G := G_1 + G_2 + \ldots + G_{\ell+1},$$

where G_i is an empty graph defined on A_i for every $1 \le i \le \ell$, and $G_{\ell+1}$ is a complete graph defined on $[n] - \cup_{i=1}^{\ell} A_i$. Note that $n - s - 1 \ge 0$ as $\alpha_\ell^+ < \alpha_{\chi_<(H)}^+ = h$. Therefore, $G_{\ell+1}$ is well-defined, and could be a null graph (only when $h = n$).

We claim that for every copy of H in G,

$$|V(H) \cap [s+1]| \le \alpha_\ell^+. \tag{3}$$

If not, then there exists a copy of H in G with the vertices $v_1 < v_2 < \ldots < v_h$ such that $v_{\alpha_\ell^+ + 1} \in [s+1]$. In particular, there exists an integer $k_\ell \le \ell$ such that $v_{\alpha_\ell^+ + 1} \in A_{k_\ell}$. By the maximality of α_ℓ^+, the vertex $v_{\alpha_\ell^+ + 1}$ has a neighbor $v_{\ell'}$ in H, where $\ell' \in [\alpha_{\ell-1}^+ + 1,\ \alpha_\ell^+]$. Since A_{k_ℓ} is an independent set of G, this implies that there exists an integer $k_{\ell-1} < k_\ell$ such that $v_{\alpha_{\ell-1}^+ + 1} \in A_{k_{\ell-1}}$. Repeat this process until we reach to A_1. Then we obtain an integer $1 \le \ell_0 \le \ell$ and a sequence of numbers $\ell \ge k_\ell > k_{\ell-1} > \ldots > k_i > \ldots > k_{\ell_0} = 1$ such that $v_{\alpha_i^+ + 1} \in A_{k_i}$. In particular, we have $v_{\alpha_{\ell_0}^+ + 1} \in A_1$. By the maximality of $\alpha_{\ell_0}^+$ and $\ell_0 \ge 1$, the vertex $v_{\alpha_{\ell_0}^+ + 1}$ has a neighbor $v_{\ell_0'}$ in H, where $v_{\ell_0'} < v_{\ell_0}$. However, we run out of the space for $v_{\ell_0'}$ as A_1 is an independent set with the smallest vertices.

Finally, suppose that G has a perfect H-tiling $\mathcal{H}$. Then by (3) we have

$$|V(\mathcal{H}) \cap [s+1]| \le \frac{n}{h} \cdot \alpha_\ell^+ = s < s+1,$$

which contradicts the definition of a perfect H-tiling. $\square$

We refer to such examples G as *space barriers* as, in this case, the obstruction to G containing a perfect H-tiling is that the vertex class $[s+1]$ is 'too big'.

2.2. **Divisibility barriers.**

Proposition 2.1. *Let H be an ordered graph on h vertices with $\chi_<(H) = 2$. Suppose that H has Property B. Then for every $n \in \mathbb{N}$ with $h|n$, there is an n-vertex graph G with $\delta(G) \ge \lfloor n/2 \rfloor - 2$ that does not contain a perfect H-tiling.*

Proof. Let k be the largest integer such that $k \le \lceil n/2 \rceil$ and k is not divisible by h. Let G be the disjoint union of two complete graphs on vertex sets $[k]$ and $[k+1, n]$.

Suppose that G has a perfect H-tiling. Then there must be at least one copy H' of H, for which both $[k] \cap V(H')$ and $[k+1, n] \cap V(H')$ are non-empty. However, this is not possible for H with Property B, as there are no edges between $[k]$ and $[k+1, n]$ in G. $\square$

Note we call such graphs G *divisibility barriers* as the obstruction to containing a perfect H-tiling is a divisibility issue (in this case, the size of each of the two cliques is not divisible by h).

2.3. **Local barriers.**

Proposition 2.2. *Let H be an ordered graph on h vertices with $\chi_<(H) = 2$. Suppose that one of the vertices $1, h$ has Property C. Then for every $n \in \mathbb{N}$ with $h|n$, there is an n-vertex graph G with $\delta(G) = \lfloor n/2 \rfloor$ that does not contain a perfect H-tiling.*

Proof. Without loss of generality, we assume that the vertex h has Property C. Recall that $s(H)$ is the smallest vertex in H that is adjacent to h. Since h has Property C, there exists an edge ab in H such that

$$s(H) \leq a < b \leq h-1. \tag{4}$$

Let $G' := G_1 + G_2$, where G_1, G_2 are empty graphs on the vertex sets $[1, \lceil n/2 \rceil - 1]$ and $[\lceil n/2 \rceil, n-1]$. Then we construct an ordered graph G from G' by adding the vertex n and all edges between n and $[\lceil n/2 \rceil, n-1]$.

Suppose that G has a perfect H-tiling. Then there must be a copy of H in G such that n plays the role of h in it. By the construction of G, we have $s(H) \in [\lceil n/2 \rceil, n-1]$. Then by (4) we have $a, b \in [\lceil n/2 \rceil, n-1]$. This contradicts the fact that $[\lceil n/2 \rceil, n-1]$ is an independent set. $\square$

We call such graphs G *local barriers* as the reason G does not contain a perfect H-tiling is a localized issue (in this case, there is a vertex that does not lie in a single copy of H).

3. Applying the regularity method in the ordered setting

In this section we explain our approach to applying the regularity lemma in the ordered graph setting. Those readers unfamiliar with this result and related concepts should first read Section 5.

Let $A_1, \dots, A_k$ be large disjoint equal size vertex classes in an (unordered) graph G so that each pair (A_i, A_j) (for distinct $i, j \in [k]$) is ε-regular of density at least d (where $0 < \varepsilon < d$). Such a structure is often found in an application of Szemerédi's regularity lemma and provides a framework for embedding subgraphs H with $\chi(H) = k$ into G. Indeed, it is well-known that such a structure contains all *fixed size* subgraphs H of chromatic number at most k (see Lemma 5.7). In fact, for any fixed subgraph H with $\chi(H) = k$, $G[A_1 \cup \dots \cup A_k]$ must contain an almost perfect H-tiling. Moreover, the famous *blow-up lemma* of Komlós, Sárközy and Szemerédi [13] allows one to embed any almost spanning, bounded degree graph F with $\chi(F) = k$ into $G[A_1 \cup \dots \cup A_k]$. These properties have been used in dozens of applications of the regularity lemma.

Ideally one would like to use such properties in the vertex ordered setting. Similarly as before, let $A_1, \dots, A_k$ be large disjoint equal size vertex classes in an n-vertex ordered graph G so that each pair (A_i, A_j) (for distinct $i, j \in [k]$) is ε-regular of density at least d. Thus now the A_is are subsets of $[n]$. Let H be a fixed ordered graph with $\chi_<(H) = k$. One can find a copy of H in $G[A_1 \cup \dots \cup A_k]$: as demonstrated by Lemma 6.2 in Section 6, one can find large subclasses $S_i \subseteq A_i$ for all $i \in [k]$ and a permutation σ of $[k]$ so that $S_{\sigma(1)} < S_{\sigma(2)} < \dots < S_{\sigma(k)}$. This allows us then to embed H into $G[S_1 \cup \dots \cup S_k]$ where the ith interval of H is embedded into $S_{\sigma(i)}$.

However, in general it is far from true that $G[A_1 \cup \dots \cup A_k]$ should contain an almost perfect H-tiling. To see this consider the case when $k = 2$ and H is the ordered path 213. Suppose $G[A_1, A_2]$ is in fact complete bipartite (so certainly ε-regular) where $A_1 < A_2$. Then each copy of H in $G[A_1, A_2]$ must have one vertex in A_1 (the vertex playing the role of 1) and two vertices in A_2; so any H-tiling in $G[A_1, A_2]$ can only cover at most half of A_1.

At first sight this suggests perhaps the regularity method is not suitable for embedding large structures in ordered graphs. However, in this paper we demonstrate a method for overcoming this difficulty. Suppose we wish to embed an (almost) perfect H-tiling in an ordered graph G where $\chi_<(H) = r$. We obtain large disjoint vertex sets $A_1, \dots, A_k$ in G so that each pair (A_i, A_j) (for distinct $i, j \in [k]$) is ε-regular of density at least d; now (i) k may be significantly bigger than r and (ii) the size of the classes A_i may be far from equal. The class sizes and k are chosen so that however the vertices from $A_1 \cup \dots \cup A_k$ are labelled in $[n]$, there is a small H-tiling $\mathcal{H}$ in $G[A_1 \cup \dots \cup A_k]$ such that $|A_i \cap V(\mathcal{H})|/|A_j \cap V(\mathcal{H})| = |A_i|/|A_j|$ for all distinct $i, j \in [k]$. With this property to hand, one can now easily find an almost perfect H-tiling in $G[A_1 \cup \dots \cup A_k]$: delete the vertices from $\mathcal{H}$. Still each pair (A_i, A_j) is 2ε-regular and the ratios of the classes have been preserved. So we can find a small H-tiling in $G[A_1 \cup \cdots \cup A_k]$ as before. Repeating this process allows us to cover almost all the vertices in $G[A_1 \cup \cdots \cup A_k]$.

The challenge is to choose k not too large (else G will not be dense enough to guarantee such an ε-regular structure $G[A_1 \cup \cdots \cup A_k]$) whilst ensuring the chosen ratios $|A_i|/|A_j|$ have the 'ratio preservation' property described above. This latter point motivates the notion of a *bottlegraph* of H introduced in the next section.

4. A general framework and the proof of Theorem 1.9

4.1. **General framework.** In this section we introduce two theorems, which as well as being tools in the proof of our main result, are applicable to the general perfect H-tiling problem for ordered graphs.

The so-called *absorbing method*, pioneered by Rödl, Ruciński and Szemerédi (see e.g. [22]) has proved an immensely powerful technique for embedding problems in graphs and hypergraphs. In particular, when one wishes to embed a spanning structure F in a (hyper)graph G, the method can provide a certain 'absorbing gadget' *Abs* in G. With this gadget to hand, one then seeks to embed only an almost spanning subgraph F' of F into G; *Abs* will then have the power to extend the subgraph F' into a copy of F in G.

In this section we adapt the absorbing method to the setting of ordered graphs. More precisely, let H be an ordered graph. Given an ordered graph G, a set $S \subseteq V(G)$ is an *H-absorbing set for $Q \subseteq V(G)$*, if both $G[S]$ and $G[S \cup Q]$ contain perfect H-tilings. In this case we say that *Q is H-absorbed by S*. Sometimes we will simply refer to a set $S \subseteq V(G)$ as an H-absorbing set if *there exists* a set $Q \subseteq V(G)$ that is H-absorbed by S. The following result provides an absorbing set *Abs* in an ordered graph G of large minimum degree, where crucially *Abs* is an H-absorbing set for *every* not too large set of vertices $Q \subseteq V(G)$.

Theorem 4.1 (Absorbing theorem)**.** *Let H be an h-vertex ordered graph and let $\eta > 0$. Then there exists an $n_0 \in \mathbb{N}$ and $\nu > 0$ so that the following holds. Suppose that G is an n-vertex ordered graph where $n \geq n_0$ and where*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n.$$

Then $V(G) = [n]$ contains a set Abs so that

- *$|Abs| \leq \nu n$;*
- *Abs is an H-absorbing set for any $W \subseteq V(G) \setminus Abs$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \nu^3 n$.*

Theorem 4.1 suffices for our applications in most cases. Indeed, it is immediately applicable to the perfect H-tiling problem for any ordered graph H where the minimum degree threshold for ensuring a perfect H-tiling in an ordered graph G is at least $(1 - \frac{1}{\chi_<(H)} + o(1))|G|$. In particular, we will use this theorem for Cases (i)–(iii) of Theorem 1.9. For Case (iv) (and we suspect at least for some special cases of the general perfect H-tiling problem) we require an absorbing theorem for ordered graphs with much smaller minimum degree. In this situation, some structural properties of H can help us improve the absorbing argument; see Theorem 4.6 below.

As indicated above, to apply the absorbing method one requires a sister *almost perfect tiling theorem*, which usually states that in a graph with large minimum degree all but $o(n)$ vertices are covered by some H-tiling. Although the variety of extremal examples indicates that proving a sharp almost perfect H-tiling theorem for an arbitrary ordered graph H seems to be very difficult, in this section we propose a *general framework* for obtaining such almost perfect tiling theorems.

Let B be a complete k-partite unordered graph with parts $U_1, \dots, U_k$, and σ be a permutation of the set $[k]$. An *interval labeling* of B with respect to σ is a mapping $\phi : V(B) \to [|B|]$ such that $\phi(U_i) < \phi(U_j)$ if $\sigma(i) < \sigma(j)$. Given $t \in \mathbb{N}$, recall that $B(t)$ is a blow-up of B with vertex set

$\bigcup_{x\in V(B)} V_x$, where the V_xs are sets of t independent vertices. Let $(B(t), \phi)$ be the ordered graph obtained from $B(t)$ by equipping $V(B(t))$ with a vertex ordering, satisfying $V_x < V_y$ for every $x, y \in V(B)$ with $\phi(x) < \phi(y)$. We refer to $(B(t), \phi)$ as an *ordered blow-up* of B.

Definition 4.2 (Bottlegraph)**.** For an ordered graph H, we say that B is a *bottlegraph* assigned to H if for every interval labeling ϕ of B, there exists a constant $t = t(B, H, \phi)$ such that the ordered blow-up $(B(t), \phi)$ contains a perfect H-tiling.

Theorem 4.3 (Almost perfect tiling framework)**.** *Let H be an ordered graph on h vertices. Suppose that B is a bottlegraph assigned to H. Then for every $\eta > 0$, there exists an $n_0 \in \mathbb{N}$ so that every ordered graph G on $n \geq n_0$ vertices with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(B)} + \eta\right) n$$

contains an H-tiling covering all but at most ηn vertices.

With Theorem 4.3 at hand, in order to prove that all ordered graphs with a given minimum degree contain an almost perfect H-tiling, it is sufficient to show that certain 'interval labelled' blow-ups of a specific graph B with a given critical chromatic number contain perfect H-tilings. The latter statement usually can be verified easily by observation or by solving a linear optimization problem. Thus, for the general perfect H-tiling problem, the heart of the problem is to choose an ordered graph B whose critical chromatic number is not too big (so that corresponding minimum degree condition in Theorem 4.3 is not too high) whilst ensuring B is indeed a bottlegraph assigned to H. These competing forces mean it is far from immediate what the correct choice of B is given an arbitrary ordered graph H. However, for a given class of ordered graphs H (as we will see in the case when $\chi_<(H) = 2$), there might be some intuitive ways to construct a 'fairly good' or even optimal bottlegraph using their structural properties.

The proofs of both Theorems 4.1 and 4.3 rely on Szemerédi's regularity lemma, which will be formally introduced in Section 5. We then prove Theorem 4.3 in Section 6, and Theorem 4.1 in Section 7.

4.2. **Graphs with interval chromatic number 2.** In this section, we illustrate how to apply our general framework to prove Theorem 1.9. The following key lemma gives a construction of the bottlegraph for graphs with interval chromatic number 2.

Lemma 4.4. *Let H be an ordered graph on h vertices with $\chi_<(H) = 2$. Recall that*

$$\alpha^*(H) = \min\left\{\frac{\alpha_1^+}{h},\ \frac{h - \alpha_1^- + 1}{h}\right\}.$$

Then there exists a bottlegraph B of H such that $\chi_{cr}(B) = 1/\alpha^*(H)$.

Proof. Let $p := \alpha^* h = \min\{\alpha_1^+,\ h - \alpha_1^- + 1\}$. By the symmetry of the argument, without loss of generality we can assume that $p = \alpha_1^+$.

We first assume that H does not have Property A. Then we have $p \leq \lfloor h/2 \rfloor$, and therefore there exist integers $a \geq 2$, $0 \leq r < p$ such that $h = ap + r$. Let B be a complete multipartite graph with classes $U_0, U_1, \ldots, U_a$, in which $|U_0| = r$, and $|U_1| = \ldots = |U_a| = p$. We will show that B is a bottlegraph assigned to H.

Let ϕ be an interval labeling of B. If $r = 0$ then (B, ϕ) immediately contains a copy of H (i.e. $(B(1), \phi)$ contains a perfect H-tiling). So assume $r \neq 0$. If there exists $i \geq 1$ such $\phi(U_i) < \phi(U_0)$, then again (B, ϕ) contains a copy of H. Therefore, without loss of generality, we can assume that $\phi(U_0) < \phi(U_1) < \ldots < \phi(U_a)$.

Let $c := \mathrm{lcm}(p, r)$, the least common multiple of p and r, and $t := c/r$. Let $B' := B(t)$ and $U_0', U_1', \ldots, U_a'$ be the partite sets of B' (where U_i' corresponds to U_i). For a set $A \subseteq V(B')$ of size

h, if $|A \cap U_i'| = p$, for all $0 \le i \le a-1$ and $|A \cap U_a'| = r$, we say A is a *type I set*; if $|V(H) \cap U_0'| = 0$, $|V(H) \cap U_i'| = p$, for all $1 \le i \le a-1$ and $|V(H) \cap U_a'| = p+r$, we say A is a *type II set*. By the definition of p, all the edges of H are between the intervals $[p]$ and $[p+1,\ h]$. Therefore, both type I and type II sets induce some complete multipartite graphs in B', which contain a copy of H. By the choice of c and t, $V(B')$ can be partitioned into $t = c/r$ disjoints sets, where c/p of them are of type I and $c/r - c/p$ of them are of type II; thus this ensures a perfect H-tiling in B'. So indeed B is a bottlegraph of H.

Now we assume that H has Property A; then we have $\alpha_1^- \le \lfloor h/2 \rfloor + 1 \le \alpha_1^+$. Observe that all the edges of H are between the intervals $[\alpha_1^- - 1]$ and $[\alpha_1^+ + 1,\ h]$. Let $r := h - p$, and take $B := K_{r,p}$. Note that $p \ge r = \max\{h - \alpha_1^+, \alpha_1^- - 1\}$. Therefore, for any interval labeling ϕ of B, the ordered graph $(B,\ \phi)$ contains a copy of H; so B is a bottlegraph assigned to H.

Finally, it is easy to check that for both bottlegraphs B, we have $\chi_{cr}(B) = h/p = 1/\alpha^*(H)$. $\square$

Applying Theorem 4.3 with Lemma 4.4, we immediately obtain a bound on the minimum degree that guarantees an almost perfect H-tiling for any H with $\chi_<(H) = 2$.

Theorem 4.5. *Let H be an ordered graph on h vertices with $\chi_<(H) = 2$. For every $\eta > 0$, there exists an $n_0 \in \mathbb{N}$ so that the following holds. Every ordered graph G on $n \ge n_0$ vertices with*

$$\delta(G) \ge (1 - \alpha^*(H) + \eta)\, n,$$

contains an H-tiling covering all but at most ηn vertices.

Proof of Theorem 1.9(i)–(iii). Let $0 < \eta' \ll \nu \ll \eta, 1/|H|$ and let n be a sufficiently large integer divisible by h. Our desired lower bounds on $\delta_<(H, n)$ follow immediately from our extremal examples in Section 2. Recall that an ordered graph H has Property A if and only if $\alpha^*(H) > 1/2$. Therefore, for the rest of the proof, it is sufficient to show that every ordered graph G on n vertices with

$$\delta(G) \ge (1 - \min\{\alpha^*(H), 1/2\} + \eta)\, n$$

contains a perfect H-tiling.

First of all, by Theorem 4.1, there exists an H-absorbing set Abs so that

- $|Abs| \le \nu n$;
- Abs is an H-absorbing set for any $W \subseteq V(G) \setminus Abs$ such that $|W| \in h\mathbb{N}$ and $|W| \le \nu^3 n$.

Set $G' := G - Abs$. Then $\delta(G') \ge (1 - \min\{\alpha^*(H), 1/2\} + \eta')\, |G'|$. Thus by Theorem 4.5, G' contains an H-tiling $\mathcal{H}_1$ covering all but a set W of vertices with $|W| \le \eta' n \le \nu^3 n$. By the definition of the H-absorbing set, $G[W \cup Abs]$ contains a perfect H-tiling $\mathcal{H}_2$. Then $\mathcal{H}_1 \cup \mathcal{H}_2$ is a perfect H-tiling of G. $\square$

The proof of Theorem 1.9(iv) is similar but requires a stronger version of the absorbing theorem.

Theorem 4.6. *Let H be an h-vertex ordered graph with $\chi_<(H) = 2$. Suppose that H has Property A but not Property B, and neither of the vertices $1, h$ has Property C. Then for every $\eta > 0$, there exists an $n_0 \in \mathbb{N}$ and $\nu > 0$ so that the following holds. Suppose that G is an n-vertex ordered graph where $n \ge n_0$ and where*

$$\delta(G) \ge \eta n.$$

Then $V(G)$ contains a set Abs so that

- *$|Abs| \le \nu n$;*
- *Abs is an H-absorbing set for any $W \subseteq V(G) \setminus Abs$ such that $|W| \in h\mathbb{N}$ and $|W| \le \nu^3 n$.*

The proof of Theorem 4.6 contains some technical arguments; we postpone it to Section 7.

Proof of Theorem 1.9(iv). The upper bound on $\delta_<(H,n)$ follows similarly from Theorems 4.5 and 4.6, while the lower bound is given by the space barriers, i.e. Proposition 1.5. $\square$

5. The regularity lemma and related tools

In the proof of our main results we will use Szemerédi's regularity lemma [23]. In this section we will introduce all the information we require about this result. We first introduce some notation. The *density* of a bipartite graph with vertex classes A and B is defined to be

$$d(A,B) := \frac{e(A,B)}{|A||B|}.$$

Given $\varepsilon > 0$, a graph G and two disjoint sets $A, B \subset V(G)$, we say that the pair $(A,B)_G$ (or simply (A,B) when the underlying graph is clear) is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have $|d(A,B) - d(X,Y)| < \varepsilon$. Given $d \in [0,1)$, the pair $(A,B)_G$ is *(ε,d)-regular* if G is ε-regular, and $d(A,B) \geq d$.

We now collect together some useful properties of ε-regular pairs.

Proposition 5.1. *For $0 < \varepsilon \ll d_2 < d_1 \leq 1$, there exists an integer $K = K(\varepsilon, d_2, d_1)$ such that the following holds. Let $(A,B)_G$ be an ε-regular pair of density d_1 in a graph G where $|A|, |B| \geq K$. Then there exist a spanning subgraph $G' \subseteq G$ such that $(A,B)_{G'}$ is a $\sqrt{\varepsilon}$-regular pair of density d, where $|d - d_2| \leq \varepsilon$.*

Proof sketch. It suffices to consider the case when $d_1 - d_2 \geq \varepsilon$ (otherwise we set $G' := G$). Let G' be the graph obtained from G by retaining each edge with probability $p := d_2/d_1$, independently of all other edges. Then $\mathbb{E}(d_{G'}(A,B)) = pd_1 = d_2$.

Further, for every $X \subseteq A$ and $Y \subseteq B$ such that $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have that

$$\mathbb{E}(e_{G'}(X,Y)) = \frac{d_2}{d_1} e_G(X,Y) \in \left((d_2 - \varepsilon/d_1)|X||Y|, (d_2 + \varepsilon/d_1)|X||Y|\right).$$

Noting that there are at most $2^{|A|+|B|}$ such pairs X, Y, we may repeatedly apply Chernoff's bound to ensure with high probability the conclusion of the proposition holds. $\square$

Proposition 5.2. *For $0 < \varepsilon \ll d_2, d_1 \leq 1/2$ with $|d_1 - d_2| \leq \varepsilon$, let $(A, B_1)_G$ and $(A, B_2)_G$ be ε-regular pairs of density d_1 and d_2 respectively in a graph G where B_1 and B_2 are disjoint. Then $(A, B_1 \cup B_2)_G$ is a $(\sqrt{\varepsilon}, \min\{d_1, d_2\})$-regular pair.*

Proof. Let $X \subseteq A$ and $Y \subseteq B_1 \cup B_2$ where $|X| \geq \sqrt{\varepsilon}|A|$ and $|Y| \geq \sqrt{\varepsilon}(|B_1| + |B_2|)$. Let $Y_1 := Y \cap B_1$ and $Y_2 := Y \cap B_2$. If both $|Y_1| \geq \varepsilon|B_1|$ and $|Y_2| \geq \varepsilon|B_2|$ then it is easy to check the pair X, Y satisfies the condition in the definition of a $\sqrt{\varepsilon}$-regular pair. So without loss of generality it suffices to check the case when $|Y_1| \geq \varepsilon|B_1|$ and $|Y_2| \leq \varepsilon|B_2|$. In this case $|Y_2|/|Y| \leq \sqrt{\varepsilon}$. Thus,

$$\frac{e(X,Y)}{|X||Y|} - d(A, B_1 \cup B_2) \geq \frac{(d_1 - \varepsilon)(|Y| - |Y_2|)}{|Y|} - \max\{d_1, d_2\} \geq (d_1 - \varepsilon) - (d_1 - \varepsilon)\sqrt{\varepsilon} - (d_1 + \varepsilon) \geq -\sqrt{\varepsilon},$$

and

$$\frac{e(X,Y)}{|X||Y|} - d(A, B_1 \cup B_2) \leq \frac{(d_1 + \varepsilon)(|Y| - |Y_2|) + |Y_2|}{|Y|} - \min\{d_1, d_2\} \leq (d_1 + \varepsilon) + (1 - d_1 - \varepsilon)\sqrt{\varepsilon} - (d_1 - \varepsilon) \leq \sqrt{\varepsilon}.$$

This proves that the pair X, Y satisfies the condition in the definition of a $\sqrt{\varepsilon}$-regular pair. $\square$

We will also make use of the following well-known property of regular pairs.

Lemma 5.3 (Slicing lemma)**.** *Let (A,B) be an ε-regular pair of density d, and for some $\alpha > \varepsilon$, let $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$. Then (A', B') is $(\varepsilon', d-\varepsilon)$-regular with $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$.* □

We will apply the following degree form of Szemerédi's regularity lemma [23].

Lemma 5.4 (Regularity lemma)**.** *For every $\varepsilon > 0$ and $\ell_0 \in \mathbb{N}$ there exists $L_0 = L_0(\varepsilon, \ell_0)$ such that for every $d \in [0,1]$ and for every graph G on $n \geq L_0$ vertices there exists a partition $V_0, V_1, \dots, V_\ell$ of $V(G)$ and a spanning subgraph G' of G, such that the following conditions hold:*

(i) $\ell_0 \leq \ell \leq L_0$;
(ii) $d_{G'}(x) \geq d_G(x) - (d+\varepsilon)n$ *for every* $x \in V(G)$;
(iii) *the subgraph* $G'[V_i]$ *is empty for all* $1 \leq i \leq \ell$;
(iv) $|V_0| \leq \varepsilon n$;
(v) $|V_1| = |V_2| = \ldots = |V_\ell|$;
(vi) *for all* $1 \leq i < j \leq \ell$ *either* $(V_i, V_j)_{G'}$ *is an* (ε, d)*-regular pair or* $G'[V_i, V_j]$ *is empty.*

We call $V_1, \dots, V_\ell$ *clusters*, V_0 the *exceptional set* and the vertices in V_0 *exceptional vertices*. We refer to G' as the *pure graph*. The *reduced graph R of G with parameters ε, d and ℓ_0* is the graph whose vertices are $V_1, \dots, V_\ell$ and in which V_iV_j is an edge precisely when $(V_i, V_j)_{G'}$ is (ε, d)-regular.

A t-partite graph with parts $W_1, \dots, W_t$ is *nearly balanced* if $||W_i| - |W_j|| \leq 1$ for every $1 \leq i, j \leq t$. We will also make use of the following multipartite version of Lemma 5.4.

Lemma 5.5 (Multipartite regularity lemma)**.** *Given any integer $t \geq 2$, any $\varepsilon > 0$ and any $\ell_0 \in \mathbb{N}$ there exists $L_0 = L_0(\varepsilon, t, \ell_0) \in \mathbb{N}$ such that for every $d \in [0,1]$ and for every nearly balanced t-partite graph $G = (W_1, \dots, W_t)$ on $n \geq L_0$ vertices, there exists an $\ell \in \mathbb{N}$, a partition $W_i^0, W_i^1, \dots, W_i^\ell$ of W_i for each $i \in [t]$ and a spanning subgraph G' of G, such that the following conditions hold:*

(i) $\ell_0 \leq \ell \leq L_0$;
(ii) $d_{G'}(x) \geq d_G(x) - (d+\varepsilon)n$ *for every* $x \in V(G)$;
(iii) $|W_0^i| \leq \varepsilon n/t$ *for every* $i \in [t]$;
(iv) $|W_i^j| = |W_{i'}^{j'}|$ *for every* $i, i' \in [t]$ *and* $j, j' \in [\ell]$;
(v) *for every* $i, i' \in [t]$ *and* $j, j' \in [\ell]$ *either* $(W_i^j, W_{i'}^{j'})_{G'}$ *is an* (ε, d)*-regular pair or* $G'[W_i^j, W_{i'}^{j'}]$ *is empty.*

Similarly as before, for $i \in [t]$ and $j \in [\ell]$ we call the W_i^j *clusters*, the W_0^j the *exceptional sets* and the vertices in the W_0^j *exceptional vertices*. We refer to G' as the *pure graph*. The *reduced graph R of G with parameters ε, d and ℓ_0* is the graph whose vertices are the W_i^j (where $i \in [t]$ and $j \in [\ell]$) and in which $W_i^j W_{i'}^{j'}$ is an edge precisely when $(W_i^j, W_{i'}^{j'})_{G'}$ is (ε, d)-regular.

The following well-known corollary of the regularity lemma shows that the reduced graph almost inherits the minimum degree of the original graph.

Proposition 5.6. *Let $0 < \varepsilon, d, k < 1$, G be an n-vertex graph with $\delta(G) \geq kn$ and R be the reduced graph of G obtained by applying the regularity lemma with parameters ε, d. Then $\delta(R) \geq (k - 2\varepsilon - d)|R|$.* □

The next key lemma allows us to use the reduced graph R of G as a framework for embedding subgraphs into G.

Lemma 5.7 (Key lemma [15])**.** *Suppose that $0 < \varepsilon < d$, that $q, t \in \mathbb{N}$ and that R is a graph with $V(R) = \{v_1, \dots, v_k\}$. We construct a graph G as follows: replace every vertex $v_i \in V(R)$ with a set V_i of q vertices and replace each edge of R with an (ε, d)-regular pair. For each $v_i \in V(R)$, let U_i*

denote the set of t vertices in $R(t)$ corresponding to v_i. Let H be a subgraph of $R(t)$ with maximum degree Δ and set $h := |H|$. Set $\delta := d - \varepsilon$ and $\varepsilon_0 := \delta^{\Delta}/(2+\Delta)$. If $\varepsilon \le \varepsilon_0$ and $t - 1 \le \varepsilon_0 q$ then there are at least

$$(\varepsilon_0 q)^h \text{ labelled copies of } H \text{ in } G$$

so that if $x \in V(H)$ lies in U_i in $R(t)$, then x is embedded into V_i in G.

In particular, note that if H is an ordered graph with $\chi_<(H) = k$, and G has a vertex ordering such that $V_1 < \ldots < V_k$, then Lemma 5.7 allows us to embed many copies of H into G.

6. Proof of Theorem 4.3

We will apply the following result of Komlós [12]; this result shows that the critical chromatic number of H governs the minimum degree threshold for the existence of almost perfect H-tilings in unordered graphs.

Theorem 6.1 (Komlós [12])**.** *Let $\mu > 0$ and let F be an unordered graph. Then there exists an $n_0 = n_0(\mu, F) \in \mathbb{N}$ such that every graph G on $n \ge n_0$ vertices with*

$$\delta(G) \ge \left(1 - \frac{1}{\chi_{cr}(F)}\right) n$$

contains an F-tiling covering all but at most μn vertices.

The next result ensures that in any k linear size disjoint vertex sets $A_1, \ldots, A_k$ of an ordered graph G, one can find 'nicely ordered' linear size subsets S_i of each A_i. As we will see shortly, this property is crucial for our application of the regularity lemma in the proof of Theorem 4.3.

Lemma 6.2. *For $n \ge k \ge 2$, let $A_1, A_2, \ldots, A_k$ be nonempty disjoint subsets of $[n]$. Then there exist sets $S_1, S_2, \ldots, S_k$, where $S_i \subseteq A_i$, and a permutation $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ of the set $[k]$, such that the following conditions hold for all $i, j \in [k]$:*

(i) $|S_i| \ge |A_i|/2^{2^{k-2}}$;
(ii) $S_i < S_j$ *if* $\sigma(i) < \sigma(j)$.

Proof. We prove the lemma by induction on k. For the base case $k = 2$, let $m(A_1)$, $m(A_2)$ denote the median values of A_1 and A_2 respectively. Without loss of generality, we assume that $m(A_1) > m(A_2)$. Let $S_1 := \{a \in A_1 \mid a \ge m(A_1)\}$ and $S_2 := \{a \in A_2 \mid a \le m(A_2)\}$; so $S_2 < S_1$. It is easy to check that conditions (i) and (ii) hold for S_1, S_2 and $\sigma = (2, 1)$.

Now assume the induction hypothesis: there exist sets $S'_1, S'_2, \ldots, S'_{k-1}$, where $S'_i \subseteq A_i$, and a permutation σ' of the set $[k-1]$, such that for all $i, j \in [k-1]$,

$$|S'_i| \ge |A_i|/2^{2^{k-3}} \tag{5}$$

and

$$S'_i < S'_j \quad \text{if} \quad \sigma'(i) < \sigma'(j).$$

Without loss of generality, we may assume that $\sigma' = (1, 2, \ldots, k-1)$, and then

$$S'_1 < S'_2 < \cdots < S'_{k-1}.$$

Next, let S'_{k-1} and A_k play the roles of A_1, A_2. From the base case, we can find two sets $S_{k-1} \subseteq S'_{k-1}$ and $S'_k \subseteq A_k$, where

$$|S_{k-1}| \ge |S'_{k-1}|/2 \quad \text{and} \quad |S'_k| \ge |A_k|/2, \tag{6}$$

such that either $S_{k-1} < S'_k$ or $S'_k < S_{k-1}$. If $S_{k-1} < S'_k$, we then have

$$S'_1 < S'_2 < \cdots < S'_{k-2} < S_{k-1} < S'_k. \tag{7}$$

Let $S_i := S'_i$ for all $i \neq k-1$, and $\sigma := (1, 2, \ldots, k-1, k)$. (5), (6) together with (7) indicate that $S_1, S_2, \ldots, S_k$ and σ satisfy (i) and (ii). Otherwise, we have

$$S'_1, \ldots, S'_{k-2}, S'_k < S_{k-1}. \tag{8}$$

Applying the induction hypothesis on the sets $S'_1, \ldots, S'_{k-2}, S'_k$, we find sets $S_1, \ldots, S_{k-2}, S_k$, where $S_i \subseteq S'_i \subseteq A_i$, and a permutation σ'' of the set $[k] \setminus \{k-1\}$ such that for $i, j \in [k] \setminus \{k-1\}$

$$|S_i| \geq |S'_i|/2^{2^{k-3}}, \tag{9}$$

and

$$S_i < S_j \quad \text{if} \quad \sigma''(i) < \sigma''(j). \tag{10}$$

Let $\sigma := (\sigma''(1), \ldots, \sigma''(k-2), k, \sigma''(k))$. By (5), (6), (8), (9) and (10), the sets $S_1, S_2, \ldots, S_k$ and σ satisfy (i) and (ii). □

Proof of Theorem 4.3. We will fix additional constants satisfying the following hierarchy

$$0 < \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon \ll d, \mu_1, \mu_2 \ll \eta, 1/|B|. \tag{11}$$

Moreover, we choose an integer ℓ_0 such that $\ell_0 \geq n_0(\mu_1, B)$, where $n_0(\mu_1, B)$ is as defined in Theorem 6.1. In what follows, we assume that the order n of our given ordered graph G is sufficiently large for our estimates to hold. We now apply the regularity lemma (Lemma 5.4) with parameters ε_1, d and ℓ_0 to G to obtain a reduced graph R, clusters $\{V_a, a \in V(R)\}$, an exceptional set V_0, and a spanning subgraph $G' \subseteq G$. (11) together with Proposition 5.6 implies that

$$\delta(R) \geq \left(1 - \frac{1}{\chi_{cr}(B)} + \frac{\eta}{2}\right) |R|. \tag{12}$$

Since $|R| \geq \ell_0 \geq n_0(\mu_1, B)$, we can apply Theorem 6.1 to R to find a B-tiling $\mathcal{B}$ covering all but at most $\mu_1|R|$ vertices. We delete all the clusters not contained in some copy of B in $\mathcal{B}$ from R and add all the vertices lying in these clusters to the exceptional set V_0. Thus, $|V_0| \leq \varepsilon_1 n + \mu_1 n \leq 2\mu_1 n$. From now on, we denote by R the subgraph of the reduced graph induced by all the remaining clusters. Thus $\mathcal{B}$ now is a perfect B-tiling of R.

Fix an arbitrary copy $B \in \mathcal{B}$ with partite sets $U_1, \ldots, U_k$, and let $A := \cup_{a \in V(B)} V_a$ and $A_i := \cup_{a \in U_i} V_a$. Since B is a complete multipartite graph, repeatedly applying Propositions 5.1 and 5.2 to $G'[A]$, we can find a spanning subgraph $G'' \subseteq G'[A]$ such that for every distinct $i, j \in [k]$, $(A_i, A_j)_{G''}$ is $(\varepsilon_2, d - \varepsilon_2)$-regular.

Let $\alpha := 2^{2^{k-2}}$. By Lemma 6.2, there exist sets $S_i \subseteq A_i$, and a permutation σ of $[k]$ such that $|S_i| \geq |A_i|/\alpha$, and $S_i < S_j$ whenever $\sigma(i) < \sigma(j)$. Moreover, by the slicing lemma (Lemma 5.3), we have that each $(S_i, S_j)_{G''}$ is $(\varepsilon, d - \varepsilon)$-regular. Now we apply the key lemma (Lemma 5.7) on $G''[\cup S_i]$, and find a blown-up copy $B_1(t)$ of B, where $|B_1(t) \cap S_i| = |U_i|t$ for every i and t is a fixed integer given by the definition of the bottlegraph. Note that by the choice of the S_i, B_1 naturally has an interval ordering with respect to the permutation σ, and therefore $B_1(t)$ has a perfect H-tiling. After that, we can delete $V(B_1(t))$ from A (and therefore from each A_i); crucially after this deletion, the ratio $|A_i|/|A_j|$ amongst all pairs of classes A_i, A_j remains the same as before. Further, still for every distinct $i, j \in [k]$, $(A_i, A_j)_{G''}$ is $(2\varepsilon_2, d - 2\varepsilon_2)$-regular.

These properties allow us to repeatedly apply this argument, thereby obtaining an H-tiling on $G[A]$ covering all but at most $\mu_2|A|$ vertices.[2] Finally, simply repeat this process for all copies of B in $\mathcal{B}$; we obtain an H-tiling of G covering all but at most $(2\mu_1 + \mu_2)n \leq \eta n$ vertices. □

[2]Note, if we did not have the ratio property above, we could have 'used up' one of the classes A_i too quickly.

7. Proof of the absorbing theorems

To prove Theorems 4.1 and 4.6, we make use of the following, now standard, lemma.

Lemma 7.1. *Let $h, s \in \mathbb{N}$ and $\gamma > 0$. Suppose that H is an ordered hypergraph on h vertices. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is an ordered hypergraph on $n \geq n_0$ vertices so that, for any $x, y \in V(G)$, there are at least γn^{sh-1} $(sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings. Then $V(G)$ contains a set M so that*

- $|M| \leq (\gamma/2)^h n/4$;
- *M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq (\gamma/2)^{2h} n/(32 s^2 h^3)$.*

Lemma 7.1 was proven in the case when G is unordered by Lo and Markström [19, Lemma 1.1]. However, the proof in the ordered setting is identical (so we do not provide a proof here).

7.1. Proof of Theorem 4.1.

To prove Theorem 4.1 we must show that the hypothesis of Lemma 7.1 is satisfied. The following lemma provides a step in that direction.

Lemma 7.2. *Let H be an h-vertex ordered graph and let $0 < \eta \ll 1/h$. Then there exists an $n_0 \in \mathbb{N}$ and $\rho, \gamma > 0$ where $1/n_0 \ll \rho \ll \gamma \ll \eta$ and so that the following holds. Suppose that G is an ordered graph with vertex set $[n]$ where $n \geq n_0$ and where*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n.$$

Given any $x \in [n]$, there are at least $(1-\gamma)|B_{[n]}(x, \eta n/16)|$ elements $y \in [n]$ so that

- *$y \in B_{[n]}(x, \eta n/16)$;*[3]
- *there are at least ρn^{h-1} $(h-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ span copies of H.*

Proof. Choose $0 < \rho \ll 1/\ell_0 \ll \varepsilon \ll \gamma \ll d \ll \eta \ll 1/h$ where $\ell_0 \in \mathbb{N}$, and let n be sufficiently large. Let G be as in the statement of the lemma. Write $r := \chi_<(H)$.

It is easy to see that there is some α with $\eta/16 \leq \alpha \leq \eta/2$ such that there is a partition $W_1, \dots, W_t$ of $[n]$ where

(i) $t := \lfloor 1/\alpha \rfloor$;
(ii) $|W_i| = \lfloor \frac{n}{t} \rfloor$ or $\lceil \frac{n}{t} \rceil$ for all $i \in [t]$;
(iii) $W_i < W_j$ for every $1 \leq i < j \leq t$;
(iv) there is some $i^* \in [t]$ so that $B_{[n]}(x, \eta n/16) \subseteq W_{i^*}$.

Note that (iii) implies that each of the W_is is an interval in $[n]$. Define $G_1 := G[W_1, W_2, \dots, W_t]$; that is we have deleted all edges from each $G[W_i]$. Hence,

$$\delta(G_1) \geq \left(1 - \frac{1}{r} + \frac{\eta}{3}\right) n. \tag{13}$$

Apply Lemma 5.5 to G_1 with parameters $\varepsilon, d, t, \ell_0$ to obtain a pure graph G_1' and reduced graph R_1 of G_1, and a partition $W_i^0, W_i^1, \dots, W_i^\ell$ of W_i for each $i \in [t]$. Crucially, we have defined G_1 so that if $W_{i_1}^{j_1} W_{i_2}^{j_2} \in E(R_1)$ then $W_{i_1}^{j_1} < W_{i_2}^{j_2}$ or $W_{i_2}^{j_2} < W_{i_1}^{j_1}$. (13) together with Proposition 5.6 imply that

$$\delta(R_1) \geq (1 - 1/r + \eta/4)|R_1|. \tag{14}$$

[3] Define $B_{[n]}(x, z)$ as the set $[n] \cap [x - z, x + z]$.

Write $N_{R_1}(x) := \{W_i^j \in V(R_1) : d_{G_1}(x, W_i^j) \geq \eta |W_i^j|/4\}$. The minimum degree condition on G_1 ensures that

$$|N_{R_1}(x)| \geq (1 - 1/r + \eta/4)|R_1|. \tag{15}$$

Recall that i^* is defined in (iv) above, and fix an arbitrary cluster $W_{i^*}^{j^*}$ for some $j^* \in [\ell]$. Combining (14) and (15) ensures we can choose clusters $W_{i_1}^{j_1}, \dots, W_{i_{r-1}}^{j_{r-1}}$ so that:

(a) $W_{i_1}^{j_1}, \dots, W_{i_{r-1}}^{j_{r-1}}$ together with $W_{i^*}^{j^*}$ form a copy of K_r in R_1;
(b) $W_{i_1}^{j_1}, \dots, W_{i_{r-1}}^{j_{r-1}} \in N_{R_1}(x)$;
(c) There is some $z^* \in \{0, \dots, r-1\}$ so that

$$W_{i_1}^{j_1} < \dots < W_{i_{z^*}}^{j_{z^*}} < (W_{i^*}^{j^*} \cup \{x\}) < W_{i_{z^*+1}}^{j_{z^*+1}} < \dots < W_{i_{r-1}}^{j_{r-1}}.$$

In particular, (c) is ensured by the choice of the subgraph G_1 of G.

By the slicing lemma (Lemma 5.3) and the fact that $W_{i_k}^{j_k} \in N_{R_1}(x)$ for every $k \in [r-1]$, the pair $(W_{i^*}^{j^*}, N_{G_1}(x) \cap W_{i_k}^{j_k})_{G_1'}$ is $(\varepsilon^{1/2}, d/2)$-regular. By the definition of $(\varepsilon^{1/2}, d/2)$-regularity, all but at most $r\varepsilon^{1/2}|W_{i^*}^{j^*}|$ vertices $y \in W_{i^*}^{j^*}$ have degree at least $(d/2 - \varepsilon^{1/2})|N_{G_1}(x) \cap W_{i_k}^{j_k}| \geq d\eta |W_{i_k}^{j_k}|/12$ into $N_{G_1}(x) \cap W_{i_k}^{j_k}$ in G_1 for every $k \in [r-1]$. Fix such a vertex y. Define

$$W_k' := N_{G_1}(x) \cap W_{i_k}^{j_k} \cap N_{G_1}(y)$$

for each $k \in [r-1]$, and note that $|W_k'| \geq d\eta|W_{i_k}^{j_k}|/12$. Given any $i \neq j \in [r-1]$, Lemma 5.3 implies that each pair $(W_i', W_j')_{G_1'}$ and $(W_i', W_{i^*}^{j^*})_{G_1'}$ are $(\varepsilon^{1/4}, d/4)$-regular. Recalling that $\chi_<(H) = r$, property (c) above together with Lemma 5.7 implies that there are at least ρn^{h-1} $(h-1)$-sets $X \subseteq W_{i^*}^{j^*} \cup \bigcup W_k'$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ span copies of H.

For each choice of the cluster $W_{i^*}^{j^*}$, there were at most $r\varepsilon^{1/2}|W_{i^*}^{j^*}|$ 'bad' selections for $y \in W_{i^*}^{j^*}$. Since $|W_{i^*}^0| \leq \varepsilon n/t$ this implies that for all but at most $(r\varepsilon^{1/2} + \varepsilon)|W_{i^*}|$ vertices $y \in W_{i^*}$, there are at least ρn^{h-1} $(h-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ span copies of H. Thus, (iv) above together with the fact that $|W_{i^*}| \leq 8|B_{[n]}(x, \eta n/16)|$, and $\varepsilon \ll \gamma$ implies that the conclusion of the lemma holds. □

With Lemma 7.2 to hand, we can now prove the following result. Note that Lemma 7.3 together with Lemma 7.1 immediately imply Theorem 4.1.

Lemma 7.3. *Let H be an h-vertex ordered graph and $0 < \eta \ll 1/h$. Then there exists an $n_0 \in \mathbb{N}$ and $\xi > 0$ where $1/n_0 \ll \xi \ll \eta \ll 1/h$ so that the following holds. Set $s := \lceil 32/\eta \rceil$. Suppose that G is an ordered graph with vertex set $[n]$ where $n \geq n_0$ and where*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n.$$

Given any $x, y \in [n]$, there are at least ξn^{sh-1} $(sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings.

Proof. Choose ξ so that $0 < \xi \ll \rho'' \ll \rho' \ll \rho \ll \gamma \ll \eta$ where ρ and γ are as in Lemma 7.2. Let G be as in the statement of the lemma.

First suppose $x, y \in [n]$ and $|x - y| \leq \eta n/16$. Then by Lemma 7.2 there are at least $\eta n/20$ vertices z in $B_{[n]}(x, \eta n/16) \cap B_{[n]}(y, \eta n/16)$ for which

- there are at least ρn^{h-1} $(h-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{z\}]$ span copies of H;

- there are at least ρn^{h-1} $(h-1)$-sets $Y \subseteq V(G)$ such that both $G[Y \cup \{y\}]$ and $G[Y \cup \{z\}]$ span copies of H.

Choose z, X and Y to be disjoint; there are at least $\eta\rho^2 n^{2h-1}/((20h)!) > \rho' n^{2h-1}$ choices for the set $S := \{z\} \cup X \cup Y$. Notice that each such set S is chosen so that both $G[S \cup \{x\}]$ and $G[S \cup \{y\}]$ contain perfect H-tilings.

Next, we assume $|x - y| \leq \eta n/9$. Then by the above argument, there are at least $\eta n/72$ vertices z such that:

- $|x - z|, |y - z| \leq \eta n/16$;
- there are at least $\rho' n^{2h-1}$ $(2h-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{z\}]$ contain perfect H-tilings;
- there are at least $\rho' n^{2h-1}$ $(2h-1)$-sets $Y \subseteq V(G)$ such that both $G[Y \cup \{y\}]$ and $G[Y \cup \{z\}]$ contain perfect H-tilings.

Similarly as before, choose disjoint z, X, Y; there are at least $\rho'' n^{4h-1}$ choices for the set $S := \{z\} \cup X \cup Y$, for which both $G[S \cup \{x\}]$ and $G[S \cup \{y\}]$ contain perfect H-tilings.

More generally, for any $x, y \in [n]$, by repeated applications of the above argument we obtain some $t \leq s$ such that there are at least $\xi^{1/2} n^{th-1}$ $(th-1)$-sets $X' \subseteq V(G)$ such that both $G[X' \cup \{x\}]$ and $G[X' \cup \{y\}]$ contain perfect H-tilings. For each such set X' we have that $G \setminus X'$ contains more than $\rho n^h/2$ copies of H. Add $s - t$ such disjoint copies of H to obtain from X' a set X. Then X is as desired and there are at least ξn^{sh-1} choices for X. $\square$

7.2. **Proof of Theorem 4.6.** To prove Theorem 4.6, we need the following two lemmas to verify the hypothesis of Lemma 7.1.

Lemma 7.4. *Let H be an h-vertex ordered graph with $\chi_<(H) = 2$, which satisfies the following properties:*

(i) $1, \lceil h/2 \rceil, h$ *are isolated vertices;*
(ii) *all edges of H are between the intervals $A := [2, \lceil h/2 \rceil - 1]$ and $B := [\lceil h/2 \rceil + 1, h - 1]$.*

Let $0 < \eta < 1$. Then there exists an $n_0 \in \mathbb{N}$ and $\xi > 0$ where $1/n_0 \ll \xi \ll \eta, 1/h$ so that the following holds. Suppose that G is an ordered graph with vertex set $[n]$ where $n \geq n_0$ and where

$$\delta(G) \geq \eta n. \tag{16}$$

Given any $x, y \in [n]$, there are at least ξn^{h-1} $(h-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ span copies of H.

Proof. Let H_0 be a complete bipartite ordered graph with parts $S_0 < L_0$, where $|S_0| = |A| + 1$ and $|L_0| = |B| + 1$. For a copy of $H_0 \subseteq G$ and a vertex $v \in V(G) \setminus V(H_0)$, we say H_0 is *good for* v if one of the following holds: (a) $v < S_0 < L_0$; (b) $S_0 < v < L_0$; (c) $S_0 < L_0 < v$. By the assumption on H, if H_0 is good for v, then $G[V(H_0) \cup v]$ contains a spanning copy of H. Therefore, it is sufficient to find ξn^{h-1} copies of H_0 in G which are good for both x and y.

Without loss of generality we assume $x < y$. Let $V_1 := \{v \in [n] \mid v < x\}$, $V_2 := \{v \in [n] \mid x < v < y\}$, and $V_3 := \{v \in [n] \mid v > y\}$. By (16) and the pigeonhole principle, there exist $1 \leq i \leq j \leq 3$ such that $e(G[V_i, V_j]) \geq \eta n^2/13$. A standard application of the regularity method shows that there are at least ξn^{h-1} copies of H_0 in $G[V_i, V_j]$. By the construction of V_is, each such copy of H_0 is good for both x and y, and this completes the proof. $\square$

Recall that $s(H)$ is the smallest vertex in H that is adjacent to h, and if h is isolated, then $s(H) = 0$.

Lemma 7.5. *Let H be an h-vertex ordered graph with $\chi_<(H) = 2$, which satisfies the following properties:*

(i) 1 *and* $\lceil h/2 \rceil$ *are isolated vertices;*
(ii) *all edges of H are between the intervals* $[2, \lceil h/2\rceil - 1]$ *and* $[\lceil h/2\rceil + 1, h]$*;*
(iii) $1 \leq s(H) < \lceil h/2 \rceil$ *and* $[s(H), h-1]$ *is an independent set.*

Let $0 < \eta \ll 1/h$. Then there exists an $n_0 \in \mathbb{N}$ and $\xi > 0$ where $1/n_0 \ll \xi \ll \eta \ll 1/h$ so that the following holds. Set $s := 2(\lceil h/2\rceil - s(H))$. Suppose that G is an ordered graph with vertex set $[n]$ where $n \geq n_0$ and

$$\delta(G) \geq \eta n. \tag{17}$$

For every $x, y \in [n]$, there are at least ξn^{sh-1} $(sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings.

Proof. Choose $0 < \xi \ll \xi_1, \xi_2 \ll \xi_3 \ll \varepsilon' \ll \varepsilon \ll d \ll d' \ll \eta \ll 1/h$, and without loss of generality we always assume $x < y$. Let $A := [2, s(H)-1]$, $B := [s(H), \lceil h/2\rceil - 1]$, and $C := [\lceil h/2\rceil + 1, h-1]$. We also write $a := |A|$, $b := |B|$ and $c := |C|$, then $h = a + b + c + 3$. Note that A and $B \cup C$ are independent sets of H, and every edge of H is either between intervals A and C, or between B and h.

Claim 7.6. *For $x, y \leq (1-\eta/3)n$, there are at least $\xi_1 n^{h-1}$ $(h-1)$-sets $X_1 \subseteq V(G)$ such that both $G[X_1 \cup \{x\}]$ and $G[X_1 \cup \{y\}]$ span copies of H.*

Proof. Let H_1 be a complete bipartite ordered graph with parts $S_1 < L_1$, where $|S_1| = a + b + 1$ and $|L_1| = c + 1$. For a copy of $H_1 \subseteq G$ and a vertex $v \in V(G) \setminus V(H_1)$, we say H_1 is *good for* v if either $v < S_1 < L_1$, or $S_1 < v < L_1$. Note that $G[V(H_1) \cup v]$ contains a spanning copy of H, if H_1 is good for v. Now let $V_1 := \{v \in [n] \mid v < x\}$, $V_2 := \{v \in [n] \mid x < v < y\}$, $V_3 := \{v \in [n] \mid y < v \leq (1-\eta/3)n\}$, and $V_0 = [(1-\eta/3)n + 1, n]$. By (17) and the pigeonhole principle, there exists $i \in [3]$ such that

$$e(G[V_i, V_0]) \geq \frac{1}{3}\left(\frac{\eta n}{3}\right)\left(\eta n - \frac{\eta n}{3}\right) \geq \frac{2\eta^2 n^2}{27}. \tag{18}$$

Notice that for any choice of $i \in [3]$, every copy of H_1 in $G[V_i, V_0]$ is good for both x and y. So, as in the proof of Lemma 7.4, (18) implies that there are $\xi_1 n^{h-1}$ copies of H_1 in $G[V_i, V_0]$ which are good for both x and y, as desired. □

Claim 7.7. *For $x \leq \eta n/3$ and $y \geq (1-\eta/3)n$, there are at least $\xi_2 n^{bh-1}$ $(bh-1)$-sets $X_2 \subseteq V(G)$ such that both $G[X_2 \cup \{x\}]$ and $G[X_2 \cup \{y\}]$ contain perfect H-tilings.*

Proof. Let $M := [x+1, y-1]$. By (17), we have $|N(y) \cap M| \geq \eta n - 2\eta n/3 = \eta n/3$. Let N be a subset of $N(y) \cap M$ of size $\eta n/6$; then

$$e(G[N, M-N]) \geq \left(\frac{\eta n}{6}\right)\left(\eta n - \frac{2\eta n}{3} - \frac{\eta n}{6}\right) = \frac{\eta^2 n^2}{36}.$$

A standard application of the regularity method shows that there exists an (ε', d')-regular pair (P', Q') in G, where $P' \subseteq N$; $Q' \subseteq M - N$; $|P'|, |Q'| \geq 2\xi_3 n$. By Lemma 6.2 and Lemma 5.3, there exist sets $P \subseteq P'$ and $Q \subseteq Q'$ such that: $|P|, |Q| \geq \xi_3 n$; (P, Q) is an (ε, d)-regular pair in G; either $P < Q$ or $Q < P$.

Case 1: $P < Q$**.**
Let H_2 be a complete bipartite ordered graph with parts $S_2 < L_2$, where $|S_2| = a + b + 1$ and $|L_2| = c + 1$. By Lemma 5.7, there are at least $\xi_2 n^{h-1}$ copies of H_2 in $G[P, Q]$, and for every such H_2, we have $x < S_2 < L_2 < y$ and $S_2 \subseteq P \subseteq N \subseteq N(y)$. Recall that each edge of H lies either between A and C, or between B and h. Therefore, $G[V(H_2) \cup \{x\}]$ contains a spanning copy of H, as H_2 is a complete bipartite graph. Similarly, $G[V(H_2) \cup \{y\}]$ also contains a spanning copy of H, as H_2 is a complete bipartite graph and $S_2 \subseteq N(y)$. Hence, there are at least $\xi_2 n^{h-1}$ $(h-1)$-sets $X' \subseteq V(G)$ such that both $G[X' \cup \{x\}]$ and $G[X' \cup \{y\}]$ contain perfect H-tilings. By adding $b-1$ additional disjoint copies of H, one can easily see that Claim 7.7 holds in this case.

Case 2: $Q < P$**.**
Let F_1 be the complete bipartite ordered graph with parts $S'_1 < L'_1$, where $|S'_1| = a + b + 1$ and $|L'_1| = c+2$. Let F_2 be a complete bipartite ordered graph with parts $S'_2 < L'_2$, where $|S'_2| = a+b+2$ and $|L'_2| = c + 1$. Note that both F_1 and F_2 contain a spanning copy of H.

Let F_3 be the complete bipartite ordered graph with parts $S'_3 < L'_3$, where $|S'_3| = a + b$ and $|L'_3| = c + 2$. We say a copy of F_3 is *good for* x if $x < S'_3 < L'_3$. Note that $G[V(F_3) \cup \{x\}]$ contains a spanning copy of H, if F_3 is good for x. Lastly, let F_4 be the complete bipartite ordered graph with parts $S'_4 < L'_4$, where $|S'_4| = a + 1$ and $|L'_4| = b + c + 1$. We say a copy of F_4 is *good for* y if $S'_4 < L'_4 < y$ and $L'_4 \subseteq N(y)$. Observe that $G[V(F_4) \cup \{y\}]$ contains a spanning copy of H, if F_4 is good for y.

Let H_3 be the complete bipartite ordered graph with parts $S_3 < L_3$, where $|S_3| = b(a+b+1)-1$ and $L_2 = b(c+2)$. By Lemma 5.7, there are at least $\xi_2 n^{bh-1}$ copies of H_3 in $G[P, Q]$, and for every such H_3, we have $x < S_3 < L_3 < y$ and $L_3 \subseteq P \subseteq N(y)$. Note that such H_3 can be decomposed into $b - 1$ copies of F_1 and one good copy of F_3. This indicates that $G[V(H_3) \cup \{x\}]$ contains a perfect H-tiling. Similarly, such H_3 can also be decomposed into $b - 1$ copies of F_2 and one good copy of F_4, which indicates that $G[V(H_3) \cup \{y\}]$ contains a perfect H-tiling. □

For every $x, y \leq (1-\eta/3)n$ or $x \leq \eta n/3$ and $y \geq (1-\eta/3)n$, simply adding enough disjoint copies of H to the sets obtained from Claims 7.6 or 7.7 completes the proof. For every $x \geq \eta n/3$ and $y \geq (1 - \eta/3)n$, there are at least $\eta n/3$ vertices z (i.e. the vertices in $[\eta n/3]$) such that: (i) $\{y, z\}$ satisfies the condition of Claim 7.7; (ii) $\{x, z\}$ either satisfies the condition of Claim 7.6 or Claim 7.7. Applying Claims 7.6 and 7.7 on pairs $\{x, z\}$ and $\{y, z\}$ produces many disjoint copies of z, X_1, X_2 (or z, X_2, X_2). Finally, adding enough extra disjoint copies of H to $z \cup X_1 \cup X_2$ (or $z \cup X_2 \cup X_2$), we show that for every $x, y \in [n]$, there are at least $\xi n^{2bh-1} = \xi n^{sh-1}$ $(sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings. □

Proof of Theorem 4.6. Since H has property A, it satisfies the following conditions:

- all edges of H are between the intervals $[1, \lceil h/2 \rceil - 1]$ and $[\lfloor h/2 \rfloor + 2, h]$.
- if h is even, then the vertices $h/2, h/2+1$ are isolated; if h is odd, then the vertex $(h+1)/2$ is isolated.

Furthermore, since H does not have property B, at least one of 1 and h must be isolated in H.

We first assume that both $1, h$ are isolated, then by Definition 1.8 neither of them has Property C. Note that H satisfies the assumptions in Lemma 7.4. Together with Lemma 7.1, this immediately implies Theorem 4.6.

Now without loss of generality, we assume that 1 is isolated in H and $s(H) > 1$. Since h does not have Property C, by Definition 1.8 $[s(H), h-1]$ is an independent set in H. Similarly, H satisfies the assumptions in Lemma 7.5, and this, together with Lemma 7.1, completes the proof. □

8. Concluding remarks

In this paper we have introduced a general framework for the perfect H-tiling problem in ordered graphs. This approach can be summarized as follows:

Step 1: Find a candidate extremal example; an n-vertex ordered graph G with minimum degree $\delta(G) = \alpha n - O(1)$ without a perfect H-tiling.
Step 2: Find a bottlegraph B assigned to H with $\alpha \geq 1 - 1/\chi_{cr}(B)$.
Step 3: If $\alpha \geq 1 - 1/\chi_{<}(H)$ then Theorems 4.1 and 4.3 now combine to yield the asymptotically exact threshold. Otherwise, one seeks an improved absorbing theorem, using structural information about H (á la Theorem 4.6).

Despite introducing this framework, we suspect determining the general perfect H-tiling threshold will be challenging in the sense that there could be a range of different extremal examples and optimal bottlegraphs, depending on the precise structure of H.

On the other hand, in the case when H is an h-vertex ordered graph and $12 \in E(H)$ or $(h-1)h \in E(H)$, it is actually straightforward to deduce from Theorem 1.1 the minimum degree threshold for forcing a perfect H-tiling.

Proposition 8.1. *Let $n, h \in \mathbb{N}$ such that $h|n$. Suppose H is an h-vertex ordered graph. If G is an n-vertex ordered graph with $\delta(G) \geq (1 - 1/h)n$, then G contains a perfect H-tiling.*

Moreover, suppose $12 \in E(H)$ or $(h-1)h \in E(H)$. Then there are n-vertex ordered graphs with $\delta(G) \geq (1 - 1/h)n - 1$ that do not contain a perfect H-tiling.

Proof. Consider the unordered underlying graph G' of any ordered n-vertex graph G with $\delta(G) \geq (1 - 1/h)n$. Theorem 1.1 implies that G' contains a perfect K_h-tiling. Since any ordered copy of K_h contains H, this ensures G contains a perfect H-tiling.

For the moreover part, notice such H satisfy $\alpha^*(H) = 1/h$. The result then follows directly from Proposition 1.5. □

In [12], Komlós determined the minimum degree threshold for an (unordered) graph to contain an H-tiling covering a given proportion of the vertices; it would be interesting to obtain an ordered analogue of this result.

Question 8.2. *Let $s \in (0, 1)$ and H be an ordered graph. What is the minimum degree threshold that ensures an ordered graph G contains an H-tiling cover at least an sth proportion of its vertices?*

There has also been interest in Ramsey and Turán properties of *edge ordered graphs* (see e.g. [3, 24]; it would be interesting to study the perfect H-tiling problem in this setting also.

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Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, IL, USA, and Moscow Institute of Physics and Technology, Russian Federation. Partially supported by NSF Grant DMS-1764123 and Arnold O. Beckman Research Award (UIUC) Campus Research Board 18132 and the Langan Scholar Fund (UIUC).
E-mail address: `jobal@illinois.edu`

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, USA.
E-mail address: `linali2@illinois.edu`

School of Mathematics, University of Birmingham, Edgbaston, Birmingham, UK.
E-mail address: `a.c.treglown@bham.ac.uk`