On degree sequences forcing the square of a Hamilton cycle

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University of Birmingham

Las Vegas, April 2015

Joint work with
Katherine Staden (University of Warwick)

Spanning subgraphs of graphs

Question

When does a graph G contain a given spanning subgraph H?

$(|G| = n)$

Natural spanning structures H:

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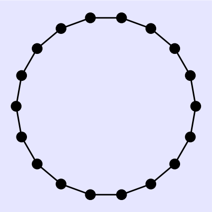

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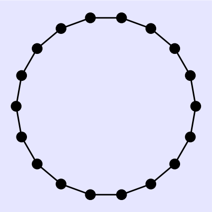
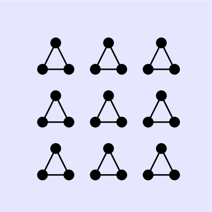


2. Perfect K_r-packing



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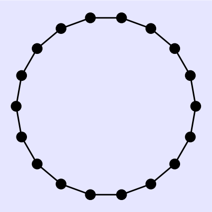

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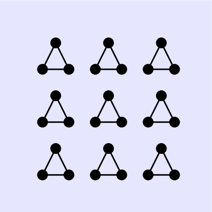

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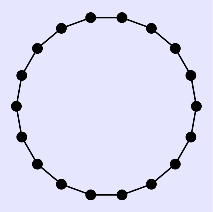

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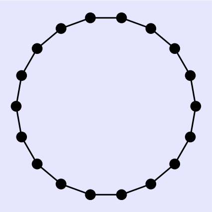

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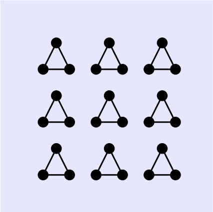

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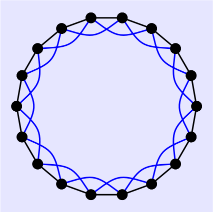

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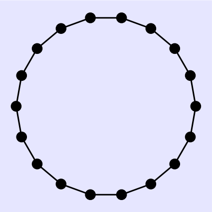

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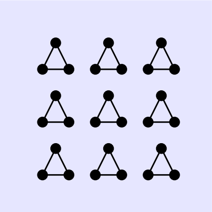

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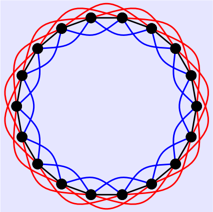

Minimum degree conditions

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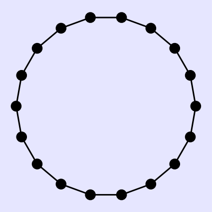

Dirac 1952

$\delta(G) \geq n/2$

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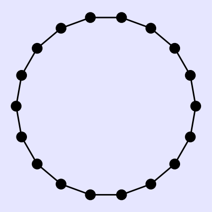

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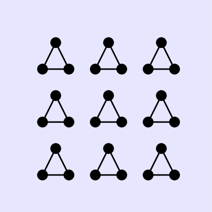

Hajnal-Szemerédi 1970

$r|n$

$\delta(G) \geq \frac{r-1}{r} n$

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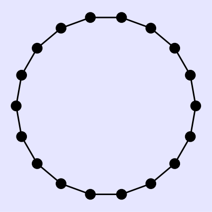

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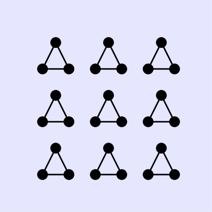

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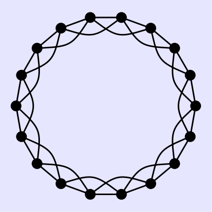

(Pósa 1962, Seymour 1974) Komlós-Sárközy-Szemerédi 1998

$\delta(G) \geq \frac{r}{r+1} n$

Degree sequence conditions

- The minimum degree condition in each of these results is best-possible.
- This does not mean we cannot strengthen these results considerably though.

Degree sequence of G: write the degrees of vertices in G as $d_1 \leq d_2 \leq \ldots \leq d_n$.

Degree sequence results

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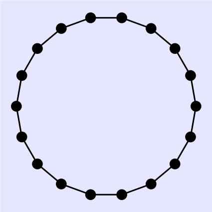
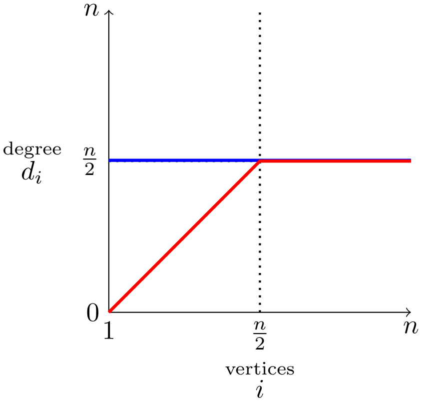


Pósa 1962

$d_i \geq i + 1$ for all $i < \frac{n}{2}$



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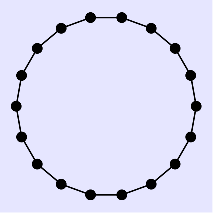
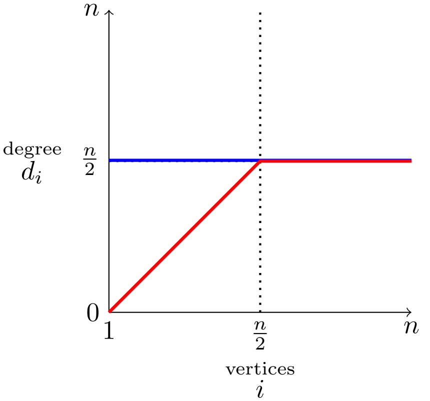


Chvátal 1972

$d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < \frac{n}{2}$



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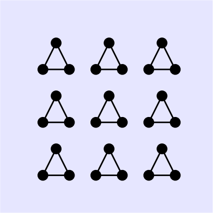
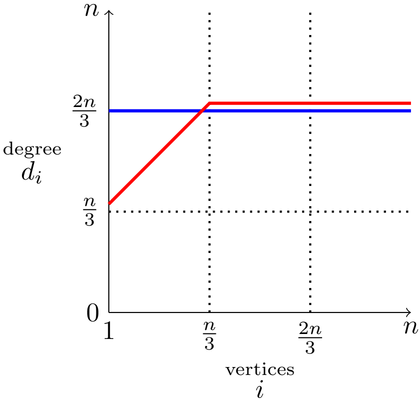


T. 2014$^+$

$$d_i \geq \left(\frac{r-2}{r} + \eta\right) n + i \quad \text{for all } i \leq \frac{n}{r}$$



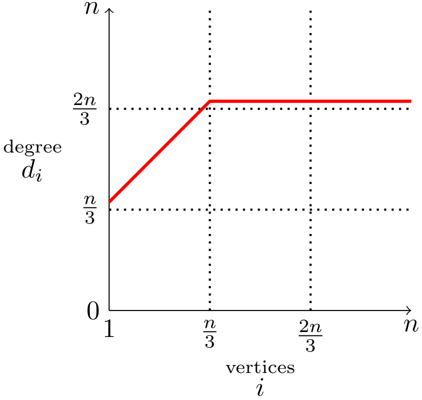
Main result

Theorem (Staden and T., 2014^+)

$\forall\ \eta > 0\ \exists\ n_0 \in \mathbb{N}$ s.t. if G on $n \geq n_0$ vertices with

$$d_i \geq \left(\frac{1}{3} + \eta\right) n + i \quad \text{for all } i \leq \frac{n}{3}$$

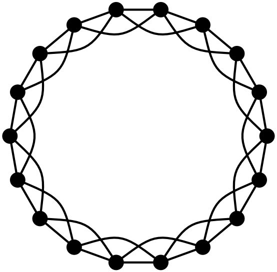
$\implies$ G contains the square of a Hamilton cycle.



Doesn't quite imply Komlós–Sárközy–Szemerédi theorem.

An extremal example

If $3|n$ and G contains the square of a Hamilton cycle, then G contains a perfect triangle packing.



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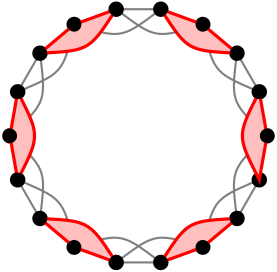
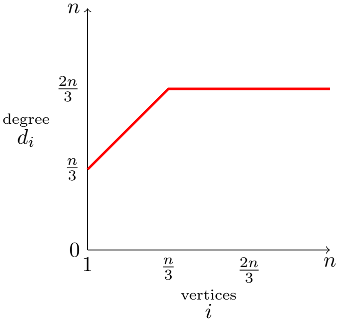


An extremal example

$d_i \geq \frac{n}{3} + i$ for all $i < \frac{n}{3}$ would be best possible for triangle packing:



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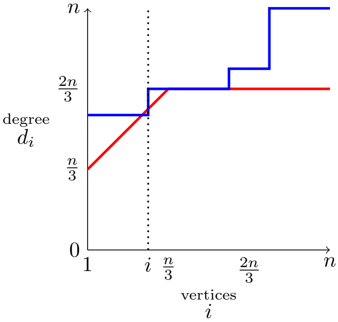

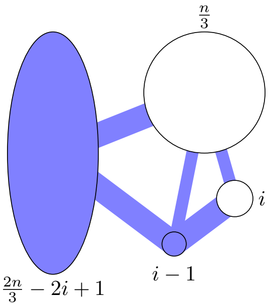

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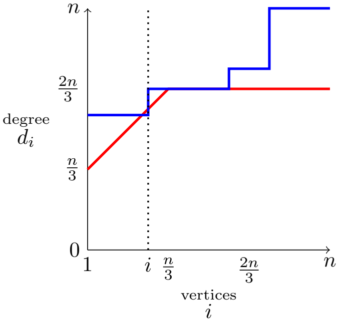

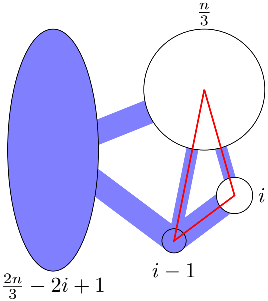

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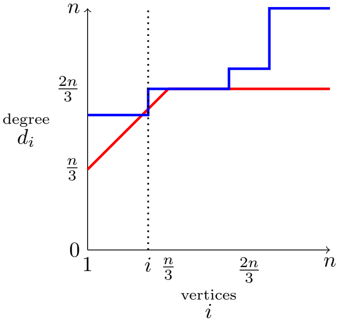

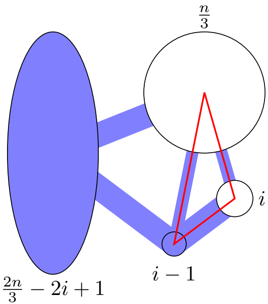

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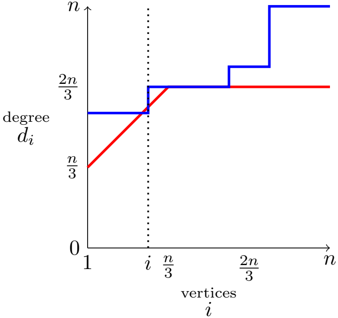

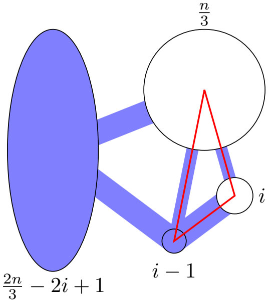

So our result is best possible up to the ηn term.
...but in fact ηn cannot be replaced by $o(\sqrt{n})$.

Proof idea

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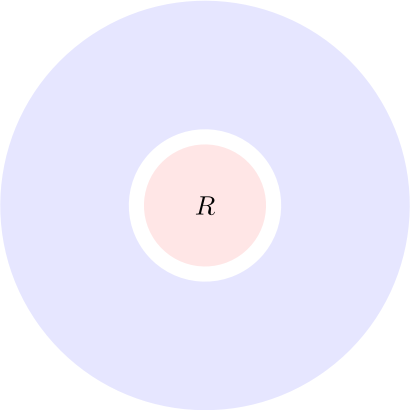

- Remove a randomly chosen **reservoir** R of order $o(n)$. Any two square paths can be connected via R.

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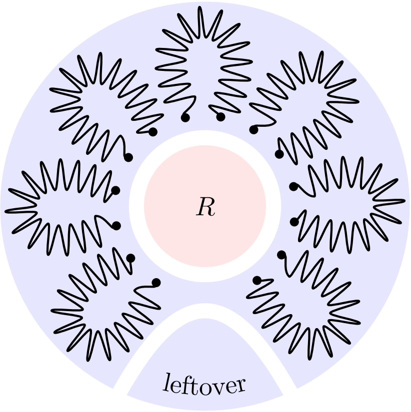

- Remove a randomly chosen **reservoir** R of order $o(n)$. Any two square paths can be connected via R.
- Find a collection of vertex-disjoint square paths which cover $(1 - o(1))n$ vertices in the remaining graph.

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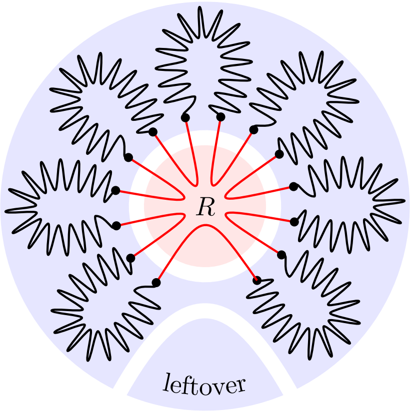

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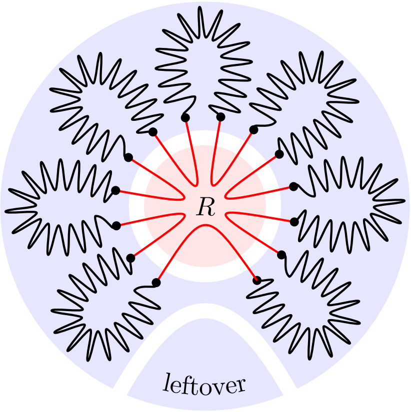

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$\longrightarrow$ Square cycle of length $(1 - o(1))n$.

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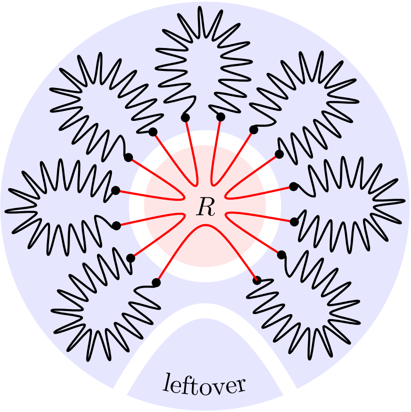

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$\ldots \longrightarrow$ Hamilton square cycle.

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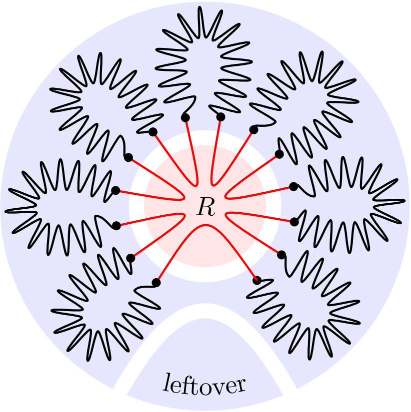

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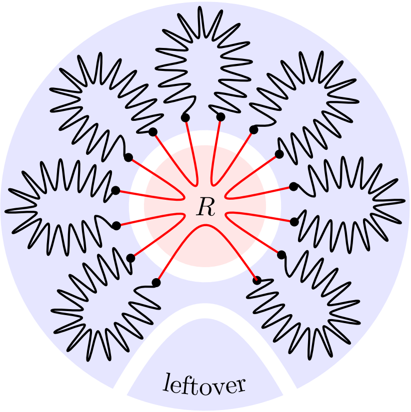

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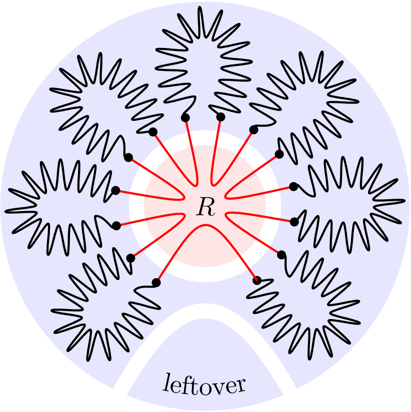

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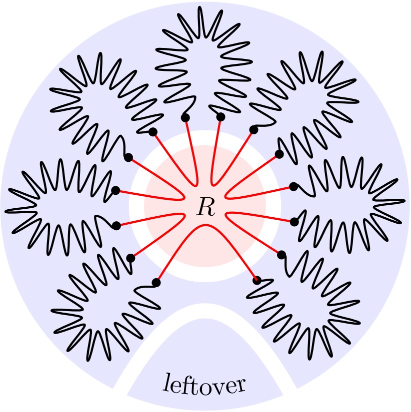

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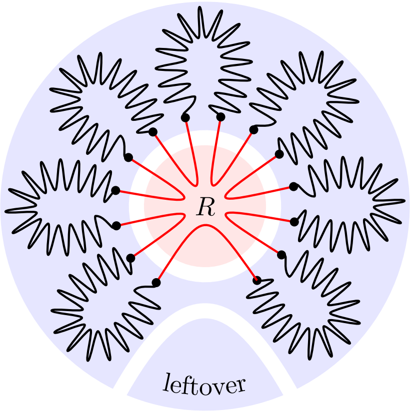

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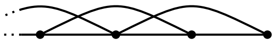

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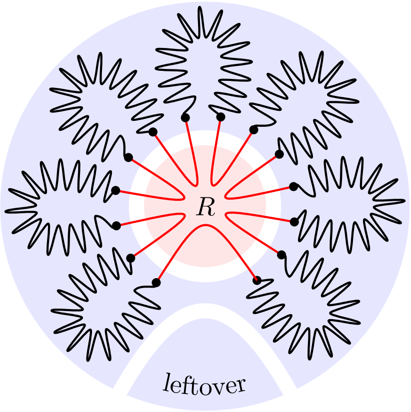

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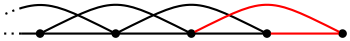

Open problems and questions

- Is it true that any (large) graph G on n vertices with degree sequence at least

$$\frac{n}{3}+1+\eta n, \frac{n}{3}+2+\eta n, \quad \dots, \quad \frac{2n}{3}, \frac{2n}{3}, \quad \dots, \quad \frac{2n}{3}$$

contains a square Hamilton cycle?
($\implies$ Komlós-Sárközy-Szemerédi)

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  contains a square Hamilton cycle?
  ($\implies$ Komlós-Sárközy-Szemerédi)

- What about higher powers of Hamilton cycles? Does the degree sequence

$$d_i \geq \left(\frac{r-1}{r+1}+\eta\right)n+i \quad \text{for } i \leq \frac{n}{r+1}$$

  guarantee the r^{th} power of a Hamilton cycle?