Cycles of every length and orientation in randomly perturbed digraphs

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Joint work with Igor Araujo, József Balogh, Robert Krueger and Simón Piga
Theorem (Dirac 1952)

A $n$-vertex graph, $\delta(G) \geq n/2 \implies G$ contains a Hamilton cycle.

- The minimum degree condition here is tight.
- Also natural to look at analogous question for digraphs.
- A digraph has at most two edges between every pair of vertices; at most one oriented in each direction.
In the setting of digraphs there are different types of Hamilton cycle and also more than one notion of minimum degree.

- **Consistently oriented Hamilton cycle**: edges oriented cyclically
The extremal problem

- Given a digraph $G$, the **minimum total degree** $\delta(G)$ is the minimum number of edges incident to a vertex.
- The **minimum semi-degree** $\delta^0(G)$ is the minimum of the minimum in- and outdegrees.

**Theorem (Ghouila-Houri 1960)**

A $n$-vertex strongly connected digraph, $\delta(G) \geq n \implies G$ contains a consistently oriented Hamilton cycle.

**Corollary**

$\delta^0(G) \geq n/2 \implies G$ contains a consistently oriented Hamilton cycle.

- Degree conditions here are tight.
The extremal problem

- **Antidirected Hamilton cycle**: edges alternate direction.

![Antidirected Hamilton cycle diagram]

**Theorem (DeBiasio and Molla, 2015)**

A sufficiently large $2m$-vertex digraph, $\delta^0(G) \geq m + 1 \implies G$ contains an antidirected Hamilton cycle.

- Minimum semi-degree condition here tight.

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The extremal problem

Theorem (DeBiasio, Kühn, Molla, Osthus and Taylor, 2015)

\( G \) sufficiently large \( n \)-vertex digraph, \( \delta^0(G) \geq n/2 \) \( \implies \) \( G \) contains every orientation of a Hamilton cycle except perhaps the antidirected Hamilton cycle.

- Minimum semi-degree condition here tight.
The random digraph problem

- binomial random digraph $D(n, p)$: digraph with vertex set $[n]$, where each of the $n(n - 1)$ possible directed edges is present with probability $p$, independently of all other edges.

**Theorem (Frieze, 1988)**

If $p = (\log n + \omega(1))/n$ then asymptotically almost surely (a.a.s.) $D(n, p)$ contains a consistently oriented Hamilton cycle.

If $p = (\log n - \omega(1))/n$ then a.a.s. $D(n, p)$ contains a vertex with no in-neighbour or no out-neighbour.

**Theorem (Montgomery, 2021+)**

Determined threshold for $D(n, p)$ containing a given orientation of a Hamilton cycle. Threshold can vary from $p = \log n/2n$ to $p = \log n/n$. 
The model of randomly perturbed (di)graphs was introduced in 2003 by Bohman, Frieze and Martin.

Given a dense (di)graph $G$, how many random edges does one need to add to $G$ so that, asymptotically almost surely, the resulting (di)graph has a given property?

This model has been studied for a range of properties including Hamilton cycles; spanning trees; $H$-factors; Ramsey properties.
### Theorem (Bohman, Frieze and Martin, 2003)

\[ \forall \alpha > 0, \exists C = C(\alpha) \text{ s.t. if } G \text{ n-vertex digraph with } \delta^0(G) \geq \alpha n, \text{ then } G \cup D(n, C/n) \text{ a.a.s. contains a consistently oriented Hamilton cycle.} \]

- Cannot lower the probability here.

The following provides a pancyclicity version of this result.

### Theorem (Krivelevich, Kwan, and Sudakov, 2016)

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Main result

Theorem (Araujo, Balogh, Krueger, Piga and T., 2022+)

∀ α > 0, ∃ C = C(α) s.t. if G n-vertex digraph with δ₀(G) ≥ αn, then G ∪ D(n, C/n) a.a.s. contains every orientation of a cycle of every possible length.

- Note this is a universality result.
- This method not so applicable for arbitrary orientations of Hamilton cycles.
- Instead we apply the absorbing method (which comes with its own challenges for arbitrary orientations).
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Final remarks

We relaxed the minimum semi-degree in our result to a minimum total degree, for most orientations of a cycle.

**Theorem (Araujo, Balogh, Krueger, Piga and T., 2022+)**

\[ \forall \alpha, \eta > 0, \exists C = C(\alpha, \eta) \text{ s.t. if } G \text{ n-vertex digraph with } \delta(G) \geq 2\alpha n, \text{ then } G \cup D(n, C/n) \text{ a.a.s. contains every orientation of a cycle of every possible length that contains at most } (1 - \eta)n \text{ vertices of indegree 1.} \]

- It would be interesting to obtain other minimum total degree results in the setting of randomly perturbed digraphs.
- For example, for the \( T_k \)-factor problem where \( T_k \) is the transitive tournament on \( k \) vertices.