Cycles of every length and orientation in randomly perturbed digraphs

Andrew Treglown

University of Birmingham

1st September 2023

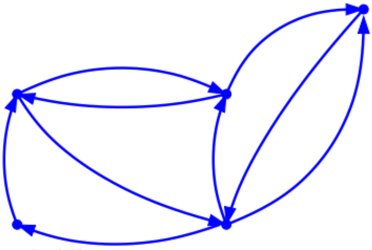
Joint work with Igor Araujo, József Balogh, Robert Krueger and Simón Piga

Introduction

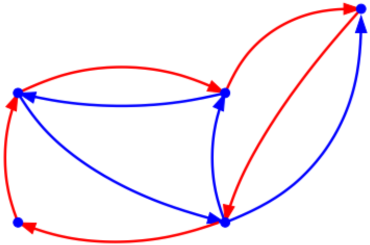
Theorem (Dirac 1952)

G n-vertex graph, $\delta(G) \geq n/2 \implies G$ contains a Hamilton cycle.

- The minimum degree condition here is tight.
- Also natural to look at analogous question for *digraphs*.
- A digraph has at most two edges between every pair of vertices; at most one oriented in each direction.



- In the setting of digraphs there are different types of Hamilton cycle and also more than one notion of minimum degree.
- Consistently oriented Hamilton cycle: edges oriented cyclically



The extremal problem

- Given a digraph G, the minimum total degree $\delta(G)$ is the minimum number of edges incident to a vertex.
- The minimum semi-degree $\delta^0(G)$ is the minimum of the minimum in- and outdegrees.

Theorem (Ghouila-Houri 1960)

G n-vertex **strongly connected digraph**, $\delta(G) \geq n \implies G$ *contains a consistently oriented Hamilton cycle.*

Corollary

$\delta^0(G) \geq n/2 \implies G$ *contains a consistently oriented Hamilton cycle.*

- Degree conditions here are tight.

The extremal problem

- Antidirected Hamilton cycle: edges alternate direction.

Theorem (DeBiasio and Molla, 2015)

G sufficiently large $2m$-vertex digraph, $\delta^0(G) \geq m+1 \implies G$ contains an antidirected Hamilton cycle.

- Minimum semi-degree condition here tight.

Theorem (DeBiasio, Kühn, Molla, Osthus and Taylor, 2015)

G sufficiently large n-vertex digraph, $\delta^0(G) \geq n/2 \implies G$ contains **every** *orientation of a Hamilton cycle except perhaps the antidirected Hamilton cycle.*

- Minimum semi-degree condition here tight.

The random digraph problem

- binomial random digraph $D(n,p)$: digraph with vertex set $[n]$, where each of the $n(n-1)$ possible directed edges is present with probability p, independently of all other edges.

Theorem (Frieze, 1988)

If $p = (\log n + \omega(1))/n$ then asymptotically almost surely (a.a.s.) $D(n,p)$ contains a consistently oriented Hamilton cycle.

- If $p = (\log n - \omega(1))/n$ then a.a.s. $D(n,p)$ contains a vertex with no in-neighbour or no out-neighbour.

Theorem (Montgomery, 2021+)

Determined threshold for $D(n,p)$ containing a given orientation of a Hamilton cycle. Threshold can vary from $p = \log n/2n$ to $p = \log n/n$.

Randomly perturbed digraphs

- The model of randomly perturbed (di)graphs was introduced in 2003 by Bohman, Frieze and Martin.
- Given a dense (di)graph G, how many random edges does one need to add to G so that, asymptotically almost surely, the resulting (di)graph has a given property?
- This model has been studied for a range of properties including Hamilton cycles; spanning trees; H-factors; Ramsey properties.

Randomly perturbed digraphs

Theorem (Bohman, Frieze and Martin, 2003)

$\forall\, \alpha > 0$, $\exists\ C = C(\alpha)$ s.t. if G n-vertex digraph with $\delta^0(G) \geq \alpha n$, then $G \cup D(n, C/n)$ a.a.s. contains a consistently oriented Hamilton cycle.

- Cannot lower the probability here.

The following provides a pancyclicity version of this result.

Theorem (Krivelevich, Kwan, and Sudakov, 2016)

$\forall\, \alpha > 0$, $\exists\ C = C(\alpha)$ s.t. if G n-vertex digraph with $\delta^0(G) \geq \alpha n$, then $G \cup D(n, C/n)$ a.a.s. contains a consistently oriented cycle of every length between 2 and n.

Randomly perturbed digraphs

Theorem (Bohman, Frieze and Martin, 2003)

$\forall\, \alpha > 0$, $\exists\ C = C(\alpha)$ s.t. if G n-vertex digraph with $\delta^0(G) \geq \alpha n$, then $G \cup D(n, C/n)$ a.a.s. contains a consistently oriented Hamilton cycle.

- Cannot lower the probability here.

The following provides a pancyclicity version of this result.

Theorem (Krivelevich, Kwan, and Sudakov, 2016)

$\forall\, \alpha > 0$, $\exists\ C = C(\alpha)$ s.t. if G n-vertex digraph with $\delta^0(G) \geq \alpha n$, then $G \cup D(n, C/n)$ a.a.s. contains a consistently oriented cycle of every length between 2 and n.

Main result

Theorem (Araujo, Balogh, Krueger, Piga and T., 2022+)

$\forall\, \alpha > 0$, $\exists\ C = C(\alpha)$ s.t. if G n-vertex digraph with $\delta^0(G) \geq \alpha n$, then $G \cup D(n, C/n)$ a.a.s. contains every orientation of a cycle of every possible length.

- Note this is a **universality** result.
- Proofs of Bohman–Frieze–Martin and Krivelevich–Kwan–Sudakov use rotation-extension type arguments.
- This method not so applicable for arbitrary orientations of Hamilton cycles.
- Instead we apply the absorbing method (which comes with its own challenges for arbitrary orientations).

Main result

Theorem (Araujo, Balogh, Krueger, Piga and T., 2022+)

$\forall\, \alpha > 0$, $\exists\ C = C(\alpha)$ *s.t. if* G *n-vertex digraph with* $\delta^0(G) \geq \alpha n$, *then* $G \cup D(n, C/n)$ *a.a.s. contains every orientation of a cycle of every possible length.*

- Note this is a **universality** result.
- Proofs of Bohman–Frieze–Martin and Krivelevich–Kwan–Sudakov use rotation-extension type arguments.
- This method not so applicable for arbitrary orientations of Hamilton cycles.
- Instead we apply the absorbing method (which comes with its own challenges for arbitrary orientations).

Final remarks

We relaxed the minimum semi-degree in our result to a minimum total degree, for most orientations of a cycle.

Theorem (Araujo, Balogh, Krueger, Piga and T., 2022+)

$\forall\, \alpha, \eta > 0$, $\exists\ C = C(\alpha, \eta)$ s.t. if G n-vertex digraph with $\delta(G) \geq 2\alpha n$, then $G \cup D(n, C/n)$ a.a.s. contains every orientation of a cycle of every possible length that contains at most $(1-\eta)n$ vertices of indegree 1.

- It would be interesting to obtain other minimum total degree results in the setting of randomly perturbed digraphs
- For example, for the T_k-factor problem where T_k is the transitive tournament on k vertices.