CYCLE TILINGS AND H-FACTORS IN DIRECTED GRAPHS

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Abstract. We prove several results concerning cycle tilings and H-factors in digraphs. We provide a minimum semi-degree condition for forcing a digraph to contain a given spanning collection of vertex-disjoint orientations of cycles. Our result is asymptotically best possible for odd cycles and can be viewed as a digraph analogue of the El-Zahar conjecture. In addition, we asymptotically determine the minimum degree threshold for forcing an H-factor in a digraph for a range of digraphs H, including the cases when H is a tree or anti-directed cycle. Furthermore, an asymptotically exact Ore-type result for forcing a transitive tournament factor in a digraph is proven. Several related open problems are also highlighted.

1. Introduction

1.1. **Minimum degree conditions forcing H-factors.** The focus of this paper is on cycle tilings and H-factors in digraphs. Unless stated otherwise, the digraphs we consider do not have loops and we allow for at most one edge in each direction between any pair of vertices. For a digraph G we write $V(G)$ for its vertex set and $E(G)$ for its edge set, and define $|G| := |V(G)|$. Given $x, y \in V(G)$, we write xy for the edge directed from x to y in G. Given two (di)graphs H and G, an *H-tiling* in G is a collection of vertex-disjoint copies of H in G. An *H-factor* in G is an H-tiling that covers all the vertices of G. We write T_r to denote the transitive tournament on r vertices.

The following celebrated result of Hajnal and Szemerédi [15] determines the minimum degree threshold for forcing a K_r-factor in a graph.

Theorem 1.1. [15] *Let $n \in \mathbb{N}$ be divisible by $r \in \mathbb{N}$. If G is an n-vertex graph with $\delta(G) \geq (1 - 1/r)n$, then G contains a K_r-factor. Moreover, the bound on $\delta(G)$ is tight.*

In the setting of digraphs, there is more than one natural version of minimum degree. The *minimum (total) degree* $\delta(G)$ of a digraph G is the minimum number of edges incident to a vertex in G. The *minimum semi-degree* $\delta^0(G)$ of a digraph G is the minimum of all the in- and outdegrees of the vertices in G. In [28], the second author determined the minimum semi-degree $\delta^0(G)$ threshold for forcing an H-factor in a digraph G for any fixed *tournament* H (for sufficiently large digraphs G). Moreover, the analogous threshold for the minimum degree $\delta(G)$ version of the problem was established through results from [5, 6, 31]. For example, in [5] it was proven that given any $n \in \mathbb{N}$ divisible by r, every n-vertex digraph G with $\delta(G) \geq 2(1 - 1/r)n - 1$ contains a T_r-factor. Viewing a graph as a digraph where every edge is a 'double edge', one immediately sees that this result implies Theorem 1.1.

Building on a number of earlier results, Kühn and Osthus [20] determined, up to an additive constant, the minimum degree threshold for forcing an H-factor in a graph G, for *any* fixed graph H. Our first new result provides a digraph analogue of the Kühn–Osthus theorem for a certain type of H-factor. Given digraphs H and F, a *homomorphism* from H into F is a mapping $\phi : V(H) \rightarrow$

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$V(F)$ such that $\phi(x)\phi(y) \in E(F)$ for every $xy \in E(G)$. A *directed path* P is a path $v_1 \dots v_k$ where $v_i v_{i+1} \in E(P)$ for all $i \in [k-1]$.

Theorem 1.2. *Let H be a digraph that has a homomorphism into a directed path. Given any $\eta > 0$, there exists $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ divisible by $|H|$. If G is an n-vertex digraph with*

$$\delta(G) \geq (1+\eta)n,$$

then G contains an H-factor.

For many choices of H, the minimum degree condition in Theorem 1.2 is asymptotically best possible. For example, suppose H is a connected digraph. Let G be the n-vertex digraph that is the disjoint union of two complete digraphs G_1 and G_2 chosen such that the sizes of G_1 and G_2 are as equal as possible under the constraint that $|H|$ does not divide $|G_1|$ and $|G_2|$. Then $\delta(G) \geq n-4$ and G does not contain an H-factor.

A digraph C is an *orientation of a cycle* (or simply a *cycle*) if its underlying graph is a cycle; in particular, there are no double edges in C. A *directed cycle* C_k on k vertices is a cycle $v_1 \dots v_k v_1$ where $v_i v_{i+1}, v_k v_1 \in E(C_k)$ for each $i \in [k-1]$. We say that a cycle C is *balanced* if, when traversing C, the number of forward edges equals the number of backward edges; thus, if C is balanced it must have even order. Well-studied balanced cycles include anti-directed cycles. It is straightforward to see that balanced cycles have homomorphisms into directed paths, and so we obtain the following corollary.

Corollary 1.3. *Let C be a balanced cycle. Given any $\eta > 0$, there exists $n_0 = n_0(\eta, C) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ divisible by $|C|$. If G is an n-vertex digraph with*

$$\delta(G) \geq (1+\eta)n,$$

then G contains a C-factor.

It would be interesting to determine the minimum degree threshold for forcing a C-factor in a digraph for other orientations of a cycle C. For directed cycles this threshold is known. Indeed, Wang [31] showed that if $n \in \mathbb{N}$ is divisible by 3, then every n-vertex digraph G with $\delta(G) \geq (3n-3)/2$ contains a C_3-factor. Zhang and Wang [34] then showed the same condition on $\delta(G)$ forces a C_4-factor (provided 4 divides n).[1] Finally, Czygrinow, Kierstead and Molla (see [24, Corollary 1.5.7]) proved that for any $k \geq 3$, there exists $n_0 \in \mathbb{N}$ such that the following holds: if G is a digraph on $n \geq n_0$ vertices, n is divisible by k and $\delta(G) \geq (3n-3)/2$, then G can be partitioned into tiles of order k such that each tile contains every orientation of a cycle on k vertices; in particular, G contains a C-factor for any orientation of a cycle C on k vertices.[2] Moreover, there are examples that show that one cannot lower the bound on $\delta(G)$ here in the case when $C = C_k$; see, e.g., [24, Example 1.3.4].

On the other hand, for orientations of a cycle C that are not directed, a result of Lo implies that the minimum degree threshold for forcing a C-factor in an n-vertex digraph 'tends to $(1+o(1))n$' as $|C|$ grows; see [21, Corollary 3.3] for the precise statement.

There has been much recent interest in minimum semi-degree conditions for forcing a fixed or spanning tree in digraphs; see, e.g., [23, 25, 26, 27]. Note that Theorem 1.2 has implications to tree factors since any tree has a homomorphism into a sufficient long directed path.

Corollary 1.4. *Let T be a tree. Given any $\eta > 0$, there exists $n_0 = n_0(\eta, T) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ divisible by $|T|$. If G is an n-vertex digraph with*

$$\delta(G) \geq (1+\eta)n,$$

[1] In fact, their result is more refined and shows there is a unique extremal example.

[2] See [32] for a related result.

then G contains a T-factor.

1.2. **Cycle tilings and an analogue of the El-Zahar conjecture.** Since $\delta(G) \geq 2\delta^0(G)$ for every digraph G, Theorem 1.2 yields the asymptotically sharp minimum semi-degree condition for forcing an H-factor in a digraph, for any connected digraph H that has a homomorphism into a directed path.[3]

The next theorem asymptotically determines the minimum semi-degree threshold for forcing a C-factor for any orientation of an *odd* cycle C.

Theorem 1.5. *Given $k \in \mathbb{N}$ and $\eta > 0$, there exists $n_0 = n_0(k, \eta) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ divisible by $2k+1$. Let C be an orientation of a cycle on $2k+1$ vertices. If G is an n-vertex digraph with*

$$\delta^0(G) \geq \frac{(k+1)n}{2k+1} + \eta n,$$

then G contains a C-factor.

Given any orientation of a cycle C of length $2k+1$ and $n \in \mathbb{N}$ divisible by $2k+1$, let G be the complete 3-partite digraph with vertex classes of sizes $\frac{n}{2k+1} - 1, \frac{kn}{2k+1} + 1$ and $\frac{kn}{2k+1}$. Then G does not contain a C-factor and $\delta^0(G) = \frac{(k+1)n}{2k+1} - 1$. Thus, the minimum semi-degree condition in Theorem 1.5 is asymptotically sharp.

Note that the underlying graph of G is an extremal example for the Kühn–Osthus theorem [20] in the case of C_{2k+1}-factors – so Theorem 1.5 aligns with the corresponding behaviour exhibited in the graph setting. Exact versions of Theorem 1.5 are known in the case when $C = T_3$ or C_3; indeed, in these cases it is known that if G is a sufficiently large n-vertex digraph with $\delta^0(G) \geq 2n/3$, then G contains a C-factor; see [5, 7, 28].[4]

Rather than proving Theorem 1.5 directly, we will prove a significant generalisation that relates to a classical problem in the graph setting. The well-known El-Zahar conjecture [10] provides a minimum degree condition for covering a graph with vertex-disjoint cycles.

Conjecture 1.6. [10] *Suppose that G is an n-vertex graph and $n_1, \dots, n_r \geq 3$ are integers such that $\sum_{i=1}^r n_i = n$. If*

$$\delta(G) \geq \sum_{i=1}^{r} \lceil n_i/2 \rceil,$$

then G contains r vertex-disjoint cycles whose lengths are $n_1, \dots, n_r$.

In 1998 the El-Zahar conjecture was proven by Abbasi [1] for sufficiently large graphs. Note that in the case when the number of odd n_is is $t \geq 2$, the minimum degree condition in Conjecture 1.6 can be seen to be best possible by considering the complete 3-partite graph with vertex classes of sizes $t-1$, $(n-t+2)/2$ and $(n-t)/2$.

Our next result gives an El-Zahar-type result in the directed setting.

Theorem 1.7. *For any $\eta > 0$, there exists $n_0 = n_0(\eta) \in \mathbb{N}$ such that the following holds. Let $\ell_1, \dots, \ell_t \geq 3$ be integers such that $n := \ell_1 + \dots + \ell_t \geq n_0$, and let $D_1, \dots, D_t$ be orientations of cycles of lengths $\ell_1, \dots, \ell_t$, respectively. If G is an n-vertex digraph with*

$$\delta^0(G) \geq \frac{n+t}{2} + \eta n,$$

then G contains vertex-disjoint copies of $D_1, \dots, D_t$.

[3]The case when H is an anti-directed cycle was already established in [9, Observation 1.1].

[4]In fact, the condition that n is sufficiently large is not needed for $C = T_3$.

Notice that Theorem 1.5 is a special case of Theorem 1.7. If each cycle D_i has odd length, then the minimum semi-degree condition in Theorem 1.7 agrees with the degree condition in Conjecture 1.6 up to the ηn term. Therefore by considering the digraph analogue of the extremal example for Conjecture 1.6, one sees that Theorem 1.7 is asymptotically best possible for odd cycles. It would be extremely interesting to establish a full digraph analogue of the El-Zahar conjecture that is sharp for all choices of the cycles D_i.

1.3. **Ore-type results for transitive tournament factors.** Given a graph G and $x \in V(G)$, we write $d_G(x)$ for the *degree* of x in G. Given a digraph G and $x \in V(G)$, we write $d^+_G(x)$ and $d^-_G(x)$ for the *out-degree* and *in-degree* of x in G, respectively.

A result of Kierstead and Kostochka [16] on equitable colourings yields the following Ore-type version of Theorem 1.1.

Theorem 1.8. [16] *Let $n \in \mathbb{N}$ be divisible by $r \in \mathbb{N}$. If G is an n-vertex graph so that*

$$d_G(x) + d_G(y) \geq 2(1 - 1/r)n - 1$$

for all non-adjacent $x \neq y \in V(G)$, then G contains a K_r-factor.

Note that Theorem 1.8 immediately implies Theorem 1.1 and the Ore-type condition cannot be lowered. The $r = 3$ case of Theorem 1.8 was established earlier by Enomoto [11] and Wang [30].

Ore-type results have also been studied for digraphs. For example, a well-known result of Woodall [33] from 1972 states the following: if G is a digraph on $n \geq 3$ vertices so that $d^+_G(x) + d^-_G(y) \geq n$ for every $x \neq y \in V(G)$ with $xy \notin E(G)$, then G contains a directed Hamilton cycle.

The next result similarly provides an Ore-type condition for forcing a T_3-factor in a digraph.

Theorem 1.9. *Let $n \in \mathbb{N}$ be divisible by 3. Suppose that G is an n-vertex digraph so that for every $x \neq y \in V(G)$ with $xy \notin E(G)$ we have*

$$d^+_G(x) + d^-_G(y) \geq 4n/3 - 1. \tag{1}$$

Then G contains a T_3-factor.

Note that the Ore-type condition in Theorem 1.9 cannot be lowered. For example, consider the digraph G consisting of vertex classes V_1, V_2 together with a vertex w such that: $|V_1| = 2n/3 - 1$ and $|V_2| = n/3$; there are all possible (double) edges (i) in V_1, (ii) between V_1 and V_2 and (iii) between w and V_2. Then for any $x \neq y \in V(G)$ such that $xy \notin E(G)$ we have $d^+_G(x) + d^-_G(y) \geq 4n/3 - 2$. However, G does not contain a T_3-factor since w does not lie in a copy of T_3.

Our next result asymptotically generalises Theorem 1.9 to T_r-factors for all $r \geq 2$.

Theorem 1.10. *Let $r \geq 2$ be an integer and let $\eta > 0$. There exists $n_0 = n_0(r, \eta) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ divisible by r. Suppose that G is an n-vertex digraph so that for any $x \neq y \in V(G)$ such that $xy \notin E(G)$ we have that*

$$d^+_G(x) + d^-_G(y) \geq 2(1 - 1/r + \eta)n. \tag{2}$$

Then G contains a T_r-factor.

Note that one cannot replace the ηn term in Theorem 1.10 with a -2. Indeed, consider the complete r-partite digraph G with $r - 2$ vertex classes of size n/r, one vertex class of size $n/r - 1$ and one vertex class of size $n/r + 1$. Then G does not contain a T_r-factor and $d^+_G(x) + d^-_G(y) \geq 2(1 - 1/r)n - 2$ for all $x, y \in V(G)$.

Notation. Given a digraph G and $x \in V(G)$, we write $N^+_G(x)$ and $N^-_G(x)$ for the *out-neighbourhood* and *in-neighbourhood* of x in G, respectively. For disjoint sets $A, B \subseteq V(G)$, we write $e_G(A, B)$ for the number of edges in G with startpoint in A and endpoint in B. Given a set $X \subseteq V(G)$, we write $G[X]$ for the subdigraph of G induced by X and let $G \setminus X := G[V(G) \setminus X]$. Given $x_1, \dots, x_t \in V(G)$, we set $G[x_1, \dots, x_t] := G[\{x_1, \dots, x_t\}]$. Given a set $X \subseteq V(G)$ and $v \in V(G)$, we define $d^+_G(v, X) := |N^+_G(v) \cap X|$ and $d^-_G(v, X) := |N^-_G(v) \cap X|$.

If G is a digraph with loops and $z \in V(G)$ we take the convention that $d^+_G(z)$ (resp. $d^-_G(z)$) is the number of out-neighbours (resp. in-neighbours) of z in G excluding z itself.

Let $\mathcal{H}$ be a collection of digraphs and G be a digraph. We say that G contains an *$\mathcal{H}$-factor* if G contains a collection of vertex-disjoint copies of elements from $\mathcal{H}$ that together cover all the vertices of G.

The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < a \ll b \ll c \leq 1$, then there are non-decreasing functions $f : (0, 1] \to (0, 1]$ and $g : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ with $b \leq f(c)$ and $a \leq g(b)$. Note that $a \ll b$ implies that we may assume in the proof that, e.g., $a < b$ or $a < b^2$.

Organisation of the paper. The paper is organised as follows. In Section 2 we introduce some useful results, including the diregularity lemma [3] and an absorbing result of Lo and Markström [22]. We then prove Theorem 1.2 in Section 3. We prove Theorem 1.7 in Section 4. In Section 5 we prove Theorem 1.9 before proving Theorem 1.10 in Section 6. Finally, we raise some open problems in Section 7.

2. Useful results

2.1. The diregularity lemma.

Let G be a digraph and $A, B \subseteq V(G)$ be disjoint. The *density of* (A, B) is defined by $d_G(A, B) := \frac{e_G(A,B)}{|A||B|}$; so $d_G(A, B)$ is not necessarily equal to $d_G(B, A)$. Given $\varepsilon > 0$ we say that (A, B) is *ε-regular (in G)* if for all subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| > \varepsilon|A|$ and $|B'| > \varepsilon|B|$ we have

$$|d_G(A, B) - d_G(A', B')| < \varepsilon \, .$$

We will make use of the following well-known property.

Proposition 2.1. *Suppose that $0 < \varepsilon < \xi \leq 1/2$. Let (A, B) be ε-regular with density d. If $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \xi|A|$ and $|B'| \geq \xi|B|$ then (A', B') is ε/ξ-regular with density at least $d - \varepsilon$.* □

We now state the *degree form* of the diregularity lemma.

Lemma 2.2 (Diregularity lemma [3])**.** *Given any $\varepsilon \in (0, 1)$ and $\ell_0 \in \mathbb{N}$, there exist $L = L(\varepsilon, \ell_0) \in \mathbb{N}$ and $n_0 = n_0(\varepsilon, \ell_0) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let G be an n-vertex digraph and let $d \in [0, 1]$. Then, there is a partition $\{V_0, V_1, \dots, V_\ell\}$ of $V(G)$ with $\ell_0 < \ell < L$ and a spanning subdigraph G' of G such that*

- (a) $|V_0| \leq \varepsilon n$;
- (b) $|V_i| = |V_1|$ *for every* $i \in [\ell]$;
- (c) *for every* $v \in V(G)$, $d^+_{G'}(v) > d^+_G(v) - (d + \varepsilon)n$ *and* $d^-_{G'}(v) > d^-_G(v) - (d + \varepsilon)n$;
- (d) $e(G'[V_i]) = 0$ *for every* $i \in [\ell]$;
- (e) *for every distinct* $i, j \in [\ell]$, *the pair* (V_i, V_j) *is ε-regular in G' with density either* 0 *or at least* d.

We call $V_1, \dots, V_\ell$ *clusters* and V_0 the *exceptional set*, and refer to G' as the *pure digraph*. The *reduced digraph R of G* with parameters ε, d and ℓ_0 is the digraph defined by

$$V(R) := \{V_1, \dots, V_t\} \qquad \text{and} \qquad E(R) := \{V_iV_j : d_{G'}(V_i, V_j) \geq d\}.$$

The *special reduced digraph R^* of G* with parameters ε, d and ℓ_0 is the digraph obtained from the reduced digraph R of G by adding a loop at every $V_i \in V(R)$ such that $G[V_i]$ is a complete digraph. The notion of the special reduced digraph will be crucial for our proof of Theorem 1.10.

The following well-known result states that the reduced digraph of G essentially 'inherits' any lower bound on the minimum degree of G.

Proposition 2.3. *Let $0 < \varepsilon \leq d/2$ and let G be an n-vertex digraph such that $\delta(G) \geq \alpha n$ for some $\alpha > 0$. Suppose we have applied Lemma 2.2 to G to obtain the reduced digraph R of G with parameters ε, d and ℓ_0. Then $\delta(R) \geq (\alpha - 4d)|R|$.* □

Given a digraph R and $t \in \mathbb{N}$, we let $R(t)$ denote the *t-blow-up of R*. More precisely, $V(R(t)) := \{v^j : v \in V(R) \text{ and } j \in [t]\}$ and $E(R(t)) := \{v^a w^b : vw \in E(R) \text{ and } a, b \in [t]\}$.

The following result is a special case of the counting lemma (often called the key lemma) from [19].

Lemma 2.4. [19] *Suppose that $0 < \varepsilon < d$, that $m, t \in \mathbb{N}$ and that $R = v_1 \dots v_k$ is a directed path. Construct a digraph G by replacing every vertex $v_i \in V(R)$ by a set V_i of m vertices, and replacing each edge v_iv_{i+1} of R with an ε-regular pair (V_i, V_{i+1}) of density at least d. For each $v_i \in V(R)$ let U_i denote the set of t vertices in $R(t)$ corresponding to v_i. Let H be a subdigraph of $R(t)$ on h vertices and maximum degree $\Delta \in \mathbb{N}$. Set $\delta := d - \varepsilon$ and $\varepsilon_0 := \delta^\Delta/(2 + \Delta)$. If $\varepsilon \leq \varepsilon_0$ and $t - 1 \leq \varepsilon_0 m$ then there are at least $(\varepsilon_0 m)^h$ copies of H in G so that if $x \in V(H)$ lies in U_i, then x is embedded into V_i in G.*

The following result will be applied in the proof of Theorem 1.10 to convert a large T_r-tiling in a special reduced digraph into a large T_r-tiling in the original digraph G. It is (for example) a special case of Corollary 2.3 in [4].

Lemma 2.5. *Let $\varepsilon, d > 0$ and $m, r \in \mathbb{N}$ such that $0 < 1/m \ll \varepsilon \ll d \ll 1/r$. Let H be a digraph obtained from T_r by replacing every vertex of T_r with m vertices and replacing each edge of T_r with an ε^2-regular pair of density at least d. Then H contains a T_r-tiling covering all but at most εmr vertices.*

2.2. **The absorbing lemma.** Let H be a digraph. Given a digraph G, a set $S \subseteq V(G)$ is called an *H-absorbing set for $Q \subseteq V(G)$*, if both $G[S]$ and $G[S \cup Q]$ contain H-factors.

The proof of Theorem 1.2 makes use of the following *absorbing lemma* of Lo and Markström [22].

Lemma 2.6. [22] *Let $h, t \in \mathbb{N}$ and let $\gamma > 0$. Suppose that H is an h-vertex digraph. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a digraph on $n \geq n_0$ vertices so that, for any $x, y \in V(G)$, there are at least γn^{th-1} $(th-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain H-factors. Then $V(G)$ contains a set M so that*

- *$|M| \leq (\gamma/2)^h n/4$;*
- *M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq (\gamma/2)^{2h} hn/32$.*

2.3. **Probabilistic estimates.** We will need the following form of the well-known Chernoff-Hoeffding inequality.

Lemma 2.7. *Let X be a nonnegative random variable with binomial or hypergeometric distribution and expected value μ. Then, for $t \geq 0$*

$$\mathbb{P}[X \leq \mu - t] \leq e^{-t^2/2\mu}. \tag{3}$$

3. Proof of Theorem 1.2

Let $P(t_1, \dots, t_k)$ denote the blow-up of the directed path $v_1 \dots v_k$ where v_i is replaced by a set of t_i vertices for each $i \in [k]$. If H is a digraph with a homomorphism into a directed path, then H is a spanning subdigraph of $P(t_1, \dots, t_k)$ for some choice of $k, t_1, \dots, t_k \in \mathbb{N}$. Thus, to prove Theorem 1.2 it suffices to prove it in the case when $H = P(t_1, \dots, t_k)$. We first prove an approximate version of Theorem 1.2 in this case.

Theorem 3.1. *Let $k \geq 2$ and $t_1, \dots, t_k \in \mathbb{N}$. Set $H := P(t_1, \dots, t_k)$. Given any $\eta > 0$, there exists $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. If G is an n-vertex digraph with*

$$\delta(G) \geq (1 + \eta)n,$$

then G contains an H-tiling covering all but at most ηn vertices.

Proof. Define constants $\varepsilon, d > 0$ and $n_0, \ell_0, L \in \mathbb{N}$ so that

$$\frac{1}{n_0} \ll \frac{1}{L} \ll \frac{1}{\ell_0} \ll \varepsilon \ll d \ll \eta, \frac{1}{|H|}. \tag{4}$$

Let G be a digraph on $n \geq n_0$ vertices as in the statement of the theorem. Apply Lemma 2.2 with parameters ε, d, ℓ_0 to obtain the reduced digraph R of G with $\ell_0 < |R| < L$, and the pure digraph G'. Let m denote the size of the clusters of G and set $\ell := |R|$.

Proposition 2.3 and (4) imply that $\delta(R) \geq (1 + \eta/2)|R|$. Therefore, Ghouila-Houri's theorem [13] implies that R contains a directed Hamilton path.

Initially, suppose R in fact contains a directed Hamilton cycle $V_1 \dots V_\ell V_1$; so the clusters of G are $V_1, \dots, V_\ell$. Set $V_{\ell+1} := V_1$.

Lemma 2.4 implies that $G[V_1 \cup \dots \cup V_k]$ contains a copy of H with precisely t_i vertices in V_i for each $i \in [k]$. Similarly, Lemma 2.4 implies that $G[V_2 \cup \dots \cup V_{k+1}]$ contains a copy of H with precisely t_i vertices in V_{i+1} for each $i \in [k]$. Repeating this process (shifting the clusters we find the copy of H in each time), one can find a collection $\mathcal{C}_1$ of ℓ vertex-disjoint copies of H in G that cover precisely the same number of vertices (i.e., $|H|$) in each of the clusters V_i (for $i \in [\ell]$).

Remove the vertices of $\mathcal{C}_1$ from the clusters $V_1, \dots, V_\ell$. Then Proposition 2.1 implies that (V_i, V_{i+1}) is still an $\varepsilon/2$-regular pair in G' with density at least $d - \varepsilon$. We may now again repeatedly apply Lemma 2.4 to find a collection $\mathcal{C}_2$ of ℓ vertex-disjoint copies of H in G that cover precisely the same number of vertices in each of the clusters V_i (for $i \in [\ell]$).

In fact, notice that we can repeat this process until we have covered all but at most $\eta m/3$ vertices from each of the clusters V_i of G with vertex-disjoint copies of H. Indeed, given any $i \in [\ell]$ and $V_i' \subseteq V_i$ and $V_{i+1}' \subseteq V_{i+1}$ such that $|V_i'|, |V_{i+1}'| \geq \eta m/3$, Proposition 2.1 implies that (V_i', V_{i+1}') is a $(3\varepsilon/\eta)$-regular pair in G' with density at least $d - \varepsilon$. Since $3\varepsilon/\eta \leq \sqrt{\varepsilon}$, this allows one to still apply Lemma 2.4 where needed (where now $\sqrt{\varepsilon}$ plays the role of ε and $d - \varepsilon$ plays the role of d).

In summary, this process ensures we can obtain an H-tiling $\mathcal{H}$ in G that does not cover at most $\eta m/3$ vertices from each cluster V_i. As $\mathcal{H}$ also does not cover any vertex from V_0, $\mathcal{H}$ is an H-tiling in G covering all but at most

$$|V_0| + \ell \cdot \frac{\eta m}{3} \leq \varepsilon n + \frac{\eta n}{3} \overset{(4)}{\leq} \frac{\eta n}{2}$$

vertices of G, as desired.

Now suppose $V_1 \dots V_\ell$ is just a directed Hamilton path in R rather than a directed Hamilton cycle. Then we can follow precisely the same procedure as above, except that at each step where we require a copy H' of H in G that uses both vertices from V_1 and V_ℓ, we cannot find such an H'. In the above procedure we only covered at most

$$|V_1| + \dots + |V_{k-1}| + |V_{\ell-k+2}| + \dots + |V_\ell| = (2k-2)m \leq (2k-2)n/\ell \overset{(4)}{\leq} \varepsilon n$$

vertices using such copies H' of H. By ignoring such copies of H in the H-tiling $\mathcal{H}$, this tells us that we can still find an H-tiling in G covering all but at most $\eta n/2 + \varepsilon n \leq \eta n$ vertices of G, as desired. □

Remark 3.2. *The reader may wonder why we did not prove Theorem 3.1 via a single application of the blow-up lemma* [18]. *Note though, in general one cannot apply the blow-up lemma to* ***spanning*** *structures of the reduced digraph (such as directed Hamilton cycles or paths). More formally, in an application of the blow-up lemma, one requires that the number of clusters T considered satisfies $\varepsilon \ll 1/T$, but in our case $T = \ell$ and $1/\ell \ll \varepsilon$.*

Next we apply Lemma 2.6 to provide an absorbing lemma for Theorem 1.2.

Lemma 3.3. *Let $k \geq 2$ and $t_1, \dots, t_k \in \mathbb{N}$. Set $H := P(t_1, \dots, t_k)$ and $h := |H|$. Given any $\eta > 0$, there exist $\xi = \xi(\eta, H) > 0$ and $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. If G is an n-vertex digraph with*

$$\delta(G) \geq (1 + \eta)n,$$

then $V(G)$ contains a set M so that

- *$|M| \leq \xi n$;*
- *M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \xi^2 hn/2$.*

Proof. Define constants $\varepsilon, d > 0$ and $n_0, \ell_0, L \in \mathbb{N}$ so that

$$\frac{1}{n_0} \ll \frac{1}{L} \ll \frac{1}{\ell_0} \ll \varepsilon \ll d \ll \eta, \frac{1}{h}. \tag{5}$$

Let G be a digraph on $n \geq n_0$ vertices as in the statement of the lemma.

Consider any $x, y \in V(G)$. As $\delta(G) \geq (1+\eta)n$, $|N^+_G(x) \cap N^+_G(y)| \geq \eta n$ or $|N^-_G(x) \cap N^-_G(y)| \geq \eta n$. We may assume that the former holds since the latter case follows analogously.

Apply Lemma 2.2 with parameters ε, d, ℓ_0 to obtain the reduced digraph R of G with $\ell_0 < |R| < L$, and the pure digraph G'. Let m denote the size of the clusters of G.

Note that there exists some cluster V_2 such that $V'_2 := V_2 \cap N^+_G(x) \cap N^+_G(y)$ satisfies $|V'_2| \geq \eta m/2$. Moreover, Proposition 2.3 and (5) imply that $\delta(R) \geq (1 + \eta/2)|R|$. Thus, there is a directed path $V_1V_2 \dots V_k$ in R on k vertices whose second vertex is V_2.

For each $i \in [k] \setminus \{2\}$, define V'_i to be a subset of V_i of size precisely $|V'_2|$. Note that (V_i, V_{i+1}) is an ε-regular pair in G' with density at least d for each $i \in [k-1]$. As $|V'_2| \geq \eta m/2$, Proposition 2.1 and (5) therefore imply that (V'_i, V'_{i+1}) is a $\sqrt{\varepsilon}$-regular pair in G' with density at least $d/2$ for each $i \in [k-1]$.

Set $\varepsilon_0 := (d/2 - \sqrt{\varepsilon})^\Delta/(2 + \Delta)$ where Δ is the maximum degree of $H' := P(t_1 - 1, t_2, \dots, t_k)$. By Lemma 2.4 there are at least $(\varepsilon_0 \eta m/2)^{h-1}$ copies of H' in $G[V'_1 \cup \dots \cup V'_k]$. Moreover, as $V'_2 \subseteq N^+_G(x) \cap N^+_G(y)$, each such copy of H' together with x forms a copy H_x of H in G, and each such copy of H' together with y forms a copy H_y of H in G.

Note that

$$(\varepsilon_0 \eta m/2)^{h-1} \overset{(5)}{\geq} (\varepsilon_0 \eta n/(4L))^{h-1}.$$

Set $\gamma := (\varepsilon_0 \eta/(4L))^{h-1}$ and $\xi := (\gamma/2)^h/4$. We have shown that given any $x, y \in V(G)$, there are at least γn^{h-1} sets $X \subseteq V(G)$ of size $h - 1$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of H. Lemma 2.6 now implies $V(G)$ contains a set M so that

- $|M| \leq \xi n$;
- M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \xi^2 hn/2$.

□

With Theorem 3.1 and Lemma 3.3 at hand, it is now straightforward to prove Theorem 1.2.

Proof of Theorem 1.2. Let H be as in the statement of the theorem and set $h := |H|$. Recall that it suffices to prove the case when $H = P(t_1, \dots, t_k)$ for some $k \geq 2$ and $t_1, \dots, t_k \in \mathbb{N}$. Define constants $\eta_1, \xi > 0$ and $n_0 \in \mathbb{N}$ so that

$$\frac{1}{n_0} \ll \eta_1 \ll \xi \ll \eta, \frac{1}{h}. \tag{6}$$

Let G be a digraph on $n \geq n_0$ vertices as in the statement of the theorem. By Lemma 3.3, $V(G)$ contains a set M so that

- $|M| \leq \xi n$;
- M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \xi^2 hn/2$.

Set $G' := G \setminus M$. By (6), $\delta(G') \geq (1+\eta_1)|G'|$. Thus, G' contains an H-tiling $\mathcal{H}_1$ covering all but at most $\eta_1|G'| \leq \xi^2 hn/2$ of its vertices. Moreover, the choice of M ensures that there is an H-tiling $\mathcal{H}_2$ covering precisely those vertices in G that are not in $\mathcal{H}_1$. Thus, $\mathcal{H}_1 \cup \mathcal{H}_2$ is our desired H-factor in G. □

4. Proof of the EL-Zahar-type result

4.1. **Auxiliary results for the proof of Theorem 1.7.** In this subsection we provide a few results that are needed for the proof of Theorem 1.7. The first result provides a minimum semi-degree condition for forcing every orientation of a Hamilton cycle. Note that an asymptotic version of this result (see [14]) is also sufficient for our purposes.

Theorem 4.1. [8] *There exists $n_{4.1} \in \mathbb{N}$ such that when $n \geq n_{4.1}$, the following holds for every n-vertex digraph G. If $\delta^0(G) > n/2$, then G contains every orientation of a Hamilton cycle.*

The following lemma is the main tool in our proof of Theorem 1.7. The iterative approach is very similar to an argument in [12].

Lemma 4.2. *For every $L \in \mathbb{N}$ and $\eta, \gamma > 0$, there exists $n_{4.2} := n_{4.2}(L, \eta, \gamma) \in \mathbb{N}$ such that for every integer $n \geq n_{4.2}$ the following holds. Let $\ell_1, \dots, \ell_t \geq 3$ be integers of size at most L such that*

$$\ell_1 + \cdots + \ell_t \leq (1-\gamma)n \tag{7}$$

and let $D_1, \dots, D_t$ be orientations of cycles of lengths $\ell_1, \dots, \ell_t$, respectively. If G is an n-vertex digraph with

$$\delta^0(G) \geq \frac{n+t}{2} + \eta n,$$

then G contains vertex-disjoint copies of $D_1, \dots, D_t$. Furthermore, if we let G' be the digraph formed by deleting the vertices of these cycles from G, then we have

$$\delta^0(G') \geq \left(\frac{1}{2} + \frac{\eta}{3}\right)|G'|.$$

Proof. For every $\ell \in [L]$, let p_ℓ be the number of cycles in $D_1, \dots, D_t$ of length ℓ. Define $\sigma := \eta/3$ and let $\beta, \gamma > 0$ and $R \in \mathbb{N}$ be such that

$$0 < \frac{1}{n} \ll \beta \ll \frac{1}{R} \ll \gamma, \sigma, \frac{1}{L}. \tag{8}$$

Let G be an n-vertex digraph as in the statement of the lemma. Then (8) implies that $\sum_{\ell \in [L]} R\ell < \sigma n$, so we can greedily construct a collection of vertex-disjoint cycles in G consisting of R cycles of every length $\ell \in [L]$ for which $p_\ell > 0$. So, for notational convenience, we can assume that p_ℓ is divisible by R for every $\ell \in [L]$ and

$$\delta^0(G) \geq \frac{n+t}{2} + 2\sigma n. \tag{9}$$

For every $\ell \in [L]$, let $q_\ell := p_\ell / R$ and let $q := (q_1, \ldots, q_L)$. Define

$$|q| := \sum_{i=1}^{L} q_i,$$

and note that $|q| = t/R$. We call a collection of vertex-disjoint cycles in G a *q-cycle tiling* if it consists of exactly q_ℓ cycles of length ℓ for every $\ell \in [L]$. We will construct the desired cycles in R rounds. In each round $r \in [R]$, we will find a q-cycle tiling that is vertex-disjoint from the q-cycle tilings constructed in the $r-1$ previous rounds.

We can assume that

$$|q| \geq \beta n, \tag{10}$$

as otherwise $\sum_{i=1}^{t} \ell_i \leq Lt = LR|q| \overset{(8)}{<} \sigma n$, and we can greedily find the desired vertex-disjoint cycles.

Let $m := \left\lfloor \frac{n}{R+1} \right\rfloor$ and note that

$$m \geq \frac{n-R}{R+1} = \frac{n}{R} - \frac{n}{R(R+1)} - \frac{R}{R+1}, \tag{11}$$

so

$$m \overset{(7),(11)}{\geq} \frac{\sum_{i=1}^{t} \ell_i}{R} + \frac{\gamma n}{R} - \frac{n}{R(R+1)} - \frac{R}{R+1} \overset{(8)}{\geq} \frac{\sum_{i=1}^{t} \ell_i}{R} = \sum_{\ell \in [L]} \ell \cdot q_\ell. \tag{12}$$

Uniformly at random select a partition of $V(G)$ into $R+1$ parts $W_1, \ldots, W_{R+1}$ so that

$$|W_1| = |W_2| = \cdots = |W_R| = m \quad \text{and} \quad |W_{R+1}| = n - Rm \geq m.$$

For every $r \in [R+1]$, we have

$$\frac{|W_r| \cdot t}{n} \geq \frac{mt}{n} \overset{(11)}{\geq} \frac{t}{R} - \frac{t}{R(R+1)} - \frac{tR}{n(R+1)} \geq |q| - \frac{n}{R(R+1)} - \frac{R}{R+1} \overset{(8)}{\geq} |q| - 0.1\sigma |W_r|,$$

so by the Chernoff and union bounds the following holds with high probability for every $v \in V(G)$:

$$d_G^{\pm}(v, W_r) \overset{(9)}{\geq} \frac{|W_r|}{n} \left(\frac{n+t}{2} + 2\sigma n \right) - 0.1\sigma |W_r| \geq \frac{|W_r| + |q|}{2} + \sigma |W_r|. \tag{13}$$

The proof of the lemma follows easily once the following claim is established.

Claim 4.3. *For $r \in [R+1]$, let $U \subseteq V(G) \setminus W_r$ where $|U| \geq m$. Suppose that for every $v \in V(G)$ we have $d_G^{\pm}(v, U) \geq (1/2 + \sigma)|U|$. Then $G[U \cup W_r]$ contains every orientation of a q-cycle tiling. Furthermore, if U' is the set of vertices in $U \cup W_r$ that are not covered by the q-cycle tiling, then for every $v \in V(G)$ we have $d_G^{\pm}(v, U') \geq (1/2 + \sigma)\, |U'|$.*

Proof of the claim. Recall that $|U| \geq m \overset{(12)}{\geq} \sum_{\ell \in [L]} \ell q_\ell$. Uniformly at random select a partition $\{U_0, U_1\}$ of U such that $|U_1| = \sum_{\ell \in [L]} (\ell - 1) q_\ell$. Note that

$$|U_1|, |U_0| \geq |q| \overset{(10)}{\geq} \beta n,$$

so the Chernoff and union bounds imply that, with high probability, for every $v \in V(G)$ and $i \in \{0, 1\}$,

$$d_G^{\pm}(v, U_i) > \left(\frac{1}{2} + \sigma \right) |U_i| - \beta \sigma n \overset{(10)}{\geq} \left(\frac{1}{2} + \sigma \right) |U_i| - \sigma |q|. \tag{14}$$

Recall that $|U_1| = \sum_{\ell\in[L]}(\ell-1)q_\ell \geq 2|q|$. So (14) implies that $\delta^0(G[U_1]) \geq (1/2+\sigma/2)|U_1|$. Theorem 4.1 then implies that $G[U_1]$ contains a vertex-disjoint collection of paths $\mathcal{P}$ such that for every $\ell \in [L]$ the collection $\mathcal{P}$ contains exactly q_ℓ paths on $\ell-1$ vertices where the paths are of any desired orientation. Note that the paths in $\mathcal{P}$ completely cover the vertices in U_1.

For every $P \in \mathcal{P}$, and every $\diamond, \circ \in \{-,+\}$, when we let u, v be the endpoints of P we have

$$|N_G^\diamond(u,W_r) \cap N_G^\circ(v,W_r)| \overset{(13)}{\geq} 2\left(\frac{|W_r|+|q|}{2} + \sigma|W_r|\right) - |W_r| = |q| + 2\sigma|W_r|.$$

Therefore, because $|\mathcal{P}| = \sum_{\ell\in[L]} q_\ell = |q|$, we can greedily extend each of the $|q|$ paths in $\mathcal{P}$ into vertex-disjoint cycles of the desired orientations using one additional vertex in W_r. Let $\mathcal{C}$ be the resulting q-cycle tiling and let $U' := (U \cup W_r) \setminus V(\mathcal{C})$. Recall that, by construction, we have $|W_r \cap U'| = |W_r| - |q|$ and $U \cap U' = U_0$. Therefore, for every $v \in V(H)$ and $\diamond \in \{-,+\}$

$$\begin{aligned}
d_G^\diamond(v,U') &= d_G^\diamond(v,U_0) + d_G^\diamond(v, W_r \cap U') \\
&\overset{(14)}{\geq} \left(\frac{1}{2}+\sigma\right)|U_0| - \sigma|q| + d_G^\diamond(v,W_r) - |q| \\
&\overset{(13)}{\geq} \left(\frac{1}{2}+\sigma\right)|U_0| - \sigma|q| + \left(\frac{|W_r|+|q|}{2} + \sigma|W_r|\right) - |q| \\
&= \frac{|U_0|}{2} + \frac{|W_r|-|q|}{2} + \sigma|U_0| - \sigma|q| + \sigma|W_r| \\
&= \left(\frac{1}{2}+\sigma\right)|U'|.
\end{aligned}$$

■

Recall that (13) implies that for every $v \in V(G)$ we have $d_G^\pm(v,W_1) \geq (1/2+\sigma)|W_1|$. Therefore, we can apply Claim 4.3 with W_1 playing the role of U and $r=2$ to construct a q-tiling $\mathcal{C}_1$ in the digraph induced by $W_1 \cup W_2$. Let $U' \subseteq W_1 \cup W_2$ be the set of vertices not covered by this q-tiling. Note that

$$|U'| \geq 2m - |V(\mathcal{C}_1)| = 2m - \sum_{\ell\in[L]} \ell q_\ell \overset{(12)}{\geq} m,$$

and recall that Claim 4.3 implies that for every $v \in V(G)$, we have $d_G^\pm(v,U') \geq (1/2+\sigma)|U'|$. Therefore, we can apply Claim 4.3 again with U' now playing the role of U and $r=3$ to construct a second q-tiling $\mathcal{C}_2$ in $U' \cup W_3 \subseteq W_1 \cup W_2 \cup W_3$. We can proceed in this manner to construct R vertex-disjoint q-cycle tilings. These R vertex-disjoint q-cycle tilings form the desired vertex-disjoint copies of $D_1, \ldots, D_t$ in G. □

We will use the following straightforward application of the Chernoff bound.

Lemma 4.4. *For every $\sigma > 0$ there exists $n_{4.4} := n_{4.4}(\sigma) \in \mathbb{N}$ such that for every $n \geq n_{4.4}$ the following holds for every n-vertex digraph G and every integer m such that $\sigma n \leq m \leq (1-\sigma)n$. There exists a partition $\{U_1, U_2\}$ of $V(G)$ such that $|U_1| = m$ and $|U_2| = n-m$ where, for $j \in [2]$, we have*

$$\delta^0(G[U_j]) \geq \frac{\delta^0(G)}{n}|U_j| - |U_j|^{2/3}.$$

Proof. We select $\{U_1, U_2\}$ uniformly at random from all partitions of $V(G)$ with $|U_1| = m$ and $|U_2| = n-m$. Then, for every $v \in V(G)$, $j \in \{1,2\}$ and $\diamond \in \{-,+\}$, the random variable $|N_G^\diamond(v) \cap U_j|$ has hypergeometric distribution with expected value $\mu := (d_G^\diamond(v)/n) \cdot |U_j|$. Since

$\mu \leq |U_j|$ and $|U_j| \geq \sigma n$, by Lemma 2.7 we have

$$\mathbb{P}\left[\left||N^{\diamond}_G(v) \cap U_j| < \mu - |U_j|^{2/3}\right]\right] \overset{(3)}{\leq} e^{-|U_j|^{4/3}/2\mu} \leq e^{-|U_j|^{1/3}/2} \leq e^{-(\sigma n)^{1/3}/2}.$$

Since $4n \cdot e^{-(\sigma n)^{1/3}/2} < 1$, the union bound implies that there exists an outcome where $|N^{\diamond}_G(v) \cap U_j| \geq \mu - |U_j|^{2/3}$ for each of the $n \cdot 2 \cdot 2$ such random variables. □

The following argument is essentially identical to the proof of Proposition 5.1 in [2] except that we use Theorem 4.1 instead of Dirac's theorem. We reproduce it here for completeness.

Lemma 4.5. *For every $\eta > 0$, there exists $L := L_{4.5}(\eta) \in \mathbb{N}$ such that the following holds. Let $\ell_1, \dots, \ell_t$ be a sequence of integers that are each at least L, let $n := \ell_1 + \cdots + \ell_t$, and let $D_1, \dots, D_t$ be orientations of cycles of lengths $\ell_1, \dots, \ell_t$, respectively. If G is an n-vertex digraph with $\delta^0(G) \geq (1/2 + \eta)n$, then G contains vertex-disjoint copies of $D_1, \dots, D_t$.*

Proof. Let $\sigma := \min\{\eta/2, 1/3\}$. We assume that

$$\frac{1}{L} \ll \sigma. \tag{15}$$

In particular, we assume that $L \geq n_{4.1}$ and $L \geq n_{4.4}(\sigma)$. Therefore, Lemma 4.4 implies that for every $U \subseteq V(G)$ with $|U| \geq L$ and for every integer m such that $\sigma|U| \leq m \leq (1 - \sigma)|U|$, there exists a partition $\{U_1, U_2\}$ of U such that $|U_1| = m$, $|U_2| = |U| - m$, and for $j \in [2]$

$$\delta^0(G[U_j]) \geq \frac{\delta^0(G[U])}{|U|}|U_j| - |U_j|^{2/3} = \left(\frac{\delta^0(G[U])}{|U|} - \frac{1}{|U_j|^{1/3}}\right)|U_j|. \tag{16}$$

The lemma follows from the following claim (with $I := [t]$, $U := V(G)$, and $k := 0$).

Claim 4.6. *Let $I \subseteq [t]$ be non-empty and let $U \subseteq V(G)$ such that $\sum_{i \in I} \ell_i = |U|$. If there exists a non-negative integer k such that*

$$\delta^0(G[U]) \geq \left(\frac{1}{2} + 2\sigma - \sum_{i=0}^{k-1}\left(\frac{(1-\sigma)^i}{|U|}\right)^{1/3}\right)|U|, \tag{17}$$

then there exists a collection $\{C_i\}_{i \in I}$ of vertex-disjoint cycles in $G[U]$ where C_i is a copy of D_i for each $i \in I$.

Proof of the claim. We will prove the claim by induction on $|I|$. Since I is nonempty we have $|U| \geq L$, so

$$\sum_{i=0}^{k-1}\left(\frac{(1-\sigma)^i}{|U|}\right)^{1/3} \leq \frac{1}{L^{1/3}}\sum_{i=0}^{\infty}\left((1-\sigma)^{1/3}\right)^i = \frac{1}{L^{1/3}\left(1 - (1-\sigma)^{1/3}\right)} \overset{(15)}{\leq} \sigma,$$

which implies

$$\delta^0(G[U]) \overset{(17)}{\geq} (1/2 + \sigma)|U|. \tag{18}$$

For convenience, we can assume $I = \{1, \dots, |I|\}$ and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_{|I|}$. If $\ell_1 > (1 - \sigma)|U|$, then $\ell_2 + \ldots + \ell_{|I|} = |U| - \ell_1 < \sigma|U|$, and we can greedily construct a (empty when $|I| = 1$) collection of vertex-disjoint copies $C_2, \dots, C_{|I|}$ of $D_2, \dots, D_{|I|}$, respectively. We can then use Theorem 4.1 to find a spanning copy C_1 of D_1 in $G[U \setminus \bigcup_{i=2}^{|I|} V(C_i)]$. Note that this proves the base case ($|I| = 1$) of the induction. So assume $\ell_{|I|} \leq \ell_{|I|-1} \leq \cdots \leq \ell_1 \leq (1 - \sigma)|U|$. Then, because $\sigma \leq 1/3$, there exists a partition $\{I_1, I_2\}$ of I such that for $j \in [2]$ we have

$$\sigma|U| \leq \sum_{i \in I_j} \ell_i \leq (1 - \sigma)|U|.$$

With (16) this implies that there exists a partition $\{U_1, U_2\}$ of U such that for $j \in [2]$

$$\sigma|U| \le |U_j| = \sum_{i \in I_j} \ell_i \le (1-\sigma)|U| \tag{19}$$

and

$$\begin{aligned}
\delta^0(G[U_j]) &\overset{(16),(17)}{\ge} \left(\frac{1}{2} + 2\sigma - \sum_{i=0}^{k-1} \left(\frac{(1-\sigma)^i}{|U|} \right)^{1/3} - \frac{1}{|U_j|^{1/3}} \right) |U_j| \\
&\overset{(19)}{\ge} \left(\frac{1}{2} + 2\sigma - \sum_{i=0}^{k-1} \left(\frac{(1-\sigma)^i}{\frac{|U_j|}{1-\sigma}} \right)^{1/3} - \frac{1}{|U_j|^{1/3}} \right) |U_j| \\
&= \left(\frac{1}{2} + 2\sigma - \sum_{i=0}^{k} \left(\frac{(1-\sigma)^i}{|U_j|} \right)^{1/3} \right) |U_j|.
\end{aligned}$$

The claim then follows by the induction hypothesis applied to both $G[U_1]$ and $G[U_2]$. ■

□

4.2. **An absorbing lemma for Theorem 1.7.** The following absorbing lemma follows quickly from Lemma 2.6.

Lemma 4.7. *Let C be an orientation of a cycle on $\ell \ge 3$ vertices. Given any $\eta > 0$, there exists $\xi_{4.7} := \xi_{4.7}(\eta, \ell) > 0$ such that, for every $0 < \xi \le \xi_{4.7}$, there exists $n_{4.7} = n_{4.7}(\xi, \eta, \ell) \in \mathbb{N}$ such that the following holds for all $n \ge n_{4.7}$ and every n-vertex digraph G. If either $\ell = 3$ and*

$$\delta^0(G) \ge \left(\frac{2}{3} + \eta \right) n,$$

or $\ell \ne 3$ and

$$\delta^0(G) \ge \left(\frac{1}{2} + \eta \right) n,$$

then $V(G)$ contains a set M so that

- *$|M| \le \xi n$;*
- *M is a C-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in \ell\mathbb{N}$ and $|W| \le \ell \cdot \xi^2 n/2$.*

Proof. Define $\gamma, \xi > 0$ so that $\gamma \ll \eta, 1/\ell$ and $\xi := (\gamma/2)^\ell/4$. Let $n \in \mathbb{N}$ be sufficiently large.

Let $x, y \in V(G)$. Let $\mathcal{X}$ be the collection of $(\ell-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain a copy of C.

We will show that

$$\ell! \cdot |\mathcal{X}| \ge (\eta n)^{\ell-1} \ge \ell! \cdot \gamma n^{\ell-1}.$$

By Lemma 2.6 (with $h := \ell$ and $t := 1$) this will prove the lemma because

$$(\gamma/2)^\ell n/4 = \xi n \qquad \text{and} \qquad (\gamma/2)^{2\ell} \ell n/32 = \ell \cdot \xi^2 n/2.$$

First suppose $\ell = 3$; so $\delta^0(G) \ge \left(\frac{2}{3} + \eta\right) n$. For every $\diamond \in \{-, +\}$, we have $|N_G^\diamond(x) \cap N_G^\diamond(y)| \ge (1/3 + 2\eta)n > \eta n$. Furthermore, for every $z \in N_G^\diamond(x) \cap N_G^\diamond(y)$ and $\circ, \bullet \in \{+, -\}$, we have $|N_G^\circ(x) \cap N_G^\circ(y) \cap N_G^\bullet(z)| > \eta n$. This implies that $2|\mathcal{X}| \ge (\eta n)^2$, which completes the proof in this case.

Now assume $\ell \ge 4$. Using the fact that $\delta^0(G) \ge \left(\frac{1}{2} + \eta\right) n$, we will show that we can construct at least $(\eta n)^{\ell-1}$ paths $v_1, \ldots, v_{\ell-1}$ in G so that $\{v_1, \ldots, v_{\ell-1}\} \in \mathcal{X}$. This will then imply that $\ell! \cdot |\mathcal{X}| \ge (\eta n)^{\ell-1}$, which will complete the proof.

For every $\diamond \in \{-, +\}$, we have $|N_G^\diamond(x) \cap N_G^\diamond(y)| \ge 2\eta n$, so we start the construction by picking v_1 and then $v_{\ell-1}$ distinct from v_1 in at least $(\eta n)^2$ ways. We can iteratively select $v_2, \ldots, v_{\ell-3}$ in at least $(\eta n)^{\ell-4}$ ways. (When $\ell = 4$, this sequence of vertices is empty.) Finally, because

$|N_G^{\diamond}(v_{\ell-3}) \cap N_G^{\circ}(v_{\ell-1})| \geq 2\eta n$, for every $\diamond, \circ \in \{-,+\}$, we can complete the construction by selecting $v_{\ell-2}$ in one of at least ηn ways. As this gives us $(\eta n)^2 \cdot (\eta n)^{\ell-4} \cdot \eta n = (\eta n)^{\ell-1}$ ways of constructing such a path, the proof is complete. $\square$

4.3. **Proof of Theorem 1.7.** Let $L \in \mathbb{N}$ and $\gamma, \xi > 0$ be such that

$$0 < \frac{1}{n} \ll \gamma \ll \xi \ll \frac{1}{L} \ll \eta. \tag{20}$$

Let G be an n-vertex digraph as in the statement of the theorem.

We will call a collection of vertex-disjoint cycles a *cycle tiling* and we will say two cycle tilings, $\mathcal{C}$ and $\mathcal{C}'$, are *isomorphic* if there exist a bijection $f : \mathcal{C} \to \mathcal{C}'$ such that C is isomorphic to $f(C)$ for every $C \in \mathcal{C}$. Let $\mathcal{C}$ be a cycle tiling consisting of vertex-disjoint copies of $D_1, \dots, D_t$. Let $\mathcal{C}_{\rm sm}$ be the collection of (small) cycles in $\mathcal{C}$ with length at most L and let $\mathcal{C}_{\rm lg} := \mathcal{C} \setminus \mathcal{C}_{\rm sm}$ be the remaining (large) cycles in $\mathcal{C}$.

Case 1: $\sum_{C \in \mathcal{C}_{\rm lg}} |C| \geq \gamma n$. In this case, we have

$$\sum_{C \in \mathcal{C}_{\rm sm}} |C| \leq n - \sum_{C \in \mathcal{C}_{\rm lg}} |C| \leq (1-\gamma)n,$$

and, by (20), we can assume $n \geq N_{4.2}(L, \eta, \gamma)$. Therefore, we can apply Lemma 4.2 to construct a cycle tiling isomorphic to $\mathcal{C}_{\rm sm}$. By Lemma 4.2, if we let G' be the digraph formed by removing the vertices of such a cycle tiling from G, we have $\delta^0(G') \geq (1/2 + \eta/3)|G'|$. By (20) we can assume that $L \geq L_{4.5}(\eta/3)$, so Lemma 4.5 implies that there exist a cycle tiling in G' that is isomorphic to $\mathcal{C}_{\rm lg}$. Together, this ensures our desired copies of $D_1, \dots, D_t$ in G.

Case 2: $\sum_{C \in \mathcal{C}_{\rm lg}} |C| < \gamma n$. Let $\mathcal{C}_{\rm tri}$ be the orientations of a triangle that appear in $\mathcal{C}_{\rm sm}$. If $|\mathcal{C}_{\rm tri}| \geq n/3 - \eta n$, then let C^* be the orientation of a triangle that occurs most frequently among the cycles in $\mathcal{C}_{\rm tri}$. Otherwise, let C^* be the orientation of a cycle that occurs most frequently among the cycles in $\mathcal{C}_{\rm sm} \setminus \mathcal{C}_{\rm tri}$. Let ℓ^* be the length of C^*.

Note that we clearly have $\delta^0(G) \geq (1/2 + \eta)n$ and if C^* is an orientation of triangle, then we have

$$\delta^0(G) \geq \frac{n + |\mathcal{C}_{\rm tri}|}{2} + \eta n \geq \frac{n + n/3 - \eta n}{2} + \eta n = \left(\frac{2}{3} + \eta/2\right) n.$$

Since $\ell^* \leq L$, (20) implies that we can assume $\xi \leq \xi_{4.7}(\eta/2, \ell^*)$ and $n \geq n_{4.7}(\xi, \eta/2, \ell^*)$, so, in all cases, Lemma 4.7 implies that there exists $M \subseteq V(G)$ such that

$$|M| \leq \xi n \tag{21}$$

and

(22) M is a C^*-absorbing set for every $W \subseteq V(G) \setminus M$ with $|W| \in \ell^* \mathbb{N}$ and $|W| \leq \ell^* \cdot \xi^2 n/2$.

If C^* is an orientation of a triangle, then C^* appears at least $(1/3 - \eta)n/2 \geq \xi n$ times in $\mathcal{C}_{\rm tri}$. Otherwise, C^* occurs at least

$$\frac{\sum_{C \in \mathcal{C}_{\rm sm} \setminus \mathcal{C}_{\rm tri}} |C|}{L^2 \cdot 2^L} = \frac{n - 3|\mathcal{C}_{\rm tri}| - \sum_{C \in \mathcal{C}_{\rm lg}} |C|}{L^2 \cdot 2^L} > \frac{3\eta n - \gamma n}{L^2 \cdot 2^L} \overset{(20)}{\geq} \xi n$$

times in $\mathcal{C}_{\rm sm}$. Therefore, in all cases, we can form $\mathcal{C}'_{\rm sm}$ by removing exactly

$$\frac{|M|}{\ell^*} + \lfloor \xi^2 n/2 \rfloor \overset{(20),(21)}{<} \xi n$$

cycles that are isomorphic to C^* from $\mathcal{C}_{\rm sm}$. (Recall that (22) implies that $|M|$ is a divisible by ℓ^*.)

By (21) and as we are in Case 2, we can greedily find a collection of disjoint cycles in $G \setminus M$ that is isomorphic to $\mathcal{C}_{\mathrm{lg}}$. Form G' be removing the vertices of these cycles from $G \setminus M$. The definitions imply that

$$\left(\sum_{C \in \mathcal{C}'_{\mathrm{sm}}} |C|\right) + \ell^* \left\lfloor \xi^2 n/2 \right\rfloor = \left(\sum_{C \in \mathcal{C}_{\mathrm{sm}}} |C|\right) - |M| = |G'|. \tag{23}$$

In particular,

$$(1-\gamma)|G'| \overset{(20)}{\geq} |G'| - \ell^* \left\lfloor \xi^2 n/2 \right\rfloor \overset{(23)}{=} \sum_{C \in \mathcal{C}'_{\mathrm{sm}}} |C|.$$

As we are in Case 2, we also have

$$\delta^0(G') \geq \frac{n+t}{2} + \eta n - |M| - \sum_{C \in \mathcal{C}_{\mathrm{lg}}} |C| \overset{(20),(21)}{\geq} \frac{n+t}{2} + \frac{\eta}{4} n. \tag{24}$$

By (20) we can assume $|G'| \geq n_{4.2}(L, \eta/4, \gamma)$, so Lemma 4.2 implies that there exists a collection of disjoint cycles in G' that is isomorphic to $\mathcal{C}'_{\mathrm{sm}}$. Let W be the set of vertices in G' that are not covered by this collection. Then,

$$|W| = |G'| - \sum_{C \in \mathcal{C}'_{\mathrm{sm}}} |C| \overset{(23)}{=} \ell^* \left\lfloor \xi^2 n/2 \right\rfloor.$$

Therefore, (22) implies that there is a C^*-factor in $G[M \cup W]$. That is, we have found a collection of exactly $(|M| + |W|)/\ell^* = |M|/\ell^* + \left\lfloor \xi^2 n/2 \right\rfloor$ additional vertex-disjoint cycles isomorphic to C^*. The union of this with the collection of previously constructed cycles is isomorphic to $\mathcal{C}$. □

5. Proof of Theorem 1.9

The proof adapts that of Theorem 4.1 from [28]. Let G be as in the statement of the theorem. Let G' denote the *graph* on $V(G)$ where $xy \in E(G')$ precisely when $xy \in E(G)$ or $yx \in E(G)$. So $d_{G'}(x) + d_{G'}(y) \geq 4n/3 - 1$ for all non-adjacent $x \neq y \in V(G')$ by (1). Theorem 1.8 therefore implies that G' contains a K_3-factor and so G contains a $\{T_3, C_3\}$-factor. Let $\mathcal{M}$ denote the $\{T_3, C_3\}$-factor in G that contains the most copies of T_3.

Suppose for a contradiction that $\mathcal{M}$ is not a T_3-factor. Then there is a copy C of C_3 in $\mathcal{M}$. Let $V(C) = \{x, y, z\}$ where $xy, yz, zx \in E(C)$.

Note that $yx, zy, xz \notin E(G)$ since otherwise there is a copy of T_3 in G on $\{x, y, z\}$, a contradiction to the choice of $\mathcal{M}$. By (1) this implies that $d^+_G(y) + d^-_G(x)$, $d^+_G(z) + d^-_G(y)$, $d^+_G(x) + d^-_G(z) \geq 4n/3 - 1$. This implies that $d^+_G(x) + d^+_G(y) + d^+_G(z) \geq 2n - 1$ or $d^-_G(x) + d^-_G(y) + d^-_G(z) \geq 2n - 1$; without loss of generality, assume the former holds.

As $yx, zy, xz \notin E(G)$, $G[x, y, z]$ contains precisely three edges. In particular, there are at least $d^+_G(x) + d^+_G(y) + d^+_G(z) - 3 \geq 2n - 4 > 6(|\mathcal{M}| - 1)$ edges in G with startpoint in $V(C)$ and endpoint in $V(G) \setminus V(C)$. This implies that there is an element $T \in \mathcal{M} \setminus \{C\}$ that receives at least 7 edges from $V(C)$ in G.

Hence, there is a vertex, say x, in $V(C)$ that sends out 3 edges to T. Furthermore, y and z have a common outneighbour in G that lies in $V(T)$. Together, this implies that $G[V(C) \cup V(T)]$ contains two vertex-disjoint copies of T_3. This yields a $\{T_3, C_3\}$-factor in G containing more copies of T_3 than $\mathcal{M}$, a contradiction. Thus, the assumption that $\mathcal{M}$ is not a T_3-factor is false, as desired. □

Remark 5.1. *Notice that we can replace (1) in Theorem 1.9 with any of the following conditions: (i) $d^+_G(x) + d^+_G(y) \geq 4n/3 - 1$; (ii) $d^-_G(x) + d^-_G(y) \geq 4n/3 - 1$; (iii) $d^-_G(x) + d^+_G(y) \geq 4n/3 - 1$. Indeed, in each case the proof proceeds analogously.*

6. Proof of Theorem 1.10

6.1. **Finding an almost spanning T_r-tiling.** Given a digraph G and $z \in V(G)$, the *dominant degree $d^*_G(z)$ of z* is defined as $d^*_G(z) := \max\{d^+_G(z), d^-_G(z)\}$. Given a set $X \subseteq V(G)$, we define $d^*_G(z, X) := \max\{|N^+_G(z) \cap X|, |N^-_G(z) \cap X|\}$.

The following theorem will ensure a digraph G as in Theorem 1.10 contains a T_r-tiling covering most of the vertices of G.

Theorem 6.1. *Let $r \geq 2$ be an integer and let $\gamma > 0$. There exists $n_0 = n_0(r, \gamma) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose that G is an n-vertex digraph so that for any $x \neq y \in V(G)$ at least one of the following conditions holds:*

(i) $xy, yx \in E(G)$;

(ii) $d^*_G(z) \geq (1 - 1/r + \gamma)n$ for some $z \in \{x, y\}$.

Then G contains a T_r-tiling covering all but at most γn vertices of $V(G)$.

Notice that a digraph G as in Theorem 1.10 satisfies the hypothesis of Theorem 6.1 with η playing the role of γ. In fact, the hypothesis of Theorem 6.1 is significantly more relaxed than that of Theorem 1.10. On the other hand, one cannot strengthen the conclusion of Theorem 6.1 to ensure a T_r-factor. For example, consider the disjoint union G of two complete digraphs A, B, so that $|A|$ and $|B|$ are not divisible by r. Then G does not contain a T_r-factor even though one can choose the sizes of A and B so that (i) and (ii) from Theorem 6.1 hold.

We will need the following simple facts.

Fact 6.2. *Let $n, r \in \mathbb{N}$ such that $n \geq r \geq 2$. Let G be an n-vertex digraph so that for each $z \in V(G)$*

$$d^*_G(z) > (1 - 1/(r-1))n.$$

Then G contains a copy of T_r. □

Fact 6.3. *Let $n, r \in \mathbb{N}$ such that $n \geq r^2$ and $r \geq 2$. Let G be an n-vertex digraph so that for any $x \neq y \in V(G)$ such that $xy \notin E(G)$ we have that*

$$d^+_G(x) + d^-_G(y) > 2(1 - 1/r)n. \tag{25}$$

Then for every $z \in V(G)$, z lies in a copy of T_r in G.

Proof. Given any $z \in V(G)$ notice that $d^+_G(z) > (1-2/r)n$. If $N^+_G(z)$ induces a complete subdigraph of G then we obtain our desired copy of T_r. Otherwise, there is some $z_1 \in N^+_G(z)$ with $d^*_G(z_1) > (1 - 1/r)n$; without loss of generality assume that $d^+_G(z_1) = d^*_G(z_1)$. If $r = 2$ then $\{z, z_1\}$ induces our desired copy of T_r in G. If $r > 2$ then $|N^+_G(z) \cap N^+_G(z_1)| > (1 - 3/r)n \geq 0$. If $N^+_G(z) \cap N^+_G(z_1)$ induces a complete subdigraph of G then we obtain our desired copy of T_r. Otherwise, there is some $z_2 \in N^+_G(z) \cap N^+_G(z_1)$ with $d^*_G(z_2) > (1 - 1/r)n$. By repeating this process we obtain our desired copy of T_r containing z. □

Remark 6.4. *Notice that we can replace (25) in Fact 6.3 with any of the following conditions: (i) $d^+_G(x) + d^+_G(y) > 2(1 - 1/r)n$; (ii) $d^-_G(x) + d^-_G(y) > 2(1 - 1/r)n$; (iii) $d^-_G(x) + d^+_G(y) > 2(1 - 1/r)n$.*

Fact 6.5. *Suppose that $r, t \in \mathbb{N}$ such that r divides t. Then both $T_r(t)$ and $T_{r+1}(t)$ contain T_r-factors.* □

To prove Theorem 6.1 we will build up the T_r-tiling using a variant of an approach that was first used in [17]. The next lemma is a key tool used for this; its proof is similar in flavour to that of Lemma 6.4 from [29].

Lemma 6.6. *Let $r \geq 2$ be an integer and let $\gamma > 0$. There exists $n_0 = n_0(r, \gamma) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose that G is an n-vertex digraph possibly with loops and such that for any $x \neq y \in V(G)$ at least one of the following conditions holds:*

(i) $xy, yx \in E(G)$;
(ii) $d^*_G(z) \geq (1-1/r+\gamma)n$ *for some* $z \in \{x,y\}$.

Further, suppose the largest T_r*-tiling in* G *covers* $n' \leq (1-\gamma)n$ *vertices of* $V(G)$*. Then* G *contains a* $\{T_r, T_{r+1}\}$*-tiling that covers at least* $n' + \gamma^2 n/3$ *vertices of* $V(G)$.

Proof. We first show that $n' \geq \gamma n/3$. Let K be the largest complete digraph in G. By conditions (i) and (ii), every set $X \subseteq V(G)$ of size at least $|K|+1$ contains a vertex $z \in X$ so that $d^*_G(z) \geq (1-1/r+\gamma)n$. If $|K| \geq \gamma n/2$ then certainly G contains a T_r-tiling covering at least $|K|-r+1 \geq \gamma n/3$ vertices. Otherwise, there is a set $Y \subseteq V(G)$ where $|Y| \leq |K| < \gamma n/2$ so that $d^*_G(z) \geq (1-1/r+\gamma)n$ for every $z \in V(G) \setminus Y$; in particular, $d^*_{G\setminus Y}(z) \geq (1-1/r+\gamma/2)n$ for every $z \in V(G) \setminus Y$. In this case we can repeatedly apply Fact 6.2 to obtain a T_r-tiling in $G \setminus Y \subseteq G$ covering at least $\gamma n/2$ vertices. In either case, this shows that $n' \geq \gamma n/3$.

Let $\mathcal{M}$ be the largest T_r-tiling in G and suppose that $n' \leq (1-\gamma)n$. Note that $G_1 := G \setminus V(\mathcal{M})$ does not contain a copy of T_r by the maximality of $\mathcal{M}$. Thus, (i) and (ii) imply that all but at most $r-1$ vertices $z \in V(G_1)$ satisfy $d^*_G(z) \geq (1-1/r+\gamma)n$.

We claim that there are at least $\gamma^2 n$ vertices $z \in V(G_1)$ that satisfy $d^*_{G_1}(z) < (1-1/r+\gamma)|G_1|$. Suppose for a contradiction that this is not the case. Then by deleting at most $\gamma^2 n \leq \gamma|G_1|$ vertices from G_1 we obtain an induced subdigraph G_2 of G_1 so that $d^*_{G_2}(z) > (1-1/r)|G_2|$ for every $z \in V(G_2)$; so Fact 6.2 implies that $G_2 \subseteq G_1$ contains a copy of T_r, a contradiction.

The last two paragraphs imply that there are at least $\gamma^2 n - (r-1) \geq \gamma^2 n/3$ vertices $z \in V(G_1)$ such that $d^*_G(z, V(\mathcal{M})) \geq (1-1/r+\gamma)n - (1-1/r+\gamma)|G_1| = (1-1/r+\gamma)|V(\mathcal{M})|$. This condition implies that for each such z, there are at least $\gamma|V(\mathcal{M})| = \gamma n' \geq \gamma^2 n/3$ copies T of T_r in $\mathcal{M}$ such that (a) z sends out all possible edges to T or (b) z receives all possible edges from T; in particular, z forms a copy of T_{r+1} with each such T.

One can now greedily assign at least $\gamma^2 n/3$ such vertices z to distinct copies of T_r in $\mathcal{M}$ to obtain a $\{T_r, T_{r+1}\}$-tiling that covers at least $n' + \gamma^2 n/3$ vertices, as desired. □

We will now combine Lemma 6.6 with the diregularity lemma to prove Theorem 6.1. The main novelty in our proof is that we need to consider the special reduced digraph R^* of G, rather than the reduced digraph R. For this, we will need to extend the notion of a t-blow-up of a digraph to digraphs with loops. Indeed, given a digraph R^* with loops, the digraph (with loops) $R^*(t)$ is defined as follows: $V(R^*(t)) := \{v^j : v \in V(R^*) \text{ and } j \in [t]\}$ and $E(R^*(t)) := \{v^a w^b : vw \in E(R^*) \text{ and } a, b \in [t]\}$. Thus, if there is a loop at $v \in V(R^*)$, then for every $a, b \in [t]$, $R^*(t)$ contains the edge $v^a v^b$ and in particular, there is a loop at v^a in $R^*(t)$.

Proof of Theorem 6.1. Define additional constants ε, d and $n_0, \ell_0 \in \mathbb{N}$ so that

$$0 < \frac{1}{n_0} \ll \frac{1}{\ell_0} \ll \varepsilon \ll d \ll \gamma, \frac{1}{r}. \tag{26}$$

Note that it suffices to prove the theorem under the assumption that $\gamma < 1/4$. Set $s := \lceil 12/\gamma^2 \rceil$. Apply Lemma 2.2 with parameters ε, d and ℓ_0 to G to obtain clusters $V_1, \dots, V_k$, an exceptional set V_0 and a pure digraph G'. Set $m := |V_1| = \dots = |V_k|$. Let R and R^* respectively denote the reduced digraph and special reduced digraph of G with parameters ε, d and ℓ_0.

Note that we may assume that for any distinct $i, j \in [k]$, if (V_i, V_j) is of density 1 in G then it is of density 1 in G'.

For any distinct $i, j \in [k]$, suppose that $V_i V_j \notin E(R)$ or $V_j V_i \notin E(R)$. Then by the remark in the last paragraph, there exist some $x \in V_i$ and $y \in V_j$ such that $xy \notin E(G)$ or $yx \notin E(G)$. Thus, for some $z \in \{x,y\}$, $d^*_G(z) \geq (1-1/r+\gamma)n$ by (ii) and so $d^*_{G'}(z) \geq (1-1/r+3\gamma/4)n$ by (26) and Lemma 2.2(c). Without loss of generality, assume that $z = x$. Note that $d^*_{G'}(x, V_1 \cup$

$\cdots \cup V_k) \geq (1-1/r+\gamma/2)n$ by Lemma 2.2(a). This together with Lemma 2.2(e) implies that $d^*_R(V_i) \geq (1-1/r+\gamma/2)n/m \geq (1-1/r+\gamma/2)k$.

In summary, for any distinct $i,j \in [k]$ at least one of the following conditions holds:

(i′) $V_iV_j, V_jV_i \in E(R)$;

(ii′) $d^*_R(V_i) \geq (1-1/r+\gamma/2)k$ or $d^*_R(V_j) \geq (1-1/r+\gamma/2)k$.

Furthermore, note that if there is no loop at some $V_i \in V(R^*)$ in R^*, then there exists a $z \in V_i$ so that $d^*_G(z) \geq (1-1/r+\gamma)n$. Arguing as before, we conclude that

(iii′) $d^*_{R^*}(V_i) \geq (1-1/r+\gamma/2)k$ for every $V_i \in V(R^*)$ that does not have a loop in R^*.

Let $t \in \mathbb{N}$. Crucially, (i′)–(iii′) ensure that for any distinct $X,Y \in V(R^*(t))$,

(i″) $XY, YX \in E(R^*(t))$, or

(ii″) $d^*_{R^*(t)}(Z) \geq (1-1/r+\gamma/2)kt = (1-1/r+\gamma/2)|R^*(t)|$ for some $Z \in \{X,Y\}$.

Claim 6.7. $R^*_s := R^*(r^s)$ *contains a* T_r*-tiling covering at least* $(1-\gamma/2)kr^s = (1-\gamma/2)|R^*_s|$ *vertices.*

Proof of the claim. If R^* contains a T_r-tiling covering at least $(1-\gamma/2)k$ vertices then Fact 6.5 implies that Claim 6.7 holds. Thus, suppose that the largest T_r-tiling in R^* covers precisely $d \leq (1-\gamma/2)k$ vertices. Then by (i′) and (ii′) we may apply Lemma 6.6 to R^* to conclude that R^* contains a $\{T_r, T_{r+1}\}$-tiling that covers at least $d+\gamma^2k/12$ vertices. Thus, by Fact 6.5, $R^*(r)$ contains a T_r-tiling covering at least $(d+\gamma^2k/12)r$ vertices. (So at least a $\gamma^2/12$-proportion of the vertices in $R^*(r)$ are covered.)

If $R^*(r)$ contains a T_r-tiling covering at least $(1-\gamma/2)kr$ vertices then again Fact 6.5 implies that the claim holds. So suppose that the largest T_r-tiling in $R^*(r)$ covers precisely $d' \leq (1-\gamma/2)kr$ vertices. Recall that $d' \geq (d+\gamma^2k/12)r$. Then by (i″) and (ii″) we may apply Lemma 6.6 to $R^*(r)$ to conclude that $R^*(r)$ contains a $\{T_r, T_{r+1}\}$-tiling that covers at least $d'+\gamma^2kr/12 \geq (d+\gamma^2k/6)r$ vertices. Thus, by Fact 6.5, $R^*(r^2)$ contains a T_r-tiling covering at least $(d+\gamma^2k/6)r^2$ vertices. (So at least a $\gamma^2/6$-proportion of the vertices in $R^*(r^2)$ are covered.) Repeating this argument at most s times we see that the claim holds. ■

We now use Claim 6.7 to prove the theorem. For each $i \in [k]$, partition V_i into classes $V_i^0, V_{i,1}, \dots, V_{i,r^s}$ where $m' := |V_{i,j}| = \lfloor m/r^s \rfloor \geq m/(2r^s)$ for all $j \in [r^s]$. Since $mk \geq (1-\varepsilon)n$ by Lemma 2.2(a),

$$m'|R^*_s| = \lfloor m/r^s \rfloor kr^s \geq mk - kr^s \geq (1-2\varepsilon)n. \tag{27}$$

Proposition 2.1 implies that, for any distinct $i_1, i_2 \in [k]$, if $(V_{i_1}, V_{i_2})_{G'}$ is ε-regular with density at least d then $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon r^s$-regular with density at least $d-\varepsilon \geq d/2$ for all $j_1, j_2 \in [r^s]$. Moreover, for any $i \in [k]$ and any distinct $j_1, j_2 \in [r^s]$, if there is a loop at V_i in $V(R^*)$, then there are all possible edges between V_{i,j_1} and V_{i,j_2} in G (and so certainly $(V_{i,j_1}, V_{i,j_2})_G$ is $2\varepsilon r^s$-regular with density at least $d/2$).

We can therefore label the vertex set of R^*_s so that $V(R^*_s) = \{V_{i,j} : i \in [k],\ j \in [r^s]\}$ where $V_{i_1,j_1}V_{i_2,j_2} \in E(R^*_s)$ implies that either (a) $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon r^s$-regular with density at least $d/2$ or (b) there are all possible edges between V_{i_1,j_1} and V_{i_2,j_2} in G.

By Claim 6.7, R^*_s has a T_r-tiling $\mathcal{M}$ that contains at least $(1-\gamma/2)|R^*_s|$ vertices. Consider any copy T of T_r in $\mathcal{M}$ and let $V(T) = \{V_{i_1,j_1}, V_{i_2,j_2}, \dots, V_{i_r,j_r}\}$. Set $V := V_{i_1,j_1} \cup V_{i_2,j_2} \cup \dots \cup V_{i_r,j_r}$. Note that $0 < 1/m' \ll 2\varepsilon r^s \ll d/2 \ll \gamma, 1/r$. Lemma 2.5 implies that $G[V]$ contains a T_r-tiling covering all but at most $\sqrt{2\varepsilon r^s}m'r \leq \gamma^2 m'r$ vertices. By considering each copy of T_r in $\mathcal{M}$ we conclude that G contains a T_r-tiling covering at least

$$(1-\gamma^2)m'r \times (1-\gamma/2)|R^*_s|/r \overset{(27)}{\geq} (1-\gamma^2)(1-\gamma/2)(1-2\varepsilon)n \overset{(26)}{\geq} (1-\gamma)n$$

vertices, as desired.

□

6.2. **Proof of Theorem 1.10.** We now have all the tools required to prove Theorem 1.10.

Proof of Theorem 1.10. Define additional constants $\gamma, \gamma_1, \gamma_2, \eta_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} \ll \gamma_2 \ll \gamma_1 \ll \gamma \ll \eta_1 \ll \eta, \frac{1}{r}. \tag{28}$$

Note that it suffices to prove the theorem under the additional assumption that $\eta \ll 1/r$.

Let G be a digraph on $n \geq n_0$ vertices as in the statement of the theorem. For any $x \neq y \in V(G)$, (2) implies that at least one of the following conditions holds:

(i) $xy, yx \in E(G)$;

(ii) $d^*_G(z) \geq (1-1/r+\eta)n$ for some $z \in \{x, y\}$.

Partition $V(G)$ into S, L where S consists precisely of those $z \in V(G)$ such that $d^*_G(z) < (1-1/r+\eta)n$. Notice (i) and (ii) imply that $G[S]$ is a complete digraph. In particular, $|S| < (1-1/r+\eta)n+1$ and so certainly $|L| \geq 2\eta_1 n$. Our argument now splits into two cases.

Case 1: $|S| \geq \eta_1 n$. Our first aim is to apply Lemma 2.6 to obtain a T_r-absorbing set.

Claim 6.8. *Given any distinct $x, y \in V(G)$ so that either $x, y \in S$ or $x, y \in L$, there are at least γn^{r-1} $(r-1)$-sets $X \subseteq V(G)$ such that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r.*

Proof of the claim. Given any distinct $x, y \in S$, since $G[S]$ is complete, every $(r-1)$-set $X \subseteq S \setminus \{x, y\}$ is such that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r. There are at least $(\eta_1 n/2)^{r-1}/(r-1)! \overset{(28)}{\geq} \gamma n^{r-1}$ such choices for X.

Next consider any distinct $x, y \in L$. Then $d^*_G(x), d^*_G(y) \geq (1-1/r+\eta)n$. Without loss of generality, suppose that $d^+_G(x) = d^*_G(x)$ and $d^-_G(y) = d^*_G(y)$; the other cases are analogous. Then $|N^+_G(x) \cap N^-_G(y)| \geq (1-2/r+2\eta)n$. If $|N^+_G(x) \cap N^-_G(y) \cap S| \geq \eta_1 n$, then as in the last paragraph we conclude that there are at least γn^{r-1} $(r-1)$-sets $X \subseteq S$ such that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r. So assume that $|N^+_G(x) \cap N^-_G(y) \cap S| < \eta_1 n$ and so $|L'| \geq (1-2/r+\eta)n$ where $L' := N^+_G(x) \cap N^-_G(y) \cap L$. As $d^*_G(z) \geq (1-1/r+\eta)n$ for all $z \in L'$, it is now straightforward to greedily construct at least γn^{r-1} $(r-1)$-sets $X \subseteq L'$ such that $G[X]$ contains a spanning copy of T_{r-1}; in particular, for each such X, both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r. ■

Claim 6.9. *Given any $x \in S$, there are at least $\eta_1 n$ vertices $y \in L$ such that the following holds: there are at least γn^{r-1} $(r-1)$-sets $X \subseteq V(G)$ such that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r.*

Proof of the claim. Notice that since $x \in S$, $xy \notin E(G)$ for at least $\eta_1 n$ vertices $y \in L$. Thus, by (2) we have that $d^+_G(x) + d^-_G(y) \geq 2(1-1/r+\eta)n$. This implies that $|N^+_G(x) \cap N^-_G(y)| \geq (1-2/r+2\eta)n$. If $|N^+_G(x) \cap N^-_G(y) \cap S| \geq \eta_1 n$, then arguing as in the proof of Claim 6.8 we conclude that there are at least γn^{r-1} $(r-1)$-sets $X \subseteq S$ such that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r. Otherwise, $|N^+_G(x) \cap N^-_G(y) \cap S| < \eta_1 n$ and so $|L'| \geq (1-2/r+\eta)n$ where $L' := N^+_G(x) \cap N^-_G(y) \cap L$. Again arguing as in the proof of Claim 6.8, we conclude that there are at least γn^{r-1} $(r-1)$-sets $X \subseteq L'$ such that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r. ■

The previous two claims allow us to prove the following.

Claim 6.10. *Given any distinct $x, y \in V(G)$, there are at least $\gamma_1 n^{2r-1}$ $(2r-1)$-sets $X_1 \subseteq V(G)$ such that $G[X_1 \cup \{x\}]$ and $G[X_1 \cup \{y\}]$ contain T_r-factors.*

Proof of the claim. Consider any $x, y \in V(G)$. First, suppose that either $x, y \in S$ or $x, y \in L$. Claim 6.8 implies that there are at least γn^{r-1} $(r-1)$-sets $X \subseteq V(G)$ such that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain spanning copies of T_r. Fix any such X. Select any r-set $X' \subseteq S \setminus (X \cup \{x, y\})$;

there are at least $(\eta_1 n/2)^r/r!$ choices for X'. Set $X_1 := X \cup X'$. As $G[X']$ is a complete digraph, $G[X_1 \cup \{x\}]$ and $G[X_1 \cup \{y\}]$ contain T_r-factors. In total, there are at least

$$\gamma n^{r-1} \times \frac{(\eta_1 n/2)^r}{r!} \times \frac{1}{(2r-1)!} \overset{(28)}{\geq} \gamma_1 n^{2r-1}$$

choices for X_1, as required.

Finally, consider the case when $x \in S$ and $y \in L$. By Claim 6.9 there are at least $\eta_1 n - 1$ vertices $z \in L \setminus \{y\}$ so that the following holds: there are at least $\gamma n^{r-1} - n^{r-2} \geq \gamma n^{r-1}/2$ $(r-1)$-sets $X \subseteq V(G) \setminus \{y\}$ such that $G[X \cup \{x\}]$ and $G[X \cup \{z\}]$ contain spanning copies of T_r. Fix any such choice of z and X. By Claim 6.8 there are at least $\gamma n^{r-1} - rn^{r-2} \geq \gamma n^{r-1}/2$ $(r-1)$-sets $X' \subseteq V(G) \setminus (X \cup \{x\})$ such that $G[X' \cup \{y\}]$ and $G[X' \cup \{z\}]$ contain spanning copies of T_r. Set $X_1 := \{z\} \cup X \cup X'$. By construction of X_1, we have that $G[X_1 \cup \{x\}]$ and $G[X_1 \cup \{y\}]$ contain T_r-factors. In total, there are at least

$$(\eta_1 n - 1) \times \frac{\gamma n^{r-1}}{2} \times \frac{\gamma n^{r-1}}{2} \times \frac{1}{(2r-1)!} \overset{(28)}{\geq} \gamma_1 n^{2r-1}$$

choices for X_1. ■

Claim 6.10 ensures that we can apply Lemma 2.6 with $T_r, r, 2, \gamma_1$ playing the roles of H, h, t, γ. Thus, $V(G)$ contains a set M so that

- $|M| \leq (\gamma_1/2)^r n/4$;
- M is a T_r-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq (\gamma_1/2)^{2r} rn/32$.

Set $G_1 := G \setminus M$. Then (i) and (ii) imply that for any $x \neq y \in V(G_1)$ at least one of the following conditions holds:

- $xy, yx \in E(G_1)$;
- $d^*_{G_1}(z) \geq (1 - 1/r + \gamma_2)|G_1|$ for some $z \in \{x, y\}$.

Theorem 6.1 therefore implies that G_1 contains a T_r-tiling $\mathcal{H}_1$ covering all but at most $\gamma_2 |G_1| \overset{(28)}{\leq} (\gamma_1/2)^{2r} rn/32$ vertices in G_1. Moreover, the choice of M ensures that there is a T_r-tiling $\mathcal{H}_2$ covering precisely the vertices in G that are not in $\mathcal{H}_1$. Thus, $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ is our desired T_r-factor in G.

Case 2: $|S| < \eta_1 n$. In this case, by repeatedly applying Fact 6.3 we greedily construct a T_r-tiling $\mathcal{S}$ in G of size at most $\eta_1 n$ that covers all of S.

Let $G_0 := G \setminus V(\mathcal{S})$. Note that as $V(G_0) \subseteq L$ we have that $d^*_{G_0}(z) \geq (1 - 1/r + \eta/2)n$ for every $z \in V(G_0)$. This condition allows us to conclude that, for every distinct $x, y \in V(G_0)$, there are at least $\gamma |G_0|^{r-1}$ $(r-1)$-sets $X \subseteq V(G_0)$ such that $G_0[X \cup \{x\}]$ and $G_0[X \cup \{y\}]$ contain spanning copies of T_r. We can now apply Lemma 2.6 and argue analogously to the last two paragraphs of Case 1 to conclude that G_0 contains a T_r-factor $\mathcal{H}$. Thus, $\mathcal{S} \cup \mathcal{H}$ is our desired T_r-factor in G. □

Remark 6.11. *Notice that we can replace (2) in Theorem 1.10 with any of the following conditions: (i) $d^+_G(x) + d^+_G(y) \geq 2(1 - 1/r + \eta)n$; (ii) $d^-_G(x) + d^-_G(y) \geq 2(1 - 1/r + \eta)n$; (iii) $d^-_G(x) + d^+_G(y) \geq 2(1 - 1/r + \eta)n$. Indeed, in each case the proof proceeds analogously.*

7. Open problems

In Theorem 1.5 we asymptotically determined the minimum semi-degree threshold for forcing a C-factor in a digraph for every orientation of an odd cycle C. The corresponding problem for even cycles seems more challenging; as a starting point, it would be interesting to resolve the following conjecture.

Conjecture 7.1. *Given any $\eta > 0$, there exists $n_0 = n_0(\eta) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ divisible by 4. If G is an n-vertex digraph with*

$$\delta^0(G) \geq (1/2 + \eta)n,$$

then G contains a C_4-factor.

The example given after the statement of Theorem 1.2 again shows that if Conjecture 7.1 is true, then the bound on $\delta^0(G)$ is asymptotically best possible. A resolution of Conjecture 7.1 could also provide insight into the question for even cycle factors more generally.

Question 7.2. *Is the following statement true? Let C be an orientation of an even cycle. Given any $\eta > 0$, there exists $n_0 = n_0(\eta, C) \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ divisible by $|C|$. If G is an n-vertex digraph with*

$$\delta^0(G) \geq (1/2 + \eta)n,$$

then G contains a C-factor.

Remark 7.3. *Note that Lemma 4.7 implies that to resolve Conjecture 7.1 and Question 7.2, it suffices to resolve the 'almost perfect' C-tiling versions of them.*

Finally, in Theorem 1.10 we asymptotically determined the Ore-type threshold for forcing a T_r-factor. It would be interesting to prove a sharp version of this result as well as to establish other Ore-type conditions for forcing H-factors in digraphs.

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