A NOTE ON COLOUR-BIAS PERFECT MATCHINGS IN HYPERGRAPHS

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Abstract. A result of Balogh, Csaba, Jing and Pluhár yields the minimum degree threshold that ensures a 2-coloured graph contains a perfect matching of significant colour-bias (i.e., a perfect matching that contains significantly more than half of its edges in one colour). In this note we prove an analogous result for perfect matchings in k-uniform hypergraphs. More precisely, for each $2 \leq \ell < k$ and $r \geq 2$ we determine the minimum ℓ-degree threshold for forcing a perfect matching of significant colour-bias in an r-coloured k-uniform hypergraph.

1. Introduction

A *perfect matching* in a hypergraph H is a collection of vertex-disjoint edges of H which covers the vertex set $V(H)$ of H. In recent decades there has been significant interest in the problem of establishing *minimum degree* conditions that force a perfect matching in a k-uniform hypergraph. More precisely, given a k-uniform hypergraph H and an ℓ-element vertex set $S \subseteq V(H)$ (where $\ell \in [k-1]$) we define $d_H(S)$ to be the number of edges containing S. The *minimum ℓ-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(S)$ over all ℓ-element sets of vertices in H. We refer to $\delta_1(H)$ as the *minimum vertex degree* of H and $\delta_{k-1}(H)$ as the *minimum codegree* of H.

Suppose that $\ell, k, n \in \mathbb{N}$ such that $\ell \leq k-1$ and k divides n. Let $m_\ell(k,n)$ denote the smallest integer m such that every k-uniform hypergraph H on n vertices with $\delta_\ell(H) \geq m$ contains a perfect matching.

A simple consequence of Dirac's theorem is that $m_1(2,n) = n/2$ for all even $n \in \mathbb{N}$. Improving earlier asymptotically exact bounds given in [12, 17], Rödl, Ruciński and Szemerédi [18] determined the minimum codegree threshold for perfect matchings in k-uniform hypergraphs. That is, they showed that if $n \in \mathbb{N}$ is sufficiently large, then $m_{k-1}(k,n) = n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k.

The value of $m_\ell(k,n)$ is known for various pairs (k,ℓ) when n is sufficiently large. For example, after an earlier asymptotic result of Pikhurko [15], Treglown and Zhao [19] determined the value of $m_\ell(k,n)$ for $\ell \geq k/2$ and n sufficiently large. However, the minimum vertex degree case of the problem is wide open in general, and the only cases where the asymptotic or exact value of $m_1(k,n)$ is known is when $k = 2, 3, 4, 5$. See, e.g., [16, 21] for discussions on further results in the area.

Given any $1 \leq \ell < k$ it is known that

$$m_\ell(k,n) \geq \max\left\{\frac{1}{2} - o(1), 1 - \left(\frac{k-1}{k}\right)^{k-\ell} - o(1)\right\}\binom{n}{k-\ell}. \tag{1}$$

See, e.g., the introduction of [20] for the two families of hypergraphs that demonstrate (1). It is widely believed that the inequality in (1) is asymptotically sharp for all choices of k, ℓ, see [11, 13].

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Moreover, Treglown and Zhao [20] gave a conjecture on the exact value of $m_\ell(k,n)$ for sufficiently large $n \in k\mathbb{N}$.

The aim of this paper is to study the *colour-bias* version of this problem. The topic of colour-bias structures in graphs was first raised by Erdős in the 1960s (see [5, 6]). Sparked by work of Balogh, Csaba, Jing and Pluhár [1], there has been renewed interest in the topic, particularly in establishing minimum degree conditions that force a colour-bias copy of a graph F. More precisely, if a graph G contains a copy of F, then however the edges of G are 2-coloured, one can clearly ensure that G contains a copy of F with at least $e(F)/2$ edges of the same colour. The question then is how large does the minimum degree $\delta(G)$ of G need to be to guarantee that G contains a copy of F with significantly more than $e(F)/2$ edges of the same colour, no matter how one 2-colours the edges of G? The following result resolves this problem in the case when F is a Hamilton cycle.

Theorem 1.1 (Balogh, Csaba, Jing and Pluhár [1]). *Let $0 < c < 1/4$ and $n \in \mathbb{N}$ be sufficiently large. If G is an n-vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-colouring of $E(G)$ there is a Hamilton cycle in G with at least $n/2 + cn/32$ edges of the same colour. Moreover, if $n \in 4\mathbb{N}$, there is an n-vertex graph G' with $\delta(G') = 3n/4$ and a 2-colouring of $E(G')$ for which every Hamilton cycle in G' has precisely $n/2$ edges in each colour.

Note that Theorem 1.1 shows that the minimum degree threshold for forcing a *colour-bias* Hamilton cycle in a graph is significantly higher than the threshold for just forcing a Hamilton cycle. Indeed, Dirac's theorem tells us that any n-vertex graph G with $\delta(G) \geq n/2$ contains a Hamilton cycle.

Since a Hamilton cycle on an even number of vertices is the union of two perfect matchings, Theorem 1.1 implies the following result.

Theorem 1.2 (Balogh, Csaba, Jing and Pluhár [1]). *Let $0 < c < 1/4$ and $n \in 2\mathbb{N}$ be sufficiently large. If G is an n-vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-colouring of $E(G)$ there is a perfect matching in G with at least $n/4 + cn/64$ edges of the same colour. Moreover, if $n \in 4\mathbb{N}$, there is an n-vertex graph G' with $\delta(G') = 3n/4$ and a 2-colouring of $E(G')$ for which every perfect matching in G' has precisely $n/4$ edges in each colour.

Let $n \in 4\mathbb{N}$. We define the graph G' in Theorem 1.2 as follows: $V(G')$ consists of the disjoint union of two vertex classes A and B of sizes $n/4$ and $3n/4$, respectively; $E(G')$ contains all possible red edges whose endpoints are both in B and all possible blue edges with one endpoint in A and one endpoint in B. Thus, $\delta(G') = 3n/4$ and every perfect matching in G' has precisely $n/4$ edges in each colour.

Since [1] appeared, a number of analogues of Theorem 1.1 have been established for other types of spanning structures. Given graphs G and F, an *F-factor* in G is a collection of vertex-disjoint copies of F in G that together cover $V(G)$. In [2], the minimum degree threshold for forcing a colour-bias K_r-factor was determined.[1] More recently, this result was extended to F-factors for every fixed graph F; see [4]. For $k \geq 2$, the minimum degree threshold for forcing a colour-bias kth power of a Hamilton cycle in a graph was established in [3].

Other variants of the problem have also been studied. In [7, 10] an r-colour version of Theorem 1.1 was proven: in this setting now one r-colours $E(G)$ and seeks a Hamilton cycle with significantly more than n/r edges of the same colour. Colour-bias problems have also been considered for random graphs [9]. Recently, Mansilla Brito [14] gave a minimum codegree result for forcing a colour-bias

[1] Recall K_r denotes the complete graph on r vertices.

copy of a tight Hamilton cycle in a 3-uniform hypergraph. We remark that all of these colour-bias results can be phrased in the equivalent language of *discrepancy*; see, e.g., [1, 2, 3, 4, 10].

Our main result determines the minimum ℓ-degree threshold for forcing a colour-bias perfect matching in a k-uniform hypergraph for all $\ell \geq 2$ and $k \geq 3$. To state our result we need the following definitions. Given integers $1 \leq \ell < k$, let $\mathcal{C}_{k,\ell}$ be the set of all $c > 0$ such that $m_\ell(k,n) \leq c\binom{n}{k-\ell}$ for all sufficiently large $n \in k\mathbb{N}$. Set $c_{k,\ell}$ to be the infimum of $\mathcal{C}_{k,\ell}$. In particular, note that the general conjecture on the asymptotic value of $m_\ell(k,n)$ equivalently states that

$$c_{k,\ell} = \max\left\{\frac{1}{2}, 1 - \left(\frac{k-1}{k}\right)^{k-\ell}\right\}.$$

Theorem 1.3. *Let $k, \ell, r \in \mathbb{N}$ where $2 \leq \ell < k$ and $r \geq 2$. Given any $\eta > 0$ where $c_{k,\ell} + \eta < 1$, there exists an $n_0 \in \mathbb{N}$ such that the following holds. Let H be a k-uniform hypergraph on $n \geq n_0$ vertices, where $n \in k\mathbb{N}$. If*

$$\delta_\ell(H) \geq (c_{k,\ell} + \eta)\binom{n}{k-\ell},$$

then given any r-colouring of $E(H)$ there is a perfect matching in H with at least $\frac{n}{rk} + \frac{\eta n}{8r(r-1)k^k(k^2+k)}$ edges of the same colour.

We remark that Theorem 1.3 holds even in the cases in which we do not know the value of $c_{k,\ell}$. By definition of $c_{k,\ell}$, the minimum ℓ-degree condition in Theorem 1.3 is essentially best possible. Indeed, for $c < c_{k,\ell}$, a minimum ℓ-degree condition of $\delta_\ell(H) \geq c\binom{n}{k-\ell}$ does not even guarantee a perfect matching, let alone one of significant colour-bias. So in this sense the colour-bias and 'standard' versions of the problem are aligned when $\ell \geq 2$.

In contrast, the same phenomenon does not occur for the minimum vertex degree version of the problem. Indeed, Theorem 1.2 tells us that the minimum degree threshold for a colour-bias perfect matching in a *graph* is different to the minimum degree threshold for a perfect matching in a graph. Furthermore, in Section 4 we describe a similar phenomenon in the 3-uniform hypergraph setting.

Remark. Whilst finalising a manuscript that gave the proof of Theorem 1.3 in the case when $\ell = k-1$ and $r = 2$, we learnt of simultaneous and independent work of Gishboliner, Glock and Sgueglia [8]. In [8] they determine the *minimum codegree threshold* for forcing a tight Hamilton cycle of significant colour-bias in an r-coloured k-uniform hypergraph (where $r \geq 2$ and $k \geq 3$). As an immediate consequence of their result they also establish the corresponding *minimum codegree threshold* for perfect matchings.

We therefore decided to seek a generalisation of our minimum codegree result to other degree conditions, i.e., Theorem 1.3. In doing so, we found an argument much cleaner than our original approach.

Notation. Let H be a hypergraph. The *neighbourhood $N_H(X)$ of a set $X \subseteq V(H)$* is the family of sets $S \subseteq V(H) \setminus X$ such that $S \cup X \in E(H)$. If $X = \{x\}$ we define $N_H(x) := N_H(X)$. Given a vertex $x \in V(H)$ and set $Y \subseteq V(H)$ we sometimes write xY or Yx to denote $\{x\} \cup Y$. Given a colouring c of $E(H)$, we call an edge $e \in E(H)$ a *C-edge* if e is coloured C in c. Given a set $X \subseteq V(H)$, we write $H[X]$ for the *induced subhypergraph of H with vertex set X*. We define $H \setminus X := H[V(H) \setminus X]$.

Given a hypergraph F with an r-colouring $c : E(F) \to \{C_1, \ldots, C_r\}$, its *colour profile* is $(x_1, \ldots, x_r)$ where x_i is the number of C_i-edges in F for each $i \in [r]$. Two colour profiles $(x_1, \ldots, x_r)$, $(y_1, \ldots, y_r)$ are said to be *different with respect to the colour C_i* if $x_i \neq y_i$.

2. Preliminaries and useful results

2.1. **Proof overview and key definitions.** Throughout this section, we will suppose that H is a k-uniform hypergraph on n vertices with an r-colouring $c : E(H) \to \{C_1, \dots, C_r\}$.

Our general strategy for the proof of Theorem 1.3 is as follows. Our aim is to find certain *gadgets* inside of H. A gadget is just a subhypergraph of H with some given structure. A gadget G is *good* if G contains two perfect matchings that have different colour profiles with respect to the r-colouring c.

For a certain well chosen $t \in \mathbb{N}$, we will prove that there are t vertex-disjoint good gadgets $G_1, \dots, G_t$ in H and a $j \in [r]$ so that, for each good gadget G_i, the two perfect matchings M_i and M_i' in G_i have colour profiles that are different with respect to the colour C_j.

We will then be able to easily find a perfect matching in H of significant colour-bias. Indeed, removing the vertices of $G_1, \dots, G_t$ from H will result in a k-uniform hypergraph H' that contains a perfect matching M. The flexibility of the good gadgets then allows us to extend M into a perfect matching in H with significant colour-bias, whatever the colour profile of M is.

We next state the definitions required to formally introduce the notion of a good gadget.

Definition 2.1. *Let $u, v \in V(H)$ be distinct and $T \in N_H(u) \cap N_H(v)$. We say uTv is*

- $\mathbf{S}$ *if $c(T \cup \{u\}) = c(T \cup \{v\})$; or*
- $\mathbf{C_iC_j}$ *if $c(T \cup \{u\}) = C_i$ and $c(T \cup \{v\}) = C_j$.*

Let $C_iC_j(uv)$ denote the collection of sets $T \in N_H(u) \cap N_H(v)$ for which uTv is C_iC_j. Define $S(uv)$ analogously.

Note that $C_iC_j(uv) = C_jC_i(vu)$ for all distinct $u, v \in V(H)$.

Definition 2.2. *Let $D > 0$ and let $u, v \in V(H)$ be distinct. We say that $N_H(u) \cap N_H(v)$ is*

- ***type*** $\mathbf{S(D)}$ *if $|S(uv)| \geq Dn^{k-2}$;*
- ***type*** $\mathbf{C_iC_j(D)}$ *if $i \neq j$ and $|C_iC_j(uv)| \geq Dn^{k-2}$.*

We remark that it may be the case that $N_H(u) \cap N_H(v)$ has more than one type.

Definition 2.3. *Let $e = \{e_1, \dots, e_k\}$ and $f = \{f_1, \dots, f_k\}$ be two edges in H. A* $\mathbf{(k^2 + k, e, f)}$**-gadget** *G is a subhypergraph of H on $k^2 + k$ vertices so that:*

- *$V(G)$ is the disjoint union of e, f and $T_1, \dots, T_k$ where $T_i \in N_H(e_i) \cap N_H(f_i)$ for each $i \in [k]$;*
- *$e, f \in E(G)$;*
- *$e_iT_i, f_iT_i \in E(G)$ for all $i \in [k]$.*

A $(k^2 + k, e, f)$-gadget in which every $e_iT_if_i$ is S will be called an ***S**-*$\mathbf{(k^2 + k, e, f)}$***-gadget***.*

A $\mathbf{(3k, e, f)}$**-gadget** *G is a subhypergraph of H on $3k$ vertices so that:*

- *$e_i = f_i$, for all $i \in \{3, \dots, k\}$;*
- *$V(G)$ is the disjoint union of e, f_1, f_2, T_1 and T_2, where $T_i \in N_H(e_i) \cap N_H(f_i)$ for each $i \in [2]$;*
- *$e, f \in E(G)$;*
- *$e_1T_1, f_1T_1, e_2T_2, f_2T_2 \in E(G)$.*

Given $t \in \{3k, k^2 + k\}$, we say that a (t, e, f)-gadget G is ***good*** *if it contains two perfect matchings with different colour profiles (with respect to the r-colouring of G induced by the r-colouring c of H).*

Note that e and f are vertex-disjoint in a $(k^2 + k, e, f)$-gadget but intersect in $k - 2$ vertices in a $(3k, e, f)$-gadget; see Figure 1.

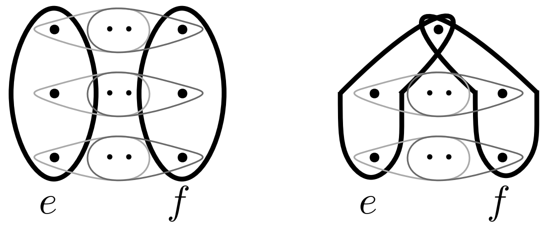

FIGURE 1. On the left, a $(12, e, f)$-gadget. On the right, a $(9, e, f)$-gadget.

2.2. **Tools for the proof of Theorem 1.3.** The following well-known result allows one to deduce a lower bound on $\delta_\ell(H)$ given a lower bound on $\delta_{\ell'}(H)$, for any $\ell \leq \ell'$.

Proposition 2.4. *Let $1 \leq \ell \leq \ell' < k$ and H be a k-uniform hypergraph on n vertices. If $\delta_{\ell'}(H) \geq x\binom{n-\ell'}{k-\ell'}$ for some $0 \leq x \leq 1$, then $\delta_\ell(H) \geq x\binom{n-\ell}{k-\ell}$.* □

The next result gives a sufficient condition for finding a good $(3k, e, f)$-gadget in a k-uniform hypergraph of large minimum 2-degree.

Lemma 2.5. *Let $k \geq 3$ and $D := 3k$. Let H be a k-uniform hypergraph on n vertices with an r-colouring $c : E(H) \to \{C_1, \dots, C_r\}$. Suppose there exists $i \neq j \in [r]$ and distinct $v_1, v_2, v_3, v_4 \in V(H)$ such that $N_H(v_1) \cap N_H(v_2)$ and $N_H(v_3) \cap N_H(v_4)$ are both type $C_iC_j(D)$. If*

$$\delta_2(H) > \frac{1}{2}\binom{n}{k-2},$$

then there exists a good $(3k, e, f)$-gadget in H, for some $e, f \in E(H)$.

Proof. By the minimum 2-degree condition, there exists a set $X \subseteq V(H)$ of size $k-2$ such that $A = X \cup \{v_1, v_3\}$ and $B = X \cup \{v_2, v_4\}$ are both in $E(H)$. We show that we can construct a $(3k, A, B)$-gadget and afterwards we prove that it is good.

Given that $N_H(v_1) \cap N_H(v_2)$ is type $C_iC_j(D)$, there are at least $3kn^{k-2}$ sets $T_{1,2} \in N_H(v_1) \cap N_H(v_2)$ such that $c(v_1T_{1,2}) = C_i$ and $c(v_2T_{1,2}) = C_j$. As $|A \cup B| = k+2 < 3k$, we may choose such a set $T_{1,2}$ so that it is also vertex-disjoint from $A \cup B$. Similarly, there is a set $T_{3,4} \in N_H(v_3) \cap N_H(v_4)$ such that $c(v_3T_{3,4}) = C_i$, $c(v_4T_{3,4}) = C_j$ and $T_{3,4}$ is vertex-disjoint from A, B and $T_{1,2}$.

Then, define a gadget G as follows:

- $V(G)$ is the union of A, B, $T_{1,2}$ and $T_{3,4}$;
- A, B, $v_1T_{1,2}$, $v_2T_{1,2}$, $v_3T_{3,4}$ and $v_4T_{3,4}$ are in $E(G)$.

By definition, G is a $(3k, A, B)$-gadget.

To prove that G is good, we need to find two perfect matchings in G with different colour profiles. Define $M_A := \{A, v_2T_{1,2}, v_4T_{3,4}\}$ and $M_B := \{B, v_1T_{1,2}, v_3T_{3,4}\}$. Both M_A and M_B are perfect matchings in G. While M_A has at least two C_j-edges ($v_2T_{1,2}$ and $v_4T_{3,4}$), M_B has at least two C_i-edges ($v_1T_{1,2}$ and $v_3T_{3,4}$). Thus, M_A and M_B have different colour profiles, as desired. □

The next lemma ensures a hypergraph H as in Theorem 1.3 contains a good gadget or a perfect matching of huge colour-bias.

Lemma 2.6. *Let $2 \leq \ell < k$ and $\eta > 0$. There exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ with $n \in k\mathbb{N}$. Let H be a k-uniform hypergraph on n vertices with an r-colouring $c : E(H) \to \{C_1, \dots, C_r\}$ and*

$$\delta_\ell(H) \geq (c_{k,\ell} + \eta)\binom{n}{k-\ell}.$$

Suppose that H does not have a perfect matching containing at least $n/k - \binom{r}{2}$ edges of the same colour. Then

- *there exists a good $(3k, e, f)$-gadget in H, for some $e, f \in E(H)$; or*
- *there exists a good $(k^2 + k, e, f)$-gadget in H, for some $e, f \in E(H)$.*

Proof. Let H and c be as in the lemma and suppose n is sufficiently large. Let $D := k^2 + k \geq 3k$. Note that, given our minimum ℓ-degree condition, Proposition 2.4 implies that

(2)

$$\delta_1(H) \geq (c_{k,\ell} + \eta)\binom{n-1}{k-1} > \left(\frac{1}{2} + \frac{\eta}{2}\right)\binom{n}{k-1} \quad \text{and} \quad \delta_2(H) \geq (c_{k,\ell} + \eta)\binom{n-2}{k-2} > \frac{1}{2}\binom{n}{k-2}.$$

Here the inequalities follow as $c_{k,\ell} \geq 1/2$ by (1).

As n is sufficiently large, and by definition of $c_{k,\ell}$, the minimum ℓ-degree condition ensures a perfect matching M in H.

Let $L := \binom{r}{2} + 1$. By the hypothesis of the lemma, M does not contain $n/k - \binom{r}{2}$ edges of the same colour; so there exist distinct edges $e_1, \dots, e_L, f_1, \dots, f_L \in M$ such that $c(e_i) \neq c(f_i)$ for each $i \in [L]$.

Given any distinct $x, y \in V(H)$, (2) implies that $|N_H(x) \cap N_H(y)| \geq \eta\binom{n}{k-1}$. In particular, this means that $N_H(x) \cap N_H(y)$ is of type $S(D)$ or of type $C_iC_j(D)$ for some distinct $i, j \in [r]$.

Suppose there exists $i \neq j \in [r]$ and distinct $x, y, z, w \in V(H)$ such that $N_H(x) \cap N_H(y)$ and $N_H(z) \cap N_H(w)$ are both type $C_iC_j(D)$. Then by Lemma 2.5, there exists a good $(3k, e, f)$-gadget in H, for some $e, f \in E(H)$.

So we may assume no such $i \neq j \in [r]$ and $x, y, z, w \in V(H)$ exist. In particular, for each of the $\binom{r}{2} = L - 1$ choices for $i \neq j \in [r]$, there is at most one pair (e_s, f_s) such that there exists a $u \in e_s$ and $v \in f_s$ so that either $N_H(u) \cap N_H(v)$ or $N_H(v) \cap N_H(u)$ is type $C_iC_j(D)$. Thus, the following claim holds.

Claim 2.7. *There is a pair (e_s, f_s) such that for each $u \in e_s$ and $v \in e_s$ we have that $N_H(u) \cap N_H(v)$ is type $S(D)$.*

Let $e_s = \{u_1, \dots, u_k\}$ and $f_s = \{v_1, \dots, v_k\}$. For each $i \in [k]$, we choose a set T_i so that

(i) $T_i \in S(u_iv_i)$;
(ii) $T_1, \dots, T_k, e_s, f_s$ are all vertex-disjoint.

Note we can guarantee (ii) since $|S(u_iv_i)| \geq Dn^{k-2} = (k^2 + k)n^{k-2}$ for each $i \in [k]$.

We construct a $(k^2 + k, e_s, f_s)$-gadget G as follows:

- $V(G)$ is the union of e_s, f_s, $T_1, \dots, T_k$;
- e_s and f_s are edges in G;
- u_iT_i, v_iT_i are edges in G for all $i \in [k]$.

By definition, G is an S-$(k^2 + k, e_s, f_s)$-gadget with $c(e_s) \neq c(f_s)$. This implies that G is a good $(k^2 + k, e_s, f_s)$-gadget. Indeed, $M_e := \{e_s, v_1T_1, \dots, v_kT_k\}$ and $M_f := \{f_s, u_1T_1, \dots, u_kT_k\}$ are perfect matchings in G with different colour profiles. $\square$

3. Proof of Theorem 1.3

Let H be a sufficiently large n-vertex k-uniform hypergraph as in the statement of the theorem. Let $c : E(H) \to \{C_1, \dots, C_r\}$ be an r-colouring of $E(H)$. If H contains a perfect matching with at least $n/k - \binom{r}{2}$ edges of the same colour, then we are done.

So, suppose no perfect matching in H contains at least $n/k - \binom{r}{2}$ edges of the same colour. By Lemma 2.6, we can find either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in H. Call this gadget G_1.

Next consider $H_1 := H \setminus V(G_1)$. Clearly $\delta_\ell(H_1) \geq (c_{k,\ell} + \eta/2)\binom{n}{k-\ell}$. Suppose H_1 contains a perfect matching M_1 with at least $|H_1|/k - \binom{r}{2}$ edges of the same colour. Thus, by taking any perfect matching in G_1 and adding it to M_1, we obtain a perfect matching in H containing at least $|H_1|/k - \binom{r}{2} \geq n/k - |G_1|/k - \binom{r}{2} \geq n/k - k - 1 - \binom{r}{2}$ edges of the same colour, as desired.

Hence, we may assume H_1 does not contain such a perfect matching M_1. By Lemma 2.6, we can find either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in H_1. Call this gadget G_2 and set $H_2 := H_1 \setminus V(G_2)$.

Repeating this argument, we either obtain a perfect matching in H of significant colour-bias, or a collection of $t := \frac{\eta n}{4k^k(k^2+k)}$ vertex-disjoint gadgets $G_1, \dots, G_t$ where, given any $i \in [t]$, G_i is either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in H. In particular, note that each gadget we select has size at most $k^2 + k$, and if one removes $t(k^2 + k)$ vertices from H one still has that $\delta_\ell(H) \geq (1/2 + \eta)\binom{n}{k-\ell} - t(k^2 + k)n^{k-\ell-1} \geq (1/2 + \eta/2)\binom{n}{k-\ell}$. Thus, we can indeed repeatedly apply Lemma 2.6 to obtain these gadgets $G_1, \dots, G_t$.

Set $\mathcal{G} := \{G_1, \dots, G_t\}$. For each colour C_i, consider the set $\mathcal{G}_i$ of all the gadgets in $\mathcal{G}$ that contain two perfect matchings with different colour profiles with respect to the colour C_i. Clearly there exists some $j \in [r]$ such that $\mathcal{G}_j$ contains at least t/r gadgets.

For each gadget G_i in $\mathcal{G}_j$ consider the perfect matching M_i in G_i with the largest possible number of edges coloured C_j; let M'_i be the perfect matching in G_i with the fewest possible edges coloured C_j. So M_i has at least one more C_j-edge than M'_i.

Let M^+ denote the union of all these M_i and let M^- denote the union of all these M'_i. So M^+ contains at least $t/r = \frac{\eta n}{4rk^k(k^2+k)}$ more C_j-edges than M^-.

Let $V(\mathcal{G}_j)$ denote the set of vertices in H that lie in one of the gadgets in $\mathcal{G}_j$. Note that $\delta_\ell(H \setminus V(\mathcal{G}_j)) \geq (c_{k,\ell} + \eta/2)\binom{n}{k-\ell}$ so there exists a perfect matching M in $H \setminus V(\mathcal{G}_j)$. Thus, $M \cup M^+$ and $M \cup M^-$ are both perfect matchings in H.

If $M \cup M^-$ contains at least $\frac{n}{rk} + \frac{\eta n}{8r(r-1)k^k(k^2+k)}$ edges of the same colour then the theorem holds. Thus, we may assume this is not the case. This immediately implies the following claim.

Claim 3.1. *For every $i \in [r]$, the number of C_i-edges in $M \cup M^-$ is at least $\frac{n}{rk} - \frac{\eta n}{8rk^k(k^2+k)}$.*

In particular, $M \cup M^-$ contains at least $\frac{n}{rk} - \frac{\eta n}{8rk^k(k^2+k)}$ C_j-edges. Since there are at least $\frac{\eta n}{4rk^k(k^2+k)}$ more C_j-edges in M^+ than in M^-, we obtain that $M \cup M^+$ contains at least $\frac{n}{rk} + \frac{\eta n}{8rk^k(k^2+k)}$ C_j-edges, as desired. $\square$

4. Concluding Remarks

In this paper we have determined the minimum ℓ-degree threshold for forcing a colour-bias perfect matching in a k-uniform hypergraph for all $2 \leq \ell < k$. The only remaining open case of the problem is the minimum *vertex* degree version.

A result of Hàn, Person and Schacht [11] yields that $m_1(3, n) = (5/9 + o(1))\binom{n-1}{2}$. The following example shows that the corresponding colour-bias problem has a significantly higher minimum vertex degree threshold.

Example 4.1. *Given any $n \in 6\mathbb{N}$, there exists an n-vertex 3-uniform hypergraph H with*

$$\delta_1(H) \geq \frac{3}{4}\binom{n-1}{2}$$

and a 2-colouring of $E(H)$ so that every perfect matching in H has precisely $n/6$ edges in each colour.

Proof. Define H so that (i) $V(H)$ is the disjoint union of two vertex classes A and B, both of size $n/2$; (ii) $E(H)$ consists of all those 3-uniform edges containing at least one vertex from each of A and B. Thus,

$$\delta_1(H) = \binom{n/2}{2} + \frac{n}{2}\left(\frac{n}{2} - 1\right) \geq \frac{3}{4}\binom{n-1}{2}.$$

Colour each edge containing 2 vertices from A red; each edge containing 2 vertices from B blue. It is easy to see that every perfect matching in H uses the same number of red and blue edges. $\square$

We suspect that this example is extremal for the minimum vertex degree problem in 3-uniform hypergraphs.

Question 4.2. *Given any $\eta > 0$ does there exists a $\gamma > 0$ so that the following holds for all sufficiently large $n \in 3\mathbb{N}$? Suppose that H is an n-vertex 3-uniform hypergraph with*

$$\delta_1(H) \geq \left(\frac{3}{4} + \eta\right)\binom{n-1}{2}.$$

Then given any 2-colouring of $E(H)$ there is a perfect matching in H with at least $n/6 + \gamma n$ edges of the same colour.

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