A NOTE ON COLOUR-BIAS PERFECT MATCHINGS IN HYPERGRAPHS

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ABSTRACT. A result of Balogh, Csaba, Jing and Pluhár yields the minimum degree threshold that ensures a 2-coloured graph contains a perfect matching of significant colour-bias (i.e., a perfect matching that contains significantly more than half of its edges in one colour). In this note we prove an analogous result for perfect matchings in $k$-uniform hypergraphs. More precisely, for each $2 \leq \ell < k$ and $r \geq 2$ we determine the minimum $\ell$-degree threshold for forcing a perfect matching of significant colour-bias in an $r$-coloured $k$-uniform hypergraph.

1. Introduction

A perfect matching in a hypergraph $H$ is a collection of vertex-disjoint edges of $H$ which covers the vertex set $V(H)$ of $H$. In recent decades there has been significant interest in the problem of establishing minimum degree conditions that force a perfect matching in a $k$-uniform hypergraph. More precisely, given a $k$-uniform hypergraph $H$ and an $\ell$-element vertex set $S \subseteq V(H)$ (where $\ell \in [k-1]$) we define $d_H(S)$ to be the number of edges containing $S$. The minimum $\ell$-degree $\delta_\ell(H)$ of $H$ is the minimum of $d_H(S)$ over all $\ell$-element sets of vertices in $H$. We refer to $\delta_1(H)$ as the minimum vertex degree of $H$ and $\delta_{k-1}(H)$ as the minimum codegree of $H$.

Suppose that $\ell, k, n \in \mathbb{N}$ such that $\ell \leq k - 1$ and $k$ divides $n$. Let $m_\ell(k, n)$ denote the smallest integer $m$ such that every $k$-uniform hypergraph $H$ on $n$ vertices with $\delta_\ell(H) \geq m$ contains a perfect matching.

A simple consequence of Dirac’s theorem is that $m_1(2, n) = n/2$ for all even $n \in \mathbb{N}$. Improving earlier asymptotically exact bounds given in [12, 17], Rödl, Ruciński and Szemerédi [18] determined the minimum codegree threshold for perfect matchings in $k$-uniform hypergraphs. That is, they showed that if $n \in \mathbb{N}$ is sufficiently large, then $m_{k-1}(k, n) = n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of $n$ and $k$.

The value of $m_\ell(k, n)$ is known for various pairs $(k, \ell)$ when $n$ is sufficiently large. For example, after an earlier asymptotic result of Pikhurko [15], Treglown and Zhao [19] determined the value of $m_\ell(k, n)$ for $\ell \geq k/2$ and $n$ sufficiently large. However, the minimum vertex degree case of the problem is wide open in general, and the only cases where the asymptotic or exact value of $m_1(k, n)$ is known is when $k = 2, 3, 4, 5$. See, e.g., [16, 21] for discussions on further results in the area.

Given any $1 \leq \ell < k$ it is known that

$$m_\ell(k, n) \geq \max \left\{ \frac{1}{2} - o(1), 1 - \left( \frac{k-1}{k} \right)^{k-\ell} - o(1) \right\} \binom{n}{k-\ell}.$$

See, e.g., the introduction of [20] for the two families of hypergraphs that demonstrate (1). It is widely believed that the inequality in (1) is asymptotically sharp for all choices of $k, \ell$, see [11, 13].

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Moreover, Treglown and Zhao [20] gave a conjecture on the exact value of \( m_t(k, n) \) for sufficiently large \( n \in 4\mathbb{N} \).

The aim of this paper is to study the colour-bias version of this problem. The topic of colour-bias structures in graphs was first raised by Erdős in the 1960s (see [5, 6]). Sparked by work of Balogh, Csaba, Jing and Pluhár [1], there has been renewed interest in the topic, particularly in establishing minimum degree conditions that force a colour-bias copy of a graph \( F \). More precisely, if a graph \( G \) contains a copy of \( F \), then however the edges of \( G \) are 2-coloured, one can clearly ensure that \( G \) contains a copy of \( F \) with at least \( e(F)/2 \) edges of the same colour. The question then is how large does the minimum degree \( \delta(G) \) of \( G \) need to be to guarantee that \( G \) contains a copy of \( F \) with significantly more than \( e(F)/2 \) edges of the same colour, no matter how one 2-colours the edges of \( G \)? The following result resolves this problem in the case when \( F \) is a Hamilton cycle.

**Theorem 1.1** (Balogh, Csaba, Jing and Pluhár [1]). Let \( 0 < c < 1/4 \) and \( n \in \mathbb{N} \) be sufficiently large. If \( G \) is an \( n \)-vertex graph with

\[
\delta(G) \geq (3/4 + c)n,
\]

then given any 2-colouring of \( E(G) \) there is a Hamilton cycle in \( G \) with at least \( n/2 + cn/32 \) edges of the same colour. Moreover, if \( n \in 4\mathbb{N} \), there is an \( n \)-vertex graph \( G' \) with \( \delta(G') = 3n/4 \) and a 2-colouring of \( E(G') \) for which every Hamilton cycle in \( G' \) has precisely \( n/2 \) edges in each colour.

Note that Theorem 1.1 shows that the minimum degree threshold for forcing a colour-bias Hamilton cycle in a graph is significantly higher than the threshold for just forcing a Hamilton cycle. Indeed, Dirac’s theorem tells us that any \( n \)-vertex graph \( G \) with \( \delta(G) \geq n/2 \) contains a Hamilton cycle.

Since a Hamilton cycle on an even number of vertices is the union of two perfect matchings, Theorem 1.1 implies the following result.

**Theorem 1.2** (Balogh, Csaba, Jing and Pluhár [1]). Let \( 0 < c < 1/4 \) and \( n \in 2\mathbb{N} \) be sufficiently large. If \( G \) is an \( n \)-vertex graph with

\[
\delta(G) \geq (3/4 + c)n,
\]

then given any 2-colouring of \( E(G) \) there is a perfect matching in \( G \) with at least \( n/4 + cn/64 \) edges of the same colour. Moreover, if \( n \in 4\mathbb{N} \), there is an \( n \)-vertex graph \( G' \) with \( \delta(G') = 3n/4 \) and a 2-colouring of \( E(G') \) for which every perfect matching in \( G' \) has precisely \( n/4 \) edges in each colour.

Let \( n \in 4\mathbb{N} \). We define the graph \( G' \) in Theorem 1.2 as follows: \( V(G') \) consists of the disjoint union of two vertex classes \( A \) and \( B \) of sizes \( n/4 \) and \( 3n/4 \), respectively; \( E(G') \) contains all possible red edges whose endpoints are both in \( B \) and all possible blue edges with one endpoint in \( A \) and one endpoint in \( B \). Thus, \( \delta(G') = 3n/4 \) and every perfect matching in \( G' \) has precisely \( n/4 \) edges in each colour.

Since [1] appeared, a number of analogues of Theorem 1.1 have been established for other types of spanning structures. Given graphs \( G \) and \( F \), an \( F \)-factor in \( G \) is a collection of vertex-disjoint copies of \( F \) in \( G \) that together cover \( V(G) \). In [2], the minimum degree threshold for forcing a colour-bias \( K_r \)-factor was determined.\(^1\) More recently, this result was extended to \( F \)-factors for every fixed graph \( F \); see [4]. For \( k \geq 2 \), the minimum degree threshold for forcing a colour-bias \( k \)th power of a Hamilton cycle in a graph was established in [3].

Other variants of the problem have also been studied. In [7, 10] an \( r \)-colour version of Theorem 1.1 was proven: in this setting now one \( r \)-colours \( E(G) \) and seeks a Hamilton cycle with significantly more than \( n/r \) edges of the same colour. Colour-bias problems have also been considered for random graphs [9]. Recently, Mansilla Brito [14] gave a minimum codegree result for forcing a colour-bias

\(^1\)Recall \( K_r \) denotes the complete graph on \( r \) vertices.
copy of a tight Hamilton cycle in a 3-uniform hypergraph. We remark that all of these colour-bias results can be phrased in the equivalent language of discrepancy; see, e.g., [1, 2, 3, 4, 10].

Our main result determines the minimum \( \ell \)-degree threshold for forcing a colour-bias perfect matching in a \( k \)-uniform hypergraph for all \( \ell \geq 2 \) and \( k \geq 3 \). To state our result we need the following definitions. Given integers \( 1 \leq \ell < k \), let \( C_{k,\ell} \) be the set of all \( c > 0 \) such that \( m_{\ell}(k,n) \leq c \left( \frac{n}{k-\ell} \right) \) for all sufficiently large \( n \in k\mathbb{N} \). Set \( c_{k,\ell} \) to be the infimum of \( C_{k,\ell} \). In particular, note that the general conjecture on the asymptotic value of \( m_{\ell}(k,n) \) equivalently states that

\[
c_{k,\ell} = \max \left\{ \frac{1}{2}, 1 - \left( \frac{k-1}{k} \right)^{k-\ell} \right\}.
\]

**Theorem 1.3.** Let \( k, \ell, r \in \mathbb{N} \) where \( 2 \leq \ell < k \) and \( r \geq 2 \). Given any \( \eta > 0 \) where \( c_{k,\ell} + \eta < 1 \), there exists an \( n_0 \in \mathbb{N} \) such that the following holds. Let \( H \) be a \( k \)-uniform hypergraph on \( n \geq n_0 \) vertices, where \( n \in k\mathbb{N} \). If

\[
d_{\ell}(H) \geq (c_{k,\ell} + \eta) \left( \frac{n}{k-\ell} \right),
\]

then given any \( r \)-colouring of \( E(H) \) there is a perfect matching in \( H \) with at least \( \frac{n}{\ell r k} + \frac{\eta n}{8(r-1)k^2(k^2+k)} \) edges of the same colour.

We remark that Theorem 1.3 holds even in the cases in which we do not know the value of \( c_{k,\ell} \). By definition of \( c_{k,\ell} \), the minimum \( \ell \)-degree condition in Theorem 1.3 is essentially best possible. Indeed, for \( c < c_{k,\ell} \), a minimum \( \ell \)-degree condition of \( d_{\ell}(H) \geq c \left( \frac{n}{k-\ell} \right) \) does not even guarantee a perfect matching, let alone one of significant colour-bias. So in this sense the colour-bias and ‘standard’ versions of the problem are aligned when \( \ell \geq 2 \).

In contrast, the same phenomenon does not occur for the minimum vertex degree version of the problem. Indeed, Theorem 1.2 tells us that the minimum degree threshold for a colour-bias perfect matching in a graph is different to the minimum degree threshold for a perfect matching in a graph. Furthermore, in Section 4 we describe a similar phenomenon in the 3-uniform hypergraph setting.

**Remark.** Whilst finalising a manuscript that gave the proof of Theorem 1.3 in the case when \( \ell = k-1 \) and \( r = 2 \), we learnt of simultaneous and independent work of Gishboliner, Glock and Sgurev [8]. In [8] they determine the minimum codegree threshold for forcing a tight Hamilton cycle of significant colour-bias in an \( r \)-coloured \( k \)-uniform hypergraph (where \( r \geq 2 \) and \( k \geq 3 \)). As an immediate consequence of their result they also establish the corresponding minimum codegree threshold for perfect matchings.

We therefore decided to seek a generalisation of our minimum codegree result to other degree conditions, i.e., Theorem 1.3. In doing so, we found an argument much cleaner than our original approach.

**Notation.** Let \( H \) be a hypergraph. The *neighbourhood* \( N_H(X) \) of a set \( X \subseteq V(H) \) is the family of sets \( S \subseteq V(H) \setminus X \) such that \( S \cup X \in E(H) \). If \( X = \{x\} \) we define \( N_H(x) := N_H(X) \). Given a vertex \( x \in V(H) \) and set \( Y \subseteq V(H) \) we sometimes write \( xy \) for \( x \in Y \). Given a colouring \( c \) of \( E(H) \), we call an edge \( e \in E(H) \) a *C-edge* if \( e \) is coloured \( C \) in \( c \). Given a set \( X \subseteq V(H) \), we write \( H[X] \) for the *induced subhypergraph* of \( H \) with vertex set \( X \). We define \( H \setminus X := H[V(H) \setminus X] \).

Given a hypergraph \( F \) with an \( r \)-colouring \( c : E(F) \to \{C_1, \ldots, C_r\} \), its *colour profile* is \( (x_1, \ldots, x_r) \) where \( x_i \) is the number of \( C_i \)-edges in \( F \) for each \( i \in [r] \). Two colour profiles \( (x_1, \ldots, x_r), (y_1, \ldots, y_r) \) are said to be different with respect to the colour \( C_i \) if \( x_i \neq y_i \).
2. Preliminaries and useful results

2.1. Proof overview and key definitions. Throughout this section, we will suppose that $H$ is a $k$-uniform hypergraph on $n$ vertices with an $r$-colouring $c : E(H) \to \{C_1, \ldots, C_r\}$.

Our general strategy for the proof of Theorem 1.3 is as follows. Our aim is to find certain gadgets inside of $H$. A gadget is just a subhypergraph of $H$ with some given structure. A gadget $G$ is good if $G$ contains two perfect matchings that have different colour profiles with respect to the $r$-colouring $c$.

For a certain well chosen $t \in \mathbb{N}$, we will prove that there are $t$ vertex-disjoint good gadgets $G_1, \ldots, G_t$ in $H$ and a $j \in [r]$ so that, for each good gadget $G_i$, the two perfect matchings $M_i$ and $M'_i$ in $G_i$ have colour profiles that are different with respect to the colour $C_j$.

We will then be able to easily find a perfect matching in $H$ of significant colour-bias. Indeed, removing the vertices of $G_1, \ldots, G_t$ from $H$ will result in a $k$-uniform hypergraph $H'$ that contains a perfect matching $M$. The flexibility of the good gadgets then allows us to extend $M$ into a perfect matching in $H$ with significant colour-bias, whatever the colour profile of $M$ is.

We next state the definitions required to formally introduce the notion of a good gadget.

Definition 2.1. Let $u, v \in V(H)$ be distinct and $T \in N_H(u) \cap N_H(v)$. We say $uTv$ is

- **S** if $c(T \cup \{u\}) = c(T \cup \{v\})$; or
- $C_iC_j$ if $c(T \cup \{u\}) = C_i$ and $c(T \cup \{v\}) = C_j$.

Let $C_iC_j(uv)$ denote the collection of sets $T \in N_H(u) \cap N_H(v)$ for which $uTv$ is $C_iC_j$. Define $S(uv)$ analogously.

Note that $C_iC_j(uv) = C_jC_i(vu)$ for all distinct $u, v \in V(H)$.

Definition 2.2. Let $D > 0$ and let $u, v \in V(H)$ be distinct. We say that $N_H(u) \cap N_H(v)$ is

- **type $S(D)$** if $|S(uv)| \geq Dn^{k-2}$; or
- **type $C_iC_j(D)$** if $i \neq j$ and $|C_iC_j(uv)| \geq Dn^{k-2}$.

We remark that it may be the case that $N_H(u) \cap N_H(v)$ has more than one type.

Definition 2.3. Let $e = \{e_1, \ldots, e_k\}$ and $f = \{f_1, \ldots, f_k\}$ be two edges in $H$. A $(k^2 + k, e, f)$-gadget $G$ is a subhypergraph of $H$ on $k^2 + k$ vertices so that:

- $V(G)$ is the disjoint union of $e_i$ and $T_i$ where $T_i \in N_H(e_i) \cap N_H(f_i)$ for each $i \in [k]$;
- $e_i$ and $f_i$ are edges in $G$;
- $e_iT_i, f_iT_i \in E(G)$ for all $i \in [k]$.

A $(k^2 + k, e, f)$-gadget in which every $e_iT_i$ is an edge will be called an **S-(k$^2$ + k, e, f)-gadget**.

A $(3k, e, f)$-gadget $G$ is a subhypergraph of $H$ on $3k$ vertices so that:

- $e_i = f_i$, for all $i \in \{3, \ldots, k\}$;
- $V(G)$ is the disjoint union of $e_1, f_1, T_1$ and $T_2$, where $T_i \in N_H(e_i) \cap N_H(f_i)$ for each $i \in \{2\}$;
- $e, f$ are edges in $G$;
- $e_1T_1, f_1T_1, e_2T_2, f_2T_2 \in E(G)$.

Given $t \in \{3k, k^2 + k\}$, we say that a $(t, e, f)$-gadget $G$ is **good** if it contains two perfect matchings with different colour profiles (with respect to the $r$-colouring of $G$ induced by the $r$-colouring $c$ of $H$).

Note that $e$ and $f$ are vertex-disjoint in a $(k^2 + k, e, f)$-gadget but intersect in $k - 2$ vertices in a $(3k, e, f)$-gadget; see Figure 1.
2.2. Tools for the proof of Theorem 1.3. The following well-known result allows one to deduce a lower bound on $\delta(H)$ given a lower bound on $\delta'_e(H)$, for any $\ell \leq \ell'$. 

**Proposition 2.4.** Let $1 \leq \ell \leq \ell' < k$ and $H$ be a $k$-uniform hypergraph on $n$ vertices. If $\delta'_e(H) \geq x^{(n-\ell')}_{k-\ell}$ for some $0 \leq x \leq 1$, then $\delta_e(H) \geq x^{(n-\ell)}_{k-\ell}$.

The next result gives a sufficient condition for finding a good $(3k, e, f)$-gadget in a $k$-uniform hypergraph of large minimum 2-degree.

**Lemma 2.5.** Let $k \geq 3$ and $D := 3k$. Let $H$ be a $k$-uniform hypergraph on $n$ vertices with an $r$-colouring $c : E(H) \to \{C_1, \ldots, C_r\}$. Suppose there exists $i \neq j \in [r]$ and distinct $v_1, v_2, v_3, v_4 \in V(H)$ such that $N_H(v_1) \cap N_H(v_2)$ and $N_H(v_3) \cap N_H(v_4)$ are both type $C_iC_j(D)$. If

$$\delta_2(H) > \frac{1}{2} \binom{n}{k-2},$$

then there exists a good $(3k, e, f)$-gadget in $H$, for some $e, f \in E(H)$.

**Proof.** By the minimum 2-degree condition, there exists a set $X \subseteq V(H)$ of size $k - 2$ such that $A = X \cup \{v_1, v_3\}$ and $B = X \cup \{v_2, v_4\}$ are both in $E(H)$. We show that we can construct a $(3k, A, B)$-gadget and afterwards we prove that it is good.

Given that $N_H(v_1) \cap N_H(v_2)$ is type $C_iC_j(D)$, there are at least $3kn^{k-2}$ sets $T_{1,2} \in N_H(v_1) \cap N_H(v_2)$ such that $c(v_1T_{1,2}) = C_i$ and $c(v_2T_{1,2}) = C_j$. As $|A \cup B| = k + 2 < 3k$, we may choose such a set $T_{1,2}$ so that it is also vertex-disjoint from $A \cup B$. Similarly, there is a set $T_{3,4} \in N_H(v_3) \cap N_H(v_4)$ such that $c(v_3T_{3,4}) = C_i$, $c(v_4T_{3,4}) = C_j$ and $T_{3,4}$ is vertex-disjoint from $A$, $B$ and $T_{1,2}$.

Then, define a gadget $G$ as follows:

- $V(G)$ is the union of $A$, $B$, $T_{1,2}$ and $T_{3,4}$;
- $A$, $B$, $v_1T_{1,2}$, $v_2T_{1,2}$, $v_3T_{3,4}$ and $v_4T_{3,4}$ are in $E(G)$.

By definition, $G$ is a $(3k, A, B)$-gadget.

To prove that $G$ is good, we need to find two perfect matchings in $G$ with different colour profiles. Define $M_A := \{A, v_2T_{1,2}, v_4T_{3,4}\}$ and $M_B := \{B, v_1T_{1,2}, v_3T_{3,4}\}$. Both $M_A$ and $M_B$ are perfect matchings in $G$. While $M_A$ has at least two $C_i$-edges ($v_2T_{1,2}$ and $v_4T_{3,4}$), $M_B$ has at least two $C_j$-edges ($v_1T_{1,2}$ and $v_3T_{3,4}$). Thus, $M_A$ and $M_B$ have different colour profiles, as desired.

The next lemma ensures a hypergraph $H$ as in Theorem 1.3 contains a good gadget or a perfect matching of huge colour-bias.

**Lemma 2.6.** Let $2 \leq \ell < k$ and $\eta > 0$. There exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ with $n \in k\mathbb{N}$. Let $H$ be a $k$-uniform hypergraph on $n$ vertices with an $r$-colouring $c : E(H) \to \{C_1, \ldots, C_r\}$ and

$$\delta_e(H) \geq (c_k, \ell + \eta) \binom{n}{k-\ell}.$$
Suppose that $H$ does not have a perfect matching containing at least $n/k - \binom{k}{2}$ edges of the same colour. Then

- there exists a good $(3k, e, f)$-gadget in $H$, for some $e, f \in E(H)$; or
- there exists a good $(k^2 + k, e, f)$-gadget in $H$, for some $e, f \in E(H)$.

**Proof.** Let $H$ and $c$ be as in the lemma and suppose $n$ is sufficiently large. Let $D := k^2 + k \geq 3k$. Note that, given our minimum $\ell$-degree condition, Proposition 2.4 implies that

\[ \delta_1(H) \geq (c_{k, \ell} + \eta) \binom{n-1}{k-1} > \left( \frac{1}{2} + \frac{\eta}{2} \right) \binom{n}{k-1} \quad \text{and} \quad \delta_2(H) \geq (c_{k, \ell} + \eta) \binom{n-2}{k-2} > \frac{1}{2} \binom{n}{k-2}. \]

Here the inequalities follow as $c_{k, \ell} \geq 1/2$ by (1).

As $n$ is sufficiently large, and by definition of $c_{k, \ell}$, the minimum $\ell$-degree condition ensures a perfect matching $M$ in $H$.

Let $L := \left( \binom{k}{2} \right) + 1$. By the hypothesis of the lemma, $M$ does not contain $n/k - \binom{k}{2}$ edges of the same colour; so there exist distinct edges $e_1, \ldots, e_L, f_1, \ldots, f_L \in M$ such that $c(e_i) \neq c(f_i)$ for each $i \in [L]$.

Given any distinct $x, y \in V(H)$, (2) implies that $|N_H(x) \cap N_H(y)| \geq \eta \binom{n-1}{k-1}$. In particular, this means that $N_H(x) \cap N_H(y)$ is of type $S(D)$ or of type $C_iC_j(D)$ for some distinct $i, j \in [r]$.

Suppose there exists $i \neq j \in [r]$ and distinct $x, y, z, w \in V(H)$ such that $N_H(x) \cap N_H(y)$ and $N_H(z) \cap N_H(w)$ are both type $C_iC_j(D)$. Then by Lemma 2.5, there exists a good $(3k, e, f)$-gadget in $H$, for some $e, f \in E(H)$.

So we may assume no such $i \neq j \in [r]$ and $x, y, z, w \in V(H)$ exist. In particular, for each of the $\binom{r}{2} = L - 1$ choices for $i \neq j \in [r]$, there is at most one pair $(e_s, f_s)$ such that there exists a $u \in e_s$ and $v \in f_s$ so that either $N_H(u) \cap N_H(v)$ or $N_H(v) \cap N_H(u)$ is type $C_iC_j(D)$. Thus, the following claim holds.

**Claim 2.7.** There is a pair $(e_s, f_s)$ such that for each $u \in e_s$ and $v \in e_s$ we have that $N_H(u) \cap N_H(v)$ is type $S(D)$.

Let $e_s = \{u_1, \ldots, u_k\}$ and $f_s = \{v_1, \ldots, v_k\}$. For each $i \in [k]$, we choose a set $T_i$ so that

1. $T_i \in S(u_i, v_i)$;
2. $T_1, \ldots, T_k, e_s, f_s$ are all vertex-disjoint.

Note we can guarantee (ii) since $|S(u_i, v_i)| \geq Dn^{k-2} = (k^2 + k)n^{k-2}$ for each $i \in [k]$.

We construct a $(k^2 + k, e_s, f_s)$-gadget $G$ as follows:

- $V(G)$ is the union of $e_s, f_s, T_1, \ldots, T_k$;
- $e_s$ and $f_s$ are edges in $G$;
- $u_i T_i, v_i T_i$ are edges in $G$ for all $i \in [k]$.

By definition, $G$ is an $S-(k^2 + k, e_s, f_s)$-gadget with $c(e_s) \neq c(f_s)$. This implies that $G$ is a good $(k^2 + k, e_s, f_s)$-gadget. Indeed, $M_e := \{e_s, v_1 T_1, \ldots, v_k T_k\}$ and $M_f := \{f_s, u_1 T_1, \ldots, u_k T_k\}$ are perfect matchings in $G$ with different colour profiles. \hfill \square

3. **Proof of Theorem 1.3**

Let $H$ be a sufficiently large $n$-vertex $k$-uniform hypergraph as in the statement of the theorem. Let $c : E(H) \to \{C_1, \ldots, C_r\}$ be an $r$-colouring of $E(H)$. If $H$ contains a perfect matching with at least $n/k - \binom{k}{2}$ edges of the same colour, then we are done.

So, suppose no perfect matching in $H$ contains at least $n/k - \binom{k}{2}$ edges of the same colour. By Lemma 2.6, we can find either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in $H$. Call this gadget $G_1$.  

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Next consider $H_1 := H \setminus V(G_1)$. Clearly $\delta_\ell(H_1) \geq (c_{k,\ell} + \eta/2)\binom{n}{k-\ell}$. Suppose $H_1$ contains a perfect matching $M_1$ with at least $|H_1|/k - \binom{n}{2}$ edges of the same colour. Thus, by taking any perfect matching in $G_1$ and adding it to $M_1$, we obtain a perfect matching in $H$ containing at least $|H_1|/k - \binom{n}{2} \geq n/k - |G_1|/k - \binom{n}{2} \geq n/k - k - 1 - \binom{n}{2}$ edges of the same colour, as desired.

Hence, we may assume $H_1$ does not contain such a perfect matching $M_1$. By Lemma 2.6, we can find either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in $H_1$. Call this gadget $G_2$ and set $H_2 := H_1 \setminus V(G_2)$.

Repeating this argument, we either obtain a perfect matching in $H$ of significant colour-bias, or a collection of $t := \frac{mn}{4rk^2(k^2+k)}$ vertex-disjoint gadgets $G_1, \ldots, G_t$ where, given any $i \in [t]$, $G_i$ is either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in $H$. In particular, note that each gadget we select has size at most $k^2 + k$, and if one removes $t(k^2 + k)$ vertices from $H$ one still has that $\delta_\ell(H) \geq (1/2 + \eta/2)\binom{n}{k-\ell} - t(k^2 + k)n^{k-\ell-1} \geq (1/2 + \eta/2)\binom{n}{k-\ell}$. Thus, we can indeed repeatedly apply Lemma 2.6 to obtain these gadgets $G_1, \ldots, G_t$.

Set $G := \{G_1, \ldots, G_t\}$. For each colour $C_i$, consider the set $G_i$ of all the gadgets in $G$ that contain two perfect matchings with different colour profiles with respect to the colour $C_i$. Clearly there exists some $j \in [r]$ such that $G_j$ contains at least $t/r$ gadgets.

For each gadget $G_i$ in $G_j$ consider the perfect matching $M_i$ in $G_i$ with the largest possible number of edges coloured $C_j$; let $M_i'$ be the perfect matching in $G_i$ with the fewest possible edges coloured $C_j$. So $M_i$ has at least one more $C_j$-edge than $M_i'$.

Let $M^+$ denote the union of all these $M_i$ and let $M^-$ denote the union of all these $M_i'$. So $M^+$ contains at least $t/r = \frac{mn}{4rk^2(k^2+k)}$ more $C_j$-edges than $M^-$. Let $V(G_j)$ denote the set of vertices in $H$ that lie in one of the gadgets in $G_j$. Note that $\delta_\ell(H \setminus V(G_j)) \geq (c_{k,\ell} + \eta/2)\binom{n}{k-\ell}$ so there exists a perfect matching $M$ in $H \setminus V(G_j)$. Thus, $M \cup M^+$ and $M \cup M^-$ are both perfect matchings in $H$.

If $M \cup M^-$ contains at least $\frac{n}{r} + \frac{mn}{8rk^2(k^2+k)}$ edges of the same colour then the theorem holds. Thus, we may assume this is not the case. This immediately implies the following claim.

**Claim 3.1.** For every $i \in [r]$, the number of $C_i$-edges in $M \cup M^-$ is at least $\frac{n}{r} - \frac{mn}{8rk^2(k^2+k)}$.

In particular, $M \cup M^-$ contains at least $\frac{n}{r} - \frac{mn}{8rk^2(k^2+k)}$ more $C_j$-edges in $M^+$ than in $M^-$, we obtain that $M \cup M^+$ contains at least $\frac{n}{r} + \frac{mn}{8rk^2(k^2+k)}$ $C_j$-edges, as desired. 

4. Concluding Remarks

In this paper we have determined the minimum $\ell$-degree threshold for forcing a colour-bias perfect matching in a $k$-uniform hypergraph for all $2 \leq \ell < k$. The only remaining open case of the problem is the minimum vertex degree version.

A result of Hán, Person and Schacht [11] yields that $m_1(3, n) = (5/9 + o(1))\binom{n-1}{2}$. The following example shows that the corresponding colour-bias problem has a significantly higher minimum vertex degree threshold.

**Example 4.1.** Given any $n \in 6\mathbb{N}$, there exists an $n$-vertex 3-uniform hypergraph $H$ with

$$\delta_1(H) \geq \frac{3}{4}\binom{n-1}{2}$$

and a 2-colouring of $E(H)$ so that every perfect matching in $H$ has precisely $n/6$ edges in each colour.

**Proof.** Define $H$ so that (i) $V(H)$ is the disjoint union of two vertex classes $A$ and $B$, both of size $n/2$; (ii) $E(H)$ consists of all those 3-uniform edges containing at least one vertex from each
of $A$ and $B$. Thus,

$$\delta_1(H) = \binom{n/2}{2} + \frac{n}{2} \binom{n/2 - 1}{2} \geq \frac{3}{4} \binom{n - 1}{2}.$$  

Colour each edge containing 2 vertices from $A$ red; each edge containing 2 vertices from $B$ blue. It is easy to see that every perfect matching in $H$ uses the same number of red and blue edges. □

We suspect that this example is extremal for the minimum vertex degree problem in 3-uniform hypergraphs.

**Question 4.2.** Given any $\eta > 0$ does there exists a $\gamma > 0$ so that the following holds for all sufficiently large $n \in 3\mathbb{N}$? Suppose that $H$ is an $n$-vertex 3-uniform hypergraph with

$$\delta_1(H) \geq \left(\frac{3}{4} + \eta\right) \binom{n - 1}{2}.$$  

Then given any 2-colouring of $E(H)$ there is a perfect matching in $H$ with at least $n/6 + \gamma n$ edges of the same colour.

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**References**


