

Two-point sets and the Axiom of Choice

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Can you draw a collection of points X on a sheet of paper with the property that any line you draw crosses X in precisely two points? As a first attempt, one might draw a circle, but there are lines which miss X completely, and any line tangent to X will cross it in precisely one point.

More formally, we consider “two-point sets”, which are defined to be subsets of the plane which meet every line in exactly two points. The existence of two-point sets was shown in 1914 by Mazurkiewicz, and the main open problem concerning these objects is to determine if there exist examples which are Borel. Loosely speaking, this problem can be interpreted as trying to determine if there exist two-point sets which can be found in a simple constructive manner. If this question has a positive answer, then we will most likely need to be able to construct a two-point set without making use of a well-ordering of the real line, as is currently the usual technique.

We discuss recent work by Robin Knight, Rolf Suabedissen and the speaker, which shows the existence of two-point sets without explicitly invoking the Axiom of Choice.