# 7. Lecture 7

**Exercise 7.1.** *Prove that if $G = G(n,p)$, then $\alpha(G) \leq \frac{3\ln n}{p} + 1$ with high probability. In particular, if $\omega(\frac{\ln n}{n}) = p = o(1)$ we have $\alpha(G) = o(n)$ and $e(G) = o(n^2)$ with high probability.*

We will present a proof of the following result. The proof of the relevant absorbing lemma follows the argument of Nenadov & Pehova in [3] and is significantly simpler than the argument originally given in [1].

**Lemma 7.1** (Balogh, Sharifzadeh, & Molla [1])**.** *Let $\frac{1}{n} \ll \beta \ll \gamma$. If $3 \mid n$, $\delta(G) \geq \left(\frac{1}{2} + \gamma\right) n$, and $\alpha(G) \leq \beta n$, then $G$ has a $K_3$-factor.*

**Example 7.2.** *Let $G$ be a graph on $n$ vertices consisting of two disjoint cliques of size $\lfloor (n-1)/2 \rfloor$ and $\lceil (n+1)/2 \rceil$. We have $\delta(G) \geq \frac{n-3}{2}$ and $\alpha(G) \leq 2$, but $G$ has no $K_3$-factor.*

**Lemma 7.3.** *Let $\frac{1}{n} \ll \beta \ll \sigma \ll \gamma$. If $\delta(G) \geq \left(\frac{1}{2} + \gamma\right) n$, and $\alpha(G) \leq \beta n$, then $G$ has a $K_3$-tiling that covers all but at most $\sigma n$ vertices.*

*Proof sketch.*

- Let $\varepsilon, d$ be such that
$$\beta \ll \varepsilon \ll d, \sigma \ll \gamma \ll 1.$$
- There exists a $\varepsilon$-regular partition $V_0, V_1, \dots, V_k$ for $G$.
- We can assume $\beta \ll 1/k$ and that $k$ is even.
- Let $m := |V_1| = \cdots |V_k|$.
- Let $R$ be the reduced graph on $V_1, \dots, V_k$ with parameter $d$.
- We have $\delta(R) \geq k/2$, so $R$ has a perfect matching $M$.
- Let $V_iV_j \in M$ and let $H = G[V_i, V_j]$, so $H$ is $\varepsilon$-regular with density at least $d$.
- We will now show that $H$ has a $K_3$-tiling that covers all but at most $4\varepsilon m$ of the vertices in $H$.
- Since this holds for every edge in $M$ and $|V_0| + (4\varepsilon m)k/2 \leq \sigma m$, this will prove the lemma.
- Assume $\mathcal{T}$ is a maximum $K_3$-tiling of $H$ subject to $||V(\mathcal{T}) \cap U_i| - |V(\mathcal{T}) \cap U_j|| \leq 1$.
- Let $U_i := V_i \setminus V(\mathcal{T})$ and $U_j := V_j \setminus V(\mathcal{T})$.
- We can assume that $|U_j| + 1 \geq |U_i| \geq |U_j|$ and for a contradiction assume that $|U_j| \geq \varepsilon m$.
- Since $H$ is $\varepsilon$-regular and $|U_i| \geq |U_j| \geq \varepsilon m$, the density of $G[U_i, U_j]$ is at least $d - \varepsilon$.
- Therefore, there exists $v \in U_i$ such that
$$\deg(v, U_j) \geq (d-\varepsilon)|U_j| \geq (d/2) \cdot \varepsilon m \geq (d/2) \cdot \varepsilon \cdot \frac{n}{2k} > \beta n > \alpha(G).$$
- Therefore, there exists a triangle with one vertex in $U_i$ and two vertices in $U_j$ which violates the maximality of $\mathcal{T}$. □

The following example shows that we cannot hope to build absorbers in the standard way.

**Example 7.4.** *Let $V_1, V_2, \dots, V_{2m+1}$ be a partition of $[n]$ such that $|V_1| \geq (1/2 - \gamma)\, n$, $|V_{2i}| = |V_{2i+1}|$ for $i \in [m]$ and $|V_2|, \dots, |V_{2m+1}| \geq 2\gamma n$. Let $G$ be the graph on $[n]$ in which $G[V_1, \overline{V_1}]$, $G[V_2, V_3], \dots, G[V_{2m}, V_{2m+1}]$ are complete bipartite graphs. Note that $\delta(G) \geq (1/2 + \gamma)n$. Let $X \subseteq V(G) \setminus V_1$ be a 3-set. Since every triangle in $G$ has exactly one vertex in $V_1$, there does not exist a set $U \subseteq V(G) \setminus X$ such that both $G[U]$ and $G[U \cup X]$ have a $K_3$-factor. By Exercise 7.1, we can add $o(n^2)$ edges to $G$ to form $G'$ so that $\alpha(G') = o(n)$. Since we only added $o(n^2)$ edges, for a fixed constant $k$, there are at most $o(n^{3k})$ $3k$-sets $U$ in $V(G')$ such that both $G'[U]$ and $G'[U \cup X]$ have a $K_3$-factor.*

The following lemma is a simple exercise.

**Lemma 7.5.** *Let $\frac{1}{n} \ll \beta \ll \gamma$. If $G$ is an $n$-vertex graph with $\delta(G) \geq \left(\frac{1}{2} + \gamma\right) n$ and $\alpha(G) \leq \beta n$, then for every 3-set $\{v_1, v_2, v_3\} \subseteq V(G)$ there exists a collection of at least $0.1\gamma n$ vertex disjoint 9-sets such that for each such 9-set $A$, both $G[A]$ and $G[A \cup \{v_1, v_2, v_3\}]$ have $K_3$-factors.*

**Lemma 7.6.** *[Montgomery (Lemma 2.8 in [2])- Robustly Matchable Bipartite Graphs] There exists $m_0$ such that for every $\ell, m$ such that $m \geq m_0$ and $m \geq \ell \geq 0$, the following holds for disjoint sets $X$, $Y$, and $Z$ with $|X| = m + \ell$, $|Y| = 2m$, and $Z = 3m$. There exists a bipartite graph $H$ with partite sets $X \cup Y$ and $Z$ such that $\Delta(H) \leq 40$ and for any $X' \subseteq X$ such that $|X'| = \ell$ the graph $H - X'$ has a perfect matching.*

*Sketch proof of Theorem 7.1.*

- Select $\nu, \sigma > 0$ so $\beta \ll \sigma \ll \nu \ll \gamma$.
- Let $m := 3\lfloor \nu n \rfloor$ and let $\ell = \lfloor \nu m \rfloor \leq \nu^2 n$.
- Uniformly at random select a set $X \subseteq V(G)$ of size $m + \ell$.
- So, by the Chernoff and Union bounds, with high probability, for every $v \in V$, we have

$$\deg(v, X) \geq m/2 \geq \nu n. \tag{1}$$

- Arbitrarily select disjoint sets $Y, Z_1, \ldots, Z_{3m} \subseteq V(G) \setminus X$ with $|Y| = 2m$ and $|Z_1| = \cdots = |Z_{3m}| = 2$ and let $Z := \{Z_1, \ldots, Z_{3m}\}$.
- By Lemma 7.6, there exists a bipartite graph $H$ with parts $X \cup Y$ and $Z$ such that for any $X' \subseteq X$ with $|X'| = \ell$ there is a perfect matching of $H - X'$.
- Note that $|E(H)| \leq 40|Z| \leq 120m \leq 480\nu n$ and $\left|X \cup Y \cup \left(\bigcup_{i\in[3m]} Z_i\right)\right| \leq (m + \ell) + 2m + 2 \cdot 3m \leq 30\nu n$.
- Therefore, Lemma 7.5 implies that, iteratively, for each edge $\{v, Z_i\} = e \in E(H)$ with $v \in X \cup Y$ and $Z_i \in Z$, we can construct an absorber $A_e$ of $\{v\} \cup Z_i$ disjoint from $V(H)$ that is also disjoint from all previously constructed absorbers.
- Let
$$A := X \cup Y \cup \left(\bigcup_{i\in[3m]} Z_i\right) \cup \left(\bigcup_{e\in E(H)} A_e\right).$$
and note that $|A| \leq 0.5\gamma n$.
- Therefore, by Lemma 7.3 (with $\gamma/2$ playing the role of $\gamma$), there exists a $K_3$-tiling $\mathcal{T}_1$ of $G - A$ such that if $W := V(G - A) \setminus V(\mathcal{T})$ we have $|W| \leq \sigma n$.
- So, since $\alpha(G) \leq \beta n$, (1) implies that there exists a $K_3$-tiling $\mathcal{T}_2$ of $G[W \cup X]$ such that $|\mathcal{T}_2| = |W|$ and $W \subseteq V(\mathcal{T}_2)$.
- Since $n \equiv m \equiv |Y \cup Z| = 0 \pmod 3$, we have
$$0 \equiv n - |\mathcal{T}_1| - |\mathcal{T}_2| \equiv |(X \cup Y \cup Z) \setminus V(\mathcal{T}_2)| \equiv |X \setminus V(\mathcal{T}_2)| \equiv |X \setminus V(\mathcal{T}_2)| - m$$
- Therefore, $\alpha(G) \leq \beta n$ (1) implies that there exists a $K_3$-tiling $\mathcal{T}_3$ of $G[X \setminus V(\mathcal{T}_2)]$ such that $|X \setminus V(\mathcal{T}_2 \cup \mathcal{T}_3)| = m$. That is, if $X' := X \cap V(\mathcal{T}_2) \cap V(\mathcal{T}_3)$, then $|X'| = \lfloor \nu m \rfloor$.
- So there exists a perfect matching $M$ of $H - X'$.
- Recall that $A \setminus X' = V(G) \setminus V(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$.
- Since, for every $\{x, Z_i\} = e \in M$, there is a $K_3$-factor of $G[\{v\} \cup Z_i \cup A_e]$ and for every $e \in E(H) \setminus M$, there is a $K_3$-factor of $G[A_e]$, there is a $K_3$-factor of $G$.

□

The argument above can be generalized in the follow way.

**Lemma 7.7** (Nenadov & Pehova (Lemma 2.2 in [3]))**.** *Let $\frac{1}{n} \ll \nu \ll \zeta \ll \frac{1}{t}$. For every vertex graph $H$ and every $n$-vertex graph $G$ the following holds. Suppose that for every $|H|$-set $S \subseteq V(G)$ there exists at least $\zeta n$ vertex disjoint sets $A_S$ of order $t$ such that both $G[A_S]$ and $G[A_S \cup S]$ contain $H$-factors. Then there exists a set $A \subseteq V(G)$ with $|A| \leq \zeta n$ such that for every $W \subseteq V(G) \setminus A$ such that $|H|$ divides $|W|$ and $|W| \leq \nu n$ there is an $H$-factor of $G[A \cup W]$.*

## References


1. J. Balogh, T. Molla, and M. Sharifzadeh, *Triangle factors of graphs without large independent sets and of weighted graphs*, Random Structures Algorithms **49** (2016), no. 4, 669–693.
2. Richard Montgomery, *Embedding bounded degree spanning trees in random graphs*, arXiv preprint arXiv:1405.6559 (2014).
3. Rajko Nenadov and Yanitsa Pehova, *On a Ramsey–Turán variant of the Hajnal–Szemerédi Theorem*, SIAM Journal on Discrete Mathematics **34** (2020), no. 2, 1001–1010.