3. Lecture 3

3.1. Introduction.

Definition 3.1 (Extremal graphs)**.** Let G be an n-vertex graph and let $\beta > 0$. For every $k \in \mathbb{N}$, we say that G is (K_k, β)-extremal if there exists $U \subseteq V(G)$ such that $|U| \geq (1/k - \beta)n$ and either $e(G[U]) \leq \beta n^2$ or $G[U, \overline{U}] \leq \beta n^2$. Note when $\delta(G)$ is close to $(k-1)n/k$, $\beta > 0$ is small, and $|U| \geq (1/k - \beta)n$, we cannot have $G[U, \overline{U}] \leq \beta n^2$ unless $k \leq 2$.

The following two lemmas together imply the Corrádi-Hajnal Theorem [1], for large graphs.

Lemma 3.2 (Extremal lemma)**.** *There exists $\beta > 0$ and n_0 such that for every $n \geq n_0$ the following holds for every n-vertex graph G. If 3 divides n, G is (K_3, β)-extremal, and $\delta(G) \geq \frac{2n}{3}$, then G has a K_3-factor.*

Lemma 3.3 (Non-extremal/stability lemma)**.** *For every $\beta > 0$, there exists $\gamma > 0$ and n_0 such that for every $n \geq n_0$ the following holds for every n-vertex graph G. If 3 divides n, G is not (K_3, β)-extremal, and $\delta(G) \geq (\frac{2}{3} - \gamma)n$, then G has a K_3-factor.*

The next two lemmas imply Lemma 3.3.

Lemma 3.4. *Let $\frac{1}{n} \ll \gamma \ll \beta$. If $\delta(G) \geq \left(\frac{2}{3} - \gamma\right)n$ and $\alpha(G) < \left(\frac{1}{3} - \beta\right)n$, then G has a K_3-tiling that covers all but at most 7 vertices.*

Proof sketch. Let $\mathcal{T}$ be a maximum K_3-tiling in G and let $W := V(G) \setminus V(\mathcal{T})$. Let M be a maximum matching in $G[W]$. We can assume that $\mathcal{T}$ was selected to maximize $|M|$. Let $Z := W \setminus V(M)$.

We have $|Z| \leq 1$. Indeed, assume for a contradiction that $|Z| \geq 2$, and let $u, v \in W$ be a pair of distinct vertices. The following observations follow from the selection of $\mathcal{T}$ and M.

- Z is an independent set.
- For every $e \in M$ we have $e(u, e), e(v, e) \leq 1$.
- For every $T \in \mathcal{T}$, we have $e(\{u, v\}, T) \leq 4$ and if $e(\{u, v\}, T) = 4$, then $N(u, T) = N(v, T)$.

Let

$$U := \{y \in V(G) : y \in V(T) \setminus N(u) \text{ for some } T \in \mathcal{T} \text{ such that } e(\{u, v\}, T) = 4\},$$

and note that

$$2\left(\frac{2}{3} - \gamma\right)n \leq d(u) + d(v) \leq 4|U| + 3(|\mathcal{T}| - |U|) + 2|M| \leq |U| + 3|\mathcal{T}| + 2\left(\frac{n - 3|\mathcal{T}|}{2}\right) = |U| + n,$$

so $|U| \geq (1/3 - 2\gamma)n \geq (1/3 - \beta)n > \alpha(G)$ which mean there exists $e \in G[U]$. This contradiction our selection of $\mathcal{T}$ and M. (How?)

We also have $|M| \leq 3$. Indeed assume for a contradiction that we have distinct $e_1, e_2, e_3, e_4 \in M$. And for $i \in [4]$, let $N_i := \bigcap_{v \in e_i} N(v)$ be the common neighbors of the endpoints of e_i. The following observations follow easily.

- For every $i \in [4]$, $|N_i| \geq (1/3 - 2\gamma)n$.
- For every $i \in [4]$, $N_i \cap W = \emptyset$.
- For every $i \in [4]$ and every $T \in \mathcal{T}$, $\sum_{i \in [4]} |N_i \cap V(T)| \leq 4$ and if $\sum_{i \in [4]} |N_i \cap V(T)| \leq 4$ we have $N_1 \cap V(T) = N_2 \cap V(T) = N_3 \cap V(T) = N_\cap V(T)$.

Let $\mathcal{T}' := \{T \in \mathcal{T} : \sum_{i \in [4]} |N_i \cap V(T)| = 4\}$. So,

$$4\left(1/3 - 2\gamma\right)n \leq |N_1| + |N_2| + |N_3| + |N_4| \leq 4|\mathcal{T}'| + 3(|\mathcal{T}| - |\mathcal{T}'|) \leq |\mathcal{T}'| + n,$$

and $|\mathcal{T}'| \geq (1/3 - 8\gamma)n$. So, if $U := \bigcup_{T \in \mathcal{T}'} V(T_i) \setminus N_1$, we have $|U| \geq (2/3 - 16\gamma)n$. Then, for every $v \in U$, we have $|N(v, U)| \geq \delta(G) + |U| - n \geq (1/3 - \beta)n > \alpha(G)$. So, there exists an edge $e \in E(G[N(v, U)])$. The fact that ve is a triangle, contradicts the maximality of $\mathcal{T}$. (How?) □

The following lemma implies that we can find a suitable absorbing set for Lemma 3.3. The proof follows easily from the definitions.

Lemma 3.5. *Let $\frac{1}{n} \ll \gamma \ll \beta$. If $\delta(G) \geq \left(\frac{2}{3} - \gamma\right)n$ and G is not (K_3, β)-extremal, then for every distinct pair of vertices $x, y \in V(G)$ there exists at least βn^2 edges e such that both xe and ye are triangles.*

We now collect a few lemmas we will need in the proof of Lemma 3.2.

Lemma 3.6. *For $\frac{1}{n} \ll \gamma \ll \beta$, the following holds for every n-vertex graph G. If G is not (K_2, β)-extremal and $\delta(G) \geq (\frac{1}{2} - \gamma)n$, then*

(a) G contains $0.1 \cdot \beta n$ vertex disjoint copies of K_3 and
(b) when n is even G contain a perfect matching.

Proof sketch. To see (a), note that, because G is not (K_2, β)-extremal, for every $u \in V(G)$ and every $U \subseteq N(v)$ such that $|U| \geq (1/2 - \beta)n$, there exists $vw \in E(G[U])$ and uvw is a triangle. To see (b), note that since G is not (K_2, β)-extremal, G is 2-connected and $\alpha(G) < (1/2 - \beta)n$. Therefore, $\delta(G) \geq \max\{\alpha(G), (n+2)/3\}$ and a result of Nash-Williams implies that G is Hamiltonian. □

Lemma 3.7. *Every graph G has a matching of size at least* $\min\left\{\delta(G), \left\lfloor \frac{|V(G)|}{2} \right\rfloor\right\}$.

Proof sketch. Let M be a maximum matching. For a contradiction, assume

$$|M| < \min\left\{\delta(G), \left\lfloor \frac{|V(G)|}{2} \right\rfloor\right\}.$$

Let $U := V(G) \setminus V(M)$. Notice that $|M| < \frac{|V(G)|-1}{2}$, so $|V(M)| = 2|M| < |V(G)| - 1$ and there exist distinct $u, u' \in U$. Since M is maximal and $|M| < \delta(G)$, we have

$$\deg(u, V(M)) + \deg(u', V(M)) = \deg(u) + \deg(u') \geq 2\delta(G) > 2|M|,$$

so the result follows. (Why?) □

While the following two lemmas might appear complicated, they actually follow easily from the minimum degree condition and the definition of extremal graphs. We prove more general versions than we will need for Lemma 3.2.

Lemma 3.8. *For every $k \geq 2$ and $\frac{1}{n} \ll \gamma \ll \beta \ll \gamma' \ll \nu_3 \ll \nu_2 \ll \nu_1 \ll \frac{1}{r}$ the following holds for every n-vertex graph G. If G is (K_k, β)-extremal and $\delta(G) \geq \left(\frac{k-1}{k} - \gamma\right)n$, then there exists a partition U_1, U_2, W of $V(G)$ and sets $U_1' \subseteq U_1$ and $U_2' \subseteq U_2$ such that the following conditions hold.*

(A) Both $|U_1| \geq \left(\frac{1}{k} - \gamma'\right)n$ and $|U_2| \geq \left(\frac{k-1}{k} - \gamma'\right)n$.
(B) For every $w \in W$, $\deg_G(w, U_2') \geq \left(\frac{k-2}{k} + \nu_2\right)n$ and $\deg_G(w, U_1') \geq \nu_2 n$.
(C) Either
(i) for every $i \in [2]$, $\Delta(\overline{G}[U_i, \overline{U_i}]) \leq \nu_1 n$ and $\Delta(\overline{G}[U_i', \overline{U_i'}]) \leq \nu_3 n$, or
(ii) $k = 2$ and for every $i \in [2]$, $\Delta(\overline{G}[U_i, \overline{U_{3-i}}]) \leq \nu_1 n$ and $\Delta(\overline{G}[U_i', \overline{U_{3-i}'}]) \leq \nu_3 n$.

Proof sketch. There exists $U \subseteq V(G)$ such that $|U| \geq (1/k - \beta)n$ and either $e(G[U]) \leq \beta n^2$ or $k = 2$ and $e(G[U, \overline{U}]) \leq \beta n^2$. First assume $e(G[U]) \leq \beta n^2$ and note that

$$e(\overline{G}([U, \overline{U}]) \leq |U|\Delta(\overline{G}) - 2e(\overline{G}[U]) \leq |U|(|U| + \gamma n + \beta n) - 2\left(\binom{|U|}{2} - \beta n^2\right) \leq 4\beta n^2$$

Therefore, if we let

$$U_1' := \{u \in U : \deg_{\overline{G}}(u, \overline{U}) \leq \nu_3 n/2\} \quad \text{and} \quad U_2' := \{u \in \overline{U} : \deg_{\overline{G}}(u, U) \leq \nu_3 n/2\},$$

and

$$U_1 := U_1' \cup \{v \in V(G) : \deg_G(v, U_1') < \nu_2 n\} \quad \text{and}$$
$$U_2 := U_2' \cup \left\{v \in V(G) : \deg_G(v, U_2') < \left(\frac{k-2}{k} + \nu_2\right)n\right\}.$$

and $W := V(G) \setminus (U_1 \cup U_2)$, then the result follows (since we can assume $(\gamma' - \beta)\nu_3/2 \gg 4\beta$). The case when $k = 2$ and $e(G[U, \overline{U}]) \leq \beta n^2$ is similar. □

Lemma 3.9. *For every $k \geq 2$ and $\frac{1}{n} \ll \gamma \ll \beta \ll \gamma' \ll \beta' \ll \nu_3 \ll \nu_2 \ll \nu_1 \ll \frac{1}{r}$ the following holds. If G is (K_k, β)-extremal and $\delta(G) \geq \left(\frac{k-1}{k} - \gamma\right)n$, then there exists $s \in \{2, \ldots, k\}$, a partition $U_1, \ldots, U_s, W$ of $V(G)$, and sets $U_i' \subseteq U_i$ for every $i \in [s]$ such that the following conditions hold.*

(A) *For every* $i \in [s-1]$, $|U_i| \geq \left(\frac{1}{k} - \gamma'\right) n$ *and* $|U_s| \geq \left(\frac{k-s+1}{k} - \gamma'\right) n$.
(B) *For every* $w \in W$, $\deg_G(w, U'_s) \geq \left(\frac{k-s}{k} + \nu_2\right) n$ *and* $\deg_G(w, U'_i) \geq \nu_2 n$ *for every* $i \in [s-1]$.
(C) *Either*
  (i) *For every* $i \in [s]$ *we have* $\Delta(\overline{G}[U_i, \overline{U_i}]) \leq \nu_1 n$ *and* $\Delta(\overline{G}[U'_i, \overline{U'_i}]) \leq \nu_3 n$, *or*
  (ii) $s = k$ *and if we let* $\sigma \in S_r$ *be the transposition of* $k-1$ *and* k, *then for every* $i \in [k]$, $\Delta(\overline{G}[U_i, \overline{U_{\sigma(i)}}]) \leq \nu_1 n$ *and* $\Delta(\overline{G}[U'_i, \overline{U'_{\sigma(i)}}]) \leq \nu_3 n$.
(D) *If* $s < k$, *then* $G[U_s]$ *is not* (K_{k-s+1}, β')*-extremal.*

Proof sketch. First note when $k = 2$ this lemma and Lemma 3.8 are equivalent. With this as the base case and Lemma 3.8, the result follows by induction on k.

Indeed, for the induction step with $k \geq 3$, let γ^{**}, γ^*, β^*, ν_1^*, ν_2^*, ν_3 be such that

$$\gamma \ll \beta \ll \gamma^{**} \ll \beta^* \ll \gamma^* \ll \gamma' \ll \beta' \ll \nu_3^* \ll \nu_3 \ll \nu_2 \ll \nu_2^* \ll \nu_1^* \ll \nu_1$$

and Lemma 3.8 holds with n, γ, β, γ^{**}, ν_3^*, ν_2^*, ν_1^*, and k playing the roles of n, γ, β, γ', ν_3, ν_2, ν_1, and k, respectively, and also Lemma 3.9 holds with $n/3$, $2\gamma^{**}$, β^*, γ^*, β', ν_3^*, ν_2^*, ν_1^*, and $k-1$ playing the roles of n, γ, β, γ', β', ν_3, ν_2, ν_1, and k, respectively.

Assume that we have applied Lemma 3.8 to G in this way, so we have U_1, U'_1, U_2, U'_2, and W that satisfy the conclusion with γ^*, ν_3^*, ν_2^*, ν_1^*, and k playing the roles of γ', ν_3, ν_2, ν_1, respectively. If $G[U_2]$ is not (K_{k-1}, β^*)-extremal, then these sets satisfy the conclusion of Lemma 3.9 and we are done. Otherwise, note that $\delta(G) \geq \left(\frac{k-1}{k} - \gamma\right) n$ and $|U_2| \geq \left(\frac{k-1}{k} - \gamma^{**}\right) n$ together imply that

$$\delta(G[U_2]) \geq \delta(G) - (n - |U_2|) \geq \left(1 - \frac{\frac{1}{k} + \gamma n}{|U_2|}\right) |U_2| \geq \left(\frac{k-2}{k-1} - 2\gamma^{**}\right) |U_2|.$$

Therefore, since $G[U_2]$ is (K_{k-1}, β)-extremal and $|U_2| \geq n/3$, we can apply Lemma 3.9 to $G[U_2]$ with $2\gamma^{**}$, β^*, γ^*, β', ν_3^*, ν_2^*, ν_1^*, and $k-1$ playing the roles of γ, β, γ', β', ν_3, ν_2, ν_1, and k, respectively. This yields the desired conclusion after an appropriate relabelling. $\square$

Proof of Lemma 3.2. Pick ν_1, ν_2, ν_3, β', γ', β, γ and n_0 so that for $n \geq n_0$.

$$\frac{1}{n} \ll \gamma \ll \beta \ll \gamma' \ll \beta' \ll \nu_3 \ll \nu_2 \ll \nu_1 \ll \frac{1}{3}.$$

Apply Lemma 3.9, so we have $s \in \{2, 3\}$, and sets W U_i, U'_i for $i \in [s]$ that satisfy the conclusion. For $i \in [s-1]$, let $c_i := n/3 - |U_i|$ and let $c_s := (4-s)n/3 - |U_s|$. Note that

$$s\gamma' n \geq |W| = c_1 + \cdots + c_s \geq 0.$$

We will now find a small K_3-tiling $\mathcal{T}$ that covers all of W and is such that the graph $G - V(\mathcal{T})$ has a K_3-factor. In an attempt to make the argument more intuitive, we can think of moving vertices in $V(\mathcal{T})$ to "balance" the sizes of the sets $U_1, \ldots, U_s$. That is, if $c_i > 0$ we want to move c_i vertices to U_i and if $c_i < 0$ we want to move $-c_i$ vertices from U_i to some other set U_j. Note that this is not quite true in Case 3 below. In this case, if $c_2 + c_3 > 0$ we need to move $c_2 + c_3$ vertices to $U_2 \cup U_3$ and if $c_2 + c_3 < 0$ we need to move $-(c_2 + c_3)$ vertices from $U_2 \cup U_3$ to U_1. On top of this, we need to ensure that both $|U_2 \setminus V(\mathcal{T})|$ and $|U_3 \setminus V(\mathcal{T})|$ are even.

In what follows, we will use the following fact repeatedly. Suppose $i \in [s]$, $c_i < 0$, $|U_i| = n/3 - c_i$, and $\Delta(\overline{G}(U_i, \overline{U_i}) \leq \nu_3 n$. Then, because $\delta(G) \geq 2n/3$, we have $\delta(G[U_i]) \geq -c_i$. Therefore, by Lemma 3.7, there is a matching of size $-c_i$ in $G[U_i]$. Furthermore, with this matching and the fact that $\Delta(\overline{G}(U_i, \overline{U_i}) \leq \nu_3 n$ we can easily find $-c_i$ disjoint triangles each with two vertices in U_i and one vertex in U_j for any $j \in [s] \setminus \{i\}$.

Case 1. ($s = 2$) Note that, in this case, because $G[U_2]$ is not (K_2, β')-extremal, Lemma 3.6 implies that we can easily find a small collection of disjoint triangles in $G[U_2]$. While this fact, it is easy to find a K_3-tiling $\mathcal{T}$ that satisfies the following.

- If $c_1, c_2 \geq 0$, then $\mathcal{T}$ contains
  - **(move c_1 vertices from W to U_1)** exactly c_1 triangles that each have one vertex in W and two vertices in U'_2 and
  - **(move c_2 vertices from W to U_2)** exactly c_2 triangles that each have one vertex in each of U'_1, U'_2, and W.

- If $c_1 < 0$ and $c_2 \geq 0$, then $\mathcal{T}$ contains
  - **(move $-c_1$ vertices from U_1 to U_2)** exactly $-c_1$ triangles that each have two vertices in U_1 and one vertex in U_2 and
  - **(move $c_1 + c_2$ vertices from W to U_2)** exactly $c_1 + c_2$ triangles that each have one vertex in each of U_1', U_2', and W.
- If $c_2 < 0$ and $c_1 \geq 0$, then $\mathcal{T}$ contains
  - **(move $-c_2$ vertices from U_2 to U_1)** exactly $-c_2$ triangles that each have three vertices in U_2 and
  - **(move $c_1 + c_2$ vertices from W to U_1)** exactly $c_1 + c_2$ triangles that each have one vertex in W and two vertices in U_2'.

In all cases, $W \setminus V(\mathcal{T}) = \emptyset$ and $|U_2 \setminus V(\mathcal{T})| = 2|U_1 \setminus V(\mathcal{T})|$. Furthermore, Lemma 3.6 implies that since $G[U_2]$ is not (K_2, β')-extremal, the graph $G[U_2 \setminus V(\mathcal{T})]$ contains a perfect matching M. Now considering the balanced bipartite graph with parts M and U_1 in which $u \in U_1$ is adjacent to $vw \in M$ if uvw is a triangle. Hall's Theorem implies that this bigraph has a perfect matching and this corresponds to a K_3-factor of G.

Case 2. ($s = 3$ and (C)(i) holds) We can assume without loss of generality that $c_1 \geq c_2 \geq c_3$. In particular, this implies that $c_1 \geq 0$. In a manner similar to Case 1, it is trivial to find a K_3-tiling $\mathcal{T}$ that satisfies the following.

- If $c_2, c_3 \geq 0$, then $\mathcal{T}$ contains
  - **(move c_1 vertices from W to U_1)** exactly c_1 triangles that each have one vertex in each of W, U_2', and U_3',
  - **(move c_2 vertices from W to U_2)** exactly c_2 triangles that each have one vertex in each of W, U_1', and U_3', and
  - **(move c_3 vertices from W to U_3)** exactly c_3 triangles that each have one vertex in each of W, U_1', and U_2'.
- If $c_2, c_3 < 0$, then $\mathcal{T}$ contains
  - **(move $c_1 + c_2 + c_3$ vertices from W to U_1)** exactly $c_1 + c_2 + c_3$ triangles that each have one vertex in each of W, U_2', and U_3',
  - **(move $-c_2$ vertices from U_2 to U_1)** exactly $-c_2$ triangles that each have one vertex in U_3, and two vertices in U_2, and
  - **(move $-c_3$ vertices from U_3 to U_1)** exactly $-c_3$ triangles that each have one vertex in U_2, and two vertices in U_3.
- If $c_2 \geq 0$ and $c_3 < 0$, then for $c_1' \leq 0$ and $c_2' \leq 0$ such that $c_1' + c_2' = c_3$ and $c_1 + c_1' \geq 0$ and $c_2 + c_2' \geq 0$ the collection $\mathcal{T}$ contains
  - **(move $c_1 + c_1'$ vertices from W to U_1)** exactly $c_1 + c_1'$ triangles that each have one vertex in each of W, U_2', and U_3',
  - **(move $c_2 + c_2'$ vertices from W to U_2)** exactly $c_2 + c_2'$ triangles that each have one vertex in each of W, U_1', and U_3',
  - **(move $-c_1'$ vertices from U_3 to U_1)** exactly $-c_1'$ triangles that each have one vertex in U_2 and two vertices in U_3, and
  - **(move $-c_2'$ vertices from U_3 to U_2)** exactly $-c_2'$ triangles that each have one vertex in U_1 and two vertices in U_3.

It is now trivial to find a K_3-factor in $G' := G - V(\mathcal{T})$, because G' contains a nearly complete spanning balanced 3-partite graph.

Case 3. ($s = 3$ and (C)(ii) holds) We can assume without loss of generality that $c_2 \geq c_3$. Let $z := 0$ if c_3 is even or $z := 1$ is c_3 is odd. Recall that in this case $G[U_2]$ and $G[U_3]$ are almost cliques, so it is trivial to find any desired small matching or small K_3-tiling in both graphs. Also note that if $c_2 + c_3 = 0$, then $|U_2| + |U_3| = 2n/3$, so

$$\delta(G[U_2 \cup U_3]) \geq \delta(G) - n/3 \geq n/3 > \min\{|U_2| - 1, |U_3| - 1\},$$

so there are edges in $G[U_2, U_3]$ and a triangle with one vertex in each of U_1, U_2, and U_3. With this, and our previous lemmas, we can find a K_3-tiling $\mathcal{T}$ that satisfies the following.

- If $c_1 \geq 0$ and $c_2 + c_3 = 0$, then $\mathcal{T}$ contains

- **(move z vertices from U_3 to U_2)** exactly z triangles that each have one vertex in each of U_1, U_2, and U_3, and
- **(move c_1 vertices from W to U_1)** exactly c_1 triangles that each have two vertices in U_2' and one vertex in W.

- If $c_1 \geq 0$ and $c_2 + c_3 > 1$, then $\mathcal{T}$ contains
  - **(move c_1 vertices from W to U_1)** exactly c_1 triangles that each have two vertices in U_2' and one vertex in W,
  - **(move $c_2 + c_3 - z$ vertices from W to U_2)** exactly $c_2 + c_3 - z$ triangles that each have one vertex in each of U_1', U_2' and W, and
  - **(move z vertices from W to U_3)** exactly z triangles that each have one vertex in each of U_1', U_3', and W.
- If $c_1 \geq 0$ and $c_2 + c_3 < 0$, then $\mathcal{T}$ contains
  - **(move $c_1 + c_2 + c_3$ vertices from W to U_1)** exactly $c_1 + c_2 + c_3$ triangles that each have two vertices in U_2' and one vertex in W,
  - **(move $-c_2 - c_3 - z$ vertices from U_2 to U_1)** exactly $-c_2 - c_3 - z$ triangles that each have three vertices in U_2, and
  - **(move z vertices from U_3 to U_1)** exactly z triangles that each have three vertices in U_3.
- If $c_1 < 0$ and $c_2 + c_3 \geq 0$, then $\mathcal{T}$ contains
  - **(move $-c_1 - z$ vertices from U_1 to U_2)** exactly $-c_1 - z$ triangles that each have two vertices in U_1 and one vertex in U_2,
  - **(move z vertices from U_1 to U_3)** exactly z triangles that each have two vertices in U_1 and one vertex in U_3, and
  - **(move $c_1 + c_2 + c_3$ vertices from W to U_2)** exactly $c_1 + c_2 + c_3$ triangles that each have one vertex in each of U_1', U_2', and W.

In all cases, $W \setminus V(\mathcal{T}) = \emptyset$ and $|\,(U_2 \cup U_3) \setminus V(\mathcal{T})| = 2|U_1 \setminus V(\mathcal{T})|$. We also have $|U_3 \setminus V(\mathcal{T})|$ is even. These two fact together imply that $|U_2 \setminus V(\mathcal{T})|$ is even. Therefore, there is a perfect matching of $G[(U_2 \cup U_3) \setminus V(\mathcal{T})]$. As in Case 1, we can use Hall's Theorem to then find a K_3-tiling of $G - V(\mathcal{T})$.

□

References

1. K. Corrádi and A. Hajnal, *On the maximal number of independent circuits in a graph*, Acta Mathematica Hungarica **14** (1963), no. 3, 423–439.