A priori error analysis of the BEM with graded meshes for the electric field integral equation on polyhedral surfaces

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Abstract

The Galerkin boundary element discretisations of the electric field integral equation (EFIE) on Lipschitz polyhedral surfaces suffer slow convergence rates when the underlying surface meshes are quasi-uniform and shape-regular. This is due to singular behaviour of the solution to this problem in neighbourhoods of vertices and edges of the surface. Aiming to improve convergence rates of the Galerkin boundary element method (BEM) for the EFIE on a Lipschitz polyhedral closed surface Γ, we employ anisotropic meshes algebraically graded towards the edges of Γ. We prove that on sufficiently graded meshes the h-version of the BEM with the lowest-order Raviart-Thomas elements regains (up to a small order of $\varepsilon > 0$) an optimal convergence rate (i.e., the rate of the h-BEM on quasi-uniform meshes for smooth solutions).

Key words: electromagnetic scattering, electric field integral equation, Galerkin discretisation, boundary element method, anisotropic elements, graded mesh, a priori error analysis
AMS Subject Classification: 65N38, 65N12, 78M15

1 Introduction

The boundary element method (BEM), known as the method of moments in the engineering literature (see, e.g., [24], [2, Chapter 12], [25, Chapter 2]), is widely used for simulation of electromagnetic phenomena and is the basis of some widespread commercial software (e.g., FEKO, WIPL-D). When simulating the scattering of time-harmonic electromagnetic waves at a perfect conductor, the underlying mathematical model can be formulated as

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the electric field integral equation (EFIE), whose solution is the electric current density induced on the surface of the scatterer (see, e.g., [18, 23, 2]).

In this note we consider the EFIE on a Lipschitz polyhedral surface Γ in $\mathbb{R}^3$ (i.e., $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron). Our goal is to establish convergence rates of the lowest-order Galerkin BEM on graded meshes for this problem.

The Galerkin BEM considered in this paper employs div_Γ-conforming lowest-order Raviart-Thomas surface elements to discretise the variational formulation of the EFIE (known as Rumsey's principle). This approach is called the natural BEM for the EFIE. Theoretical aspects of the natural Galerkin BEM for the EFIE on Lipschitz surfaces have been extensively studied over the last decade. These included quasi-optimal convergence and a priori error estimates for the lowest-order h-BEM, see [20, 13, 10, 16, 15], as well as for high-order methods (p- and hp-BEM), see [4, 7, 3, 6]. In all these studies, however, the underlying surface meshes on Γ were assumed shape-regular.

It is now well known that convergence rates of the h-BEM with quasi-uniform and shape-regular meshes are bounded by the poor regularity of solutions to the EFIE on non-smooth surfaces. For example, on a closed polyhedral surface $\Gamma = \partial\Omega$, the solution may be only $\mathbf{H}^\varepsilon(\Gamma)$-regular (with a small $\varepsilon > 0$ in the case of non-convex polyhedron Ω, cf. [19, Section 4.4.2]), and convergence rate of the h-BEM is only $\frac{1}{2} + \varepsilon$ in this case, whereas in the case of smooth solutions the lowest-order h-BEM converges with the optimal rate of $\frac{3}{2}$ (see [20, Theorem 8.2] and [3, Theorem 2.2]). Motivated by the desire to regain the optimal convergence rate for the h-BEM on non-smooth surfaces, in [8] we studied the Galerkin BEM on graded meshes with highly anisotropic elements along the edges of Γ. Our expectation was that, similar to the h-BEM for the Laplacian (see [26, 27]), one could recover the optimal convergence rate of the h-BEM for the EFIE by employing graded meshes with sufficiently high strength of grading. It turned out, however, that we were able to prove asymptotic quasi-optimality of the Galerkin h-BEM only under a mild restriction on the strength of grading (see Proposition 3.1 below). The question then arises whether this restriction prevents one from recovering the optimal convergence rate of the h-BEM. We address this issue in the present note by considering explicit expressions for singularities in the solution to the EFIE. We prove that the strength of grading can be selected depending on the strength of singularities such that the h-version of the BEM indeed regains a suboptimal convergence rate of $\frac{3}{2} - \varepsilon$ (for any $\varepsilon > 0$). To the best of our knowledge, theoretical error analysis of the Galerkin BEM with graded meshes for the EFIE is not available in the literature, and with this article we fill this gap.

The rest of the article is structured as follows. In the next section, we introduce necessary notation, formulate the EFIE in its variational form, and recall the typical structure of the solution to this problem. In Section 3, we describe the construction of graded meshes on Γ, introduce the boundary element space, and formulate the main result of the paper, Theorem 3.1. Technical details and the proof of Theorem 3.1 are included in Section 4.

2 The electric field integral equation

The variational formulation of the EFIE is posed on the Hilbert space

$$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$$

equipped with its graph norm $\|\cdot\|_{\mathbf{X}}$. Here, div_Γ denotes the surface divergence operator, $\mathbf{H}_{\|}^{-1/2}(\Gamma)$ is the dual space of $\mathbf{H}_{\|}^{1/2}(\Gamma)$ (the tangential trace space of $\mathbf{H}^1(\Omega)$ on Γ, see [11, 14]), and $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$. The space $\mathbf{X}$ is the natural tangential trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$, see [11, 14]. We refer to [11, 12, 14, 15] for definitions and properties of $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and other involved trace spaces and differential operators on Γ.

In the present article, we use a traditional notation for the Sobolev spaces (of scalar functions) H^s ($s \geq -1$) and their norms on Lipschitz domains and surfaces (see [21, 22]). The norm and inner product in $L^2(\Gamma) = H^0(\Gamma)$ will be denoted by $\|\cdot\|_{0,\Gamma}$ and $(\cdot,\cdot)_{0,\Gamma}$, respectively. The notation $(\cdot,\cdot)_{0,\Gamma}$ will be used also for appropriate duality pairings extending the $L^2(\Gamma)$-pairing for functions on Γ. For vector fields we will use boldface symbols (e.g., $\mathbf{u} = (u_1, u_2)$), and the spaces (or sets) of vector fields are also denoted in boldface (e.g., $\mathbf{H}^s(D) = (H^s(D))^2$ with $D \subset \mathbb{R}^2$). The norms and inner products in these spaces are defined componentwise. The notation for Sobolev spaces of tangential vector fields on Γ follows [11, 12, 14]. This notation is summarised in [3, Section 3.1]. In particular, $\mathbf{L}^2_t(\Gamma)$ denotes the space of two-dimensional, tangential, square integrable vector fields on Γ with the norm $\|\cdot\|_{0,\Gamma}$ and inner product $(\cdot,\cdot)_{0,\Gamma}$ (the similarity of this notation with the one for scalar functions should not lead to any confusion, as the meaning will always be clear from the context). We will also use the space

$$\mathbf{H}^\tau_-(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2_t(\Gamma);\ \mathbf{u}|_F \in \mathbf{H}^\tau(F) \text{ for each face } F \subset \Gamma\}, \quad \tau \geq 0$$

with the norm $\|\mathbf{u}\|_{\mathbf{H}^\tau_-(\Gamma)} := \left(\sum\limits_{F\subset\Gamma} \|\mathbf{u}|_F\|^2_{\mathbf{H}^\tau(F)} \right)^{1/2}$.

For a fixed wave number $k > 0$ and for a given source functional $\mathbf{f} \in \mathbf{X}'$, the variational formulation for the EFIE reads as: *find a complex tangential field* $\mathbf{u} \in \mathbf{X}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) := \langle \Psi_k \mathrm{div}_\Gamma\, \mathbf{u}, \mathrm{div}_\Gamma\, \mathbf{v}\rangle - k^2 \langle \boldsymbol{\Psi}_k \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}. \tag{2.1}$$

Here, Ψ_k (resp., $\boldsymbol{\Psi}_k$) denotes the scalar (resp., vectorial) single layer boundary integral operator on Γ for the Helmholtz operator $-\Delta - k^2$, see [13, Section 4.1] (resp., [15, Section 5]). To ensure the uniqueness of the solution to (2.1) we always assume that k^2 is not an electrical eigenvalue of the interior problem in Ω.

We will now recall the typical structure of the solution $\mathbf{u}$ to problem (2.1). Let $V = \{v\}$ and $E = \{e\}$ denote the sets of vertices and edges of Γ, respectively. For $v \in V$, let $E(v)$ denote the set of edges with v as an end point. Then, for sufficiently smooth source functional $\mathbf{f}$ (e.g., with $\mathbf{f}$ representing the excitation by an incident plane wave), the solution $\mathbf{u}$ of (2.1) can be written as (see [4, Appendix A])

$$\mathbf{u} = \mathbf{u}_{\mathrm{reg}} + \mathbf{u}_{\mathrm{sing}}, \tag{2.2}$$

where

$$\mathbf{u}_{\text{reg}} \in \mathbf{X}^\tau := \{\mathbf{u} \in \mathbf{H}_-^\tau(\Gamma);\ \operatorname{div}_\Gamma \mathbf{u} \in H_-^\tau(\Gamma)\} \quad \text{with} \quad \tau > 0$$

(here, the space $H_-^\tau(\Gamma)$ is defined similarly to the space $\mathbf{H}_-^\tau(\Gamma)$ in a piecewise fashion by localisation to each face of Γ, and the space $\mathbf{X}^\tau$ is equipped with its graph norm $\|\cdot\|_{\mathbf{X}^\tau}$, see [3]),

$$\mathbf{u}_{\text{sing}} = \sum_{e\in E} \mathbf{u}^e + \sum_{v\in V} \mathbf{u}^v + \sum_{v\in V} \sum_{e\in E(v)} \mathbf{u}^{ev}, \tag{2.3}$$

and $\mathbf{u}^e$, $\mathbf{u}^v$, and $\mathbf{u}^{ev}$ are the edge, vertex, and edge-vertex singularities, respectively.

Explicit formulas for singularities in (2.3) were derived in [4, Appendix A] from the regularity theory developed in [19, Section 4.4] for boundary value problems for Maxwell's equations in 3D. For our purposes in this note, it is sufficient to provide a qualitative snapshot of the results in [4, Appendix A]. In particular, we will write out the generic singular terms for $\mathbf{u}^e$, $\mathbf{u}^v$, and $\mathbf{u}^{ev}$ omitting cut-off functions and smooth factors (we refer to [4, Appendix A] for complete expansions and to [6, Section II] for leading singularities). Let r be the distance to a vertex $v \in \Gamma$, and let ρ be the distance to one of the edges $e \subset \partial\Gamma$ such that $\bar{e} \ni v$. Then any singular vector field $\mathbf{u}^s$ in (2.3) ($s = e$, v, or ev) can be written as

$$\mathbf{u}^s = \mathbf{curl}_\Gamma\, w^s + \mathbf{v}^s = \mathbf{curl}_\Gamma\, w^s + (v_1^s,\, v_2^s), \tag{2.4}$$

where the typical (scalar) edge singularities w^e, v_j^e ($j = 1, 2$) are of the type

$$\rho^\gamma |\log \rho|^{\sigma_1}, \quad \gamma \ge \gamma_0 > 1/2, \quad \sigma_1 \ge 0 \text{ is integer}; \tag{2.5}$$

the typical (scalar) vertex singularities w^v, v_j^v ($j = 1, 2$) are of the type

$$r^\lambda |\log r|^{\sigma_2}, \quad \lambda \ge \lambda_0 > 0, \quad \sigma_2 \ge 0 \text{ is integer}; \tag{2.6}$$

and the typical (scalar) edge-vertex singularities w^{ev}, v_j^{ev} ($j = 1, 2$) are of the type

$$r^{\lambda-\gamma} \rho^\gamma |\log r|^{\sigma_3}, \quad \lambda \ge \lambda_0 > 0, \quad \gamma \ge \gamma_0 > 1/2, \quad \sigma_3 \ge 0 \text{ is integer}. \tag{2.7}$$

Let us denote

$$\alpha_0 := \min\{\gamma, \lambda + 1/2\}, \tag{2.8}$$

where $\gamma > 1/2$ and $\lambda > 0$ are the exponents in (2.5) and (2.6), respectively.

Remark 2.1 *A few important observations should be made here.*

(i) The exponents $\gamma_0 > 1/2$ and $\lambda_0 > 0$ correspond to the strongest edge and vertex singularities, respectively. This implies, in particular, that all scalar singularities in (2.4) *belong to $H^1(\Gamma)$ (and hence $\mathbf{v}^s \in \mathbf{H}^1(\Gamma)$ for $s = e$, v, ev). The singular terms of higher regularity (i.e., the terms with exponents $\gamma > \gamma_0$ and $\lambda > \lambda_0$) are necessary to obtain the smooth remainder $\mathbf{u}_{\text{reg}}$ as regular as needed. This can be done by considering decomposition* (2.3) *with sufficiently many singularity terms of each type.*

(ii) The functions $w^s \in H^1(\Gamma)$ ($s = e$, v, or ev) in (2.4) *are typical singularities inherent to solutions of the boundary integral equations with hypersingular operator for the*

Laplacian on Γ *(with sufficiently smooth right-hand side). On the other hand, the functions* $\mathrm{div}_\Gamma\,\mathbf{v}^e$, $\mathrm{div}_\Gamma\,\mathbf{v}^v$, $\mathrm{div}_\Gamma\,\mathbf{v}^{ev} \in L^2(\Gamma)$ *behave as*

$$\rho^{\gamma-1}|\log\rho|^{\tilde\sigma_1}, \quad r^{\lambda-1}|\log r|^{\tilde\sigma_2}, \quad r^{\lambda-\gamma}\rho^{\gamma-1}|\log r|^{\tilde\sigma_3},$$

respectively, with the same γ, λ *as in* (2.5)–(2.6) *and with integers* $\tilde\sigma_i \geq 0$ *(*$i = 1,2,3$*). These functions therefore coincide with typical singularities in solutions to the boundary integral equations with weakly singular operator on* Γ *(again, with sufficiently smooth right-hand side). Appropriate polynomial approximations of singularities inherent to solutions of the boundary integral equations with hypersingular and weakly singular operators have been extensively studied. Of our particular interest in this article are the sharp error bounds for such approximations on graded meshes that were established in* [26, 27] *(see Lemmas* 4.2 *and* 4.3 *below). These error bounds will be used to prove the main result of this article.*

3 Galerkin BEM on graded meshes. The main result.

To approximate the solution of (2.1) we apply the natural BEM based on Galerkin discretisations with lowest-order Raviart-Thomas spaces on graded meshes.

The construction of graded meshes on individual faces of Γ was described in [8, Section 3] by following [27, Section 3]. We reproduce this construction here for completeness. For simplicity, we can assume that all faces of Γ are triangles. On general polygonal faces the construction is similar, or one can first subdivide the polygon into triangles. On a triangular face $F \subset \Gamma$, we first draw three lines through the centroid and parallel to the sides of F. This makes F divided into three parallelograms and three triangles (see Figure 1). Each of the three parallelograms can be mapped onto the unit square $\widehat{Q} = (0,1)^2$ by a linear transformation such that the vertex $(0,0)$ of $\widehat{Q}$ is the image of a vertex of F. Analogously, each of the three subtriangles can be mapped onto the unit triangle $\widehat{T} = \{\mathbf{x} = (x_1, x_2);\ 0 < x_1 < 1,\ 0 < x_2 < x_1\} \subset \widehat{Q}$ such that the vertex $(1,1)$ of $\widehat{T}$ is the image of the centroid of F. Next, the graded mesh on $\widehat{Q}$ (and hence on $\widehat{T}$) is generated by the lines

$$x_1 = \left(\frac{i}{N}\right)^\beta, \quad x_2 = \left(\frac{j}{N}\right)^\beta, \quad i,j = 0,1,\ldots,N.$$

Here, $\beta \geq 1$ is the grading parameter (which defines the strength of grading), and $N \geq 1$ corresponds to the level of refinement. Mapping each cell of these meshes back onto the face F, we obtain a graded mesh $\Delta_h^\beta(F)$ made of triangles and parallelograms on F (see Figure 1). Note that the diameter of the largest element of this mesh is proportional to βN^{-1}. Hence, $h = 1/N$ defines the mesh parameter, and we will denote by $\mathcal{T} = \{\Delta_h^\beta\}$ a family of graded meshes $\Delta_h^\beta = \bigcup\limits_{F\subset\Gamma} \Delta_h^\beta(F)$ generated on Γ by following the procedure described above.

We will denote by $\mathbf{X}_h \subset \mathbf{X}$ the div_Γ-conforming boundary element space over the graded mesh Δ_h^β. On each element $K \in \Delta_h^\beta$, the restriction $\mathbf{X}_h|_K$ is obtained from the

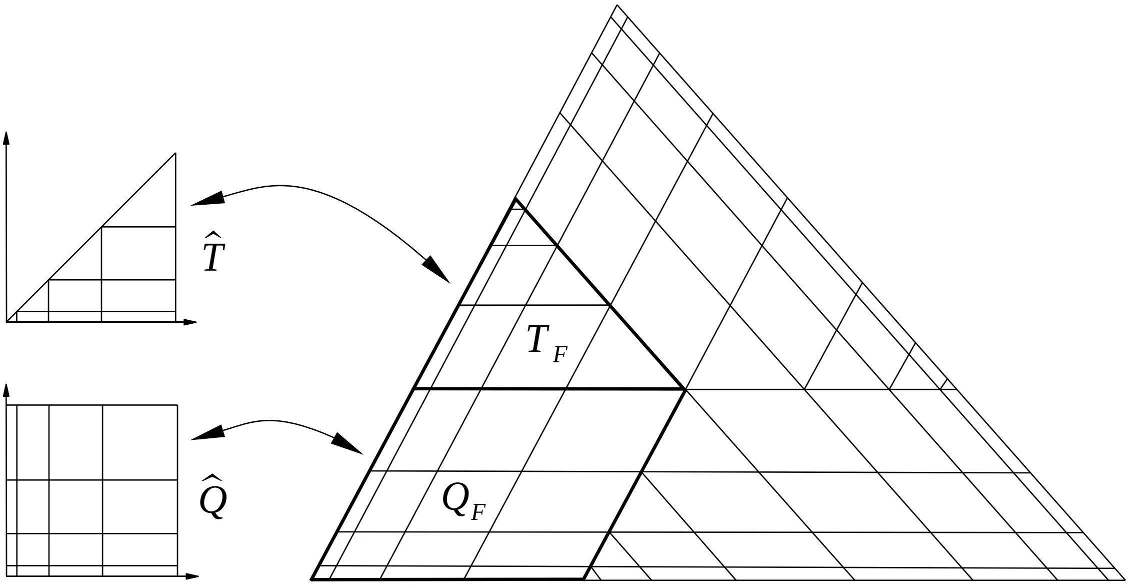

Figure 1: Graded mesh on the triangular face $F \subset \Gamma$. The triangular (resp., parallelogram) block of elements T_F (resp., Q_F) is the image of the graded mesh on the unit triangle $\widehat{T}$ (resp., the unit square $\widehat{Q}$).

lowest-order Raviart-Thomas space on the reference triangle (square) by using the Piola transformation, see [9, Section III.3].

The following result states the unique solvability of the Galerkin boundary element discretisation of (2.1) on graded meshes as well as quasi-optimal convergence of the Galerkin approximations under a mild restriction on the grading parameter β.

Proposition 3.1 [8, Theorem 3.1] *There exists $h_0 < 1$ such that for any $\mathbf{f} \in \mathbf{X}'$ and for any graded mesh Δ_h^β with $h \le h_0$ and $\beta \in [1, 3)$, the Galerkin boundary element discretisation of* (2.1) *admits a unique solution $\mathbf{u}_h \in \mathbf{X}_h$ and the h-version of the Galerkin BEM on graded meshes Δ_h^β converges quasi-optimally, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}} \le C \inf_{\mathbf{v} \in \mathbf{X}_h} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}}, \tag{3.1}$$

where the constant C may depend only on the geometry of Γ and the grading parameter β.

If some information about the regularity of the solution $\mathbf{u}$ to (2.1) is available, then convergence result of Proposition 3.1 translates into an a priori error estimate in the natural $\mathbf{X}$-norm. For scattering problems with sufficiently smooth source functional $\mathbf{f}$, the regularity of the solution depends only on the geometry of Γ. In particular, nonsmoothness of Γ leads to singularities in the solution of the EFIE, that can be explicitly described using a finite set of functions (edge-, vertex-, and edge-vertex singularities (2.5)–(2.7)). The following theorem states that by selecting the grading parameter $\beta \in [1,3)$ sufficiently large (depending on the strength of singularities in $\mathbf{u}$), the h-version of the BEM on graded meshes Δ_h^β regains (up to a small order of ε) an optimal convergence rate (i.e., the rate of the h-BEM on quasi-uniform meshes in the case of a smooth solution).

Theorem 3.1 *Let $\mathbf{u} \in \mathbf{X}$ be the solution of* (2.1) *with sufficiently smooth source functional $\mathbf{f}$, and let α_0 be defined by* (2.8)*. Then the solution $\mathbf{u}_h$ to the Galerkin boundary element discretisation of* (2.1) *on the graded mesh Δ_h^β with $\beta = \max\left\{\frac{3}{2\alpha_0}, 1\right\} \in [1,3)$ satisfies the following error estimate for any $\varepsilon > 0$*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}} \le C\, h^{3/2-\varepsilon}. \tag{3.2}$$

The constant C is independent of h but may depend on the geometry of Γ, the source functional $\mathbf{f}$, the grading parameter β, and on ε.

We prove this theorem in Section 4.2 below.

Throughout the rest of the article, C denotes a generic positive constant that is independent of the mesh parameter h and involved functions but may depend on the geometry of Γ and the grading parameter β.

4 Technical details and the proof of Theorem 3.1

Let us first introduce necessary notation. We will denote by $\Pi_{\rm RT}$ the (classical) Raviart-Thomas interpolation operator, $\Pi_{\rm RT} : \mathbf{H}_-^s(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_h$ $(s > 0)$, see [9]. By Π_0 we denote the L^2-projection onto the space $S_0(\Delta_h^\beta)$ of piecewise constant functions over the mesh Δ_h^β. We also denote by $S_1(\Delta_h^\beta)$ the set of continuous functions on Γ that are linear on triangular elements of the graded mesh Δ_h^β and bilinear on parallelogram elements of Δ_h^β.

4.1 Auxiliary results

Let us collect some technical lemmas that will be used to prove the main result of the article.

The first lemma concerns a specially designed projection onto $\mathbf{X}_h$ that proves useful when deriving the error bounds for approximations of vector fields in dual spaces. In this lemma, $\mathbf{H}_-^{-1/2}(\Gamma)$ denotes the dual space of $\mathbf{H}_-^{1/2}(\Gamma)$ (with $\mathbf{L}_t^2(\Gamma)$ as pivot space).

Lemma 4.1 *There exists an operator $\mathcal{Q}_h : \mathbf{H}_-^s(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_h$ $(s > 0)$ such that*

$$\mathrm{div}_\Gamma \circ \mathcal{Q}_h = \Pi_0 \circ \mathrm{div}_\Gamma = \mathrm{div}_\Gamma \circ \Pi_{\rm RT} \tag{4.1}$$

and for any $\mathbf{u} \in \mathbf{H}_-^s(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ and arbitrary $\varepsilon > 0$

$$\|\mathbf{u} - \mathcal{Q}_h \mathbf{u}\|_{\mathbf{H}_-^{-1/2}(\Gamma)} \le C \left(h^{1/2-\varepsilon} \|\mathbf{u} - \Pi_{\rm RT}\mathbf{u}\|_{0,\Gamma} + \sum_{F \subset \Gamma} \|\mathrm{div}_\Gamma(\mathbf{u} - \Pi_{\rm RT}\mathbf{u})\|_{\tilde{H}^{-1/2}(F)} \right), \tag{4.2}$$

where $C > 0$ is independent of h and $\mathbf{u}$ but may depend on β and ε.

The projection operator $\mathcal{Q}_h$ satisfying (4.1) has been constructed in [8] (see Proposition 6.1 therein). The upper bound in (4.2), however, is new and provides an improvement on the estimate established in [8, Proposition 6.1]. This improvement has turned out to be crucial for the proof of the main result of this article. We will recall the construction of the operator $\mathcal{Q}_h$ and prove estimate (4.2) in the Appendix.

The next two lemmas establish the error bounds for piecewise polynomial approximations of scalar singularities on graded meshes. The following result has been established in [26, Section 3.6] (see also [27, Lemma 3.1]).

Lemma 4.2 *Let $w^s \in H^1(\Gamma)$ be a singular scalar function in representation* (2.4) ($s = e$, v, *or* ev). *Then there exists $w_h^s \in S_1(\Delta_h^\beta)$ such that for any $\varepsilon > 0$*

$$\|w^s - w_h^s\|_{H^{1/2}(\Gamma)} \le C\, h^{\min\{\alpha_0\beta, 3/2\}-\varepsilon}, \tag{4.3}$$

where α_0 is defined by (2.8) *and the positive constant C is independent of h but may depend on β and ε.*

The following lemma concerns piecewise constant approximations of typical (scalar) singularities inherent to solutions to the boundary integral equations with weakly singular operator on Γ (with sufficiently smooth right-hand sides). The corresponding result has been proved in [26, Section 3.4] (see also [27, Lemma 3.1]). Here, we use the fact that the operator div_Γ reduces all singularity exponents by one, while preserving the structure of the corresponding singularity (see Remark 2.1(ii)).

Lemma 4.3 *Let $\mathbf{v}^s \in \mathbf{H}^1(\Gamma)$ be a singular vector field in representation* (2.4) ($s = e$, v, *or* ev). *Then for each face $F \subset \Gamma$ and for any $\varepsilon > 0$*

$$\|\mathrm{div}_\Gamma\, \mathbf{v}^s - \Pi_0(\mathrm{div}_\Gamma\, \mathbf{v}^s)\|_{\tilde{H}^{-1/2}(F)} \le C\, h^{\min\{\alpha_0\beta, 3/2\}-\varepsilon},$$

where α_0 is defined by (2.8) *and the positive constant C is independent of h but may depend on β and ε.*

Finally, the following result holds for the L^2-projection Π_0 onto the space $S_0(\Delta_h^\beta)$.

Lemma 4.4 *Let F be a face of Γ. For all $s \in [0, 1]$, there exists $C > 0$ (depending only on s, β and F) such that*

$$\|v - \Pi_0 v\|_{0,F} \le C\, h^s\, \|v\|_{H^s(F)} \quad \forall v \in H^s(F). \tag{4.4}$$

Proof. For $s = 0$, since Π_0 is the L^2-projection, we trivially have

$$\|v - \Pi_0 v\|_{0,F} \le \|v\|_{0,F} \quad \forall v \in L^2(F).$$

On the other hand, by Theorem 3.1.4 of [17], there exists $C > 0$ depending on β and F such that

$$\|v - \Pi_0 v\|_{0,F} \le C\, h\, \|v\|_{H^1(F)} \quad \forall v \in H^1(F),$$

which yields (4.4) for $s = 1$. For any $s \in (0, 1)$ the result then follows by interpolation. □

4.2 Proof of Theorem 3.1

Due to the quasi-optimality result in (3.1), we will prove Theorem 3.1 by finding discrete vector fields belonging to $\mathbf{X}_h$ and approximating the smooth and singular parts of $\mathbf{u}$ (see (2.2)) such that the approximation errors satisfy the upper bound in (3.2). In the rest of this section, we will write $a \lesssim b$, which means the existence of a generic positive constant C such that $a \le Cb$.

We start with approximating the singular part $\mathbf{u}_{\rm sing}$ in decomposition (2.2). Recalling (2.3)–(2.4), we can write $\mathbf{u}_{\rm sing} = \mathbf{curl}_\Gamma\, w + \mathbf{v}$, where w (resp., $\mathbf{v}$) is represented, similarly to (2.3), as the finite sum of singular functions (resp., singular vector fields).

Approximation of $\mathbf{curl}_\Gamma\, w$. We use Lemma 4.2 for individual singularities in the representation of w to find a function $w_h \in S_1(\Delta_h^\beta)$ such that the norm $\|w - w_h\|_{H^{1/2}(\Gamma)}$ is bounded as in (4.3). Then, recalling the fact that the the operator $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}_\parallel^{-1/2}(\Gamma)$ is continuous (see [12]), we obtain for any $\varepsilon > 0$

$$\begin{aligned}\|\mathbf{curl}_\Gamma\, w - \mathbf{curl}_\Gamma\, w_h\|_{\mathbf{X}} &= \|\mathbf{curl}_\Gamma (w - w_h)\|_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} \\ &\lesssim \|w - w_h\|_{H^{1/2}(\Gamma)} \lesssim h^{\min\{\alpha_0\beta, 3/2\} - \varepsilon}. \end{aligned} \tag{4.5}$$

It is easy to see that $\mathbf{curl}_\Gamma\, w_h \in \mathbf{X}_h$.

Approximation of $\mathbf{v}$. We use the operator $\mathcal{Q}_h$ from Lemma 4.1 to define $\mathbf{v}_h := \mathcal{Q}_h \mathbf{v} \in \mathbf{X}_h$. First, since $\|\cdot\|_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} \lesssim \|\cdot\|_{\mathbf{H}_-^{-1/2}(\Gamma)}$ (see, e.g., [11, Proposition 2.6]), we estimate

$$\begin{aligned}\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{X}} &\simeq \|\mathbf{v} - \mathcal{Q}_h\mathbf{v}\|_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma\, \mathbf{v} - \mathrm{div}_\Gamma\, \mathcal{Q}_h\mathbf{v}\|_{H^{-1/2}(\Gamma)} \\ &\lesssim \|\mathbf{v} - \mathcal{Q}_h\mathbf{v}\|_{\mathbf{H}_-^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma\, \mathbf{v} - \mathrm{div}_\Gamma\, \mathcal{Q}_h\mathbf{v}\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

Then, using the commuting diagram property in (4.1) and estimate (4.2), we deduce

$$\begin{aligned}\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{X}} &\lesssim \|\mathbf{v} - \mathcal{Q}_h\mathbf{v}\|_{\mathbf{H}_-^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma\, \mathbf{v} - \Pi_0(\mathrm{div}_\Gamma\, \mathbf{v})\|_{H^{-1/2}(\Gamma)} \\ &\lesssim h^{1/2-\varepsilon}\|\mathbf{v} - \Pi_{\rm RT}\mathbf{v}\|_{0,\Gamma} + \sum_{F\subset\Gamma} \|\mathrm{div}_\Gamma\, \mathbf{v} - \Pi_0(\mathrm{div}_\Gamma\, \mathbf{v})\|_{\tilde{H}^{-1/2}(F)}.\end{aligned} \tag{4.6}$$

Recall that $\mathbf{v} \in \mathbf{H}^1(\Gamma)$ (see Remark 2.1(i)). Hence, using Theorem 4.1 in [1] elementwise we estimate the L^2-error of the Raviart-Thomas interpolation:[1]

$$\|\mathbf{v} - \Pi_{\rm RT}\mathbf{v}\|_{0,\Gamma} \lesssim h\, \|\mathbf{v}\|_{\mathbf{H}^1(\Gamma)}. \tag{4.7}$$

The $\tilde{H}^{-1/2}$-error of the L^2-projection of $\mathrm{div}_\Gamma\, \mathbf{v}$ is estimated by applying Lemma 4.3 to individual singular vector fields in the representation of $\mathbf{v}$ (on individual faces of Γ). As a result, we obtain for any $\varepsilon > 0$

$$\|\mathrm{div}_\Gamma\, \mathbf{v} - \Pi_0(\mathrm{div}_\Gamma\, \mathbf{v})\|_{\tilde{H}^{-1/2}(F)} \lesssim h^{\min\{\alpha_0\beta, 3/2\} - \varepsilon} \quad \forall\, F \subset \Gamma. \tag{4.8}$$

[1] This also follows from our result in [8, Lemma 7.1] by using a standard Bramble-Hilbert-type argument.

Collecting estimates (4.7) and (4.8) for the corresponding terms in the right-hand side of (4.6), we obtain

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{X}} \lesssim h^{\min\{\alpha_0\beta, 3/2\}-\varepsilon}. \tag{4.9}$$

We will now approximate the regular term in decomposition (2.2).

Approximation of the smooth remainder $\mathbf{u}_{\rm reg}$. Considering enough singularity terms in representation (2.3) we obtain $\mathbf{u}_{\rm reg} \in \mathbf{X}^1$. Then, proceeding in the same way as for the vector field $\mathbf{v}$ above (that is, using Lemma 4.1 and estimating as in (4.6)–(4.7)) we find $\mathbf{u}^h_{\rm reg} := \mathcal{Q}_h \mathbf{u}_{\rm reg} \in \mathbf{X}_h$ such that

$$\|\mathbf{u}_{\rm reg} - \mathbf{u}^h_{\rm reg}\|_{\mathbf{X}} \lesssim h^{3/2-\varepsilon}\, \|\mathbf{u}_{\rm reg}\|_{\mathbf{H}^1(\Gamma)} + \sum_{F\subset\Gamma} \|(Id - \Pi_0)\, \mathrm{div}_\Gamma\, \mathbf{u}_{\rm reg}\|_{\tilde{H}^{-1/2}(F)}. \tag{4.10}$$

The $\tilde{H}^{-1/2}$-norm of the error of the L^2-projection of $\mathrm{div}_\Gamma\, \mathbf{u}_{\rm reg}$ on each face $F \subset \Gamma$ is now estimated as follows. First, we use a standard duality argument to obtain

$$\begin{aligned}
\|(Id - \Pi_0)\, \mathrm{div}_\Gamma\, \mathbf{u}_{\rm reg}\|_{\tilde{H}^{-1/2}(F)} &= \sup_{\varphi\in H^{1/2}(F)\setminus\{0\}} \frac{\big((Id-\Pi_0)\,\mathrm{div}_\Gamma\,\mathbf{u}_{\rm reg}, \varphi\big)_{0,F}}{\|\varphi\|_{H^{1/2}(F)}} \\
&= \sup_{\varphi\in H^{1/2}(F)\setminus\{0\}} \inf_{\varphi_h\in S_0(\Delta^\beta_h(F))} \frac{\big((Id-\Pi_0)\,\mathrm{div}_\Gamma\,\mathbf{u}_{\rm reg}, \varphi-\varphi_h\big)_{0,F}}{\|\varphi\|_{H^{1/2}(F)}} \\
&\lesssim h^{1/2}\, \|(Id-\Pi_0)\,\mathrm{div}_\Gamma\,\mathbf{u}_{\rm reg}\|_{0,F},
\end{aligned} \tag{4.11}$$

where at the last step we applied Lemma 4.4 with $s = 1/2$. Since $\mathrm{div}_\Gamma\, \mathbf{u}_{\rm reg} \in H^1(\Gamma)$, the right-hand side of (4.11) is estimated by applying Theorem 3.1.4 of [17] to obtain

$$\|(Id-\Pi_0)\,\mathrm{div}_\Gamma\,\mathbf{u}_{\rm reg}\|_{\tilde{H}^{-1/2}(F)} \lesssim h^{3/2}\, \|\mathrm{div}_\Gamma\,\mathbf{u}_{\rm reg}\|_{H^1(F)} \quad \forall\, F \subset \Gamma.$$

Hence, we deduce from (4.10)

$$\|\mathbf{u}_{\rm reg} - \mathbf{u}^h_{\rm reg}\|_{\mathbf{X}} \lesssim h^{3/2-\varepsilon}\, \|\mathbf{u}_{\rm reg}\|_{\mathbf{X}^1(\Gamma)}. \tag{4.12}$$

Approximation of the solution $\mathbf{u}$. We use the approximations of $\mathbf{curl}_\Gamma\, w$, $\mathbf{v}$, and $\mathbf{u}_{\rm reg}$ found above to define

$$\widetilde{\mathbf{u}}_h := \mathbf{u}^h_{\rm reg} + \mathbf{curl}_\Gamma\, w_h + \mathbf{v}_h \in \mathbf{X}_h.$$

Then combining estimates (4.5), (4.9), (4.12) and applying the triangle inequality, we derive

$$\|\mathbf{u} - \widetilde{\mathbf{u}}_h\|_{\mathbf{X}} \lesssim h^{\min\{\alpha_0\beta, 3/2\}-\varepsilon}, \qquad \varepsilon > 0. \tag{4.13}$$

Recalling that $\alpha_0 > 1/2$ (see (2.8)), we set the grading parameter to $\beta = \max\left\{\frac{3}{2\alpha_0}, 1\right\} \in [1, 3)$. Then the error estimate in (3.2) immediately follows from (4.13) due to the quasi-optimal convergence (3.1) of the h-BEM on graded meshes Δ^β_h with $\beta \in [1, 3)$. This finishes the proof.

Remark 4.1 *If Γ is a piecewise plane orientable open surface, then Proposition* 3.1 *remains valid (cf.,* [8, Remark 3.1]*). However, open surfaces represent the least regular case, where the solution* $\mathbf{u}$ *to the EFIE exhibits strong singularities at the edges of* Γ *such that* $\gamma_0 = 1/2$ *in* (2.5) *and* (2.7)*. As a result,* $\mathbf{u} \notin \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ *and* $w^s \notin H^1(\Gamma)$*,* $\mathbf{v}^s \notin \mathbf{H}^1(\Gamma)$*,* $\mathrm{div}_\Gamma \mathbf{v}^s \notin L^2(\Gamma)$ *for* $s = e, ev$*. Hence, the proof of Theorem* 3.1 *does not extend trivially to this case. In particular, one cannot apply the operator* $\mathcal{Q}_h$ *from Lemma* 4.1 *to the vector field* $\mathbf{v}$ *(see* (4.6)*) as* $\mathrm{div}_\Gamma \mathbf{v} \notin L^2(\Gamma)$*. It is therefore an open problem, whether a suboptimal convergence rate as in Theorem* 3.1 *can be restored in the case of open surfaces.*

5 Appendix: proof of Lemma 4.1

In this section, we prove Lemma 4.1. First, let us recall the construction of the operator $\mathcal{Q}_h\mathbf{u}$ from [8, Section 7]. For any $\mathbf{u} \in \mathbf{H}^s_-(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ we construct $\mathcal{Q}_h\mathbf{u}$ in the Raviart-Thomas spaces on individual faces of Γ. Let F be a single face of Γ. For the sake of simplicity of notation we will omit the subscript F for differential operators over this face, e.g., we will write div for div_F. We will also write $(\cdot,\cdot)$ for the $L^2(F)$- and $\mathbf{L}^2(F)$-inner products, and similarly $\|\cdot\|$ for the corresponding norms of scalar functions and vector fields.

Given $\mathbf{u} \in \mathbf{H}^s(F) \cap \mathbf{H}(\mathrm{div}, F)$, $s > 0$, we consider the following mixed problem: *Find* $(\mathbf{z}, f) \in \mathbf{H}(\mathrm{div}, F) \times L^2_*(F)$ *such that*

$$\begin{aligned} (\mathbf{z}, \mathbf{v}) + (\mathrm{div}\, \mathbf{v}, f) &= (\mathbf{u}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{div}, F), \\ (\mathrm{div}\, \mathbf{z}, g) &= (\mathrm{div}\, \mathbf{u}, g) && \forall g \in L^2_*(F), \\ \mathbf{z} \cdot \tilde{\mathbf{n}} &= \mathbf{u} \cdot \tilde{\mathbf{n}} && \text{on } \partial F. \end{aligned} \tag{5.1}$$

Here, $L^2_*(F) := \big\{v \in L^2(F);\ (v, 1) = 0\big\}$, $\tilde{\mathbf{n}}$ is the unit outward normal vector to ∂F, and $\mathbf{H}_0(\mathrm{div}, F) := \{\mathbf{v} \in \mathbf{H}(\mathrm{div}, F);\ \mathbf{v} \cdot \tilde{\mathbf{n}}|_{\partial F} = 0\}$.

The unique solvability of (5.1) is proved by standard techniques (see [9, Chapter II]). In fact, it is clear that the pair $(\mathbf{u}, 0)$ solves (5.1).

A conforming Galerkin approximation of problem (5.1) with Raviart-Thomas elements on the graded mesh $\Delta_h^\beta(F)$ reads as: *Find* $(\mathbf{z}_h, f_h) \in \mathbf{X}_h(F) \times R_h(F)$ *such that*

$$\begin{aligned} (\mathbf{z}_h, \mathbf{v}) + (\mathrm{div}\, \mathbf{v}, f_h) &= (\mathbf{u}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F), \\ (\mathrm{div}\, \mathbf{z}_h, g) &= (\mathrm{div}\, \mathbf{u}, g) && \forall g \in R_h(F), \\ \mathbf{z}_h \cdot \tilde{\mathbf{n}} &= \Pi_{\mathrm{RT}}\mathbf{u} \cdot \tilde{\mathbf{n}} && \text{on } \partial F. \end{aligned} \tag{5.2}$$

Here, $\mathbf{X}_h(F)$ denotes the restriction of $\mathbf{X}_h$ onto the face F, and $R_h(F) := \{g \in L^2(F);\ g|_K = \text{const},\ \forall K \in \Delta_h^\beta(F) \text{ and } (g, 1) = 0\}$.

The unique solvability of (5.2) is proved in [8, Section 7]. Note that the third equation in (5.2) implies $(\mathrm{div}(\mathbf{u} - \mathbf{z}_h), 1) = 0$. Hence, the second identity in (5.2) holds for any piecewise constant function $g \in \mathrm{div}\, \mathbf{X}_h(F)$. Thus, $\mathrm{div}\, \mathbf{z}_h$ is the $L^2(F)$-projection of $\mathrm{div}\, \mathbf{u}$ onto $\mathrm{div}\, \mathbf{X}_h(F)$, and therefore,

$$\mathrm{div}\, \mathbf{z}_h = \mathrm{div}\, \Pi_{\mathrm{RT}}\mathbf{u}. \tag{5.3}$$

This fact allows us to prove the following inequality:

$$\|\mathbf{u} - \mathbf{z}_h\| \leq \|\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}\|. \tag{5.4}$$

Indeed, using (5.1), (5.2) (with $\mathbf{v} = \mathbf{z}_h - \Pi_{\mathrm{RT}}\mathbf{u} \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F)$) and recalling that $\mathbf{z} = \mathbf{u}$, $f = 0$, we obtain

$$\begin{aligned} \|\mathbf{u} - \mathbf{z}_h\|^2 &= (\mathbf{u} - \mathbf{z}_h, \mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) - (\mathbf{u} - \mathbf{z}_h, \mathbf{z}_h - \Pi_{\mathrm{RT}}\mathbf{u}) = \\ &= (\mathbf{u} - \mathbf{z}_h, \mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) - (\mathrm{div}\, \mathbf{z}_h - \mathrm{div}\, \Pi_{\mathrm{RT}}\mathbf{u}, f_h) \\ &\overset{(5.3)}{=} (\mathbf{u} - \mathbf{z}_h, \mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}). \end{aligned}$$

Then (5.4) follows by applying the Cauchy-Schwarz inequality.

We now estimate $\|\mathbf{u} - \mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2}(F)}$. One has for any $\varepsilon \in (0, \frac{1}{2})$

$$\|\mathbf{u} - \mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2}(F)} \leq \|\mathbf{u} - \mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2+\varepsilon}(F)} = \sup_{\mathbf{w} \in \mathbf{H}^{1/2-\varepsilon}(F) \setminus \{\mathbf{0}\}} \frac{|(\mathbf{u} - \mathbf{z}_h, \mathbf{w})|}{\|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)}}. \tag{5.5}$$

For a given $\mathbf{w} \in \mathbf{H}^{1/2-\varepsilon}(F)$, we solve the following problem: *Find* $\varphi \in H^1_*(F) := \{\phi \in H^1(F);\ (\phi, 1) = 0\}$ *such that*

$$(\nabla \varphi, \nabla \phi) = -(\mathbf{w}, \nabla \phi) \quad \forall \phi \in H^1_*(F). \tag{5.6}$$

The regularity result for φ reads as

$$\varphi \in H^{3/2-\varepsilon}(F), \quad \|\varphi\|_{H^{3/2-\varepsilon}(F)} \lesssim \|\tilde{f}\|_{(H^{1/2+\varepsilon}(F))'} \lesssim \|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)}, \tag{5.7}$$

where $\tilde{f} \in (H^{1/2+\varepsilon}(F))'$ is defined by $\tilde{f}(\phi) = -(\mathbf{w}, \nabla \phi)$, $\forall \phi \in H^{1/2+\varepsilon}(F)$.

Then we set

$$\mathbf{q} := \mathbf{w} + \nabla \varphi \in \mathbf{H}^{1/2-\varepsilon}(F) \cap \mathbf{H}_0(\mathrm{div}, F). \tag{5.8}$$

It also follows from (5.6) that $\mathrm{div}\, \mathbf{q} = \mathrm{div}\, \mathbf{w} + \mathrm{div}\, \nabla \varphi = 0$. Furthermore, we have by (5.7)–(5.8) that

$$\|\mathbf{q}\|_{\mathbf{H}^{1/2-\varepsilon}(F)} \lesssim \|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)} + \|\varphi\|_{H^{3/2-\varepsilon}(F)} \lesssim \|\mathbf{w}\|_{\mathbf{H}^{1/2-\varepsilon}(F)}. \tag{5.9}$$

We now use (5.8) and integration by parts to represent the numerator in (5.5) as

$$\begin{aligned} (\mathbf{u} - \mathbf{z}_h, \mathbf{w}) &= (\mathbf{u} - \mathbf{z}_h, \mathbf{q}) - (\mathbf{u} - \mathbf{z}_h, \nabla \varphi) \\ &= (\mathbf{u} - \mathbf{z}_h, \mathbf{q}) + (\mathrm{div}\, (\mathbf{u} - \mathbf{z}_h), \varphi) - ((\mathbf{u} - \mathbf{z}_h) \cdot \tilde{\mathbf{n}}, \varphi)_{0,\partial F}. \end{aligned}$$

Hence, using (5.1), (5.2) and recalling that $\mathbf{z} = \mathbf{u}$, $f = 0$, we find for any $\mathbf{q}_h \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F)$

$$\begin{aligned} |(\mathbf{u} - \mathbf{z}_h, \mathbf{w})| &= |(\mathbf{u} - \mathbf{z}_h, \mathbf{q} - \mathbf{q}_h) + (\mathbf{u} - \mathbf{z}_h, \mathbf{q}_h) \\ &\quad + (\mathrm{div}\, (\mathbf{u} - \mathbf{z}_h), \varphi) - ((\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) \cdot \tilde{\mathbf{n}}, \varphi)_{0,\partial F}| \\ &= |(\mathbf{u} - \mathbf{z}_h, \mathbf{q} - \mathbf{q}_h) + (\mathrm{div}\, \mathbf{q}_h, f_h) \\ &\quad + (\mathrm{div}\, (\mathbf{u} - \mathbf{z}_h), \varphi) - ((\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) \cdot \tilde{\mathbf{n}}, \varphi)_{0,\partial F}| \\ &\leq \|\mathbf{u} - \mathbf{z}_h\|\, \|\mathbf{q} - \mathbf{q}_h\| + |(\mathrm{div}\, \mathbf{q}_h, f_h)| + \|\mathrm{div}\, (\mathbf{u} - \mathbf{z}_h)\|_{\tilde{H}^{-1/2}(F)}\, \|\varphi\|_{H^{1/2}(F)} \\ &\quad + \|(\mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u}) \cdot \tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)}\, \|\varphi\|_{H^{1-\varepsilon}(\partial F)}. \end{aligned} \tag{5.10}$$

Let $\Pi_{\rm RT}^{\rm q/u}$ denote the Raviart-Thomas interpolation operator on the 'coarse' quasi-uniform and shape-regular mesh $\Delta_h^{\rm q/u}(F)$ obtained from the graded mesh $\Delta_h^\beta(F)$ by patching together long and thin elements. We also denote by $\Pi_0^{\rm q/u}$ the $L^2(F)$-projector onto the space of piecewise constant functions on $\Delta_h^{\rm q/u}(F)$. Then we set

$$\mathbf{q}_h := \Pi_{\rm RT}^{\rm q/u}\mathbf{q} \in \mathbf{X}_h(F) \cap \mathbf{H}_0(\mathrm{div}, F).$$

By the standard properties of the Raviart-Thomas interpolation on quasi-uniform and shape-regular meshes, we have

$$\mathrm{div}\,\mathbf{q}_h = \Pi_0^{\rm q/u}\mathrm{div}\,\mathbf{q} = 0, \tag{5.11}$$

$$\|\mathbf{q} - \mathbf{q}_h\| \lesssim h^{1/2-\varepsilon}\,\|\mathbf{q}\|_{H^{1/2-\varepsilon}(F)} \overset{(5.9)}{\lesssim} h^{1/2-\varepsilon}\,\|\mathbf{w}\|_{H^{1/2-\varepsilon}(F)}. \tag{5.12}$$

To estimate $\|(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)}$ we recall that $\int_{e_h}(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\cdot\tilde{\mathbf{n}} = 0$ for any element edge $e_h \subset \partial F$. Therefore, we can use a standard duality argument to prove (cf. [10, p. 259])

$$\|(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)} \lesssim \Big(\max_{e_h\subset\partial F}|e_h|\Big)^{1-\varepsilon}\|(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{L^2(\partial F)}.$$

Then by interpolation we obtain

$$\begin{aligned}\|(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1+\varepsilon}(\partial F)} &\lesssim h^{1/2-\varepsilon}\,\|(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1/2}(\partial F)} \\ &\lesssim h^{1/2-\varepsilon}\Big(\|\mathbf{u}-\Pi_{\rm RT}\mathbf{u}\| + \|\mathrm{div}\,(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\|_{\tilde{H}^{-1/2}(F)}\Big),\end{aligned} \tag{5.13}$$

where, at the last step, we used the continuity of the normal trace operator $\mathbf{v}\mapsto\mathbf{v}\cdot\tilde{\mathbf{n}}|_{\partial F}$, see [5, Lemma 2.1].

Furthermore, one has

$$\|\varphi\|_{H^{1-\varepsilon}(\partial F)} \lesssim \|\varphi\|_{H^{3/2-\varepsilon}(F)} \overset{(5.7)}{\lesssim} \|\mathbf{w}\|_{H^{1/2-\varepsilon}(F)}. \tag{5.14}$$

Now, using (5.7), (5.11)–(5.14) in (5.10) and recalling (5.3) and (5.4), we find

$$|(\mathbf{u}-\mathbf{z}_h, \mathbf{w})| \lesssim \Big(h^{1/2-\varepsilon}\,\|\mathbf{u}-\Pi_{\rm RT}\mathbf{u}\| + \|\mathrm{div}\,(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\|_{\tilde{H}^{-1/2}(F)}\Big)\,\|\mathbf{w}\|_{H^{1/2-\varepsilon}(F)}.$$

Using this estimate in (5.5) we obtain

$$\|\mathbf{u}-\mathbf{z}_h\|_{\tilde{\mathbf{H}}^{-1/2}(F)} \lesssim h^{1/2-\varepsilon}\,\|\mathbf{u}-\Pi_{\rm RT}\mathbf{u}\| + \|\mathrm{div}\,(\mathbf{u}-\Pi_{\rm RT}\mathbf{u})\|_{\tilde{H}^{-1/2}(F)}. \tag{5.15}$$

Now we can prove the desired result.

Proof of Lemma 4.1. For any $\mathbf{u}\in\mathbf{H}^s_-(\Gamma)\cap\mathbf{H}(\mathrm{div}_\Gamma,\Gamma)$, we define $\mathcal{Q}_h\mathbf{u}\in\mathbf{X}_h$ face by face as $\mathcal{Q}_h\mathbf{u}|_F := \mathbf{z}_h$ for any face $F\subset\Gamma$, where $\mathbf{z}_h$ is a unique (vectorial) solution to (5.2). Then the commuting diagram property (4.1) follows from the second identity in (5.2) (see also (5.3)), and inequality (5.15) yields estimate (4.2). □

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