Solutions to Problem Sheet 1. MSM3A05/MSM4A05.

Question 1.

(a)

$$\begin{aligned} \sinh z &= \frac{1}{2}(e^z - e^{-z}) = \frac{1}{2}\left(\left(1+z+\frac{z^2}{2!}+\cdots\right) - \left(1-z+\frac{z^2}{2!}-\cdots\right)\right), \\ &= \frac{1}{2}(2z+\cdots) = z+\cdots, \quad \text{hence } m=1 \end{aligned}$$

(b)

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} = \frac{z-\frac{z^3}{3!}+\cdots}{1-\frac{z^2}{2!}+\cdots}, \\ &= z\left(1-\frac{z^2}{3!}+\cdots\right)\left(1-\frac{z^2}{2!}+\cdots\right)^{-1}, \\ &= z\left(1-\frac{z^2}{3!}+\cdots\right)\left(1+\frac{z^2}{2!}+\cdots\right), \\ &= z+O(z^2), \quad \text{hence } m=1\ . \end{aligned}$$

(c) $\sinh\left(\frac{1}{z}\right) = \frac{1}{2}\left(e^{\frac{1}{z}} - e^{-\frac{1}{z}}\right)$ and we have that $e^{\frac{1}{z}}$ is not bounded by any polynomial of z as $z \to 0$. Hence no such value of m exists.

(d)

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \cdots$$

Hence m=0.

(e)

$$\begin{aligned} \cot z &= \frac{\cos z}{\sin z} = \frac{1-\frac{z^2}{2!}+\cdots}{z-\frac{z^3}{3!}+\cdots}, \\ &= \frac{1}{z}\left(1-\frac{z^2}{2!}+\cdots\right)\left(1-\frac{z^2}{3!}+\cdots\right)^{-1}, \\ &= z^{-1}+o(z^{-1}), \quad \text{hence } m\text{= - 1.} \end{aligned}$$

(f)

$$\log(1+z) = z - \frac{z^2}{2!} + \frac{z^3}{3!} - \cdots$$

Hence $m = 1$.

(g)

$$\begin{aligned} \frac{1-\cos z}{1+\cos z} &== \frac{1-\left(1-\frac{z^2}{2!}+\cdots\right)}{1+\left(1-\frac{z^2}{2!}+\cdots\right)}, \\ &= \frac{z^2}{4}\left(1-\frac{z^2}{2!}+\cdots\right)\left(1-\frac{z^2}{4}+\cdots\right)^{-1}, \\ &= \frac{z^2}{4}+o(z^2), \quad \text{hence } m\text{= 2.} \end{aligned}$$

(h)

$$\begin{aligned} e^{-\cosh\left(\frac{1}{z}\right)} &= e^{-\frac{1}{2}(e^{\frac{1}{z}}+e^{-\frac{1}{z}})}, \\ &\sim e^{-\frac{1}{2}(e^{\frac{1}{z}})}. \end{aligned}$$

Hence this function is exponentially small so no such m exists.

Question 2.

In each case we are looking to establish an appropriate limit

1. $$\lim_{z\to 0}\frac{1-\cos^2 z}{z^2}=\lim_{z\to 0}\frac{1-\left(1-\frac{z^2}{2}+\cdots\right)^2}{z^2}=1.$$

2. $$\lim_{z\to 0}\frac{1-\cos^2 z}{z}=\lim_{z\to 0}\frac{1-\left(1-\frac{z^2}{2}+\cdots\right)^2}{z}=0.$$

3. $$\lim_{z\to 0}\frac{\cos z}{z^{-\frac{1}{3}}}=\lim_{z\to 0}\frac{1-\frac{z^2}{2}+\cdots}{z^{-\frac{1}{3}}}=0.$$

4. $$\lim_{z\to\infty}\frac{(\ln z)^2}{z^{\frac{1}{3}}}=\lim_{z\to\infty}\frac{\frac{3}{z}}{z^{-\frac{2}{3}}}=0.$$

5. See notes

6. $$\lim_{z\to\infty}\frac{\sinh z^{-1}}{1}=\lim_{z\to\infty}\sinh z^{-1}=0.$$

Question 3.

We have as $z\to 0$

$$\ln z^{-1}>\ln(\ln z^{-1})>1>z^{\frac{1}{2}}\ln z^{-1}>z^{\frac{1}{2}}>z\ln z^{-1}>z>z^{\frac{3}{2}}>z^2\ln z^{-1}>z^2>e^{-\frac{1}{z}}.$$

Question 4

Show that the following are asymptotic sequences:

1. $$\lim_{z\to\infty}\frac{\left(\sin\dfrac{1}{z}\right)^{n+1}}{\left(\sin\dfrac{1}{z}\right)^{n}}=\lim_{z\to\infty}\left(\sin\frac{1}{z}\right)=0.$$

2. $$\lim_{z\to 0}\frac{\ln(1+z^{1+n})}{\ln(1+z^n)}=\lim_{z\to 0}\frac{z^{n+1}+\frac{1}{2}z^{2n+2}+\cdots}{z^n+\frac{1}{2}z^{2n}+\cdots}=\lim_{z\to 0}z=0.$$

3. $$\lim_{z\to z_0}\frac{(z-z_0)^{n+1}}{(z-z_0)^n}=\lim_{z\to z_0}(z-z_0)=0.$$

Question 5

We have as $\epsilon\to 0$

$$e^{e^{\frac{1}{\epsilon}}}>\epsilon^{-\frac{3}{2}}>\epsilon^{-0.0001}>-\ln\epsilon>\epsilon^{\frac{1}{2}}\ln\epsilon>e^{-\frac{1}{\epsilon}}>5^{-\frac{1}{\epsilon}}.$$

Question 6

Let

$$e^x = a_0 + a_1 \sin x + a_2 \sin^2 x + a_3 \sin^3 x + \cdots$$

then by observation $a_0 = 1$ and

$$a_1 = \lim_{x \to 0} \frac{e^x - 1}{\sin x} = \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1}{x - \frac{x^3}{3!} + \cdots} = 1,$$

and

$$a_2 = \lim_{x \to 0} \frac{e^x - 1 - \sin x}{\sin^2 x} = \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - \left(x - \frac{x^3}{3!} + \cdots\right)}{\left(x - \frac{x^3}{3!} + \cdots\right)^2} = \frac{1}{2},$$

and

$$\begin{aligned} a_3 &= \lim_{x \to 0} \frac{e^x - 1 - \sin x - \frac{1}{2}\sin^2 x}{\sin^3 x} \\ &= \lim_{x \to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - \left(x - \frac{x^3}{3!} + \cdots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2}{\left(x - \frac{x^3}{3!} + \cdots\right)^3} = \frac{1}{3}, \end{aligned}$$

JU 13/10/10.