QUESTION 1. For small ϵ we have the following terms in descending order

$$\epsilon^{-\mu} > \epsilon^{-\nu} > \ln\left(\frac{1}{\epsilon}\right) > \epsilon^{\nu} > \epsilon > \epsilon^{\mu} > e^{-\frac{1}{\epsilon}}.$$

If you're not sure whether one term is bigger or smaller than the other you can always evaluate the limit of the ratio of the two terms as $\epsilon \to 0$ - if you get ∞ then you know that the denominator is smaller than the numerator and if you get 0 then it is vice versa. Also you should know by now that $e^{-\frac{1}{\epsilon}}$ is smaller than any power of ϵ as $\epsilon \to 0$ and clearly negative powers of ϵ are larger than positive powers of ϵ.

QUESTION 2. Let

$$\ln(1+x) = a_0 + a_1 \sin x + a_2 \sin^2 x + a_3 \sin^3 x + \cdots$$

then by observation $a_0 = 0$ and

$$a_1 = \lim_{x\to 0} \frac{\ln(1+x) - 0}{\sin x} = \lim_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots - 0}{x - \frac{x^3}{3!} + \cdots} = 1,$$

and

$$a_2 = \lim_{x\to 0} \frac{\ln(1+x) - \sin x}{\sin^2 x} = \lim_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots - \left(x - \frac{x^3}{3!} + \cdots\right)}{\left(x - \frac{x^3}{3!} + \cdots\right)^2} = -\frac{1}{2},$$

and

$$\begin{aligned} a_3 &= \lim_{x\to 0} \frac{\ln(1+x) - \sin x + \frac{1}{2}\sin^2 x}{\sin^3 x} \\ &= \lim_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots - \left(x - \frac{x^3}{3!} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2}{\left(x - \frac{x^3}{3!} + \cdots\right)^3} = \frac{1}{2}, \end{aligned}$$

and hence

$$\ln(1+x) = \sin x - \frac{1}{2}\sin^2 x + \frac{1}{2}\sin^3 x + \cdots$$

as $x \to 0$.

QUESTION 3. If we let

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{\nu t - z\cosh t} dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{f(t)} dt, \qquad (1)$$

then $f(t) = \nu t - z\cosh t$, $f'(t) = \nu - z\sinh t$, $f''(t) = -z\cosh t$. So $f(t)$ has a unique local maximum at $t = t_{max} = \sinh^{-1}\left(\frac{\nu}{z}\right)$. Using the logarithm expansion for $\sinh^{-1}$ we have

$$t_{\max} = \ln\left(\frac{\nu}{z} + \sqrt{1 + \frac{\nu^2}{z^2}}\right) = \ln\left(\frac{\nu}{z} + \frac{\nu}{z}\left(1 + \frac{z^2}{\nu^2}\right)^{\frac{1}{2}}\right)$$

$$
\begin{aligned}
&\sim \ln\left(\frac{\nu}{z}+\frac{\nu}{z}\left(1+\frac{z^2}{2\nu^2}\right)\right)\\
&\sim \ln\left(\frac{2\nu}{z}+\frac{z}{2\nu}\right)=\ln\left(\frac{2\nu}{z}\right)+\ln\left(1+\frac{z^2}{4\nu^2}\right)\\
&\sim \ln\left(\frac{2\nu}{z}\right)+\frac{z^2}{4\nu^2}
\end{aligned}
$$

as $\nu \to \infty$.
Now using Laplace's method we have

$$
K_\nu(z)=\frac{1}{2}\int_{-\infty}^{\infty} e^{f(t_{max})}e^{\frac{1}{2}f''(t_{max})(t-t_{max})^2}dt. \tag{2}
$$

Also note that $\cosh t_{max} \sim \frac{1}{2}e^{t_{max}} \sim \frac{\nu}{z}+o(1)$ so we have

$$
f(t_{max}) \sim \nu \ln\left(\frac{2\nu}{z}-\nu\right), \qquad f''(t_{max}) \sim -\nu.
$$

Let $\alpha = t - t_{max}$ then substituting into we have

$$
\begin{aligned}
K_\nu(z) &\sim \frac{1}{2}\int_{-\infty}^{\infty} e^{\nu\ln(2\nu/z)-\nu}e^{-\nu\alpha^2/2}d\alpha,\\
&\sim \frac{1}{2}\left(\frac{2\nu}{z}\right)^{\nu} e^{-\nu}\int_{-\infty}^{\infty} e^{-\nu\alpha^2/2}d\alpha.
\end{aligned}
$$

If we now let $\hat{\alpha}=\sqrt{(\nu/2)}\alpha$ so that

$$
\begin{aligned}
K_\nu(z) &\sim \frac{1}{2}\left(\frac{2\nu}{z}\right)^{\nu} e^{-\nu}\sqrt{\frac{2}{\nu}}\int_{-\infty}^{\infty} e^{-\hat{\alpha}^2}d\hat{\alpha},\\
&\sim \sqrt{\frac{\pi}{2\nu}}\left(\frac{2\nu}{z}\right)^{\nu} \quad \text{as} \quad \nu \to \infty.
\end{aligned}
$$

Question 4. Using an extension of Watson's lemma, we have that

$$
\int_0^\infty e^{-sx}\left(1+\frac{is}{5}\right)^{-\frac{1}{2}} ds \sim \frac{1}{x}-\frac{i}{10x^2}.
$$