Solutions to Problem Sheet 2. MSM3A05/MSM4A05.

Question 1.

We wish to express $\sin 2x$ in terms of

$$\left\{ \frac{x}{(1-x^2)^{\frac{1}{2}}}, \frac{x^2}{(1-x^2)^{\frac{1}{2}}}, \frac{x^3}{(1-x^2)^{\frac{1}{2}}}, \cdots \right\}$$

as $x \to 0$.
Expanding the above functions out using the binomial theorem and then expanding out $\sin(2x)$ yields the following result

$$\sin 2x = \frac{2x}{(1-x^2)^{\frac{1}{2}}} - \frac{7x^3}{3(1-x^2)^{\frac{1}{2}}} + \frac{41x^5}{30(1-x^2)^{\frac{1}{2}}} + \cdots$$

Question 2.

The conditions for Watson's lemma are met so we have

$$I = \int_0^4 t^2\sqrt{1+t}\ e^{-xt}dt \sim \int_0^\infty e^{-xt}t^2(1+\frac{1}{2}t - \frac{1}{8}t^2 + \cdots)dt.$$

These individual integrals are easy to evaluate and we have

$$I \sim \frac{\Gamma(3)}{x^3} + \frac{1}{2}\frac{\Gamma(4)}{x^4} - \frac{1}{8}\frac{\Gamma(5)}{x^5} + \cdots$$

Question 3.

The conditions for Watson's lemma are met so we have

$$I = \int_0^3 \frac{1}{\sqrt{t}}\ln(1+t^2)\ e^{-xt}dt \sim \int_0^\infty e^{-xt}\frac{1}{\sqrt{t}}\left(t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \cdots\right)dt.$$

These individual integrals are easy to evaluate and we have

$$I \sim \frac{\Gamma(5/2)}{x^{5/2}} - \frac{1}{2}\frac{\Gamma(9/2)}{x^{9/2}} + \frac{1}{3}\frac{\Gamma(13/2)}{x^{13/2}} + \cdots$$

Question 4.

Here $g(t) = -\cosh t$ and $f(t) = \cosh(\nu t)$. The maximum value of $g(t)$ occurs when $t = 0$ and we have

$$K_\nu(x) \sim \int_0^\infty e^{-x(1+t^2/2)}dt = e^{-x}\int_0^\infty e^{-xt^2/2}dt.$$

If we use the substitution $u = \sqrt{x/2}t$ we have

$$K_\nu(x) \sim \sqrt{\frac{2}{x}}e^{-x}\int_0^\infty e^{-u^2}du = \sqrt{\frac{\pi}{2x}}e^{-x}.$$

Question 5.

The maximum value of $g(t) = -\cosh(t)$ occurs at $t = \pi/4$ and hence we have

$$I = \int_{\pi/4}^{\pi/2} \cos(t)\, e^{-x\cosh t} dt \sim \int_{\pi/4}^{\pi/4+\epsilon} \exp\left(x\left[-\cosh(\pi/4) - \sinh(\pi/4)(t-(\pi/4))\right]\right)\cdot\cos(\pi/4)dt,$$

which can be simplified to

$$I \sim \cos\left(\frac{\pi}{4}\right) e^{-x\cosh(\pi/4)} \int_0^\infty e^{-x\sinh(\pi/4)(t-(\pi/4))} dt.$$

which upon using the substitution $u = \sinh(\pi/4)(t-(\pi/4))$ yields

$$I \sim \cos\left(\frac{\pi}{4}\right) \frac{e^{-x\cosh(\pi/4)}}{\sinh(\pi/4)} \int_0^\infty e^{-ux} du \sim \cos\left(\frac{\pi}{4}\right) \frac{e^{-x\cosh(\pi/4)}}{x\sinh(\pi/4)}.$$

JU 18/10/12.