Revision Notes For Chaos: Summer 2013

Cantor Sets

The Cantor middle-thirds set $\mathcal{C}$ cannot be defined in simple terms. Rather we describe this set as a limiting example of an iterative process. Firstly, we consider the closed unit interval $I = [0, 1]$. To begin the process of generating the Cantor middle-thirds set we use the simple rule
- "Every time we see a closed interval we remove the open middle third interval" thus after the first such 'iteration' we remove $(1/3, 2/3)$ which leaves us with two closed intervals, $[0, 1/3]$ and $[2/3, 1]$. If we denote this as $\mathcal{C}_1 = [0, 1/3] \cup [2/3, 1]$. Next we can think of how $\mathcal{C}_2$ will look like, we need to remove the middle third of each of the two closed intervals from $\mathcal{C}_1$. This leaves us with 4 closed intervals such that $\mathcal{C}_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. We can continue in this fashion so that $\mathcal{C}_3$ will have 2^3 intervals each of length 3^{-3} and so on till we have $\mathcal{C}_n$ having 2^n intervals each of length 3^{-n}. The above process may be thought of a generating two closed intervals (children) from an original closed interval (the parent) at each iteration. The Cantor middle-thirds set $\mathcal{C}$ can be thought of as being the set $\mathcal{C}_n$ as $n \to \infty$.

Logistic Map

Now we consider the Logistic map which has fixed points at $x_1^* = 0$ and $x_2^* = 1 - \frac{1}{\mu}$. We have $F'_\mu(x) = \mu(1 - 2x)$ and so $F'_\mu(x_1^*) = \mu$. Recall that when $|F'(x^*)| < 1$ we have stability and when $|F'(x^*)| > 1$ we have instability. Therefore x_1^* is stable for $\mu \in (0, 1)$ and unstable for $\mu \in (1, 4)$. What about x_2^*? In this case we have $F'_\mu(x_2^*) = 2 - \mu$ and so x_2^* is likewise stable for $\mu \in (1, 3)$ and unstable for $\mu \in (3, 4)$.

Now let is investigate whether a 2-cycle or orbit with period-2 exists. Lets assume that they do exists and let us call them p and q such that $F_\mu(p) = q$ and $F_\mu(q) = p$. Consequently, we must have $F_\mu^2(p) = F_\mu(F_\mu(p)) = F_\mu(q) = p$. We therefore require a solution to $F_\mu^2(x) - x = 0$ which is a quartic given by

$$\mu^2 x(1-x)(1-\mu x(1-x)) = \mu x \left(x - \frac{(\mu-1)}{\mu} \right) \left(\mu^2 x^2 - \mu(1+\mu)x + (1+\mu) \right) = 0.$$

This leads to 2-cycles

$$p, q = \frac{(\mu+1) \pm \sqrt{(\mu-3)(\mu+1)}}{2\mu}.$$

We note that the equation we needed to solve was quartic and thus it will have **four** solutions in general - however we already know that there are two fixed points to this map and so these will **appear** to be like period-2 orbits - just like impostors - they are not period-2 orbits. However, knowing that they are solutions of the equation allows us to find the two period-2 orbits. Once we have found these period-2 orbits we need to assess their stability. We can do this by using our criterion for stability. The **multiplier**

$$\lambda = F_\mu^{(2)'}(p) = \frac{d}{dx} \left(F_\mu(F_\mu(x)) \right)_{x=p} = F'_\mu(F_\mu(p)) F'_\mu(p) = F'_\mu(q) F'_\mu(p).$$

We need to check whether $|\lambda| < 1$ or > 1. The algebra can be made easier by writing $p = a + b$ and $q = a - b$ and noting that

$$\begin{aligned}\lambda = \mu^2(1-2p)*(1-2q) &= \mu^2(1-2p-2q+4pq) = \mu^2(1-2a-2b-2a+2b+4(a^2-b^2)) \\ &= \mu^2(1-4a+4a^2-4b^2) \\ &= \mu^2\left(1-\frac{2(\mu+1)}{\mu}+\frac{(\mu+1)^2}{\mu^2}-(\mu-3)(\mu+1)\right) \\ &= \mu^2-2\mu^2-2\mu+\mu^2+2\mu+1-\mu^2+2\mu+3 = -\mu^2+2\mu+4.\end{aligned}$$

We have stability when $|4+2\mu-\mu^2|<1$ which is given by $3<\mu<1+\sqrt{6}$.

The Feigenbaum number is defined as

$$\delta = \lim_{n\to\infty}\frac{\mu_n-\mu_{n-1}}{\mu_{n+1}-\mu_n}.$$

The Feigenbaum number is the limiting ratio of each bifurcation interval to the next between every period doubling, of a one-parameter map. It was shown to hold for all one-dimensional maps with a single quadratic maximum. As a consequence of this generality, every chaotic system that corresponds to this description will bifurcate at the same rate.

Sensitivity to initial conditions

We consider a general map $f(x)$ and consider two initial points x_0 and $x_0+\delta x_0$ where $0<\delta\ll 1$. Assuming that $f(x)$ is sufficiently well behaved we may use Taylor's theorem to write

$$f(x_0+\delta x_0)\sim f(x_0)+\delta x_0 f'(x_0).$$

If we have $\{x_n\}_{n\geq 0}=\{f^{(n)}(x_0)\}_{n\geq 0}$ then we have

$$\begin{aligned}f(x_0+\delta x_0) &\sim f(x_0)+\delta x_0 f'(x_0) \\ &\sim x_1++\delta x_0 f'(x_0),\end{aligned}$$

and

$$\begin{aligned}f^{(2)}(x_0+\delta x_0) &\sim f(x_1+\delta x_0 f'(x_0)) \\ &\sim f(x_1)+\delta x_0 f'(x_0)f'(x_1), \\ &\sim x_2+\delta x_0 f'(x_0)f'(x_1)\end{aligned}$$

and similarly,

$$\begin{aligned}f^{(3)}(x_0+\delta x_0) &\sim f(x_2+\delta x_0 f'(x_0)f'(x_1)) \\ &\sim f(x_2)+\delta x_0 f'(x_0)f'(x_1)f'(x_2), \\ &\sim x_3+\delta x_0 f'(x_0)f'(x_1)f'(x_2).\end{aligned}$$

Consequently, we have

$$f^{(n)}(x_0+\delta x_0)\sim x_n+\delta x_0 f'(x_0)f'(x_1)f'(x_2)\cdots f'(x_{n-1}),$$

and so we have

$$\left|f^{(n)}(x_0+\delta x_0)-x_n\right|\sim|\delta x_0|\cdot|f'(x_0)|\cdot|f'(x_1)|\cdots|f'(x_{n-1})|. \tag{1}$$

The above equation demonstrates an important point about why $f'(x)$ is so crucial in determining what happens to the difference between orbits - since after the first orbit the the distance grows by $|f'(x_0)|$ and after two iterations by $|f'(x_0)f'(x_1)|$. Imagine now that we are interested in determining the behaviour of points starting close to period-k orbits. Let us assume that we have the following orbit $\{x_1, x_2, \cdots x_k\}$. We may then apply the Taylor expansion to $f^{(k)}(x)$ for x near one of the points in the period-k orbit say x_1. The rate at which the distance grows by after k iterations is then $|f^{(k)'}(x_1)|$. However, this is the same value that would be obtained had we started from any other point in the period-k orbit (Why?). If we are interested in the *average* rate of increase then we can obtain this by taking the kth root of $|f^{(k)'}(x_1)|$ which may be written as

$$\left|f^{(k)'}(x_1)\right|^{\frac{1}{k}} = \left|\prod_{j=1}^{k} f'(x_j)\right|^{\frac{1}{k}} .$$

In plain english we would say that since $|f'(x_j)|$ is the factor at which points infinitesimally close to each other separate after one iteration the above equation states that the average factor of separation per iteration of a complete k-cycle is the geometric mean of the individual factor of separations making up the period-k orbit.

The ideas above lead to a more notion of how the difference between two points separate. If we let $\{x_j\}_{j\geq 0} = \{f^j(x_0)\}_{J\geq 0}$ be the orbit of x_0 under the map $f(x)$ then the Lyapunov[1] number $L(x_0)$ is defined as

$$L(x_0) = \lim_{n\to\infty}\left|\prod_{j=0}^{n-1} f'(x_j)\right|^{\frac{1}{n}},$$

provided this limit exists. The Lyapunov exponent $\lambda(x_0)$ of the orbit x_0 under $f(x)$ is defined as

$$\lambda(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\ln\left|f'(x_j)\right|,$$

provided this limit exists. It should be clear that $L(x_0) = e^{\lambda(x_0)}$ and a little thought demonstrates that if $\lambda(x_0) > 0$ then distances between two nearby points under the map $f(x)$ get stretched out whereas if $\lambda(x_0) < 0$ then a contraction occurs and the rate at which these occur depend on the magnitude of $\lambda(x_0)$.

Definition of Chaos

There are many definitions of chaos in the literature and this course would become very cumbersome if we attempted to go through all of these and make a point of understanding the finer details. The definitions with which we started this course; namely that chaos has three properties : nonlinearity, deterministic and sensitive dependence will now be formalised in a little more detail. For a map $F : I \to I$ where I is an interval then, f is said to be chaotic if

- f is transitive.
- The set of periodic points P is dense in I.
- f has sensitive dependence on initial conditions.

We will not look into detail at what all these concepts mean. We will just touch briefly on them.

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[1] Aleksandr Mikhailovich Lyapunov, 1857 - 1918, Lyapunov is known for his development of the stability theory of a dynamical system, as well as for his many contributions to mathematical physics and probability theory. Incidentally, he shot himself!!!

Components of Chaos

Transitive

A formal definition of transitivity of maps is given by

> A map f is topologically transitive if for any pair of nonempty open intervals J_1 and J_2 in I there exists a positive integer k such that $f^{(k)}(J_1) \cap J_2 \neq 0$.

In everyday terminology this translates to saying that under a transitive map a point in I wanders all over I and its orbit gets as close as we wish to every other point in I.

Density of a set of periodic points

> A set A is dense in I if for any $x \in I$ any open interval containing x must intersect A. Specifically, for each $\delta > 0$ the open interval $J = (x - \delta, x + \delta)$ contains a point in A.

We now show that the set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$.
To prove this we consider $x \in \mathbb{R}$ and we write it in its decimal expansion

$$x = \sum_{n=0}^{\infty} \frac{d_n}{10^n},$$

where $d_n = \{0, 1, 2....9\}$. Let $\delta > 0$ then there exists a positive number m such that $10^{-m} < \delta$. Then consider the rational number

$$y = \sum_{n=0}^{m} \frac{d_n}{10^n}.$$

Then

$$|x - y| = \sum_{n=m+1}^{\infty} \frac{d_n}{10^n} \leq \sum_{n=m+1}^{\infty} \frac{9}{10^n} = \frac{\frac{9}{10^{m+2}}}{1 - \frac{1}{10}} = \frac{1}{10^m}.$$

Hence $|x - y| < \delta$. This proves that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Now, consider the Tent map

$$T(x) = \begin{cases} 2x; & 0 \leq x < \frac{1}{2} \\ 2(1-x); & \frac{1}{2} < x \leq 1. \end{cases}$$

We will show that a point $a \in (0, 1)$ is eventually periodic under T if and only if it is rational.

Part 1 : If $a \in (0, 1)$ and $a \in \mathbb{Q}$ then it is eventually periodic under T.
We begin by letting $a = \frac{p}{q}$ be in its reduced form. There are two cases to consider. Firstly, when $q = 2k + 1$ is an odd integer. In this case we have

$$T^n\left(\frac{p}{q}\right) = \frac{\text{even integer}}{q},$$

for all $n \in \mathcal{Z}^+$. Since there are only k numbers in the interval $[0, 1]$ of the form $\frac{\text{even integer}}{q}$, namely, $\frac{2}{q}$, $\frac{4}{q}$, $\frac{6}{q}$..., $\frac{2k}{q}$. Hence the orbit of a has at most k elements and so it is eventually periodic. In the second case when we have $q = 2k$ as an even integer then a little thought reveals that for a positive integer m we must either have

- $T^m(a) = \frac{\text{integer}}{\text{odd integer}}$, which has been considered above, or
- $T^m(a) = 1$ and thus $T^{m+n}(a) = 0$ for all $n \in \mathcal{Z}^+$ which means that a is eventually periodic.

Lets try to understand the above argument heuristically. Lets assume we start of the with a number of the form

$$\frac{\text{integer}}{q} = \frac{\text{integer}}{2k},$$

then after one iteration of the Tent Map we will have either

$$\frac{2^*\text{integer}}{2k} = \frac{\text{integer}}{k},$$

or

$$2\left(1 - \frac{\text{integer}}{2k}\right) = \left(\frac{2k - \text{integer}}{k}\right)$$

Remember both of these will be less than 1 and if k is odd then we are done - that is the first statement mentioned above is true, i.e., $T^m(a) = \frac{\text{integer}}{\text{odd integer}}$. If not, then k must be even so let $k = 2f$ then applying the Tent Map again yields a similar thing to the above except with a f appearing and a factor of 2 in the numerators. This may be continued indefinitely until you reach an odd number in the denominator or you end up with a number equal to 1 which then leads to the second statement above. Ultimately, a is eventually periodic.

Part 2 : If $a \in \mathbb{R}$ is eventually periodic under T then $a \in \mathbb{Q}$.

Next we consider the case where a is eventually periodic and show that it must be rational. In general we must have

$$T^n(a) = t_n \pm 2^n a,$$

for some positive integer t_n. Since we have assumed that a is eventually periodic we must have $T^n(a) = T^{n+k}(a)$ for some positive integer k. Thus

$$t_{n+k} \pm 2^{n+k} a = t_n \pm 2^n a \qquad \Rightarrow \qquad a = \frac{t_{n+k} - t_n}{\pm 2^n \mp 2^{n+k}},$$

which shows that a is rational.

Lyapunov Stability

Consider a point x_0 and a neighbouring point $x_0 + \delta$. After n iterations of the map f let the error be defined as

$$e_n = |f^n(x_0 + \delta) - f(x_0)|.$$

We expect the relative error

$$\left|\frac{e_n}{\delta}\right| = \frac{|f^n(x_0 + \delta) - f(x_0)|}{\delta},$$

to grow exponentially with n such that

$$e^{n\lambda} = \lim_{\delta \to 0} \frac{e_n}{\delta} = \lim_{\delta \to 0} \frac{|f^n(x_0 + \delta) - f(x_0)|}{\delta},$$

for some $\delta > 0$. Hence

$$e^{\lambda n} = \left|\frac{d}{dx} f^n(x_0)\right| = \left|f'(x_0) f'(x(1)) f'(x(2)) \ldots . f'(x(n-1))\right|.$$

We therefore have, the Lyapunov exponent,

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| f'(x(k)) \right|.$$

Let $x_0 \in (0,1)$. Evaluate $\lambda(x_0)$ for the Tent map presented above. We have that $|T'(x(k))| = 2$ and so

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln 2 = \ln 2.$$

Sensitive Dependence

A map of the interval I is said to be possess sensitive dependence on initial conditions if there exists a $M > 0$ such that for any $x_0 inI$ and $\delta > 0$ there exists $y_0 \in (x_0 - \delta, x_0 + \delta)$ and a positive integer k such that

$$\left| f^{(k)}(x_0) - f^{(k)}(y_0) \right| \geq M,$$

where the number M is usually called the sensitivity constant of f.